CTI ONE

May 4th (Wed), 22

Coordinate Systems.

1. Establish 3 coordinate systems. (see Definition & Life cycle in Table 1 & Figure 1)

Table 1.

	Definition	Life Cycle
x_u-y_u	defines everythings	Long Lasting
x_v-y_v	for $\vec{P_i}$ motion from $\vec{P_A}$ to $\vec{P_B}$	for $\vec{P_A}$ to $\vec{P_B}$
x_r-y_r	for Robot Orientation at $\vec{P_i}$	Same as x_v-y_v

x_u-y_u : local1 ~; x_v-y_v : local2 ~;

x_r-y_r : Robot ~

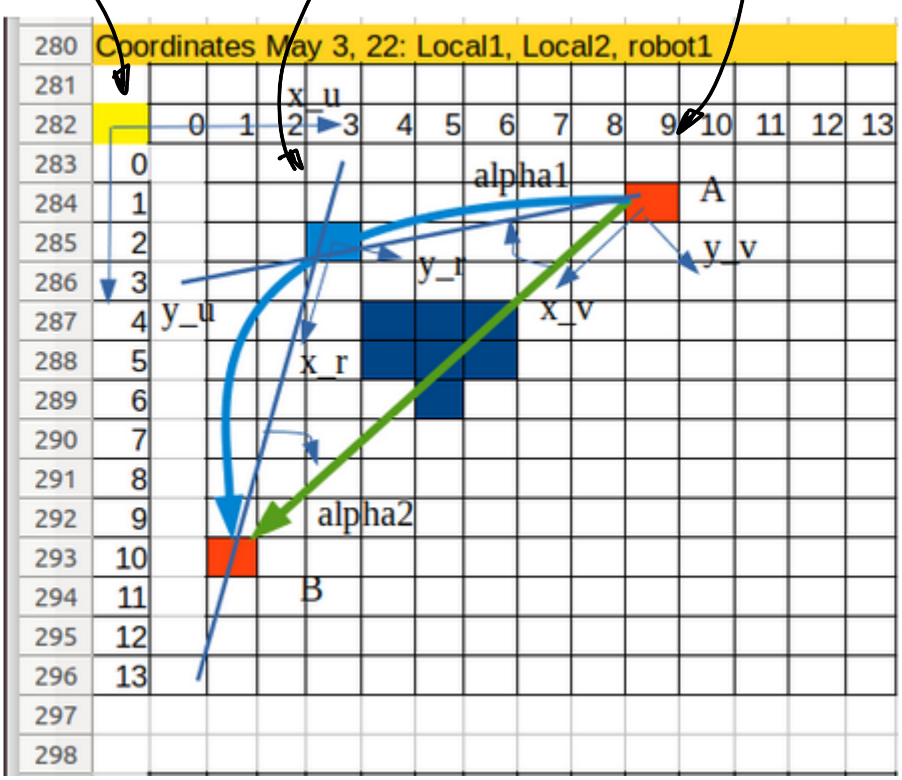

Fig 1.

Note 1: x_u-y_u is defined to match x-y Coordinate System in Computer Vision, also to match our previous definition and calculations of Angle, Perpendicular distance etc.

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Clear View of x_u-y_u, x_v-y_v, x_r-y_r. in Fig. 2 & 3

