CHOOSING FROM GRAPHICAL CHOICE ARCHITECTURES

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Ver: Jun 2020

Abstract. The revealed preference theory often considers choice from sets and introduces subjective notions, such as attention, to explain choice behavior. However, when we only observe choice from sets, we might fail to separate the preference from the effects of those subjective concepts and to answer whether the behavioral properties that are observed in experiments translate to the choice problems studied in the relevant applications. In the real world, the choice often involves, in our terminology, the *choice architectures*, which comprises the available alternatives and the observable information about alternatives, or the circumstances under which choice would be made. In this paper, we specify choice architectures as directed graphs on sets of alternatives and propose a model of choice from such choice architectures. We study both the choice function and correspondence and show that, under two sets of axioms that are similar in nature to the classical axioms, our choice function and correspondence are described, respectively, by two choice procedures incorporating *sorting* alternatives into preference maximization. The sorting procedure can be represented by the *topological sorting* of directed graphs, hence being independent of the preference. Our model is of interest since it separates the physical architecture dependency of choice from the preference maximization, which has substantial implications for the translation of choice, the formation mechanism of consideration sets, and stochastic choice. Later in the paper, we also present the application to shaping demand and revealing equilibrium. In particular, we study how the choice architectures can be an alternative language of game theory, where our choice correspondence reveals the Nash equilibrium. (JEL. D01, D11, D80)

Keywords. Choice, Choice Architectures, Digraphs, Topological Sorting, Leading Decision

I am indebted to Takashi Hayashi for his invaluable advice and guidance. Gratitude are also due to Michele Lombardi and Anna Bogomolnaia for their helpful comments. Financial support from the University of Glasgow through the Adam Smith Business School Scholarship is gratefully acknowledged.

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1. Introduction

The standard revealed preference theory considers choice from *sets*.[1] It evinces a parsimonious stance on what should be observed in the model, yet it has led us to experience identification problems when we try to understand bounded rationality, procedural rationality, and attention structures (e.g., Manzini and Mariotti (2012a); Masatlioglu, Nakajima, and Ozbay (2012)). For example, once attention structures are employed to explain choice data, we often fail to separate preference and the strength of attention-grabbing. Even when the choice data satisfy the Weak Axiom of Revealed Preference (WARP), we still have at least two explanations: (i) the choice is obtained by a preference maximization (Richter (1966)), or (ii) an exogenous order on alternatives describing their strength of attention-grabbing determines the choice while preference is empty. Such identification problems seem to be inevitable when we insist on choice from sets, and introduce subjective concepts to explain choice data. Moreover, since choice problems are entirely described by the given sets of alternatives (i.e., menus), the standard theory suggests only one-dimensional identification of choice problems. That is, the choice problems, where a decision maker (DM) chooses separately from the same menu under different information, become indistinguishable in the standard framework. Hence, it cannot explain whether and how the behavioral properties that we observed in a choice problem translate to the relevant applications. For instance, in game-theoretical context, rules of a game influence players' choice only through determining the way of strategical interaction among players, while the underlying preference of each player is fixed. However, when multiple games are considered, we are unable to tell whether the rules of games modify each player's preference as well, or solely changes the strategical interactions.

In the real world, a DM's choice often involves the physical and observable architectures that comprise alternatives and the choice-relevant information, hence *choice architectures* (Thaler, Sunstein, and Balz (2010)). For example, a DM might browse items through hyperlinks on a shopping site. In task scheduling, the designated rules in which one task necessitates another constrain the arrangements. Choice architectures give us various choice observations even with a fixed set of alternatives, hence providing rich choice data that allow us to isolate more accurate descriptions of choice than what we used to on the domain of sets. That is, if choice architectures are given general structures and employed as primitives, then in this rich domain, we

[1] For a comprehensive summary, see Chambers and Echenique (2016).

can separate the unconditional choices and explain how others depend on exogenous information. Moreover, once we unveil such an architecture dependency, it becomes possible to *lead the choice* via shaping the choice architectures to achieve desirable outcomes. Thus, why not utilize the richness?

We propose a model of choice from a class of choice architectures and study both choice functions and correspondences. Our model specifies choice architectures as directed graphs (digraphs) on sets of alternatives. This is because, apart from those architectures that have the form of digraphs by nature (hyperlink connections, store layout, etc.), we might find that some architectures suggest specific orders over the alternatives (e.g., logical connectives, eligibility in task scheduling or public services), and such orders can be represented by directed graphs. A choice function hence singles out a vertex from every digraph, whereas a choice correspondence assigns a set of vertices to every digraph.

