时间序列 HW03

18300290007 加兴华

3.2

a

$$\forall t \in N, x_t = \phi x_{t-1} + w_t = \phi\{\phi x_{t-2} + w_{t-1}\} + w_t = \cdots = \phi^{t-1} x_0 + \sum_{i=0}^{t-1} \phi^i w_{t-i} = \sum_{i=0}^{t} \phi^i w_{t-i}$$

b

$$E\left(x_t\right) = \sum_{i=0}^{t} \phi^i E(w_{t-i}) = 0$$

c

$$\begin{aligned} & \forall t \in N, \\ & var\left(x_t\right) = \textstyle\sum_{i=0}^{t} \phi^{2i} var(w_{t-i}) = \sum_{i=0}^{t} \phi^{2i} \sigma_w^2 \\ \because \quad & |\phi| < 1 \\ \therefore \quad & \textstyle\sum_{i=0}^{t} \phi^{2i} = \frac{1-\phi^{2(t+1)}}{1-\phi^2} \\ \therefore \quad & var\left(x_t\right) = \frac{1-\phi^{2(t+1)}}{1-\phi^2} * \sigma_w^2 \end{aligned}$$

d

$$cov(x_{t+h}, x_t) = \phi^h var(x_t) + \sum_{i=0}^{h-1} cov(\phi^i w_{t-i}, x_t) = \phi^h var(x_t)$$

e

x_t 不平稳, 因为$var(x_t)$ 不是t的常值函数, 从而自相关函数也与t相关, 并非只与h相关, 因此不平稳.

f

当$t \to \infty$, $var(x_t) \to \frac{1}{1-\phi^2} * \sigma_w^2$ 是t的常值函数, 从而自相关函数只与相关; 另一方面$E(x_t)$ 也为t的常值函数, 因此x_t 渐进平稳.

g

将生成的标准正态序列乘以σ_w^2 即可得到白噪声序列w_t.

然后可根据公式$x_t = \sum_{i=0}^{t} \phi^i w_{t-i}$ 获得一串非平稳AR(1)模型序列,

再截取t>N(N为相当大的正数)的序列作为平稳AR(1)模型序列.

h

$$
\begin{aligned}
x_t &= \phi x_{t-1} + w_t = \phi\{\phi x_{t-2} + w_{t-1}\} + w_t = \cdots = \phi^{t-1}x_0 + \sum_{i=0}^{t-1}\phi^i w_{t-i} \\
&= \frac{w_0\phi^{t-1}}{\sqrt{1-\phi^2}} + \sum_{i=0}^{t-1}\phi^i w_{t-i} \\
E(x_t) &= 0, \\
var(x_t) &= \frac{var(w_0)\phi^{2t-2}}{1-\phi^2} + \sum_{i=0}^{t-1}\phi^{2i}var(w_{t-i}) \\
&= \sigma_w^2 \cdot (\frac{\phi^{2t-2}}{1-\phi^2} + \sum_{i=0}^{t-1}\phi^{2i}) \\
&= \sigma_w^2 \cdot \frac{1}{1-\phi^2} \quad \text{是t的常值函数,} \\
cov(x_{t+h}, x_t) &= \phi^h var(x_t) + \sum_{i=0}^{h-1} cov(\phi^i w_{t-i}, x_t) = \phi^h var(x_t) \quad \text{是只和h相关的函数}
\end{aligned}
$$

由以上推导知: 新定义的x_0 能让过程平稳.

3.3

a

$$
\begin{aligned}
& x_t = \phi x_{t-1} + w_t \\
\Rightarrow \quad & x_t = \phi^{-1}x_{t+1} - w_{t+1} \\
&\quad = \ldots \\
&\quad = \phi^{-\infty}x_{+\infty} - \sum_{i=1}^{+\infty}\phi^{-i}w_{t+i} \\
&\quad = -\sum_{i=1}^{+\infty}\phi^{-i}w_{t+i}, \\
\Rightarrow \quad & E(x_t) = -\sum_{i=1}^{+\infty}\phi^{-i}E(w_{t+i}) = 0, \\
& var(x_t) = \sum_{i=1}^{+\infty}\phi^{-2i}var(w_{t+i}) \\
&\quad = \sigma_w^2 \cdot \frac{\phi^{-2}}{1-\phi^{-2}}, \\
\Rightarrow\ & cov(x_{t+h}, x_t) = \phi^{-h}var(x_{t+h}) + \sum_{i=1}^{h-1} cov(\phi^{-i}w_{t+i}, x_{t+h}) \\
&\quad = \phi^{-h}var(x_{t+h}) \\
&\quad = \sigma_w^2\phi^{-2}\phi^{-h}/\left(1-\phi^{-2}\right)
\end{aligned}
$$

