时间序列 hw4

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Pb 3.5

假定AR(2)方程形如$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + W_t$, 需证:

$\phi(z) = 1 - \phi_1 z - \phi_2 z^2$的两根$|z_1| > 1, |z_2| > 1 \Leftrightarrow \phi_1 + \phi_2 < 1, \quad \phi_2 - \phi_1 < 1,$ 且 $|\phi_2| < 1$

1. $\Rightarrow$:

$$
\begin{aligned}
& \phi(z) = 1 - \phi_1 z - \phi_2 z^2 = (1 - z_1^{-1} z)(1 - z_2^{-1} z) \\
\Rightarrow \quad & \begin{cases} \phi_1 = z_1^{-1} + z_2^{-1}, \\ \phi_2 = -(z_1 z_2)^{-1} \end{cases} \\
\therefore \quad & |\phi_2| = \frac{1}{|z_1 z_2|} < 1;
\end{aligned}
$$

$$
\begin{aligned}
& \phi(1) = (1 - z_1^{-1})(1 - z_2^{-1}) = \frac{(z_1 - 1)(z_2 - 1)}{z_1 z_2} \\
& \phi(-1) = (1 + z_1^{-1})(1 - z_2^{-1}) = \frac{(z_1 + 1)(z_2 + 1)}{z_1 z_2}
\end{aligned}
$$

$$
\begin{aligned}
& case1 : z_1, z_2 \text{为实根}, \ i.e. \ z_i > 1 \ or \ z_i < -1 \\
\Rightarrow \quad & \frac{z_i - 1}{z_i} > 0, \frac{z_i + 1}{z_i} > 0; \\
\Rightarrow \quad & \phi(-1) = 1 + \phi_1 - \phi_2 > 0, \phi(1) = 1 - \phi_1 - \phi_2 > 0
\end{aligned}
$$

$$
\begin{aligned}
& case1 : z_1, z_2 \text{为共轭复根}, \ i.e. \ z_1 = \overline{z_2} \\
\Rightarrow \quad & z_1 \pm 1, z_2 \pm 1 \text{也为共轭复根} \\
\Rightarrow \quad & \phi(-1), \phi(1) \text{分子分母均} > 0, \\
\Rightarrow \quad & \phi(-1) = 1 + \phi_1 - \phi_2 > 0, \phi(1) = 1 - \phi_1 - \phi_2 > 0
\end{aligned}
$$

2. $\Leftarrow$:

$$case1: z_1, z_2\text{为共轭复根}$$

$$\text{由韦达定理得}: z_1 z_2 = \frac{1}{-\phi_2} \Rightarrow |z_1 z_2| = \frac{1}{|\phi_2|} > 1$$

$$\therefore |z_i| = \sqrt{|z_1 z_2|} > 1$$

$$case2: z_1, z_2\text{为实根}(\phi(0) = 1\text{因而} \neq 0)$$

$$\phi(1) = 1 - \phi_1 - \phi_2 = \frac{(z_1 - 1)(z_2 - 1)}{z_1 z_2} > 0$$

$$\phi(-1) = 1 + \phi_1 - \phi_2 = \frac{(z_1 + 1)(z_2 - 1)}{z_1 z_2} > 0$$

$$\therefore \frac{(z_1 - 1)(z_2 - 1)}{z_1 z_2} / \frac{(z_1 + 1)(z_2 + 1)}{z_1 z_2} > 0$$

