时间序列 HW06

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反证：假定 Γ_n 正定 且 Γ_{n+1} 非正定. $\{X_t\}$ 为经中心化的平稳序列. $\mathbf{X}_n \triangleq (X_1, \cdots X_n)'$

则 $\forall\, b=(b_1,\cdots b_n)' \neq 0$. s.t. $E(b'\mathbf{X}_n)^2 = b'\Gamma_n b \neq 0$. $\Rightarrow b'\mathbf{X}_n \neq 0$.

且 $\exists\, a=(a_1,\cdots,a_n,a_{n+1})' \neq 0$. s.t. $E(a'\mathbf{X}_{n+1})^2 = a'\Gamma_{n+1}a = 0$. $\Rightarrow a'\mathbf{X}_{n+1} = 0$.

$\Rightarrow$ X_{n+1} 可被 $\mathbf{X}_n$ 线性表示. 而 $\{X_t\}$ 平稳 $\Rightarrow$ X_{n+k+1} 可被 $\mathbf{X}_{n+k}$ 线性表示

$\Rightarrow$ X_{n+k} 可被 $\mathbf{X}_n$ 线性表示. $\forall k \geq 1$.

不妨设 $X_{n+k} = \alpha'\mathbf{X}_n$. $\alpha' = (\alpha_1, \cdots, \alpha_n)$. $\alpha \neq 0$.

①: $\gamma(0) = Cov(X_{n+k}, X_{n+k}) = E(\alpha'\mathbf{X}_n)^2 = \alpha'\Gamma_n\alpha$

$\geq \lambda_1 \cdot \|\alpha\|_2^2 > 0$ λ_1 为 Γ_n 最小特征值.

②: $\gamma(0) = Cov(\alpha'\mathbf{X}_n, X_{n+k}) = E(\alpha'\mathbf{X}_n X_{n+k}) = \alpha' \begin{bmatrix} \gamma(n+k-1) \\ \vdots \\ \gamma(k-1) \end{bmatrix}$

加入 $\gamma(h) \xrightarrow{h\to\infty} 0$ 条件. s.t. $\gamma(0) \to 0$. 与 ① $\gamma(0) > 0$ 矛盾. 原假设错误

$\therefore$ 若 Γ_n 正定 $\Rightarrow$ Γ_{n+1} 正定

又 $\Gamma_1 = \gamma(0) > 0$ 正定 $\Rightarrow$ Γ_n 正定 $\forall n \in N^*$

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$$\frac{\Gamma_n \varphi_n}{\gamma(0)} = \frac{\gamma_n}{\gamma(0)} \Rightarrow \begin{bmatrix} R_{n-1} & \tilde{\rho}_{n-1} \\ \tilde{\rho}_{n-1}' & \rho(0) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_{nn} \end{bmatrix} = \begin{bmatrix} \rho_{n-1} \\ \rho(h) \end{bmatrix}$$

$$\begin{cases} R_{h-1}\cdot\alpha_1 + \tilde{\rho}_{h-1}\cdot\alpha_{hh} = \rho_{h-1} & ① \\ \tilde{\rho}_{h-1}'\cdot\alpha_1 + \rho(0)\cdot\alpha_{hh} = \rho(h) & ② \end{cases}$$

① : $\alpha_1 = R_{h-1}^{-1}(\rho_{h-1} - \tilde{\rho}_{h-1}\alpha_{hh})$

② : $\tilde{\rho}_{h-1}' R_{h-1}^{-1}\rho_{h-1} - \tilde{\rho}_{h-1}' R_{h-1}^{-1}\tilde{\rho}_{h-1}\alpha_{hh} + \rho(0)\alpha_{hh} = \rho(h)$

$$\Rightarrow \alpha_{hh} = \frac{\rho(h) - \tilde{\rho}_{h-1}' R_{h-1}^{-1}\rho_{h-1}}{\rho(0) - \tilde{\rho}_{h-1}' R_{h-1}^{-1}\tilde{\rho}_{h-1}} . \quad \#$$

