Determination of Pinger Location

A. Problem Statement

Figure 1 shows four hydrophone locations, marked by points 0 through 3, and one pinger emitter, marked by point P. The objective is to determine the location of point P based on the time difference that a sound pulse arrives at each of the hydrophone points.

A coordinate system is established with its origin at point 0. Point 1 lies along the *x* axis and point 2 lies in the *xy* plane. The coordinates of points 0 through 3 are assumed known. The distance between point P and point *i* is written as d_i. The detailed problem statement is now presented as follows:

Given:

- coordinate system with origin at point 0, point 1 along *x* axis, and point 2 in *xy* plane
- coordinates of points 0 through 3, all measured in the reference coordinate system,
- the difference in distance from point P to point *i*, *i* =1..3, as compared to the distance of point P to point 0, i.e. δ_i where $d_i = d_0 + \delta_i$, $i = 1..3$.

Find:

- all possible locations for point P

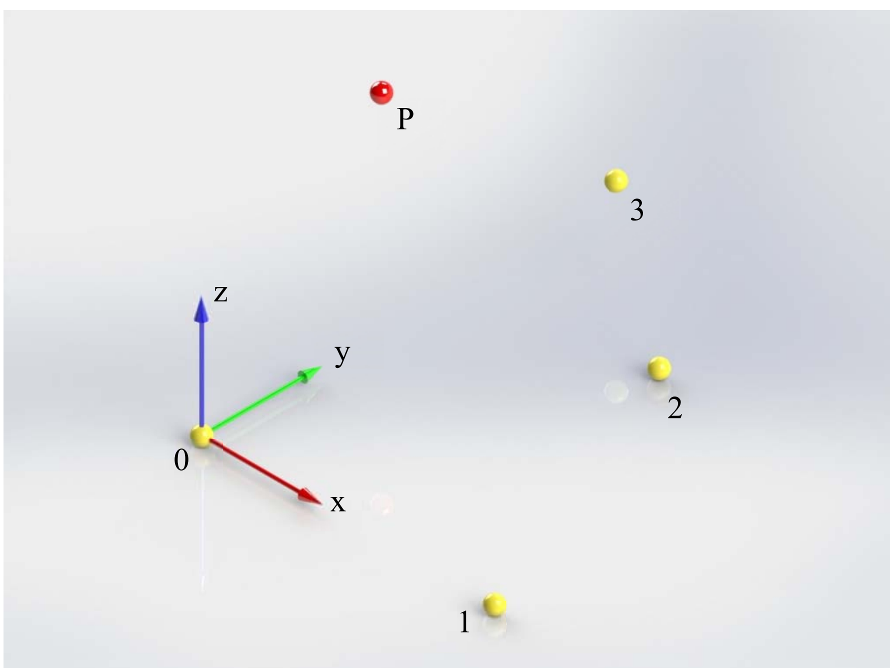

Fig. 1: Definition of Coordinate System

B. Problem Formulation

The coordinates for the four hydrophone points may be written as

$$P_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, P_1 = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix}, P_2 = \begin{bmatrix} x_2 \\ y_2 \\ 0 \end{bmatrix}, P_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}. \tag{1}$$

The coordinates of point P will be written as

$$P_P = \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{2}$$

The equations for spheres centered at 0, 1, 2 and 3 may be written as

$$x^2 + y^2 + z^2 = d_0^{\ 2} \tag{3}$$

$$(x - x_1)^2 + y^2 + z^2 = \left(d_0 + \delta_1\right)^2 \ . \tag{4}$$

$$(x - x_2)^2 + (y - y_2)^2 + z^2 = \left(d_0 + \delta_2\right)^2 \ . \tag{5}$$

$$(x - x_3)^2 + (y - y_3)^2 + (z - z_3)^2 = \left(d_0 + \delta_3\right)^2 \ . \tag{6}$$

Subtracting (3) from (4) gives

$$(x - x_1)^2 - x^2 = \left(d_0 + \delta_1\right)^2 - d_0^{\ 2} \ . \tag{7}$$

Solving this equation for d_0 gives

$$d_0 = x\left(\frac{-x_1}{\delta_1}\right) + \left(\frac{x_1^{\ 2} - \delta_1^{\ 2}}{2\delta_1}\right) \ . \tag{8}$$

