1. **Vector Autoregression (VAR)**

    a. *Description:*

The Vector Autoregression (VAR) method models the next step in each time series using an AR model. It is the generalization of AR to multiple parallel time series.

Like the autoregressive model, each variable has an equation modelling its evolution over time. This equation includes the variable's **lagged (past) values**, the **lagged values of the other variables in the model**, and an **error term**. The only prior knowledge required is a list of variables which can be hypothesized to affect each other over time.

    b. *Parameters and equations:*

- A VAR model describes the evolution of a set of k variables, called endogenous variables, over time. Each period of time is numbered, t = 1, ..., T. The variables are collected in a vector, yt, which is of length k.
- The vector is modelled as a linear function of its previous value. The vector's components are referred to as y(i,t), meaning the observation at time t of the i-th variable. For example, if the first variable in the model measures the price of wheat over time, then y(1,1998) would indicate the price of wheat in the year 1998.
- The past values of the series are used to forecast the current and future.
- A typical AR(p) model equation looks something like this:

$$Y_t = \alpha + \beta_1 Y_{t-1} + \beta_2 Y_{t-2} + \cdots + \beta_p Y_{t-p} + \epsilon_t$$

- where α is the intercept, a constant and β1, β2 till βp are the coefficients of the lags of Y till order p.
- Order 'p' means, up to p-lags of Y is used and they are the predictors in the equation. The ε_{t} is the error, which is considered as white noise.
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    c. *A simple example:*

A VAR(1) in two variables can be written in matrix form (more compact notation) as

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \end{bmatrix} + \begin{bmatrix} e_{1,t} \\ e_{2,t} \end{bmatrix},$$

Equations:

$$y_{1,t} = c_1 + a_{1,1} y_{1,t-1} + a_{1,2} y_{2,t-1} + e_{1,t}$$

$$y_{2,t} = c_2 + a_{2,1} y_{1,t-1} + a_{2,2} y_{2,t-1} + e_{2,t}.$$

Each variable in the model has one equation. The current (time t) observation of each variable depends on its own lagged values as well as on the lagged values of each other variable in the VAR.

d. GitHub / Code snippet

Python API:
https://www.statsmodels.org/dev/generated/statsmodels.tsa.vector_ar.var_model.VAR.html

2. **Vector Autoregression Moving-Average (VARMA)**

    a. *Description*:

The Vector Autoregression Moving-Average (VARMA) method models the next step in each time series using an ARMA model. It is the generalization of ARMA to multiple parallel time series.

Given a time series of data Xt , the ARMA model is a tool for understanding and predicting future values in this series. The **AR part involves regressing the variable on its own lagged** (i.e., past) **values**. The **MA part involves modeling the error term as a linear combination of error terms** occurring contemporaneously and at various times in the past.

    b. *Parameters and equations:*

The general form of a VARMA(p,q) process with VAR order p and MA order q is

$$\mathbf{y}_t = \mathbf{a}_0 + \mathbf{A}_1\mathbf{y}_{t-1} + \cdots + \mathbf{A}_p\mathbf{y}_{t-p} + \mathbf{u}_t + \Theta_1\mathbf{u}_{t-1} + \Theta_2\mathbf{u}_{t-2} + \cdots + \Theta_q\mathbf{u}_{t-q},$$

where ε_t is a white noise process with nonsingular covariance matrix Σ_u.

mu is the mean of the series.

Thetas are the parameters of the model.

    c. *GitHub / Code snippet:*

Documentation of a VARMAX Python library:
https://www.statsmodels.org/dev/generated/statsmodels.tsa.statespace.varmax.VARMAX.html

3. **Vector Autoregression Moving-Average with Exogenous Regressors (VARMAX)**

    Description:

The Vector Autoregression Moving-Average with Exogenous Regressors (VARMAX) is an extension of the VARMA model that also includes the modeling of exogenous variables.

Exogenous variables are also called covariates and can be thought of as parallel input sequences that have observations at the same time steps as the original series. The primary series(es) are referred to as endogenous data to contrast it from the exogenous sequence(s). The observations for exogenous variables are included in the model directly at each time step and are not modeled in the same way as the primary endogenous sequence (e.g. as an AR, MA, etc. process).

The method is suitable for multivariate time series without trend and seasonal components with exogenous variables.

GitHub / Code snippet

Documentation of a VARMAX Python library:

https://www.statsmodels.org/dev/generated/statsmodels.tsa.statespace.varmax.VARMAX.html