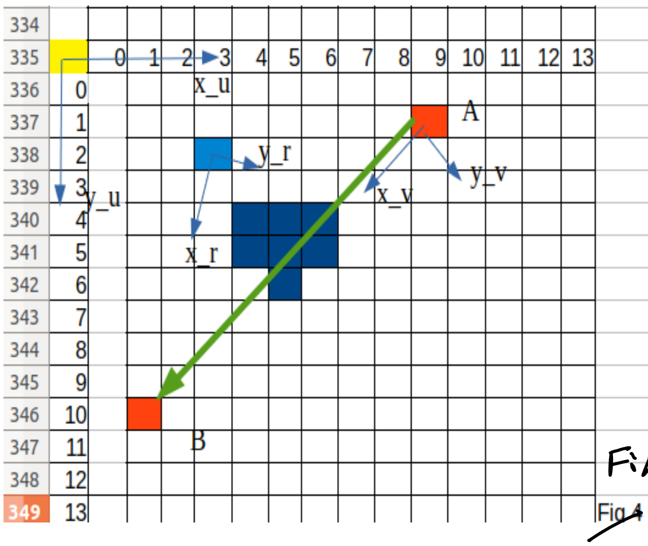

Fig.2

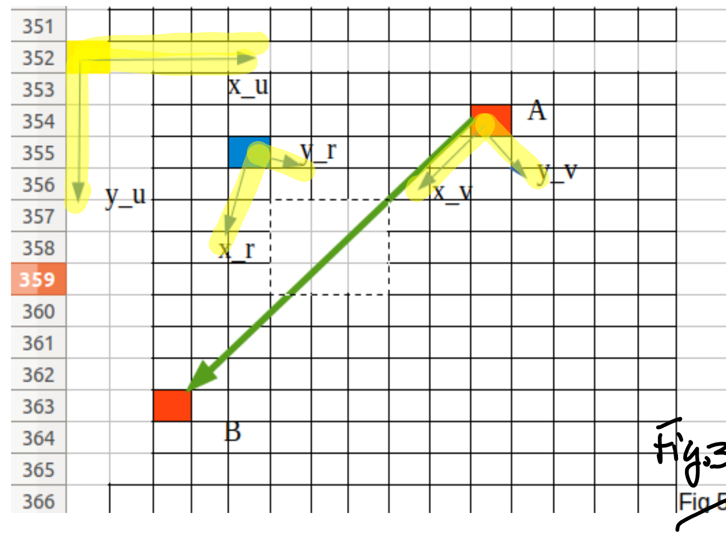

Fig.3

Now, Define mapping functions

$$\phi_1 : (x_u, y_u) \to (x_v, y_v) \quad \cdots (1)$$

$$\phi_2 : (x_u, y_u) \to (x_r, y_r) \quad \cdots (2)$$

$$\phi_3 : (x_r, y_r) \to (x_v, y_v) \quad \cdots (3)$$

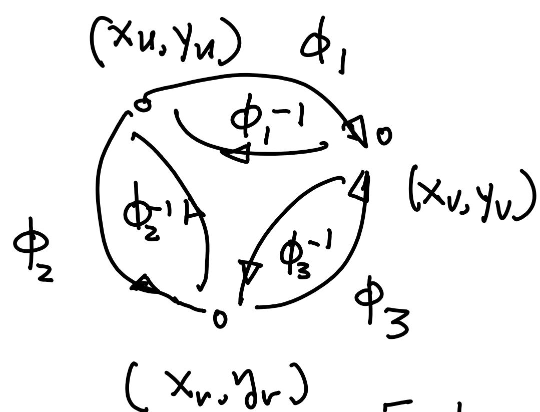

Fig 4.

