Homework 1

Deadline: Apr 16th, 2024

1. For any $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}^d$, we define

$$\mathcal{X} + \mathcal{Y} = \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in \mathcal{X}, \mathbf{y} \in \mathcal{Y}\}.$$

   Suppose sets $\mathcal{A}, \mathcal{B}, \mathcal{C} \subseteq \mathbb{R}^d$ satisfy

$$\mathcal{A} + \mathcal{C} \subseteq \mathcal{B} + \mathcal{C}.$$

   (a) If $\mathcal{A}$ and $\mathcal{B}$ are convex, $\mathcal{B}$ is closed, and $\mathcal{C}$ is bounded, prove $\mathbf{A} \subseteq \mathbf{B}$.

   (b) Show above result can fail if $\mathcal{B}$ is not convex.

   (Hint: Observe that $2\mathcal{A} + \mathcal{C} = \mathbf{A} + (\mathbf{A} + \mathcal{C}) \subseteq 2\mathcal{B} + \mathcal{C}$.)

2. Suppose a convex function $g : [0, 1] \to \mathbb{R}$ satisfies $g(0) = 0$.

   (a) Prove the function $h(t) = g(t)/t$ is non-decreasing on $t \in (0, 1]$.

   (b) Prove that for a convex function $f : \mathcal{C} \to \mathbb{R}$ and points $\mathbf{x}_0, \mathbf{x} \in \mathcal{C} \subseteq \mathbb{R}^d$, the quotient

$$\frac{f(\mathbf{x}_0 + t(\mathbf{x} - \mathbf{x}_0)) - f(\mathbf{x}_0)}{t}$$

   is non-decreasing as a function of t in $(0, 1]$.

3. Suppose $\mathcal{C}$ is a nonempty, closed and convex subset of $\mathbb{R}^d$ and define the projection for any $\mathbf{y} \in \mathbb{R}^d$ onto $\mathcal{C}$ as

$$\mathrm{proj}_{\mathcal{C}}(\mathbf{y}) = \arg\min_{\mathbf{x} \in \mathcal{C}} \|\mathbf{y} - \mathbf{x}\|_2^2.$$

   Prove that $\|\mathrm{proj}_{\mathcal{C}}(\mathbf{y}_1) - \mathrm{proj}_{\mathcal{C}}(\mathbf{y}_2)\|_2 \leq \|\mathbf{y}_1 - \mathbf{y}_2\|_2$.

4. Suppose that function $f : \mathbb{R}^d \to \mathbb{R}$ is differentiable. Prove f is convex if and ony if

$$\langle \nabla f(\mathbf{x}) - \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle \geq 0$$

   for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$.