时间序列hw5

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1

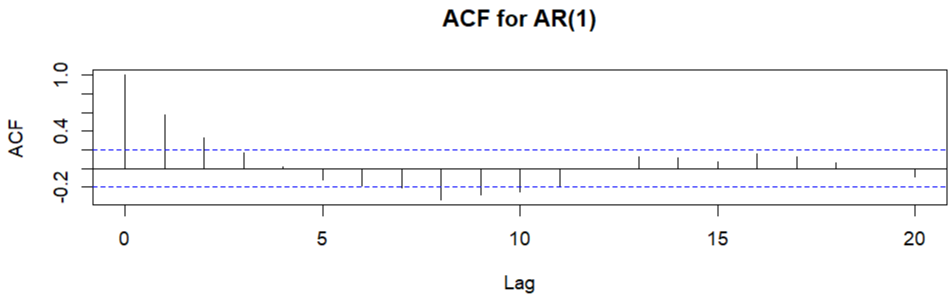
Generate

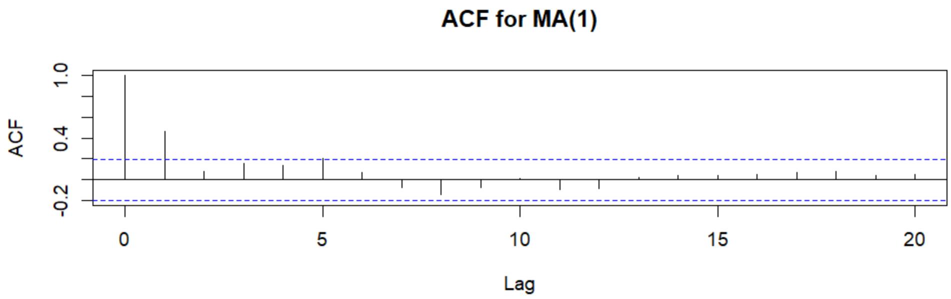
```r
n <- 100
ar <- arima.sim(list(order=c(1,0,0), ar=c(0.6)), n=n)
ma <- arima.sim(list(order=c(0,0,1), ma=c(0.9)), n=n)
arma <- arima.sim(list(order=c(1,0,1), ar=c(0.6), ma=c(0.9)), n=n)
```

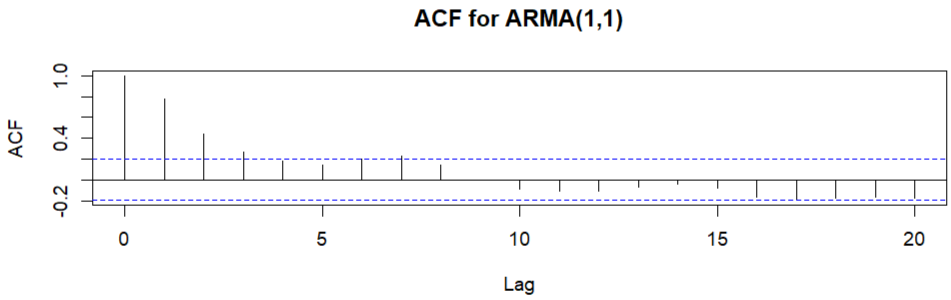
参数均小于1, 因此模型causal 和 invertible.

ACF

```r
par(mfrow=c(3,1))
acf(ar,main='ACF for AR(1)')
acf(ma,main='ACF for MA(1)')
acf(arma,main='ACF for ARMA(1,1)')
```







AR(1):$\rho(h)=\phi^h, \quad h\geq 0$

指数衰减趋于0, 符合理论值.

MA(1):$\rho(0)=1; \quad \rho(1)=\frac{\theta}{1+\theta^2}; \quad \rho(h)=0, \quad h\geq 2$

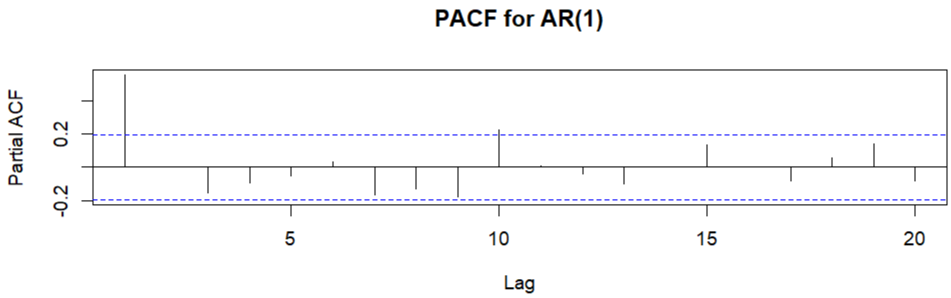
只在0和1处显著,符合理论值.