Consider ϕ_1 mapping function, in Fig 4

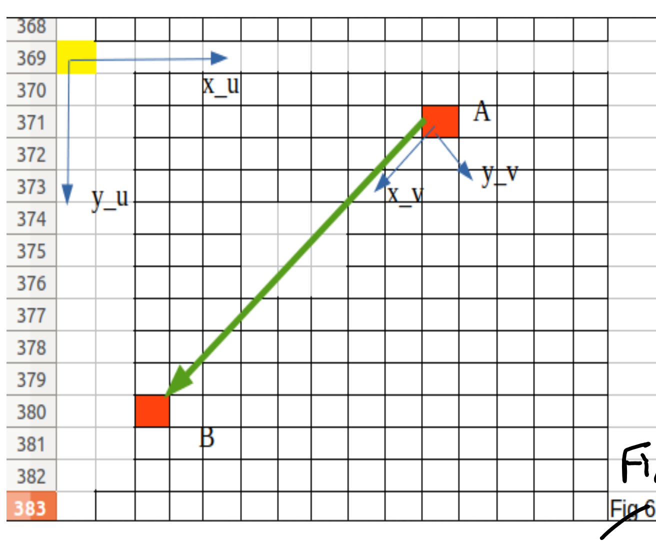

Fig 4

Let $\vec{P_i}(x_i, y_i)$ be an arbitrary point defined under (x_u, y_u), with ϕ_1, we would like to define it under (x_v, y_v).

Step 1. Translate the origin $(0,0)$ of (x_u, y_u) to the origin $(0,0)$ of (x_v, y_v),

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So, the translation is defined as

$$T_{3\times3} = \begin{pmatrix} 1 & 0 & \Delta X \\ 0 & 1 & \Delta Y \\ 0 & 0 & 1 \end{pmatrix} \quad \dots (4)$$

Hence

$$\begin{pmatrix} x_i' \\ y_i' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \Delta X \\ 0 & 1 & \Delta Y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \\ 1 \end{pmatrix} \quad \dots (4\text{-}a)$$

Since starting Point $\vec{P_A}(x_a, y_a)$ is Known, so

$$\Delta X = -x_a \quad \dots (4\text{-}b)$$

$$\Delta y = -y_a \quad \dots (4\text{-}c)$$

After Translation, the intermmediate result is shown in Fig.5

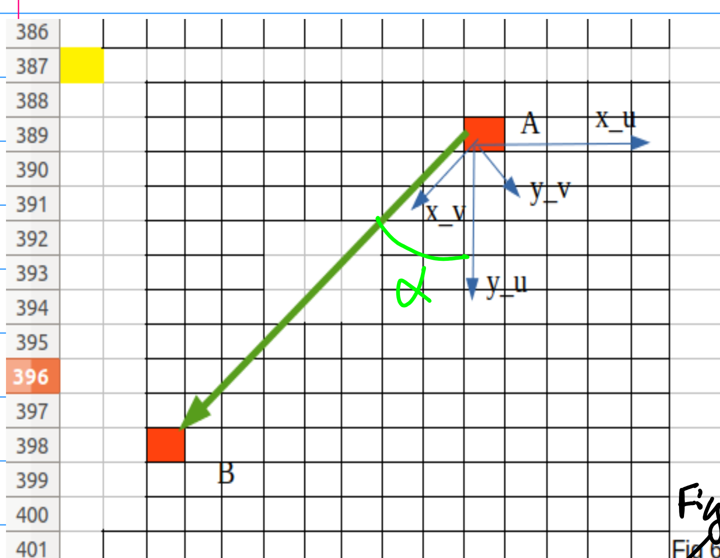

Fig.5

Now, Step 2. Rotation $R_{3\times3}$

$$R_{3\times3} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \dots (5)$$

Hence

$$\begin{pmatrix} x_i'' \\ y_i'' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_i' \\ y_i' \\ 1 \end{pmatrix} \quad \dots (5\text{-}a)$$

where the Angle α is marked in Fig.5 it is the Angle Between y_u-axis and the green line ($\vec{P_A}$ to $\vec{P_B}$)

So, write y_u-axis in a unit vector form:

$$\vec{j} = (0, 1) \quad \dots (6)$$

$$\vec{j} \cdot (\vec{P_B} - \vec{P_A}) = \|\vec{j}\| \|\vec{P_B} - \vec{P_A}\| \cos\alpha$$

$$\cos\alpha = \frac{\vec{j} \cdot (\vec{P_B} - \vec{P_A})}{\|\vec{j}\| \|\vec{P_B} - \vec{P_A}\|} \quad \dots (7)$$

$$= \frac{(0,1)\cdot(x_B - x_a, y_b - y_a)}{\sqrt{0^2+1^2} \cdot \sqrt{(x_b - x_a)^2 + (y_b - y_a)^2}}$$

$$= \frac{y_b - y_a}{\sqrt{(x_b - x_a)^2 + (y_b - y_a)^2}} \quad \dots (7\text{-}a)$$