Subtracting (3) from (5) and solving for d_0 gives

$$d_0 = x\left(\frac{-x_2}{\delta_2}\right) + y\left(\frac{-y_2}{\delta_2}\right) + \left(\frac{x_2^{\ 2} + y_2^{\ 2} - \delta_2^{\ 2}}{2\delta_2}\right) \ . \tag{9}$$

Subtracting (3) from (6) and solving for d_0 gives

$$d_0 = x\left(\frac{-x_3}{\delta_3}\right) + y\left(\frac{-y_3}{\delta_3}\right) + z\left(\frac{-z_3}{\delta_3}\right) + \left(\frac{x_3^{\ 2} + y_3^{\ 2} + z_3^{\ 2} - \delta_3^{\ 2}}{2\delta_3}\right) \ . \tag{10}$$

Equating equations (8) and (9) and regrouping gives

$$A_1 x + B_1 y + D_1 = 0 \tag{11}$$

where

$$
\begin{aligned}
A_1 &= \frac{x_1}{\delta_1} - \frac{x_2}{\delta_2}, \\
B_1 &= -\frac{y_2}{\delta_2}, \\
D_1 &= \frac{x_2^{\,2} + y_2^{\,2} - \delta_2^{\,2}}{2\delta_2} - \frac{x_1^{\,2} - \delta_1^{\,2}}{2\delta_1}.
\end{aligned}
\tag{12}
$$

Equating equations (8) and (10) and regrouping gives

$$
A_2 x + B_2 y + C_2 z + D_2 = 0 \tag{13}
$$

where

$$
\begin{aligned}
A_2 &= \frac{x_1}{\delta_1} - \frac{x_3}{\delta_3}, \\
B_2 &= -\frac{y_3}{\delta_3}, \\
C_2 &= -\frac{z_3}{\delta_3}, \\
D_2 &= \frac{x_3^{\,2} + y_3^{\,2} + z_3^{\,2} - \delta_3^{\,2}}{2\delta_3} - \frac{x_1^{\,2} - \delta_1^{\,2}}{2\delta_1}.
\end{aligned}
\tag{14}
$$

Equations (11) and (13) represent two equations in the three variables x, y, and z.

A third equation will be obtained by squaring (8) to obtain an expression for d_0^2 which is then substituted into (3). This gives

$$
x^2\left(1 - \frac{x_1^{\,2}}{\delta_1^{\,2}}\right) + x\left(\frac{x_1^{\,3}}{\delta_1^{\,2}} - x_1\right) + y^2 + z^2 + \left(\frac{x_1^{\,2}}{2} - \frac{\delta_1^{\,2}}{4} - \frac{x_1^{\,4}}{4\delta_1^{\,2}}\right) = 0 \tag{15}
$$

Equation (11) can be solved for y and this expression substituted into (13) and (15) to give

$$
Q_1 z + Q_2 = 0 \tag{16}
$$

where

$$
\begin{aligned}
Q_1 &= -\frac{z_3}{\delta_3}, \\
Q_2 &= Q_{2a} x + Q_{2b} \\
&\quad Q_{2a} = \frac{A_1 y_3}{B_1 \delta_3} - \frac{x_3}{\delta_3} + \frac{x_1}{\delta_1} \\
&\quad Q_{2b} = \frac{\delta_1^{\,2} - x_1^{\,2}}{2\delta_1} + \frac{D_1 y_3}{B_1 \delta_3} - \frac{\delta_3^{\,2} - x_3^{\,2} - y_3^{\,2} - z_3^{\,2}}{2\delta_3}
\end{aligned}
\tag{17}
$$

and

$$z^2 + R_1 = 0 \tag{18}$$

where

$$\begin{aligned} R_1 &= R_{1a}x^2 + R_{1b}x + R_{1c} , \\ R_{1a} &= \frac{A_1^2}{B_1^2} + 1 - \frac{x_1^2}{\delta_1^2} \\ R_{1b} &= \frac{x_1^3}{\delta_1^2} - x_1 + \frac{2A_1D_1}{B_1^2} \\ R_{1c} &= \frac{x_1^2}{2} - \frac{\delta_1^2}{4} + \frac{D_1^2}{B_1^2} - \frac{x_1^4}{4\delta_1^2} \end{aligned} \tag{19}$$