$$\left.\begin{aligned} \hat{X}_t &= \sum_{i=1}^{h-1} a_i X_{t-i} \\ \hat{X}_{t-h} &= \sum_{i=1}^{h-1} b_i X_{t-i} \end{aligned}\right\} \Rightarrow a = \tilde{b} \quad 且 \quad \left.\begin{aligned} &\min E[\varepsilon_t^2] \\ &\min E[\delta_{t-h}^2] \end{aligned}\right\} \Rightarrow \begin{aligned} \Gamma_{h-1} a &= \gamma_{h-1} \\ \Gamma_{h-1} b &= \tilde{\gamma}_{h-1} \end{aligned}$$

$$E(\varepsilon_t^2) = E[(X_t - \sum_{i=1}^{h-1} a_i X_{t-i})^2] = E[(1,-a')\tilde{X}\cdot\tilde{X}'\cdot\begin{bmatrix}1\\-a\end{bmatrix}] = (1,-a)\Gamma_h(1,-a)'$$

$$= (1,-a')\begin{bmatrix}\gamma(0) & \gamma_{h-1}' \\ \gamma_{h-1} & \Gamma_{h-1}\end{bmatrix}\cdot\begin{bmatrix}1\\-a\end{bmatrix}$$

$$= (\gamma(0) - a'\gamma_{h-1}, \gamma_{h-1}' - a'\Gamma_{h-1})\begin{bmatrix}1\\-a\end{bmatrix}$$

$$= \gamma(0) - a'\gamma_{h-1} - \gamma_{h-1}'a + a'\Gamma_{h-1}a \quad \Leftarrow (a = \Gamma_{h-1}^{-1}\gamma_{h-1})$$

$$= \gamma(0) - \gamma_{h-1}'\Gamma_{h-1}^{-1}\gamma_{h-1} = E(\delta_{t-h}^2)$$

$$\varepsilon_t = (1,-a')\begin{bmatrix}X_t\\ | \\ X_{t-h+1}\end{bmatrix} \qquad \delta_{t-h} = (\tilde{a}', 1)\cdot\begin{bmatrix}X_{t-1}\\ \vdots \\ X_{t-h}\end{bmatrix}$$

$$\Rightarrow Cov(\varepsilon_t, \delta_{t-h}) = (1,-a')\cdot E\{\begin{bmatrix}X_t\\ | \\ X_{t-h+1}\end{bmatrix}\cdot[X_{t-1}, -X_{t-h}]\}\cdot\begin{bmatrix}-\tilde{a}\\1\end{bmatrix}$$

$$= (1,-a')\cdot E\{\begin{bmatrix}X_t\\ \tilde{X}_{h-1}\end{bmatrix}[\tilde{X}_{h-1}', X_{t-h}]\}\cdot\begin{bmatrix}-\tilde{a}\\1\end{bmatrix}$$

$$= (1,-a')\cdot\begin{bmatrix}\gamma_{h-1}' & \gamma(h) \\ \Gamma_{h-1} & \gamma_{h-1}\end{bmatrix}\begin{bmatrix}-\tilde{a}\\1\end{bmatrix} \quad \Leftarrow a = \Gamma_{h-1}^{-1}\gamma_{h-1},\ \tilde{a} = \Gamma_{h-1}^{-1}\tilde{\gamma}_{h-1}$$

$$= (1, \gamma_{h-1}'\Gamma_{h-1}^{-1})\cdot\begin{bmatrix}\gamma_{h-1}' & \gamma(h) \\ \Gamma_{h-1} & \gamma_{h-1}\end{bmatrix}\cdot\begin{bmatrix}-\Gamma_{h-1}^{-1}\cdot\tilde{\gamma}_{h-1}\\1\end{bmatrix}$$