$$\Rightarrow \frac{(z_1 - 1)(z_2 - 1)}{(z_1 + 1)(z_2 + 1)} > 0$$

$$\text{假设}\frac{z_i - 1}{z_i + 1} < 0, \text{则} z_i \in (-1, 1)$$

$$|z_1 z_2| = \frac{1}{|\phi_2|} \in (-1, 1)\text{与}|\phi_2| < 1\text{矛盾},$$

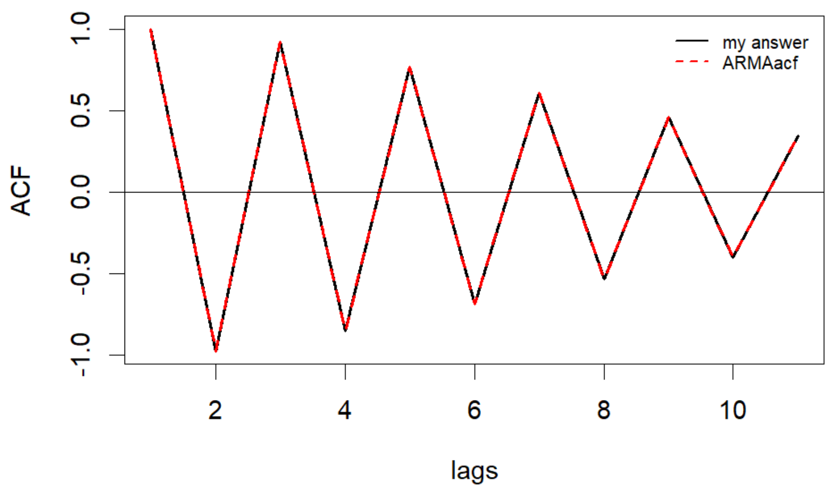
$$\therefore \frac{z_i - 1}{z_i + 1} > 0, \text{则}|z_i| > 1.$$

Pb 3.7

a

$$
\begin{aligned}
& \phi(z) && = 1 + 1.6z + 0.64z^2 = (1+0.8z)^2 \\
\therefore \quad & \phi(z) && \text{有两个相同实根} z = -1.25, \\
\Rightarrow \quad & \rho(h) && = (-1.25)^{-h}(c_1 + c_2 h) \\
& \rho(h) && = \begin{cases} 1, & h = 0 \\ \frac{\phi_1}{1-\phi_2} = \frac{-1.6}{1+0.64}, & h = \pm 1 \end{cases} \\
\Rightarrow \quad & \text{参数} && \begin{cases} c_1 & = 1 \\ c_2 & = \frac{1.6*1.25}{1.64} - 1 \approx 0.2195 \end{cases} \\
\Rightarrow \quad & \rho(h) && = (-1.25)^{-h}(1 + 0.2195h)
\end{aligned}
$$

```
x=c(0:10)
y<-(-1.25)**(-x)*(1+(1.25*1.6/1.64-1)*x)
plot.ts(y,ylab='ACF',xlab='lags',lwd=2)
lines(ARMAacf(c(-1.6,-0.64),lag.max = 10 ),col='red',lty=2,lwd=2)
legend('topright', inset=.01,bty='n', legend=c("my answer","ARMAacf"),lty=c(1,
2),cex=0.7,col=c('black','red'),lwd=c(1.5,1.5))
abline(h=0)
```

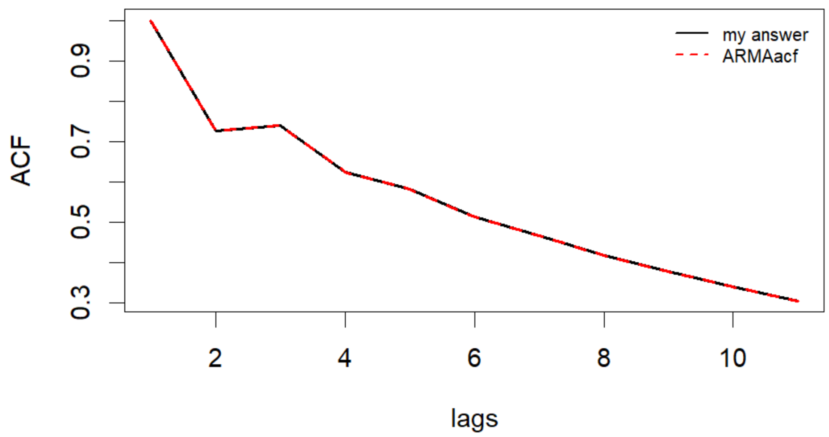


从图像上看检查计算ACF结果正确。

b

$$
\begin{aligned}
& & \phi(z) &= 1-0.4z-0.45z^2=(1-0.9z)(1+0.5z) \\
&\therefore & \phi(z) & \text{有两个不同实根} z_1=1/0.9, z_2=-2, \\
&\Rightarrow & \rho(h) &= c_1 0.9^h + c_2(-0.5)^h \\
& & \rho(h) &= \begin{cases} 1, & h=0 \\ \frac{\phi_1}{1-\phi_2}=\frac{0.4}{1-0.45}, & h=\pm 1 \end{cases} \\
&\Rightarrow & \text{参数} & \begin{cases} c_1+c_2 &= 1 \\ 0.9c_1-0.5c_2 &= \frac{8}{11} \end{cases} \Rightarrow \begin{cases} c_1 &= 0.8766 \\ c_2 &= 0.1234 \end{cases} \\
&\Rightarrow & \rho(h) &= 0.8766\cdot 0.9^h + 0.1234\cdot(-0.5)^h
\end{aligned}
$$

```
x=c(0:10)
y<-0.8766*0.9**x+0.1234*(-0.5)**x
plot.ts(y,ylab='ACF',xlab='lags',lwd=2)
lines(ARMAacf(c(0.4,0.45),lag.max = 10 ),col='red',lty=2,lwd=2)
legend('topright', inset=.01,bty='n', legend=c("my answer","ARMAacf"),lty=c(1,
2),cex=0.7,col=c('black','red'),lwd=c(1.5,1.5))
```

