金融计量学HW2

23210980044 加兴华

Problem 1 (Example 1.7)

Suppose B is a Brownian motion with $\mathcal{T} = [0, \infty)$. Brownian bridge is defined as $X_t = B_t - tB_1, t \in [0, 1]$. Note that $X_0 = X_1 = 0$. Then, show that X is a Gaussian process with $\mu(t) = 0$ and $c(t, s) = t \wedge s - ts$.

$$
\begin{aligned}
E(X_t) &= E(B_t - tB_1) \\
&= E(B_t) - tE(B_1) \\
&= 0 - t \cdot 0 \\
&= 0, \\
c(t, s) &= Cov(X_t, X_s) \\
&= Cov(B_t - tB_1, B_s - sB_1) \\
&= Cov(B_t, B_s) - tCov(B_1, B_s) - sCov(B_1, B_t) + stCov(B_1, B_1) \\
&= t \wedge s - t \cdot (s \wedge 1) - s \cdot (t \wedge 1) + ts \\
&= t \wedge s - ts
\end{aligned}
$$

Problem 2 (Example 1.8)

Verify the following collections are σ-filed.

$$
\mathcal{F}_1 = \{\emptyset, \Omega\}, \quad \mathcal{F}_2 = \{\emptyset, A, A^c, \Omega\}, \quad \mathcal{F}_3 = \{A \,|\, A \subseteq \Omega\}.
$$

(1) $\mathcal{F}_1$

- $\emptyset \in \mathcal{F}_1$
- $\emptyset, \Omega \in \mathcal{F}_1, \ \therefore \forall A \in \mathcal{F}_1, \ s.t. A^c \in \mathcal{F}_1$
- $\forall A_i \in \mathcal{F}_1, \ \bigcup_i^\infty A_i = \emptyset$ 或 $\Omega, \ \therefore \bigcup_i^\infty A_i \in \mathcal{F}_1$

(2) $\mathcal{F}_2$

- $\emptyset \in \mathcal{F}_2$
- $\emptyset, \Omega, A, A^c \in \mathcal{F}_2, \quad \therefore \forall x \in \mathcal{F}_2, \ s.t. x^c \in \mathcal{F}_2$
- $\forall A_i \in \mathcal{F}_2, \ \bigcup_i^\infty A_i = \emptyset(A_i\text{全为}\emptyset)$ or $\Omega(\text{存在}A_i = \Omega$ 或 $\text{存在}A_i, A_j = A, A^c)$ or $A(A_i\text{全为}A)$ or $A^c(A_i\text{全为}A^c)$, $\therefore \bigcup_i^\infty A_i \in \mathcal{F}_2$

(3) $\mathcal{F}_3$

- $\emptyset \subseteq \Omega, \ \therefore \emptyset \in \mathcal{F}_3$
- $\forall A \subseteq \Omega, \ s.t. A^c \subseteq \Omega \therefore \forall A \in \mathcal{F}_3, \ s.t. A^c \in \mathcal{F}_3$
- $\forall A_i \in \mathcal{F}_3, s.t. A_i \subseteq \Omega$, $\therefore \bigcup_i^\infty A_i \subseteq \Omega$, $\therefore \bigcup_i^\infty A_i \in \mathcal{F}_3$

Problem 3 (Example 1.15)

Suppose $B = (B_t, t \geq 0)$ is a Brownian motion and denote $\mathcal{F}_t = \sigma(B_s, s \in [0,t])$ for any $t \geq 0$. Define $X_t = B_t^2 - t$. Show that $\mathbb{E}[X_t|\mathcal{F}_s] = X_{s\wedge t}$.

$$\sigma(X_t) = \sigma(f(B_t, t)) \subseteq \mathcal{F}_t,$$

(1) t>s时:

$$Note: E[X_t] = E[B_t^2 - t] = E[B_t^2] - t = t - t = 0,$$
$$\therefore E[X_t|\mathcal{F}_s] = E[X_t - X_s|\mathcal{F}_s] + E[X_s|\mathcal{F}_s] = E[X_{t-s}] + X_s = X_s$$

(2) t<s时:

$$X_t \in \sigma(X_t) \subseteq \mathcal{F}_t \subseteq \mathcal{F}_s,$$
$$\therefore E[X_t|\mathcal{F}_s] = X_t$$

综上: $E[X_t|\mathcal{F}_s] = X_{t\wedge s}$.

