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Coordinate Systems.

1. Establish 3 coordinate Systems. (see Definition & Life cycle in Table 1 & Figure 1)

Table 1.

	Definition	Life Cycle
x_u-y_u	defines everythings	Long Lasting
x_v-y_v	for $\vec{P_i}$ motion from $\vec{P_A}$ to $\vec{P_B}$	for $\vec{P_A}$ to $\vec{P_B}$
x_r-y_r	for Robot Orientation at $\vec{P_i}$	Same as x_v-y_v

x_u-y_u : local1~; x_v-y_v : local2~;

x_r-y_r : Robot~

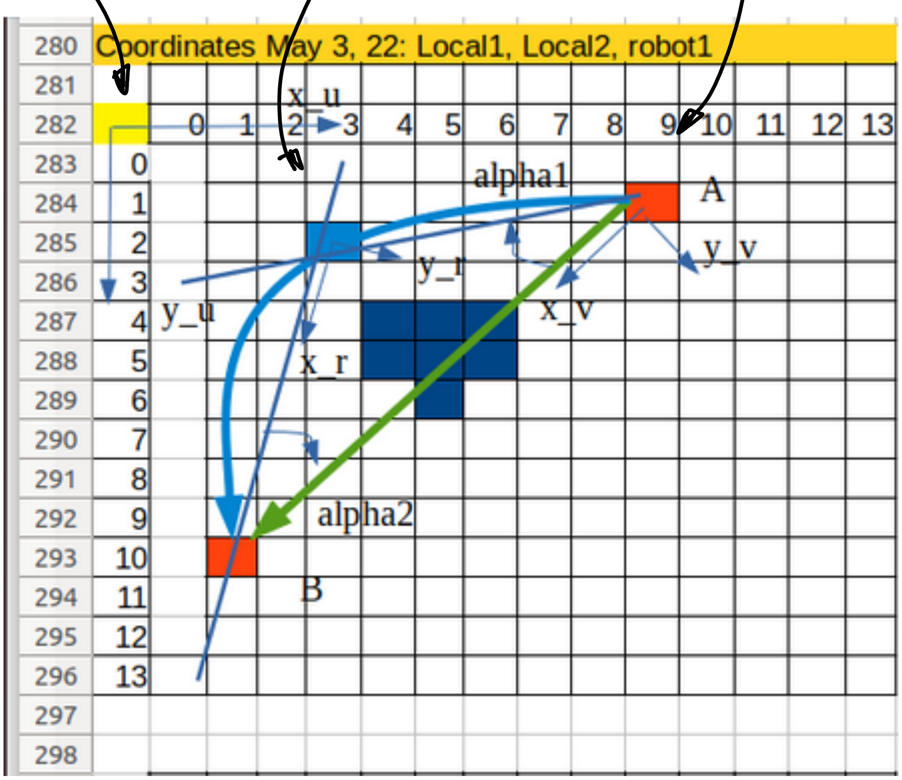

Fig1.

Note1: x_u-y_u is defined to match x-y Coordinate System in Computer Vision, also to match our previous definition and calculations of Angle, Perpendicular distance etc.

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Clear View of X_u-Y_u, X_v-Y_v, X_r-Y_r. in Fig. 2 & 3