In spirits of the Monotonicity axiom and the Independence of Irrelevant Alternatives (IIA) axiom in the context of choice from sets, we impose two sets of analogous axioms on our choice function and correspondence. They jointly characterize the choice function and correspondence as two specific procedures, both of which involve *sorting* alternatives. Such sorting obeys the acyclic part of the order induced by each given digraph and hence coincides with the *topological sorting* of each digraph with a specific transformation. Intuitively, the choice function is a position-based selection from the sorted lists that admits the choice function from lists studied by Rubinstein and Salant (2006) (RS). That is, given a choice architecture, the DM sorts the alternatives following the order induced by the architecture. He/She then picks the first or last most preferred alternative from the sorted list according to his/her preference. Meanwhile, the choice correspondence is described by the union of alternatives that survive a position-based elimination from each possible sorted list. The elimination is specified by a pair of transitive binary relations $(\succsim_{\mathbf{0}}, \succsim_{\mathbf{1}})$ and incorporates the maximization of a partial order $\succ^*$ given by $\succsim_{\mathbf{0}} \cap \succsim_{\mathbf{1}}$. Given a choice architecture, the DM considers all possible sorted lists that obey the order induced by the digraph. From each sorted list, he/she eliminates the alternatives that are weakly dominated either by $\succsim_{\mathbf{0}}$ from the front or by $\succsim_{\mathbf{1}}$ from behind. The potential choices are then obtained by gathering the remaining alternatives from each sorted list.

Notably, since the topological sorting of digraphs represents the sorting procedure, the sorted lists depend only on each given choice architecture, hence being objective and independent of the preference. Thus, our model achieves the physical separation of the preference maximization and the architecture dependency, and we subsequently explore the significance of such a separation.

In Section 5, we discuss the implications of our model. It is proved that the selection procedure of our choice function is compatible with the elimination procedure of our choice correspondence, which suggests two translation properties that link our choice function and correspondence. Moreover, our model implies the formation mechanism of consideration sets, which can be viewed as an alternative mechanism, yet is closely related to that implied by the iterative search in Masatlioglu and Nakajima (2013). We also signify a possible source of the stochastic description of choice that can unify the approaches of preference maximization and stochastic choice (Manzini and Mariotti (2014)). We then present two strands of applications in Section 6, in terms of shaping demand and revealing equilibrium. We show that (i) the interested party can lead the DM's choice via shaping choice architectures and (ii) our model can be introduced as an alternative formalization of game theory with discrete payoffs, where our choice correspondence reveals the pure-strategy Nash equilibrium as in Chambers, Echenique, and Shmaya (2017).

The rest of the paper is organized as follows. Section 2 includes the formal notations. In Section 3, we present the axioms and the characterization of the choice function. Section 4 provides the full characterization of the choice correspondence and investigates the rationalization of the induced choice correspondence on sets. The implications and applications are studied in Section 5 and Section 6. Section 7 discusses related literature. Proofs are concluded in Appendix A.

2. Preliminaries

Let X be a finite set of alternatives and 2^X denote its power set. For a given positive integer $n \leq \#X$, the set of all n-element subsets of X is denoted by $[X]^n$.

Let $D = D(V, E, \iota, \tau)$, simply $D = D(V, E)$, be the typical digraph on vertices $V \in 2^X \setminus \{\emptyset\}$ with edges $E \subset V \times V$, where E defines an irreflexive relation on V.[2] Whenever $E \neq \emptyset$, the mapping $\iota : E \to V$ maps each $e \in E$ to its first coordinate, while τ maps to the second coordinate. Let $\mathcal{D}$ be the set of all digraphs on the nonempty subsets of X. Given D, the set of vertices and set of edges are denoted as $V(D), E(D)$, respectively. Moreover, let $\mathcal{C}(D), \mathcal{P}(D)$ denote the sets of all cycles and paths in D. We write $uPv \subseteq D$ when the path $P \in \mathcal{P}(D)$ goes through u, v following

[2] That is, $D(V, E, \iota, \tau)$ does not allow any loops or multiple edges.

the directions of every $e \in P$. Unless otherwise stated, we do not distinguish a set V from the digraph $D(V, \emptyset)$, and simply write V. Similarly, given $u, v \in X$, simply write (u, v) instead of $D(\{u, v\}, \{(u, v)\})$.