$$= \gamma(h) - \gamma_{h-1}'\Gamma_{h-1}^{-1}\tilde{\gamma}_{h-1}$$

$$\therefore \phi_{hh} = \frac{Cov(\varepsilon_t, \delta_{t-h})}{\sqrt{E(\varepsilon_t^2)\cdot E(\delta_{t-h}^2)}} = \frac{\gamma(h) - \gamma_{h-1}'\Gamma_{h-1}^{-1}\tilde{\gamma}_{h-1}}{\gamma(0) - \gamma_{h-1}'\Gamma_{h-1}^{-1}\gamma_{h-1}} = \frac{\gamma(h) - \tilde{\gamma}_{h-1}'\Gamma_{h-1}^{-1}\gamma_{h-1}}{\gamma(0) - \tilde{\gamma}_{h-1}'\Gamma_{h-1}^{-1}\tilde{\gamma}_{h-1}} \Leftarrow \begin{cases}\tilde{\gamma}_{h-1}'a_{h-1} = \gamma_{h-1}'\tilde{a}_{h-1} \\ \tilde{\gamma}_{h-1}'\tilde{a}_{h-1} = \gamma_{h-1}'a_{h-1}\end{cases}$$

$$= \frac{\rho(h) - \tilde{\rho}_{h-1}'\Gamma_{h-1}^{-1}\rho_{h-1}}{\rho(0) - \tilde{\rho}_{h-1}'\Gamma_{h-1}^{-1}\tilde{\rho}_{h-1}} \quad \#$$

3.14

(a). $MSE = E[(y-g(x))^2]$

$= E[(y-E(y|x)+E(y|x)-g(x))^2]$

$= E[(y-E(y|x))^2] + E[(E(y|x)-g(x))^2] + 2E[(y-E(y|x))(E(y|x)-g(x))]$

第三项 $= E_X E_{Y|X}[(y-E(y|x))(E(y|x)-g(x))|x]$

$= E_X\{E_{Y|X}[(y|x-E(y|x))]\cdot(E(y|x)-g(x))\}$

$= E_X\{0\cdot(E(y|x)-g(x))\} = 0.$

第一项与 $g(x)$ 无关

$\therefore \arg\min\limits_g MSE = \arg\min\limits_g E[(E(y|x)-g(x))^2] = E(y|x).$

i.e. $g(x) = E(y|x)$. #

(b.) $X, Z \overset{iid}{\sim} N(0,1)$

$g(x) = E(y|x) = E(x^2+z|x) = E(x^2|x) + E(z|x) = X^2$

$MSE = E[(y-g(x))^2] = E[z^2] = Var\, z = 1$

(c). $MSE = E[(y-g(x))^2] = E[(y-a-bx)^2]$

$$\begin{cases} \dfrac{\partial MSE}{\partial a} = E[2(y-a-bx)] = -2[Ey-a-bEX] = 0 & ① \\ \dfrac{\partial MSE}{\partial b} = E[-2x(y-a-bx)] = -2[E(XY)-aEX-bE(X^2)] = 0 & ② \end{cases}$$

①: $a = Ey - bEX$. 代入②

②: $bEX^2 - E(XY) + (Ey-bEX)EX = 0.$

$\Rightarrow b = \dfrac{E(XY)-EXEY}{E(X^2)-(EX)^2} = \dfrac{cov(X,Y)}{Var(X)}$

若 $y = x^2 + z$ $\mathrm{cov}(x, y) = Ex^3 = 0$ $\mathrm{Var}\,x = 1$ $\Rightarrow b = 0$ (如果没有(b)中y的形式感觉做不了)

$\Rightarrow a = Ey - bEx = E(x^2 + z) - 0 = E(x^2) = 1$

$\Rightarrow MSE = E[(y-1)^2] = E[y^2] - 2Ey + 1 = E[(x^2+z)^2] - 1$

$= E[x^4 + 2x^2 z + z^2] - 1$

$= E(x^4) + 2E(x^2)E(z) + E(z^2) - 1$

由 $M_x(\theta) = e^{\frac{\theta^2}{2}} = (1 + \frac{\theta^2}{2} + \frac{1}{2!}\cdot\frac{\theta^4}{4} + \cdots)$

得 $E(x^4) = M_x^{(4)}(0) = 3$

$\therefore MSE = 3 + 0 + 1 - 1 = 3.$