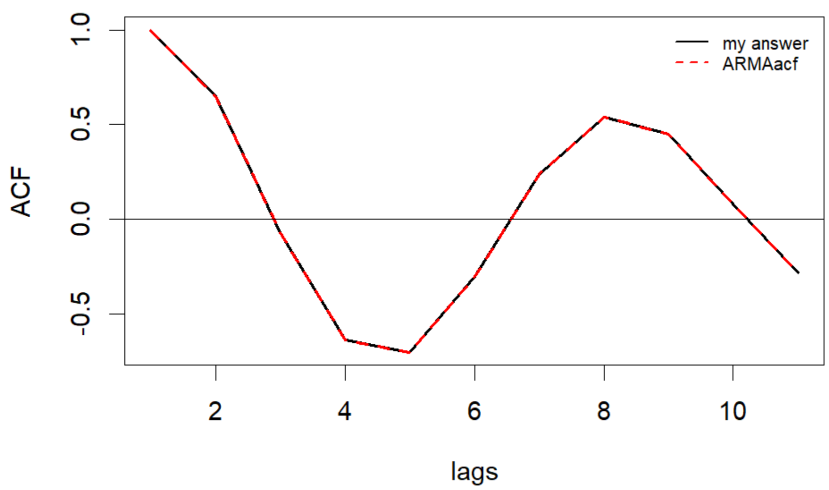


从图像上看检查计算ACF结果正确。

c

$$
\begin{aligned}
& \phi(z) && = 1 - 1.2z + 0.85z^2 = (\tfrac{0.6}{\sqrt{0.85}} - \sqrt{0.85}z)^2 + \tfrac{0.49}{0.85} \\
\therefore\quad & \phi(z) && \text{有虚根} z = \tfrac{0.6}{0.85} \pm \tfrac{0.7}{0.85} i, \text{令} z_1 = \tfrac{0.6}{0.85} + \tfrac{0.7}{0.85} i \\
\Rightarrow\quad & \rho(h) && = c_1 z_1^{-h} + \overline{c_1}(\overline{z_1})^{-h} \\
& \rho(h) && = \begin{cases} 1, & h = 0 \\ \frac{\phi_1}{1-\phi_2} = \frac{1.2}{1+0.85}, & h = \pm 1 \end{cases} \\
\Rightarrow\quad & \text{参数} && \begin{cases} 2Re(c_1) & = 1 \\ 2Re(c_1 z_1) & = \frac{1.2}{1.85} \end{cases} \Rightarrow \begin{cases} Re(c_1) & = 0.5 \\ Im(c_1) & = 0.045/1.295 \end{cases} \\
\Rightarrow\quad & \rho(h) && = (0.5 + \tfrac{0.045}{1.295} i) z_1^{-h} + (0.5 - \tfrac{0.045}{1.295} i)(\overline{z_1})^{-h}
\end{aligned}
$$

```r
x=c(0:10)
y<-(0.5+0.045/1.295i)*(6/8.5+7/8.5i)**(-x)+(0.5-0.045/1.295i)*(6/8.5-7/8.5i)**(-x)
plot.ts(y,ylab='ACF',xlab='lags',lwd=2)
lines(ARMAacf(c(1.2,-0.85),lag.max = 10 ),col='red',lty=2,lwd=2)
legend('topright', inset=.01,bty='n', legend=c("my answer","ARMAacf"),lty=c(1,
2),cex=0.7,col=c('black','red'),lwd=c(1.5,1.5))
abline(h=0)
```



从图像上看检查计算ACF结果正确。

Pb 3.8

i

$x_t = \phi x_{t-1} + \theta w_{t-1} + w_t, \text{ where } |\phi| < 1$
$\because |\phi| < 1$
$\therefore$ 序列$causal.\ x_t = \sum_{k=0}^{\infty} \psi_k w_{t-k}$
$\therefore (1-\phi z)(\psi_0 + \psi_1 z + \cdots) = 1 + \theta z$
$\therefore \psi_0 = 1; \psi_1 = \phi + \theta; \psi_2 = \phi(\phi+\theta);$
另外$E(x_{t+k} w_t) = \begin{cases} \psi_k \sigma_w^2 & k \geq 0 \\ 0 & k < 0 \end{cases}$