Given any D, its induced subgraph on $A \subseteq V(D)$ is denoted by $D[A]$.[3] Let $T : \mathcal{D} \to \mathcal{D}$ be the transitive closure operator. Denote by $\mathcal{A} : \mathcal{D} \to \mathcal{D}$ the mapping, which, from every $D \in \mathcal{D}$, deletes all the edges contained in cycles in D. That is,

$$\mathcal{A}(D) := (V(D), E(D) \setminus \{e \mid \exists C \in \mathcal{C}(D), e \in E(C)\}).$$

A digraph D is a *directed acyclic graph* (DAG) if $\mathcal{C}(D) = \emptyset$. A DAG S is a *string* if S is connected and satisfies $\#\{e \mid \iota(e) = v\} \leq 1$, $\#\{e \mid \tau(e) = v\} \leq 1$ for all $v \in V(S)$. The set of all strings in $\mathcal{D}$ is denoted by $\mathcal{S}$. Given a DAG D, a *topological sorting* $\varphi(D)$ is a string on $V(D)$ such that $\iota(e) P \tau(e) \subseteq \varphi(D)$ for any $e \in E(D)$. We identify each topological sorting by a mapping $\varphi : \mathcal{D} \to \mathcal{S}$, and denote by $\Phi(D)$ the set of all topological sorting of a given DAG D (see Figure 1).

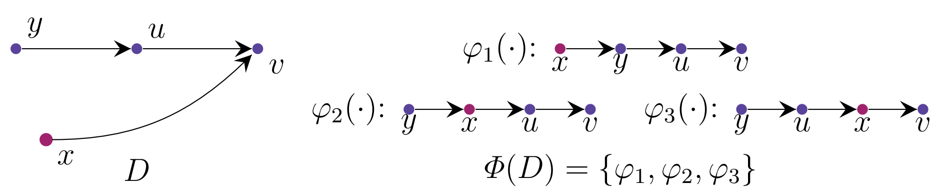

FIGURE 1 – Topological Sorting

Unless otherwise stated, the induced subgraph operator always acts last for any combination of operators. For instance, $\mathcal{A} \circ TD[V] = (\mathcal{A} \circ TD)[V]$.

3. Choice Function

In a collection of non-repeated observations, choice behavior is described by the choice function $z : \mathcal{D} \to X$ such that $z(D) \in V(D)$ for all $D \in \mathcal{D}$. We impose the following two axioms on z.

Dominance of Acyclic Connectivity (DAC)— *For any $V \in 2^X \setminus \{\emptyset\}$ and $v \in V$, if $z(D) \neq v$ for any connected DAG D on V, then $z(\tilde{D}) \neq v$ for all $\tilde{D}$ with $V \subseteq V(\tilde{D})$.*

We refer to the connected DAGs as "simple" architectures, since every connected DAG represents a nonempty partial order on the underlying set of vertices. DAC states that, given a menu, if an alternative is not chosen from all possible simple

[3] $D[A]$ contains all the edges of D which connects the vertices in A.

architectures on the menu, it would never be chosen from extensive menus regardless of the choice architectures. Conversely, if an alternative is chosen from a particular choice architecture, then for any of its sub-menus, it is possible to design an appropriate architecture from which this alternative would be chosen. Hence, DAC implies that the DM's choice must reveal his/her preference if it exists.

Given D, we call a collection of digraphs $\{D_i\}_i$ an *induced partition* of D if (i) $\{V(D_i)\}_i$ defines a partition of $V(D)$ and (ii) $D_j = TD[V(D_j)]$ for every $D_j \in \{D_i\}_i$.[4] Note that every member of an induced partition ($D_j \in \{D_i\}_i$) preserves all the information given in D since, for any $u, v \in V(D_j)$, if there is a $P \in \mathcal{P}(D)$ such that $uPv \subset D$, then $(u, v) \in E(D_j)$.

Independence of Induced Partition (IIP)— *For any $D \in \mathcal{D}$ and any of its induced partitions $\{D_i\}_i$, if $v = z(D) \neq z\left(TD[\{z(D_i)\}_i]\right) = u$, then*
(i) $v \in \{z(D_i)\}_i \implies \{(u,v) \in E\left(TD\right) \Leftrightarrow (v,u) \in E\left(TD\right)\}$;
(ii) $v \in D_j \setminus \{z(D_j)\} \implies \{(z(D_j), v) \in E\left(TD\right) \Leftrightarrow (v, z(D_j)) \in E\left(TD\right)\}$.