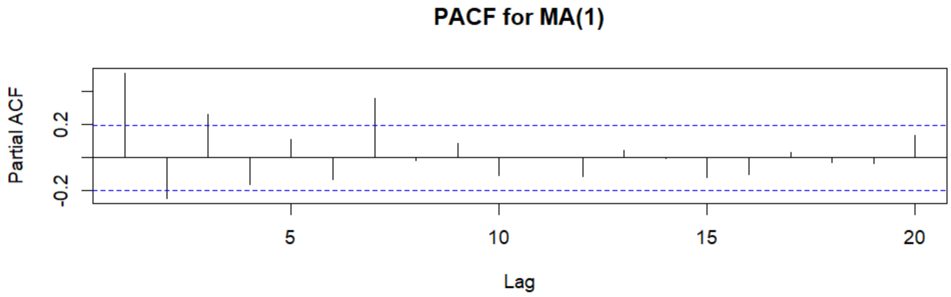
ARMA(1,1):$\rho(0)=1; \quad \rho(h)=\frac{(1+\theta\phi)(\phi+\theta)}{1+2\theta\phi+\theta^2}\phi^{h-1}, \quad h\geq 1$

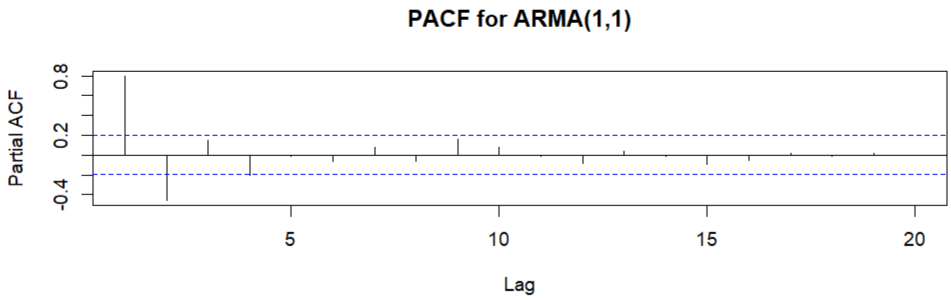
指数衰减趋于0, 符合理论值.

PACF

```
par(mfrow=c(3,1))
acf(ar,main='ACF for AR(1)')
acf(ma,main='ACF for MA(1)')
acf(arma,main='ACF for ARMA(1,1)')
```







AR(1):$\phi_{11} = \phi; \quad \phi_{ii} = 0, \quad i \geq 1$

只在1处显著,符合理论值.

MA(1):$\phi_{hh} = -\frac{(-\theta)^h(1-\theta^2)}{1-\theta^{2h+2}}, \quad h \geq 1$

来回震荡幅度衰减到0,符合理论值.

ARMA(1,1):可逆ARMA模型有一个AR(∞)表示,从而理论上不会截尾.

来回震荡幅度衰减到0,符合理论值.

和表 3.1 对比

纵观所有图,只有AR(1)的样本PACF和MA(1)的样本ACF在lag=1发生截尾, 其余图都算tails off, 和表结论一致.

2

2. $\hat{X}_{t+h} = \beta_1 X_{t+h-1} + \cdots \beta_{h-1} X_{t+1}$

$L(\alpha) \triangleq E(X_{t+h} - \alpha_1 X_{t+h-1} - \cdots - \alpha_{h-1} X_{t+1})^2$

$\Rightarrow \begin{cases} \beta = \arg\min L(\alpha). \\ \dfrac{\partial L}{\partial \alpha_i} = -E\, 2X_{t+h-i}(X_{t+h} - \alpha_1 X_{t+h-1} - \cdots - \alpha_{h-1} X_{t+1}) \end{cases}$

$\Rightarrow \quad E X_{t+h-i}(X_{t+h} - \beta_1 X_{t+h-1} - \cdots - \beta_{h-1} X_{t+1}) = 0. \quad i = 1, 2, \cdots, h-1$

i.e. $\langle X_{t+h-i}, X_{t+h} - \hat{X}_{t+h} \rangle = 0. \quad i = 1, 2, \cdots, h-1$

$\therefore \quad X_{t+h} - \hat{X}_{t+h} \perp \mathrm{span}\{X_{t+h-1}, \cdots, X_{t+1}\}.$

同理可得 $X_t - \hat{X}_t \perp \mathrm{span}\{X_{t+1}, \cdots, X_{t+h-1}\}.$

3

3、 $\begin{cases} \hat{X}_{t+h} = \beta_1 X_{t+h-1} + \cdots + \beta_{h-1} X_{t+1} \\ \hat{X}_t = \beta_1' X_{t+1} + \cdots + \beta_{h-1}' X_{t+h-1} \end{cases}$ 且 $\{X_i\}$ 为平稳序列

须证 $\beta_i = \beta_i'$ $i = 1, 2, \cdots, h-1$.