$E(x_t) = 0$
$\gamma(1) = E(x_t x_{t-1}) = \phi\gamma(0) + \theta E(w_{t-1} x_{t-1}) + E(w_t x_{t-1})$
$\gamma(1) = \phi\gamma(0) + \psi_0 \theta \sigma_w^2$

$\gamma(0) = E(x_t x_t) = E(x_t(\phi x_{t-1} + \theta w_{t-1} + w_t)) = \phi\gamma(-1) + \theta E(w_{t-1} x_t) + E(w_t x_t)$
$\gamma(0) = \phi\gamma(1) + \theta\sigma_w^2 \psi_1 + \psi_0 \sigma_w^2$

解得$\gamma(0) = \sigma_w^2 \frac{1+2\theta\phi+\theta^2}{1-\phi^2} \quad \text{and} \quad \gamma(1) = \sigma_w^2 \frac{(1+\theta\phi)(\phi+\theta)}{1-\phi^2}$

当$h > 1$:
$\gamma(h) = E(x_t x_{t-h}) = E((\phi x_{t-1} + \theta w_{t-1} + w_t) x_{t-h}) = \phi\gamma(h-1) + 0$
$\gamma(h) = \gamma(1) \cdot \phi^{h-1} = \sigma_w^2 \frac{(1+\theta\phi)(\phi+\theta)}{1-\phi^2} \phi^{h-1}$

$\therefore \rho(h) = \gamma(h)/\gamma(0) = \frac{(1+\theta\phi)(\phi+\theta)}{1+2\theta\phi+\theta^2} \phi^{h-1}, \quad h \geq 1$

ii

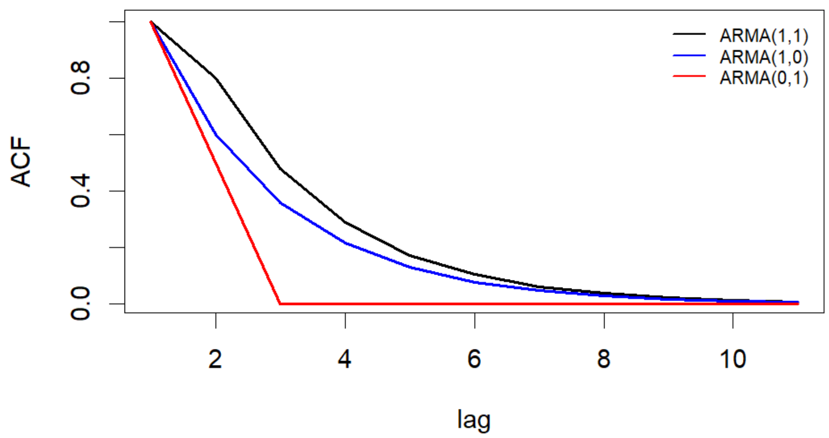
$$ARMA(1,0) : (\theta = 0)$$
$$\rho(h) = \phi^h, \quad h \geq 1$$

$$ARMA(0,1) : (\phi = 0)$$
$$\rho(h) = \frac{\theta}{1+\theta^2} \phi^{h-1}, \quad h \geq 1$$

ARMA(1,0)与ARMA(0,1)是ARMA(1,1)中某个参数取零的特殊情况.

iii

```
plot.ts(ARMAacf(0.6,0.9,lag.max = 10 ),col='black',lwd=2,ylab='ACF',xlab='lag')
lines(ARMAacf(ar=0.6,lag.max = 10 ),col='blue',lwd=2)
lines(ARMAacf(ma=0.9,lag.max = 10 ),col='red',lwd=2)
legend('topright', inset=.01,bty='n',
legend=c("ARMA(1,1)","ARMA(1,0)","ARMA(0,1)"),,cex=0.7,col=c('black','blue','red'),lwd=c(1.5,1
.5,1.5))
```



从图上可见ACF诊断能力由强到弱依次为ARMA(0,1), ARMA(1,0), ARMA(1,1). 定性的话，ARMA(0,1)诊断能力强而后两者诊断能力弱。

2

2. In Example 3.10 case (iii) of Shumway's book, when the AR(2) polynomial has two conjugate complex roots, prove the following result.

   (a) The constants c_1, c_2 in general solution form should satisfy $c_2 = \bar{c}_1$.

   (b) Write z_1 in polar coordinates, show that

   $$\rho(h) = a|z_1|^{-h}\cos(h\theta + b).$$

   where real constants a, b are to be determined by initial conditions.