IIP considers a compound choice that divides a choice architecture into several sub-architectures then chooses from the alternatives which are chosen from each sub-architecture and compares such a compound choice with that from the original architecture. It states that, *without loss of information*, the choice reversal is allowed only among those pairs of alternatives that are not simply intervened ($(u, v) \in E(TD)$ or $(v, u) \in E(TD)$ exclusively). Intuitively, for some pairs of alternatives, when a given choice architecture is restrictive or stimulative to neither of the alternatives ($(u, v), (v, u) \notin E(TD)$), then the choice might depend on how they are related to the others in the architecture and what is available on the menu. On the other hand, when the given choice architecture is stimulative to both of the alternatives ($(u, v), (v, u) \in E(TD)$), then the DM should be free to implement either of those interventions. In connection with the IIA axiom, IIP requires that omitting the alternatives that are rejected under simple interventions from a choice architecture does not alter the choice in a given instance. The following example gives a simple illustration of IIP.

Example 1 (IIP)**.** Consider the digraph D in Figure 2, where TD is given by including the dashed edges and $\{C_1, S_1\}, \{\{y\}, D_1\}$ define two induced partitions of D.

[4] Caution should be exercised in the definition of an induced partition. In particular, each $D_j \in \{D_i\}_i$ is not necessarily the induced subgraph of D, as the induced subgraph operation is taken after the transitive closure. Hence, D_j becomes the induced subgraph only if $TD = D$.

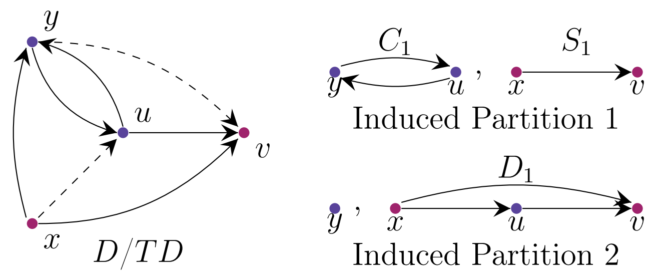

FIGURE 2 – Independence of Induced Partition

(i) Suppose $z(C_1) = y$ and $z(S_1) = v$ in $\{C_1, S_1\}$. Then IIP requires that $z(D) \neq x$ since x is not chosen from $\{v, x\}$ under a single edge. While it does allow $z(D) = u$ because the choice from cycles is not necessarily consistent. (ii) Suppose $z(D_1) = v$, then it must hold that $z(D) = z((y, v))$ for the induced partition 2.

Let $\succsim \subset X \times X$ be a connex and transitive binary relation and let $\delta : X \to \{\mathbf{0}, \mathbf{1}\}$ be an indicator function such that $\delta(x) = \delta(y)$ whenever $x \sim y$. Denote by $z_{\succsim,\delta} : \mathcal{S} \to X$ the choice function that picks the first or the last $\succsim$-maximal vertex from every $S \in \mathcal{S}$, when $\delta = \mathbf{0}$ or $\mathbf{1}$, respectively (Rubinstein and Salant (2006)).

Theorem 1. *A choice function $z : \mathcal{D} \to X$ satisfies DAC and IIP if and only if there exist a unique connex and transitive binary relation $\succsim \subset X \times X$, a unique function $\delta : X \to \{\mathbf{0}, \mathbf{1}\}$, and for each $D \in \mathcal{D}$, there is a topological sorting $\varphi_D \in \Phi(\mathcal{A} \circ TD)$ such that, for every $D \in \mathcal{D}$, $z(D) = z_{\succsim,\delta}(\varphi_D \circ \mathcal{A} \circ TD)$. Moreover, given $\succsim \subset X \times X$, this choice procedure is unique.*

Proof. See Appendix A.2.

Remark 1. In the theorem, the binary relation $\succsim$ and the function $\delta : X \to \{\mathbf{0}, \mathbf{1}\}$ are given in accordance with RS. That is, $\succsim := \succ \cup \sim$ and $\sim := \sim_{\mathbf{0}} \cup \sim_{\mathbf{1}}$, where

$$\begin{aligned}
\succ &:= \{(u, v) \mid z((u, v)) = z((v, u)) = u\}\,;\\
\sim_{\mathbf{0}} &:= \{(u, v) \mid z((u, v)) = u \wedge z((v, u)) = v\}\,;\\
\sim_{\mathbf{1}} &:= \{(u, v) \mid z((u, v)) = v \wedge z((v, u)) = u\}\,.
\end{aligned}$$

Accordingly, for any $v \in X$, we assign $\delta(v) := \mathbf{1}$ if a $u \in X$ exists such that $v \sim_{\mathbf{1}} u$, and $\delta(v) := \mathbf{0}$, otherwise. Then, $\delta : X \to \{\mathbf{0}, \mathbf{1}\}$ is well-defined under IIP.