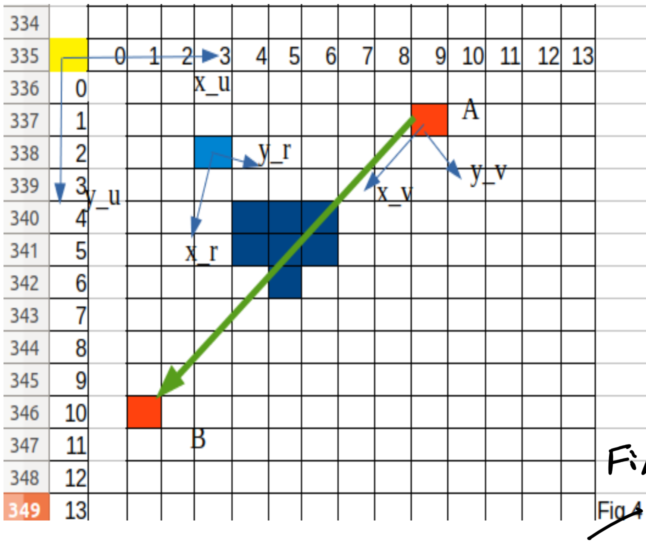

Fig. 2

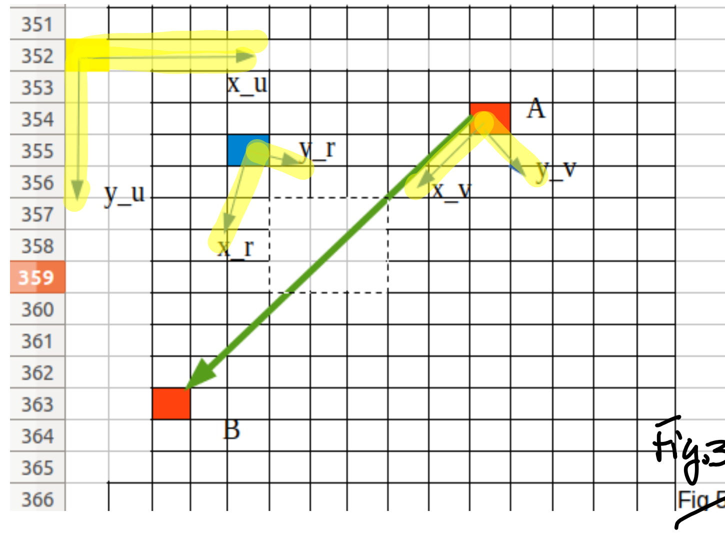

Fig. 3

Now, Define mapping functions

$$\phi_1 : (x_u, y_u) \to (x_v, y_v) \quad \cdots (1)$$

$$\phi_2 : (x_u, y_u) \to (x_r, y_r) \quad \cdots (2)$$

$$\phi_3 : (x_r, y_r) \to (x_v, y_v) \quad \cdots (3)$$

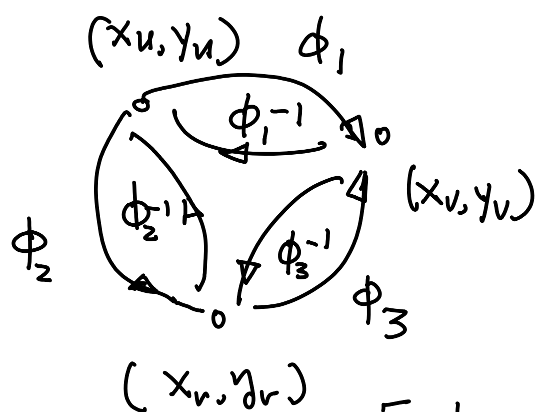

Fig 4.

Consider ϕ_1 mapping function, in Fig 4

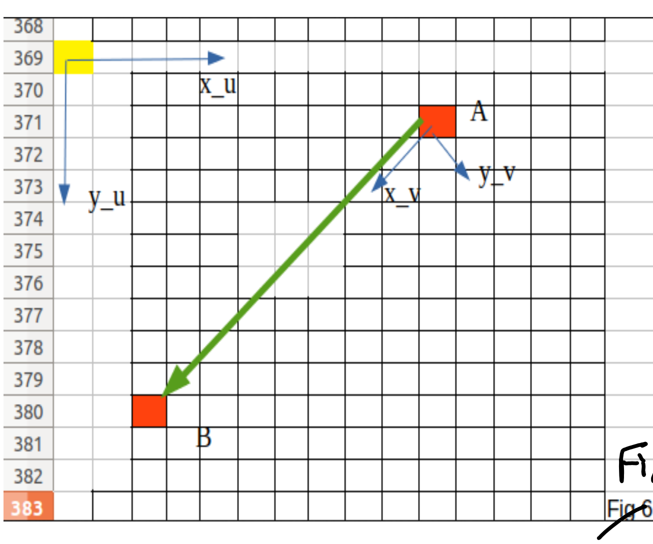

Fig 4

Let $\vec{P_i}(x_i, y_i)$ be an arbitrary point defined under (x_u, y_u), with ϕ_1, we would like to define it under (x_v, y_v).

Step 1. Translate the origin $(0,0)$ of (x_u, y_u) to the origin $(0,0)$ of (x_v, y_v),

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So, the translation is defined as

$$T_{3\times3} = \begin{pmatrix} 1 & 0 & \Delta X \\ 0 & 1 & \Delta Y \\ 0 & 0 & 1 \end{pmatrix} \quad \ldots (4)$$

Hence

$$\begin{pmatrix} x_i' \\ y_i' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \Delta X \\ 0 & 1 & \Delta Y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \\ 1 \end{pmatrix} \quad \ldots (4\text{-}a)$$