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$$\alpha = \cos^{-1}\left(\frac{y_b - y_a}{\sqrt{(x_b - x_a)^2 + (y_b - y_a)^2}}\right) \quad \cdots (7\text{-}b)$$

Therefore,

Property 1: The mapping ϕ_1 (from Local1 Coordinate System X_u-Y_u to Local2 Coordinate System X_v-Y_v) is defined as

$$\begin{pmatrix} X_i'' \\ y_i'' \\ 1 \end{pmatrix} = R_{3\times3}\, T_{3\times3} \begin{pmatrix} X_i \\ y_i \\ 1 \end{pmatrix} \quad \cdots (8)$$

After   Before

where

$$T_{3\times3} = \begin{pmatrix} 1 & 0 & \Delta X \\ 0 & 1 & \Delta Y \\ 0 & 0 & 1 \end{pmatrix} \quad \cdots (8\text{-}a)$$

with

$$\Delta X = -X_a \quad \cdots (8\text{-}b)$$

$$\Delta y = -y_a \quad \cdots (8\text{-}c)$$

And

$$R_{3\times3} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \cdots (8\text{-}c)$$

With

$$\alpha = \cos^{-1}\left(\frac{y_b - y_a}{\sqrt{(x_b - x_a)^2 + (y_b - y_a)^2}}\right) \quad \cdots (8\text{-}d)$$

Write the Above equations in terms of X and y for Python or C/C++ Implementation. we have Lemma 1.

Lemma 1: ϕ_1 mapping (from X_u-Y_u to X_v-Y_v) Can be Written as

$$\begin{cases} X_i'' = (X_i + \Delta X)\cos\alpha - (y_i + \Delta y)\sin\alpha & \cdots (9\text{-}a) \\ y_i'' = (x_i + \Delta x)\sin\alpha + (y_i + \Delta y)\cos\alpha & \cdots (9\text{-}b) \end{cases}$$

Proof:

$$\begin{pmatrix} X_i'' \\ y_i'' \\ 1 \end{pmatrix} = R_{3\times3}\, T_{3\times3} \begin{pmatrix} X_i \\ y_i \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \Delta X \\ 0 & 1 & \Delta Y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_i \\ y_i \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_i + \Delta X \\ y_i + \Delta y \\ 1 \end{pmatrix}$$

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$$= \begin{pmatrix} (x_i+\Delta x)\cos\alpha - (y_i+\Delta y)\sin\alpha \\ (x_i+\Delta x)\sin\alpha + (y_i+\Delta y)\cos\alpha \\ 1 \end{pmatrix}$$

Q.E.D. $\quad \dots (a)$

Now, Hand Calculation to Verify mapping function $\phi 1 : (X_u, Y_u) \to (X_v, Y_v)$

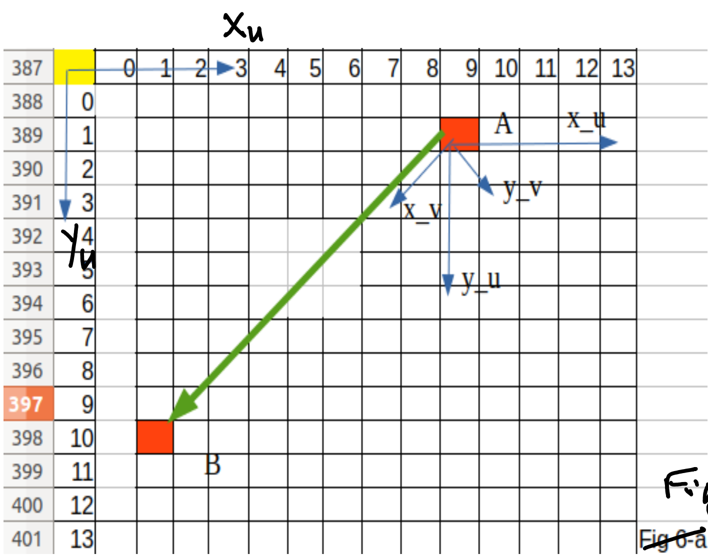


Fig.1

Given 1. (X_u, Y_u)'s origin as shown in Fig.1 (0,0), Top left corner

2. (X_v, Y_v)'s origin is located at (9,1).