   (c) Then repeat Example 3.11, display the ACF plot to see the exponential decay pattern with sinusoid pattern.

a

反证法.假设$c_2 \neq \bar{c}_1$, 也即$c_1 = a + bi, c_2 = d + ei$, a=d和b=-e不同时满足:

当h=0: $\rho(0) = c_1 + c_2 \in R \Rightarrow b = -e$;

当h=1: $\rho(1) = c_1 z_1^{-1} + c_2 z_2^{-1} = \frac{c_1 z_2 + c_2 z_1}{z_1 z_2}$

令$z_1 = x + yi, z_2 = x - yi$,则:

$$\begin{aligned}
& \rho(1) z_1 z_2 = (ax + by) + (bx - ay)i + (dx - ey) + (dy + ex)i \in R \\
\Rightarrow\quad & bx - ay + dy + ex = 0 \\
\Rightarrow\quad & (b + e)x - (a - d)y = 0 \\
\because\quad & b = -e \text{且该通解形势下} y \neq 0 \\
\therefore\quad & a = d
\end{aligned}$$

产生矛盾,原假设错误,因此$c_2 = \bar{c}_1$.

b

设$z_1 = x + yi, z_2 = x - yi$,

则$z_1 = |z_1|e^{i\theta}, z_2 = |z_1|e^{-i\theta}, \quad \theta = argcos(\frac{x}{\sqrt{x^2+y^2}})$

进而设$c_1 = a + di, c_2 = a - di$,

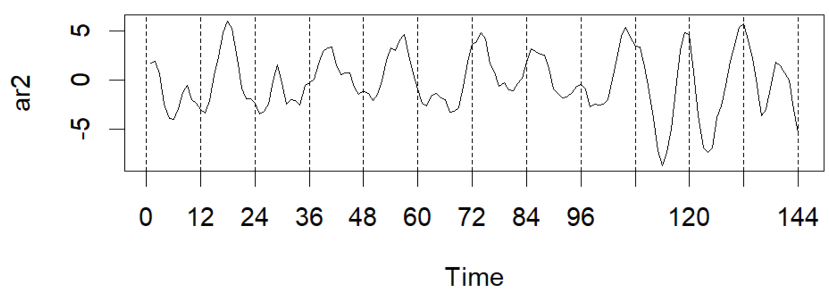
则$c_1 = Ke^{ib}, c_2 = Ke^{-ib}, \quad K = |c_1|, b = argcos(\frac{a}{\sqrt{a^2+d^2}})$

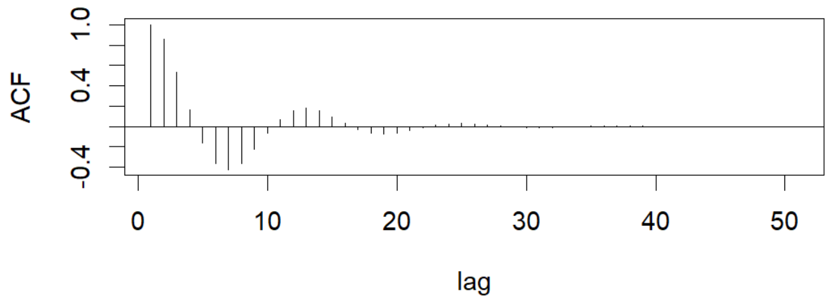
因此$\rho(h) = K|z_1|e^{i(\theta h)+ib} + K|z_1|e^{-i(\theta h)-ib} = 2K|z_1|^{-h}\cos(h\theta + b)$

从而只需求出c_1即可算出h和b的值,而c_1 可从初始条件中求出,得证.

c

```r
z = c(1,-1.5,.75) # coefficients of the polynomial
(a = polyroot(z)[1]) # print one root = 1 + i/sqrt(3)
par(mfrow=c(2,1))
arg = Arg(a)/(2*pi) # arg in cycles/pt
set.seed(8675309)
ar2 = arima.sim(list(order=c(2,0,0), ar=c(1.5,-.75)), n = 144)
plot(ar2, axes=FALSE, xlab="Time")
axis(2); axis(1, at=seq(0,144,by=12)); box()
abline(v=seq(0,144,by=12), lty=2)
ACF = ARMAacf(ar=c(1.5,-.75), ma=0, 50)
plot(ACF, type="h", xlab="lag")
abline(h=0)
```





ACF在幅度方面始终衰减;而衰减的同时体现出了比较强的余弦周期性.