Multiplying (16) by *z* gives

$$Q_1 z^2 + Q_2 z = 0 \ . \tag{20}$$

Equations (16), (18), and (20) can be written in matrix form as

$$\begin{bmatrix} 0 & Q_1 & Q_2 \\ 1 & 0 & R_1 \\ Q_1 & Q_2 & 0 \end{bmatrix} \begin{bmatrix} z^2 \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} . \tag{21}$$

A common solution for *z* will exist only if the three equations are linearly dependent and thus

$$\begin{vmatrix} 0 & Q_1 & Q_2 \\ 1 & 0 & R_1 \\ Q_1 & Q_2 & 0 \end{vmatrix} = 0 \ . \tag{22}$$

Expanding the determinant gives a 2nd order polynomial in *x* which is written as

$$\left(Q_1^2 R_{1a} + Q_{2a}^2\right)x^2 + \left(Q_1^2 R_{1b} + 2Q_{2a}Q_{2b}\right)x + \left(Q_1^2 R_{1c} + Q_{2b}^2\right) = 0 \ . \tag{23}$$

Corresponding values for *z* can be obtained from (16). Corresponding values for *y* can be obtained from (11).

C. Numerical Example

The following data is given:

- $P_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 8.5 \\ 0 \\ 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 5.65 \\ 4.75 \\ 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} -1.75 \\ 3.5 \\ 1.75 \end{bmatrix}$ in
- $\delta_1 = 2.053896$, $\delta_2 = -1.349004$, $\delta_3 = -2.817121$ in

The terms Q_1, Q_{2a}, Q_{2b}, R_{1a}, R_{1b}, and R_{1c} are calculated from (17) and (19) as

$$Q_1 = 0.6212, \quad Q_{2a} = 0.5792, \; Q_{2b} = -5.6832$$

$$R_{1a} = -10.5347, \; R_{1b} = 88.6141, \; R_{1c} = -169.2806 \; .$$

Equation (23) is evaluated as

$$-3.7297\, x^2 + 27.6117\, x - 33.0256 = 0 \; .$$

The two solutions for x are

$$x_1 = 1.500$$

$$x_2 = 5.9031 \; .$$

Corresponding values for y and z are obtained from (11) and (16) as

$$y_1 = 6.7000$$

$$y_2 = -3.7125$$

$$z_1 = 7.7500$$

$$z_2 = 3.6443 \; .$$

Thus the two possible locations for the pinger are

$$P_{P1} = \begin{bmatrix} 1.5 \\ 6.7 \\ 7.75 \end{bmatrix} \text{ in} \; ,$$

$$P_{P2} = \begin{bmatrix} 5.903120445 \\ -3.712519821 \\ 3.644346747 \end{bmatrix} \text{ in} \; .$$

The results are checked by evaluating the distances of the proposed locations for point P from the four hydrophone locations to see if the difference in distances to the points matches the input values.

For case 1, the distances from the point P_1 to each of the hydrophone points are calculated as

$$\mathrm{d}_0 = 10.353864\,, \quad \mathrm{d}_1 = 12.407760\,, \quad \mathrm{d}_2 = 9.004860\,, \quad \mathrm{d}_3 = 7.536743 \; .$$

The differences $d_i - d_0$, $i = 1..3$ do match the input values.

For case 2, the distances from the point P_2 to each of the hydrophone points are calculated as

$$\mathrm{d}_0 = 7.868348\,, \quad \mathrm{d}_1 = 5.814452\,, \quad \mathrm{d}_2 = 9.217352\,, \quad \mathrm{d}_3 = 10.685469 \; .$$

The differences $d_i - d_0$, $i = 1..3$ appear to equal the negative of the input values. The reason for this is that equation (15) was obtained by squaring and equating (3) and (8). Thus a constraint has been set on d_0^2, not d_0. For case 2, the input differences $d_i - d_0$, $i = 1..3$ are satisfied if the distances from point P_2 to each of the hydrophone points are evaluated as their negative, i.e.

$$d_0 = -7.868348\,, \quad d_1 = -5.814452\,, \quad d_2 = -9.217352\,, \quad d_3 = -10.685469\ .$$

Each of the two solutions for the coordinates for point P can be readily checked to see if it is indeed a valid solution for the pinger location.