So From Eqn (8-b), (8-c)

$\Delta X = -X_a \quad \dots (8\text{-}b)$

$\Delta y = -y_a \quad \dots (8\text{-}c)$

$\Delta X = -9, \ \Delta y = -1$

3. The Angle Between Axis Y_u and axis X_v is defined by Equation (7-b), Same as Eqn (8-d), e.g.

$$\alpha = \cos^{-1}\left(\frac{y_b - y_a}{\sqrt{(x_b-x_a)^2+(y_b-y_a)^2}}\right) \quad \dots (7\text{-}b)$$

Note: $\vec{P_a}(x_a, y_a) = (9,1)$,

$\vec{P_b}(x_b, y_b) = (1,10)$,

from Fig.1.

So,

$$\alpha = \cos^{-1}\frac{10-1}{\sqrt{(-8)^2+9^2}} = \cos^{-1}\frac{9}{\sqrt{8^2+9^2}}$$

$$= \cos^{-1}\frac{9}{\sqrt{64+81}} = \cos^{-1}\frac{9}{\sqrt{145}} =$$

$$= \cos^{-1}\frac{9}{12.042} = \cos^{-1} 0.7474$$

$= 41.634$ degree $(= 0.7267$ Radians$)$

4. $\sin\alpha$, $\cos\alpha$ for the Rotation matrix

$\cos\alpha = 0.7474$,

$\sin\alpha \big|_{\alpha=41.634} = 0.6644$

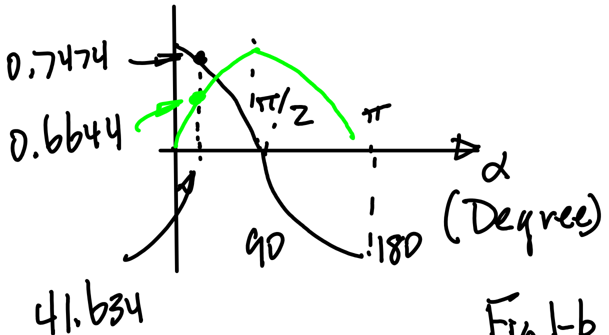


Fig 1-b

5. The Rotation Matrix $R_{3\times3}$ can now be defined with Calculated $\sin\alpha$ & $\cos\alpha$.

Hence, from Eqn. (1a), we have

$$\begin{pmatrix} x_i'' \\ y_i'' \\ 1 \end{pmatrix} = \begin{pmatrix} (x_i+\Delta x)\cos\alpha - (y_i+\Delta y)\sin\alpha \\ (x_i+\Delta x)\sin\alpha + (y_i+\Delta y)\cos\alpha \\ 1 \end{pmatrix}$$

Or,

$$x_i'' = (x_i+\Delta x)\cos\alpha - (y_i+\Delta y)\sin\alpha \quad \cdots (10\text{-}a)$$

$$y_i'' = (x_i+\Delta x)\sin\alpha + (y_i+\Delta y)\cos\alpha \quad \cdots (10\text{-}b)$$

Substitute the given condition

$$x_i'' = (x_i-9)\,0.7474 - (y_i-1)\,0.6644 \quad \cdots (11\text{-}a)$$

$$y_i'' = (x_i-9)\,0.6644 + (y_i-1)\,0.7474 \quad \cdots (11\text{-}b)$$

To Verify Eqn (11-a) & (11-b).