Since starting Point $\vec{P_A}(x_a, y_a)$ is Known, so

$$\Delta X = -x_a \quad \ldots (4\text{-}b)$$

$$\Delta y = -y_a \quad \ldots (4\text{-}c)$$

After Translation, the intermmediate result is shown in Fig.5

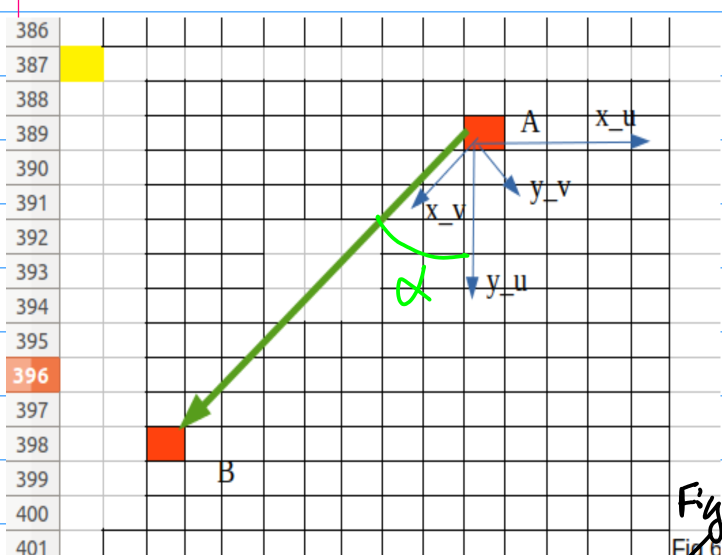

Fig.5

Now, Step 2. Rotation $R_{3\times3}$

$$R_{3\times3} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \ldots (5)$$

Hence

$$\begin{pmatrix} x_i'' \\ y_i'' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_i' \\ y_i' \\ 1 \end{pmatrix} \quad \ldots (5\text{-}a)$$

where the Angle α is marked in Fig.5 it is the Angle Between y_u-axis and the green line ($\vec{P_A}$ to $\vec{P_B}$)

So, write y_u-axis in a unit vector form:

$$\vec{j} = (0, 1) \quad \ldots (6)$$

$$\vec{j} \cdot (\vec{P_B} - \vec{P_A}) = \|\vec{j}\| \|\vec{P_B} - \vec{P_A}\| \cos\alpha$$

$$\cos\alpha = \frac{\vec{j} \cdot (\vec{P_B} - \vec{P_A})}{\|\vec{j}\| \|\vec{P_B} - \vec{P_A}\|} \quad \ldots (7)$$

$$= \frac{(0,1) \cdot (x_B - x_a, y_b - y_a)}{\sqrt{0^2 + 1^2} \cdot \sqrt{(x_b - x_a)^2 + (y_b - y_a)^2}}$$

$$= \frac{y_b - y_a}{\sqrt{(x_b - x_a)^2 + (y_b - y_a)^2}} \quad \ldots (7\text{-}a)$$

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$$\alpha = \cos^{-1}\left(\frac{y_b - y_a}{\sqrt{(x_b - x_a)^2 + (y_b - y_a)^2}}\right) \quad \cdots (7\text{-}b)$$

Therefore,

Property 1: The mapping ϕ_1 (from Local1 Coordinate System X_u-Y_u to Local2 Coordinate System X_v-Y_v) is defined as

$$\begin{pmatrix} x_i'' \\ y_i'' \\ 1 \end{pmatrix} = R_{3\times3}\, T_{3\times3} \begin{pmatrix} x_i \\ y_i \\ 1 \end{pmatrix} \quad \cdots (8)$$

After　　　　Before

where

$$T_{3\times3} = \begin{pmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1 \end{pmatrix} \quad \cdots (8\text{-}a)$$

with

$$\Delta x = -x_a \quad \cdots (8\text{-}b)$$

$$\Delta y = -y_a \quad \cdots (8\text{-}c)$$

And

$$R_{3\times3} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \cdots (8\text{-}c)$$

with

$$\alpha = \cos^{-1}\left(\frac{y_b - y_a}{\sqrt{(x_b - x_a)^2 + (y_b - y_a)^2}}\right) \quad \cdots (8\text{-}d)$$

Write the Above equations in terms of x and y for Python or C/C++ Implementation. we have Lemma 1.

Lemma 1: ϕ_1 mapping (from X_u-Y_u to X_v-Y_v) Can be Written as

$$\begin{cases} x_i'' = (x_i + \Delta x)\cos\alpha - (y_i + \Delta y)\sin\alpha & \cdots (9\text{-}a) \\ y_i'' = (x_i + \Delta x)\sin\alpha + (y_i + \Delta y)\cos\alpha & \cdots (9\text{-}b) \end{cases}$$

Proof:

$$\begin{pmatrix} x_i'' \\ y_i'' \\ 1 \end{pmatrix} = R_{3\times3}\, T_{3\times3} \begin{pmatrix} x_i \\ y_i \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_i + \Delta x \\ y_i + \Delta y \\ 1 \end{pmatrix}$$

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$$= \begin{pmatrix} (x_i+\Delta x)\cos\alpha - (y_i+\Delta y)\sin\alpha \\ (x_i+\Delta x)\sin\alpha + (y_i+\Delta y)\cos\alpha \\ 1 \end{pmatrix}$$

Q.E.D.