b

$$
\begin{aligned}
y_t &= \phi^{-\infty} y_{-\infty} + \sum_{i=0}^{+\infty} \phi^{-i} w_{t-i} \\
&= \sum_{i=0}^{+\infty} \phi^{-i} w_{t-i}, \\
\Rightarrow \quad E(y_t) &= \sum_{i=0}^{+\infty} \phi^{-i} E(w_{t-i}) = 0, \\
var(y_t) &= \sum_{i=0}^{+\infty} \phi^{-2i} var(w_{t-i}) \\
&= \sigma_w^2 \phi^{-2} \cdot (\frac{1}{1-\phi^{-2}}), \\
\Rightarrow \quad cov(y_{t+h}, y_t) &= \phi^{-h} var(y_t) + \sum_{i=0}^{h-1} cov(\phi^i w_{t+i}, y_t) \\
&= \phi^{-h} var(y_t) \\
&= \sigma_w^2 \phi^{-2} \phi^{-h} / \left(1 - \phi^{-2}\right)
\end{aligned}
$$

因此y_t 的期望和自相关函数都与x_t 的相同.

3.4

a

$$
\left.\begin{array}{c}
A(z) = 1 - 0.8z + 0.15z^2 = (1-0.3z)(1+0.5z) \\
B(z) = 1 - 0.3z
\end{array}\right\} \Rightarrow \quad (1 + 0.5\mathscr{B})(x_t) = w_t
$$

$\therefore$该模型是AR(1)causal的、MA(0)invertible的

b

$$
\left.\begin{array}{c}
A(z) = 1 - z + 0.5z^2 \Rightarrow z = 1 \pm i\text{在单位圆外} \\
B(z) = 1 - z \Rightarrow z = 1\text{在单位圆上}
\end{array}\right\} \Rightarrow \quad (1 - \mathscr{B} + 0.5\mathscr{B}^2)(x_t) = (1 - \mathscr{B})(w_t)
$$

$\therefore$该模型是AR(2)causal的、MA(1)非invertible的

附加题

假设y_t 为满足该等式的平稳解, 则:

$$y_t = \phi^n y_{t-n} + \sum_{i=0}^{n-1} \phi^i w_{t-i}, \quad \forall n \in N^*$$

$$\Rightarrow \quad var(y_t - \phi^n y_{t-n}) = var(\sum_{i=0}^{n-1} \phi^i w_{t-i}),$$

$$\Rightarrow \quad var(y_t) + \phi^{2n} var(y_{t-n}) - 2\phi^n cov(y_t, y_{t-n}) = \sum_{i=0}^{n-1} \phi^{2i} var(w_{t-i}),$$

$$\Rightarrow \quad 2\phi^n cov(y_t, y_{t-n}) + \sigma^2 \sum_{i=0}^{n-1} \phi^{2i} = var(y_t) + \phi^{2n} var(y_{t-n}),$$

$$\because \quad |\phi| = 1, \ \therefore \phi^{2n} = \phi^{2i} = 1, \ \forall i \in N$$

$$\Rightarrow \quad 2\gamma(0) - 2\phi^n \gamma(n) = n\sigma^2, \qquad (y_t\text{平稳})$$

$$\Rightarrow \quad |2\gamma(0)| + |2\phi^n \gamma(n)| \geq |n\sigma^2|,$$

$$\text{又}\because \quad |\gamma(n)| \leq |\gamma(0)|,$$

$$\Rightarrow \quad 2|2\gamma(0)| \geq |2\gamma(0)| + |2\phi^n \gamma(n)| \geq |n\sigma^2|$$

因为y_t 平稳，不等式左边为常数；但右边随n增大趋于正无穷。因此不等式不恒成立，假设矛盾.

综上, 不存在平稳解.