Problem 4 (Example 1.19)

Let $B = (B_t, t \geq 0)$ be Brownian motion. Then, find the A_t and D_t such that $(B_t^3 + A_t, t \geq 0)$ and $(B_t^4 + D_t, t \geq 0)$ are also martingales.

(1) $B_t^3 + A_t$:

$$\begin{aligned}
E[B_t^3|\mathcal{F}_s] &= E[(B_t - B_s + B_s)^3|\mathcal{F}_s] \\
&= E[(B_t - B_s)^3 + B_s^3 + 3(B_t - B_s)^2 B_s + 3(B_t - B_s)B_s^2|\mathcal{F}_s] \\
&= E[(B_t - B_s)^3] + B_s^3 + 3B_s E[(B_t - B_s)^2] + 3B_s^2 E[(B_t - B_s)] \\
&= 0 + B_s^3 + 3B_s \cdot (t - s) + 3B_s^2 \cdot 0 \\
&= (B_s^3 - 3B_s \cdot s) + 3B_s \cdot t \\
&= (B_s^3 + A_s) - E[A_t|\mathcal{F}_s]
\end{aligned}$$

假设 $A_t = -3tB_t$, 需验证 $E[A_t|\mathcal{F}_s] = -3tB_s$:

$$\begin{aligned}
E[A_t|\mathcal{F}_s] &= -3tE[B_t - B_s + B_s|\mathcal{F}_s] \\
&= -3tE[B_{t-s}] - 3tB_s \\
&= 0 - 3tB_s \\
&= -3tB_s
\end{aligned}$$

综上, $A_t = -3tB_t$

(2) $B_t^4 + D_t$:

$$
\begin{aligned}
E[B_t^4|\mathcal{F}_s] &= E[(B_t - B_s + B_s)^4|\mathcal{F}_s] \\
&= E[(B_t - B_s)^4 + B_s^4 + 4(B_t - B_s)^3 B_s + 4(B_t - B_s)B_s^3 + 6(B_t - B_s)^2 B_s^2|\mathcal{F}_s] \\
&= E[B_{t-s}^4] + B_s^4 + 4B_s E[B_{t-s}^3] + 4B_s^3 E[B_{t-s}] + 6B_s^2 E[B_{t-s}^2] \\
&= 3(t-s)^2 + B_s^4 + 4B_s E \cdot 0 + 4B_s^3 \cdot 0 + 6B_s^2 \cdot (t-s) \\
&= (B_s^4 - 6sB_s^2 + 3s^2) + 3t^2 - 6ts + 6tB_s^2 \\
&= (B_s^4 + D_s) - E[D_t|\mathcal{F}_s]
\end{aligned}
$$

假设 $D_t = 3t^2 - 6tB_t^2$, 需验证 $E[D_t|\mathcal{F}_s] = -3t^2 + 6ts - 6tB_s^2$:

$$
\begin{aligned}
E[D_t|\mathcal{F}_s] &= 3t^2 - 6tE[(B_t - B_s + B_s)^2|\mathcal{F}_s] \\
&= 3t^2 - 6tE[(B_t - B_s)^2 + B_s^2 - 2(B_t - B_s)B_s|\mathcal{F}_s] \\
&= 3t^2 - 6t\{E[(B_t - B_s)^2] + B_s^2 - 2B_s E[B_{t-s}]\} \\
&= 3t^2 - 6t\{t - s + B_s^2 - 2B_s \cdot 0\} \\
&= -3t^2 + 6ts - 6tB_s^2
\end{aligned}
$$

综上，$D_t = 3t^2 - 6tB_t^2$