2321098004 加兴华 HW1

1. (a). 反证. 若 $A \not\subseteq B$. 则必 $\exists a \in A$. $a \notin B$.

对于闭凸集B. 则存在经过a的超平面使B完全位于一侧

以a为原点, 超平面方向定为x轴, 与B异侧的法向量 $\vec{n}$ 为y轴.

易见 $a=(0,0)$ $\forall b \in B$ s.t. $y_b < 0$.

$\therefore$ 取 $c \in C$ s.t. $y_c \geq c_i$ $\forall c_i \in C$.

s.t. $y_{(a+c)} > y_{(b+c)}$

$\therefore$ $a+c \notin B+C$. 与 $A+C \subseteq B+C$ 矛盾.

$\therefore$ 不存在 $a \in A$ 且 $a \notin B$.

$\therefore$ $A \subseteq B$

(b) $A=\{2.5\}$. $B=[1,2] \cup [3,4]$.

$C=[0.5,1.5]$.

$A+C=[3,4]$. $B+C=[1.5,5.5]$.

满足 $A+C \subseteq B+C$. 且 $A \not\subseteq B$.

2. (a) $f(x) := \ln g(x)$. $\begin{cases} g(x) \text{ 在 } [0,1] \text{ 为凸}. \\ \ln x \text{ 在 } (0,1] \text{ 为凸}. \end{cases}$

$\therefore$ $f(x)$ 在 $(0,1]$ 为凸

$\Leftrightarrow$ $f''(x) \geq 0$. for $x \in (0,1]$

i.e. $\left[\frac{g(t)}{t}\right]' \geq 0$. $\Leftrightarrow$ $\frac{g(t)}{t}$ 在 $(0,1]$ 非减

(b) 令 $g(t) = f(x_0 + t(x-x_0)) - f(x_0)$

s.t. $g(0) = 0$.

$$g(\alpha t_1 + (1-\alpha)t_2) = f(x_0 + \alpha t_1(x-x_0) + (1-\alpha)t_2(x-x_0)) - f(x_0)$$
$$\geq \alpha f(x_0 + t_1(x-x_0)) - \alpha f(x_0) + (1-\alpha) f(x_0 + t_2(x-x_0)) - (1-\alpha) f(x_0)$$
$$= \alpha g(t_1) + (1-\alpha) g(t_2) \quad \forall \alpha \in [0,1]$$

$\therefore$ $g(t)$ 为凹函数 for $t \in [0,1]$.

$\therefore$ 由(a)可知 $\frac{g(t)}{t}$ 非减对于 $t \in (0,1]$ 得证.

3. 记 $x_1 = \arg\min \|y_1 - x\|_2^2$ $\quad x_2 = \arg\min \|y_2 - x\|_2^2$.

s.t. $\forall x \in C$
$$\begin{cases} \langle y_1 - x_1, x - x_1 \rangle \leq 0. & (1) \\ \langle y_2 - x_2, x - x_2 \rangle \leq 0 & (2). \end{cases}$$

$\therefore$ (1) x 取 x_2 + (2) x 取 x_1 得: $\langle y_1 - x_1 - (y_2 - x_2), x_2 - x_1 \rangle \leq 0$.

$$\Rightarrow \langle x_2 - x_1, x_2 - x_1 \rangle \leq \langle y_2 - y_1, x_2 - x_1 \rangle$$
$$\leq \|y_2 - y_1\| \cdot \|x_2 - x_1\|$$

$\therefore \|y_2 - y_1\| \geq \|x_2 - x_1\|$

4. (only if). $f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle$ (1)

$f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle$ (2)

(1) + (2): $f(x) + f(y) \geq f(x) + f(y) + \langle \nabla f(x) - \nabla f(y), y - x \rangle$

$\therefore \langle \nabla f(x) - \nabla f(y), x - y \rangle \geq 0$. $\quad \forall x, y \in \mathbb{R}^d$

(if). 令 $g(t) = f(x + t(y-x))$ $\quad t \in [0,1]$.

$$g'(t) = \langle \nabla f(x+t(y-x)), y-x \rangle$$

设 $z = x + t(y-x)$.

$$\langle \nabla f(x) - \nabla f(z), x - z \rangle \geq 0$$

$$\Rightarrow -t \langle \nabla f(x) - \nabla f(z), y-x \rangle \geq 0.$$

$$\therefore \langle \nabla f(z), y-x \rangle \geq \langle \nabla f(x), y-x \rangle.$$

$$\begin{aligned} \therefore f(y) = g(1) &= g(0) + \int_0^1 g'(t)\,dt \\ &\geq g(0) + \langle \nabla f(x), y-x \rangle \\ &= f(x) + \langle \nabla f(x), y-x \rangle. \end{aligned}$$

$\therefore f$ 为凸函数