时间序列 HW06

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3.11

(a) $\tilde{X}_{n+1} = E[X_{n+1} \mid X_n, X_{n-1}, \cdots X_0, \cdots]$

$= P_{\mathcal{H}} X_{n+1} \qquad \mathcal{H} = \text{span}\{X_n, X_{n-1}, \cdots\}$

$\because X_t = W_t + \theta W_{t-1}$

$\therefore \tilde{X}_{n+1} = \tilde{W}_{n+1} + \theta \tilde{W}_n = 0 + \theta W_n$

$= \theta (X_{t-1} - \theta W_{t-2}) = \theta X_{t-1} - \theta^2 X_{t-2} + \theta^3 X_{t-3} \cdots\cdots$

$= \sum_{i=1}^{\infty} -(-\theta)^i X_{n+1-i}$

$P_{n+1}^n = E[(X_{n+1} - \tilde{X}_{n+1})^2] = E[W_t^2] = \sigma_w^2$

(b) $\tilde{X}_{n+1}^n = \tilde{W}_{n+1}^n + \theta \tilde{W}_n^n$

$$\begin{cases} \tilde{W}_{n+1}^n = E[W_{n+1} \mid X_n, \cdots, X_1] = 0 \\ \tilde{W}_n^n = E[W_n \mid X_n, \cdots, X_1] = E[W_n \mid W_n, \cdots, W_0] = W_n \end{cases}$$

$\tilde{X}_{n+1}^n = \theta W_n = -\sum_{i=0}^{n-1} (-\theta)^{i+1} X_{n-i}$

$X_{n+1}^n - \tilde{X}_{n+1}^n = W_{n+1} + \theta W_n + \sum_{i=0}^{n-1} (-\theta)^{i+1} (W_{n-i} + \theta W_{n-1-i})$

$= W_{n+1} + \theta W_n + \sum_{i=0}^{n-1} (-\theta)^{i+1} W_{n-i} - \sum_{i=1}^{n} (-\theta)^{i+1} W_{n-i}$

$= W_{n+1} + \theta W_n + (-\theta) W_n - (-\theta)^{n+1} W_0$

$= W_{n+1} - (-\theta)^{n+1} W_0$

$\because W_{n+1} \perp W_0$

$\therefore E[(X_{n+1}-\tilde{X}_{n+1}^{n})^2] = E[W_{n+1}^2] + (-\theta)^{2n+2} E[W_0^2]$

$= \sigma_w^2 (1+\theta^{2+2n})$

当 MA invertible. i.e., $|\theta|<1$.

$E[(X_{n+1}-\tilde{X}_{n+1}^{n})^2] \xrightarrow{n\uparrow\infty} E[(X_{n+1}-\tilde{X}_{n+1})^2] = \sigma_w^2$

当 n 越大，有限近似效果越好

3.15 $AR(1): X_t = \phi X_{t-1} + W_t$

$X_{t+m}^{t} = E[X_{t+m} \mid X_t, \cdots, X_1]$

$= E[\phi^m X_t + \sum_{i=0}^{m-1} \phi^i W_{t+m-i} \mid X_t, \cdots, X_1]$

$= \phi^m X_t \quad (W_{t+m-i} \perp X_t, \cdots, X_1 \ \forall i \in [0, m-1])$

$X_{t+m} - X_{t+m}^{t} = \sum_{i=0}^{m-1} \phi^i W_{t+m-i}$

$\therefore P_{t+m}^{t} = \sigma_w^2 \sum_{i=0}^{m-1} \phi^{2i} = \frac{1-\phi^{2m}}{1-\phi^2} \sigma_w^2$

3.16 ARMA(1,1): $X_t = 0.9 X_{t-1} + 0.5 W_{t-1} + W_t$

$\tilde{X}_{n+m}^{n} = -\sum_{j=1}^{m-1} \pi_j \tilde{X}_{n+m-j}^{n} - \sum_{j=m}^{n+m-1} \pi_j X_{n+m-j}$, 由 $W_{n+m} = \sum_{j=0}^{\infty} \pi_j X_{n+m-j} = \frac{(1-\phi B)}{(1+\theta B)} X_{n+m}$ 可得 π_j (3.91)

$\tilde{X}_{n+m}^{n} = \phi \tilde{X}_{n+m-1}^{n} + \theta \tilde{W}_{n+m-1}^{n}$. 其中 $\tilde{X}_t^n = \begin{cases} 0 & t\le 0 \\ X_t & 1\le t\le n \end{cases}$, $\tilde{W}_t^n = \begin{cases} 0 & t\le 0 \\ W_t & 1\le t\le n \\ \tilde{X}_t^n - \phi \tilde{X}_{t-1}^n - \theta \tilde{W}_{t-1}^n & t>n \end{cases}$ (3.92)

$(3.92) \Rightarrow \tilde{X}_{n+m}^{n} = \phi \tilde{X}_{n+m-1}^{n} + \theta \tilde{X}_{n+m-1}^{n} - \phi\theta \tilde{X}_{n+m-2}^{n} - \theta^2 \tilde{W}_{t-1}^{n}$