$\Leftrightarrow$ 证 $\arg\min L_1(\alpha) = \arg\min L_2(\alpha)$ 式中 $\begin{cases} L_1(\alpha) = E(X_{t+h} - \sum_{i=1}^{h-1} \alpha_i X_{t+h-i})^2 \\ L_2(\alpha) = E(X_{t+1} - \sum_{i=1}^{h-1} \alpha_i X_{t+i})^2 \end{cases}$

$\begin{cases} \dfrac{dL_1}{d\alpha_i} = E\, 2X_{t+h-i}(X_{t+h} - \sum_{i=1}^{h-1} \alpha_i X_{t+h-i}) = 2\gamma(i) - 2\sum_{j=1}^{h-1} \alpha_j \gamma(|i-j|) \\ \dfrac{dL_2}{d\alpha_i} = E\, 2X_{t+i}(X_t - \sum_{i=1}^{h-1} \alpha_i X_{t+i}) = 2\gamma(i) - 2\sum_{j=1}^{h-1} \alpha_j \gamma(|i-j|) \end{cases}$ $\because \{X_t\}$ 平稳.

$\therefore \dfrac{dL_1}{d\alpha_i} = 0 \Leftrightarrow \dfrac{dL_2}{d\alpha_i} = 0.$ $i = 1, 2, \cdots, h-1$

$\therefore \arg\min L_1(\alpha) = \arg\min L_2(\alpha) \Rightarrow \beta = \beta'$.

4

4.

(a) $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|. \quad \forall x, y \in \mathcal{H}$

$\langle tx+y, tx+y\rangle \geq 0. \quad \forall t$

$\Rightarrow \langle x, x\rangle t^2 + 2\langle x, y\rangle t + \langle y, y\rangle \geq 0. \quad \forall t.$

$\Rightarrow \Delta = 4\langle x, y\rangle^2 - 4\langle x, x\rangle\langle y, y\rangle \leq 0.$

$\therefore |\langle x, y\rangle| \leq \|x\| \cdot \|y\|.$

(b) $\|x+y\| \leq \|x\| + \|y\|. \quad \forall x, y \in \mathcal{H}$

$LHS = \langle x+y, x+y\rangle = \langle x, x\rangle + \langle y, y\rangle + 2\langle x, y\rangle$

$RHS = \langle x, x\rangle + \langle y, y\rangle + 2\|x\| \cdot \|y\|$

由(a)知 $\langle x, y\rangle \leq \|x\| \cdot \|y\|.$

$\therefore$ $LHS \leq RHS$ 得证.

5

5、 $\langle X, Y \rangle \triangleq Cov(X, Y)$.

便用第2题结论可得 $\hat{X}_{t+h} = P_{span\{X_{t+h-1}, \cdots X_{t+1}\}} \cdot X_{t+h}$

$\Rightarrow \hat{X}_{t+h} \in span\{X_{t+h-1}, \cdots, X_{t+1}\}$

$\subseteq span\{W_{t+h-1}, W_{t+h-2}, \cdots\}$ ← Causality

$\therefore \hat{X}_{t+h} \perp W_{t+h}$

i.e. $\langle \hat{X}_{t+h}, X_{t+h} - \sum_{i=1}^{p} \phi_i X_{t+h-i} \rangle = 0$. $\sum_{i=1}^{p} \phi_i X_{t+h-i} \in span\{X_{t+h-i}, \cdots X_{t+1}\}$.

& $\langle \hat{X}_{t+h}, X_{t+h} - \hat{X}_{t+h} \rangle = 0$

由投影唯一性

$\Rightarrow \hat{X}_{t+h} = \sum_{i=1}^{p} \phi_i X_{t+h-i}$

6

6. $\phi_{hh} = -\frac{(-\theta)^h(1-\theta^2)}{1-\theta^{2(h+1)}}$ $h \geq 1$. $MA(1)$: $X_t = W_t + \theta W_{t-1}$, $|\theta| < 1$

$\phi_{hh} = cor(X_{t+h} - \hat{X}_{t+h}, X_t - \hat{X}_t)$.

$\hat{X}_{t+h} = \sum_{i=1}^{h-1} \beta_i X_{t+h-i}$. $\hat{X}_t = \sum_{i=1}^{h-1} \beta_i X_{t+i}$.

