时间序列 HW09

18300290007　　加兴华

3.27

当 $k=1$ 时，$\nabla X_t = X_t - X_{t-1}$

$\because$ X_t 为平稳序列

$\therefore \begin{cases} EX_t \equiv C \\ Var(X_t, X_s) =: \gamma(|t-s|) \end{cases}$

$\therefore \begin{cases} E\ \nabla X_t \equiv 0. \\ Var(\nabla X_t, \nabla X_s) = Var(X_t - X_{t-1}, X_s - X_{s-1}) \end{cases}$

$= 2\gamma(|t-s|) - \gamma(|t-s-1|) - \gamma(|t-s+1|)$

$=: \gamma_1(|t-s|)$

$\because$ 期望固定且协方差只与相对位置有关

$\therefore$ ∇X_t 平稳

假设 $k=i$ 时，$\nabla^i X_t$ 平稳

可以类似求得 $\begin{cases} E\ \nabla^{i+1} X_t \equiv 0. \\ Cov(\nabla^{i+1} X_t, \nabla^{i+1} X_s) =: \gamma_{i+1}(|t-s|) \end{cases}$

$\therefore$ $\nabla^{i+1} X_t$ 平稳

由上述可递推获得 $\nabla^k X_t$ 平稳，$k=1, 2, \cdots$

注意到 $\nabla^{n+i} \alpha^n = \nabla^{n+i-1}[\alpha^n - (\alpha-1)^n] = \nabla^{n+i-1} P_{n-1}(\alpha)$，$i \geq 1$ 式中 $P_{n-1}(\alpha)$ 为 α 的 $n-1$ 阶多项式，最高阶非零

① $n=1$ 时，$\nabla^{i+1}(at) = \nabla^i(at - at + a) = \nabla^i(a) = 0$. $\forall a$

② 假设 $n \leq k$ 时，s.t. $\nabla^{k+i}(at^k) = 0$.

则 $\nabla^{k+1+i}(at^{k+1}) = \nabla^{k+i} P_k(t) = 0$.

由①、② $\Rightarrow$ $\nabla^{n+i} t^n = 0$. $\forall n \in N^*$. $i \geq 1$.

同时，$\nabla^n t^n = \nabla^{n-1}(t^{n-1} + P_{n-2}(t)) = \nabla^{n-1} t^{n-1} = \cdots = \nabla t = 1$.

$\therefore$ 当 $i \leq k < q$，$\nabla^k y_t = \beta_1 \nabla^k t + \cdots \beta_q \nabla^k t^q + \nabla^k X_t = \beta_i + \beta_{i+1} \nabla^k t^{i+1} \cdots + \beta_q \nabla^k t^q + \nabla^k X_t$

$$E\nabla^k y_t = E[\beta_q \cdot \nabla^k t^q] + \cdots = P_{q-k}(t) + P_{q-k-1}(t) + \cdots P_0(t) + 0 = P_{q-k}(t)$$ 不恒为常数. ∴不平稳

当 $k=q$. $\nabla^q y_t = \beta_q + \nabla^q x_t$.

$$\begin{cases} E\nabla^q y_t = \beta_q \equiv C \\ Cov(\nabla^q y_t, \nabla^q y_s) = Cov(\nabla^q x_t, \nabla^q x_s) = \gamma_q(|t-s|) \end{cases}$$

∴ $\nabla^q y_t$ 平稳

当 $k>q$. $\nabla^k y_t = \nabla^k x_t$. 平稳

综上得证.

3.28 证明: $x_t = x_{t-1} + w_t - \lambda w_{t-1} \Rightarrow x_t = \sum_{j=1}^{\infty}(1-\lambda)\lambda^{j-1}x_{t-j} + w_t$ given $x_0 = 0$. $|\lambda|<1$

设 $y_t = w_t - \lambda w_{t-1}$ 可看作 invertible MA(1)

$\Rightarrow x_t = x_{t-1} + y_t$ (1)

∵ y_t invertible

∴ $y_t = \sum_{j=1}^{\infty}\lambda^j y_{t-j} + w_t$ 代入(1)式

$\Rightarrow x_t = x_{t-1} + \sum_{j=1}^{\infty}\lambda^j y_{t-j} + w_t$ 用(1)消除 y_{t-j}

$$\Rightarrow x_t = x_{t-1} + \sum_{j=1}^{\infty}\lambda^j(x_{t-j} - x_{t-j-1}) + w_t$$

$$= \sum_{j=1}^{\infty}(1-\lambda)\lambda^{j-1}x_{t-j} + w_t$$

3.29. (a) ∵ y_t 是 AR(1)

$$\therefore y_{n+j}^n = E[y_{n+j} \mid y_1, \cdots y_n]$$

$$= E[y_{n+j} \mid y_n]$$

$$= E[\phi y_{n+j-1} + \delta + w_t \mid y_n]$$

$$= E[\phi^j y_n + \sum_{i=0}^{j-1}\phi^i(\delta + w_{t-i}) \mid y_n]$$

$$= \phi^j y_n + \delta(1 + \cdots + \phi^{j-1}) + 0.$$ 得证

(b) ∵ $y_t = (1-B)x_t$.

$$\therefore y_{n+j}^n = x_{n+j}^n - x_{n+j-1}^n = \phi^j y_n + \delta(1+\phi+\cdots+\phi^{j-1}) = \phi^j(x_n - x_{n-1}) + \delta\cdot\frac{1-\phi^j}{1-\phi}$$

$$\therefore x_{n+j}^n = x_{n+j-1}^n + \phi^j(x_n - x_{n-1}) + \delta\cdot\frac{1-\phi^j}{1-\phi}$$

$$= x_n^n + (x_n - x_{n-1})\cdot\sum_1^j \phi^k + \frac{\delta}{1-\phi}\cdot\sum_1^j(1-\phi^k)$$

$$= X_n + (X_n - X_{n-1}) \cdot \frac{\phi - \phi^{j+1}}{1-\phi} + \frac{\delta}{1-\phi}\left(j - \frac{\phi - \phi^{j+1}}{1-\phi}\right)$$

令 $j = m$ 得证.

(c) $$P^n_{n+m} \doteq \sigma_w^2 \sum_{j=0}^{m-1} \psi_j^{*2}$$

$$\sum \psi_j^* z^j = (1-\phi z)^{-1}(1-z)^{-1} = (1+\phi z + (\phi z)^2 + \cdots)(1 + z + \cdots)$$

$$\Rightarrow \begin{cases} \psi_0^* = 1 \\ \psi_1^* = 1+\phi \\ \psi_j^* = [1, \phi, \phi^2, \cdots \phi^j] \cdot [1, 1, \cdots, 1]' = \frac{1-\phi^{j+1}}{1-\phi} \end{cases}$$

$j \geq 2$. $j=0,1$ 的情况可以与该式合并. $\Rightarrow \forall j \in N$.

$$\Rightarrow P^n_{n+m} \doteq \sigma_w^2 \cdot \sum_{i=0}^{m-1} \left(\frac{1-\phi^{i+1}}{1-\phi}\right)^2$$

$$= \frac{\sigma_w^2}{(1-\phi)^2} \cdot \sum_{i=1}^{m} (1 - 2\phi^i + \phi^{2i})$$

$$= \frac{\sigma_w^2}{(1-\phi)^2} \cdot \left[m - 2 \cdot \frac{\phi - \phi^{m+1}}{1-\phi} + \frac{\phi^2 - \phi^{2m+2}}{1-\phi^2}\right]$$