$= (\phi+\theta) \tilde{X}_{n+m-1}^{n} - \theta(\phi+\theta) \tilde{X}_{n+m-2}^{n} + \theta^2\phi \tilde{X}_{n+m-3}^{n} + \theta^3 \tilde{W}_{n+m-3}^{n}$

$\vdots$

$= \sum_{j=1}^{m-1} (-\theta)^{j-1} (\phi+\theta) \tilde{X}_{n+m-j}^{n} + (-\theta)^{m-1} [\phi \tilde{X}_n^n + \theta \tilde{W}_n^n]$

$= \sum_{j=1}^{m-1} (-\theta)^{j-1} (\phi+\theta) \tilde{X}_{n+m-j}^{n} + (-\theta)^{m-1} [\phi X_n + \theta W_n]$ 代入 $W_t = X_t - \phi X_{t-1} - \theta W_{t-1}$,

$= \sum_{j=1}^{m-1} (-\theta)^{j-1} (\phi+\theta) \tilde{X}_{n+m-j}^{n} + \sum_{j=m}^{m-1+n} (-\theta)^{j-1} (\theta+\phi) X_{n+m-j}$

(3.91) $\Rightarrow$ $1+\pi_1 B+\pi_2 B^2+\cdots = (1-\phi B)(1-\theta B+\theta B^2-\theta B^3\cdots\cdots)$

$$\Rightarrow \quad \pi_i = (-\theta)^{i-1}(\phi+\theta)$$

$\therefore$ 在ARMA(1,1)下，3.91与3.92预测公式表述等价

3.46. 令 $X_1^0=0$. $P_1^0=\gamma(0)$.

为方便书写记 $e_i=X_i-X_i^{i-1}$. $\langle a,b\rangle =: E(ab)$.

1°: 假设 $t<k$ 时：$X_{t+1}^t=\sum_{j=1}^t \theta_{tj}\, e_j$. i.e., $E[X_{t+1}|X_1,\cdots X_t]=E[X_{t+1}|e_1,\cdots e_t]$

则 $X_{k+1}^k = E[X_{k+1}\,|\,X_1,\cdots X_k]$

$$= E[X_{k+1}\,|\,e_1,\cdots e_{k-1}, X_k]$$

$\because \langle X_k-X_k^{k-1}, e_i\rangle=0.\quad i=1,2,\cdots,k-1$

$\Rightarrow \mathrm{span}\{e_1,e_2,\cdots e_{k-1},X_k\}=\mathrm{span}\{e_1,\cdots,e_k\}$

$\therefore X_{k+1}^k=E[X_{k+1}\,|\,e_1,\cdots e_k]=\sum_{j=1}^k \theta_{k,k-j}\cdot e_j$

$\therefore$ 归纳得 $X_{t+1}^t=\sum_{j=1}^t \theta_{t,t-j}\, e_j \quad j=1,2,\cdots$ 也即 $X_t^{t-1}=\sum_{j=1}^t \theta_{tj}\, e_{t+1-j}$ #.

2°: $P_t^{t-1}=E[e_t^2]=\langle e_t,e_t\rangle=\langle X_t-X_t^{t-1}, X_t-X_t^{t-1}\rangle$

$$=\gamma(0)+\langle X_t^{t-1},X_t^{t-1}\rangle-2\langle X_t, X_t^{t-1}\rangle$$

$$=\gamma(0)-\langle X_t^{t-1}, X_t^{t-1}\rangle-2\overbrace{\langle e_t, X_t^{t-1}\rangle}^{=0}$$

$$=\gamma(0)-\sum_{j=1}^{t-1}\theta_{t,t-j}^2\langle e_j,e_j\rangle$$

$$=\gamma(0)-\sum_{j=1}^{t-1}\theta_{t,t-j}^2\, P_j^{j-1} \quad \#$$

3°: $E[e_t\cdot e_j]=\langle X_t,X_j\rangle-\langle X_t^{t-1},X_j\rangle-\langle X_t,X_j^{j-1}\rangle+\langle X_t^{t-1},X_j^{j-1}\rangle$

$$=\gamma(t-j)-\langle X_t^{t-1},\ e_j+X_j^{j-1}\rangle-\langle e_t+X_t^{t-1}, X_j^{j-1}\rangle+\langle X_t^{t-1},X_j^{j-1}\rangle$$

$$=\gamma(t-j)-\langle X_t^{t-1},e_j\rangle-\langle e_t, X_j^{j-1}\rangle-\langle X_t^{t-1},X_j^{j-1}\rangle$$

$$= r(t-j) - \theta_{t,t-j}\cdot\langle e_j, e_j\rangle - 0 - \langle X_t^{t-1}, X_j^{j-1}\rangle$$

$$= 0. \quad (\text{因 } e_t \perp e_j)$$

$$\Rightarrow \theta_{t,t-j}\cdot P_j^{j-1} = r(t-j) - \langle X_t^{t-1}, X_j^{j-1}\rangle = r(t-j) - \sum_{k=0}^{j-1}\theta_{j,j-k}\,\theta_{t,t-k}\,P_{k+1}^{k}$$

$$\therefore \theta_{t,t-j} = \left\{ r(t-j) - \sum_{k=0}^{j-1}\theta_{j,j-k}\,\theta_{t,t-k}\,P_{k+1}^{k} \right\} / P_j^{j-1} \quad \#$$