Choose A Known point (Know for Both x_u-y_u, x_v-y_v System). Let Choose Point $\vec{P}_1(x_1, y_1) = (9,1)$ (Known on x_u-y_u),

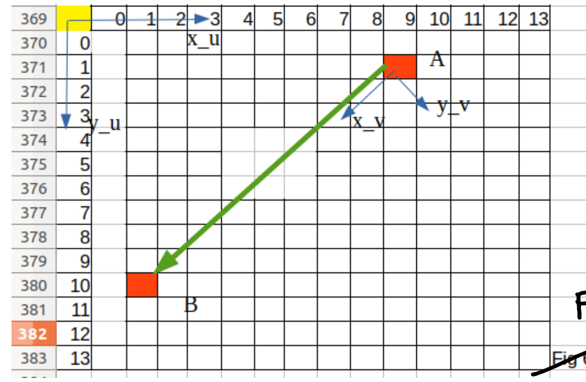


Fig.2

Point $\vec{P}_1$ is also Known to x_v-y_v, which is (0,0).

So, We can test Eqn(11-a), (11-b).

Substitute $x=9, y=1$ into Eqn (11-a)

$$x_i'' = (x_i-9)\,0.7474 - (y_i-1)\,0.6644$$
$$= (9-9)\,0.7474 - (1-1)\,0.6644$$
$$= 0 \times 0.7474 - 0 \times 0.6644$$
$$= 0$$

Similarly,

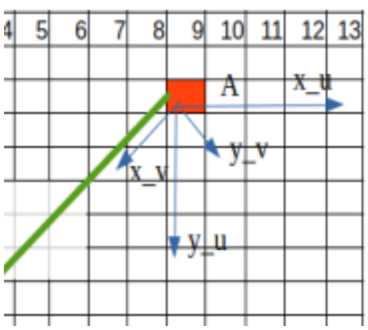
$$y_i'' = (x_i-9)\,0.6644 + (y_i-1)\,0.7474$$
$$= (9-9)\,0.6644 + (1-1)\,0.7474$$
$$= 0 \times 0.6644 + 0 \times 0.7474 = 0$$

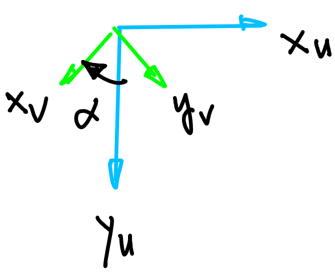
The Result (0,0) matches the actual setup of (x_v, y_v) System. Therefore Formula in Eqn (11-a), (11-b) is confirmed.

Appendix A (Errata)

Note: Translation & Rotation of Local (x_u-y_u) Coordinate System, Leads:

1° Clockwise Rotation to make y_u to overlap with x_v.





Define Clockwise Rotation as a Rotation with negative angle. so

$$R_{3\times3} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \ldots(8\text{-}c)$$

Becoms

$$R_{3\times3} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \ldots(8\text{-}c')$$

Note in general, we do not change Equation, Because (8-c) is general enough to handle α positive or negative.

2° After the Rotation

$y_u = x_v$ and $x_u = y_v$

Therefore,

$$x_i'' = (x_i+\Delta x)\cos\alpha - (y_i+\Delta y)\sin\alpha \quad \ldots(12\text{-}a)$$

$$y_i'' = (x_i+\Delta x)\sin\alpha + (y_i+\Delta y)\cos\alpha \quad \ldots(12\text{-}b)$$

Becomes

$$x_i'' = (x_i+\Delta x)\sin\alpha + (y_i+\Delta y)\cos\alpha \quad \ldots(12\text{-}a)'$$

$$y_i'' = (x_i+\Delta x)\cos\alpha - (y_i+\Delta y)\sin\alpha \quad \ldots(12\text{-}b)'$$

which is the final updated equations. //

please see the spreadsheet Calculation.

(END)