$\beta = \arg\min E(X_{t+h} - \sum_{i=1}^{h-1} \alpha_i X_{t+h-i})^2$

$= \arg\min E(W_{t+h} + \theta W_{t+h-1} - \sum_{i=1}^{h-1} \alpha_i (W_{t+h-i} + \theta W_{t+h-1-i}))^2$

$= \arg\min (1 + (\alpha_1 - \theta)^2 + \sum_{i=1}^{h-2} (\alpha_i \theta + \alpha_{i+1})^2 + (\theta \alpha_{h-1})^2)$

$$\therefore \begin{cases} 2(\alpha_1 - \theta) + 2(\alpha_1\theta + \alpha_2)\theta = 0. \Rightarrow \theta\beta_2 + \beta_1 = -\theta(\theta\beta_1 - 1) \\ \vdots \\ 2(\alpha_{i-1}\theta + \alpha_i) + 2(\alpha_i\theta + \alpha_{i+1})\theta = 0. \\ \vdots \\ 2(\alpha_{h-2}\theta + \alpha_{h-1}) + 2\theta^2\alpha_{h-1} = 0. \end{cases}$$

$\Rightarrow \theta\beta_{i+1} + \beta_i = -\theta(\theta\beta_i + \beta_{i-1})$. $\beta_h = 0$. $i = 2, \cdots h-1$.

$\Rightarrow \beta_i = -\theta\beta_{i+1} + (-\theta)^i (\theta\beta_1 - 1) = [(-\theta)^i (1+\theta^2)(\theta\beta_1 - 1)] + (-\theta)^2 \beta_{i+2}$

$\Rightarrow \beta_i = (-\theta)^i \cdot (1 + \theta^2 + \cdots + \theta^{2(h-1-i)}) \cdot (\theta\beta_1 - 1) + 0.$

令 $i=1$: $\beta_1 = -\theta\cdot(1+\theta^2+\cdots\theta^{2(h-2)})\cdot(\theta\beta_1-1)$

$$= -\theta\cdot\frac{1-\theta^{2(h-1)}}{1-\theta^2}\cdot(\theta\beta_1-1)$$

$\Longrightarrow \beta_1\cdot\left[1+\theta^2\frac{1-\theta^{2h-2}}{1-\theta^2}\right] = \theta\cdot\frac{1-\theta^{2h-2}}{1-\theta^2}$. 可解 β_1

$\Longrightarrow \beta_1 = \theta\,\frac{1-\theta^{2h-2}}{1-\theta^{2h}}$; $\theta\beta_1 - 1 = -\frac{1-\theta^2}{1-\theta^{2h}}$

$\Longrightarrow \beta_i = -(-\theta)^i\,\frac{1-\theta^{2(h-i)}}{1-\theta^{2h}}$

$\therefore \mathrm{Cov}(X_{t+h}-\hat{X}_{t+h},\, X_t-\hat{X}_t)$

$= \mathrm{Cov}\left(W_{t+h}+\theta W_{t+h-1}-\sum_{i=1}^{h-1}(W_{t+h-i}+\theta W_{t+h-i-1})\cdot\beta_i,\; W_t+\theta W_{t-1}-\sum_{i=1}^{h-1}(W_{t+i}+\theta W_{t+i-1})\cdot\beta_i\right)$

$= \sigma_w^2\cdot[\beta_1+\theta\beta_0, \beta_2+\theta\beta_1, \cdots \beta_{h-1}+\theta\beta_{h-2}, \beta_{h-1}\theta]\cdot[\beta_{h-1}, \beta_{h-2}+\theta\beta_{h-1}, \cdots, \beta_0+\theta\beta_1]^T$ 令 $\beta_0=-1$

$= \sigma_w^2\cdot\frac{-(-\theta)^h(1-\theta^2)}{1-\theta^{2h}}$

$\mathrm{Var}(X_{t+h}-\hat{X}_{t+h})$

$= \mathrm{Var}\left(W_{t+h}-W_{t+h-1}\cdot\theta-\sum_{i=1}^{h-1}\beta_i(W_{t+h-i}-\theta\cdot W_{t+h-1-i})\right)$

$= \mathrm{Var}\left(W_{t+h}-\sum_{i=1}^{h-1}(\theta\beta_{i-1}+\beta_i)W_{t+h-i}+\beta_{h-1}\cdot\theta\cdot W_t\right)$ 令 $\beta_0=1$

$= \sigma_w^2\cdot\left(1+\sum_{i=1}^{h-1}(\theta\beta_{i-1}+\beta_i)^2+\beta_{h-1}^2\theta^2\right)$

$= \sigma_w^2\cdot\frac{1-2^{2h+2}}{1-\theta^{2h}} = \mathrm{Var}(X_t-\hat{X}_t)$.

$\therefore \phi_{hh} = \frac{-(-\theta)^h(1-\theta^2)}{1-\theta^{2h+2}}$