DAC and IIP characterize the choice function by the procedure that comprises the maximization of $\succsim \subset X \times X$ and the resolution of indifference. Concretely, the selection of topological sorting φ_D and the priority indicator δ characterize the resolution of indifference. Given a choice architecture, the DM sorts the alternatives in a linear order, which is compatible with the information given by the architecture. He/She then picks the most preferred alternative, which presents in the first or last in the sorted list, according to the preference and priority indicator. The following example provides a simple demonstration of this procedure.

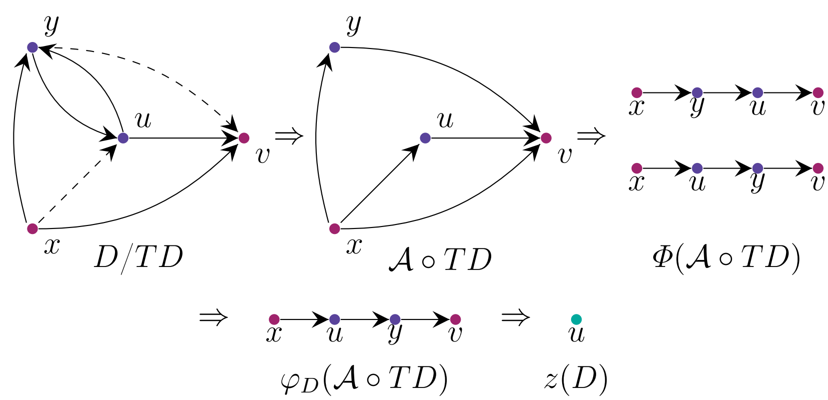

FIGURE 3 – Choice Function from Architectures

Example 2 (Choice Function)**.** Suppose $y \sim_{\mathbf{0}} u \succ v \sim_{\mathbf{1}} x$ and consider the choice architecture D in Figure 3. Since D induces two partial orders, the DM sorts $\{x, y, u, v\}$ into φ_D or $\overrightarrow{xyuv}$. Then, from one of the sorted lists, say φ_D, the DM picks the first $\succsim$-maximal alternative u as the indifferent class $\{u, y\}$ is endowed with $\delta(\cdot) = \mathbf{0}$.

For a given D, $\Phi(\mathcal{A} \circ TD)$ is predetermined yet non-singleton in general. As the choice procedure is not informative about the realization of topological sorting $\varphi_D \in \Phi(\cdot)$, it involves a systematic indeterminacy. The implication of such indeterminacy is discussed in Section 5.3.

4. Choice Correspondence

In this section, we consider the cumulative observation in which choice behavior is described by a choice correspondence $Z : \mathcal{D} \twoheadrightarrow X$ such that $\emptyset \neq Z(D) \subseteq V(D)$ for any $D \in \mathcal{D}$. We postulate the following axioms on the choice correspondence.

Dominance of Acyclic Connectivity* (DAC*)— *For any $V \in 2^X \setminus \{\emptyset\}$ and any $u \in V$, if $u \notin Z(D)$ for all connected DAG D on V, then $u \notin Z(\tilde{D})$ for all $\tilde{D} \in \mathcal{D}$ such that $V \subseteq V(\tilde{D})$.*

Difference in Acyclic Difference (DAD)— *For any $V \in 2^X \setminus \{\emptyset\}$ and any connected DAGs $D, \tilde{D}$ on V, $u \in Z(D), v \in Z(\tilde{D})$ implies $u, v \in Z(D \cap \tilde{D})$.*

The first two axioms regard the variation of choice architectures on a fixed menu, where DAC* is clearly the natural extension of DAC.

If we suppose that two different simple choice architectures $(D, \tilde{D})$ on a fixed menu (V) yield different choices ($u \in Z(D), v \in Z(\tilde{D})$ and $u \neq v$), then it is reasonable to attribute the cause to the difference between those architectures. DAD hence states that, if we remove such difference and preserve only the interventions that those architectures have in common, then the new choice architecture $(D \cap \tilde{D})$ must admit all the alternatives that are chosen separately $(u, v \in Z(D \cap \tilde{D}))$. That is because, a plainer choice architecture must not contradict the reasons that lead to the choices from finer architectures, meaning that choices from a plainer architecture should be as flexible as those from finer architectures.

Independence of Induced Partition* (IIP*)— *For any $D \in \mathcal{D}$, (i) $Z(D) = Z\left(TD\left[\bigcup_i Z(D_i)\right]\right)$ for all induced partition $\{D_i\}_i$; (ii) if $\#V(D) > 2$, and $\bigcup_i Z(D_i) = V(D)$ for every induced partition $\{D_i\}_i$, then $Z(D) = V(D)$.*

IIP* regards the variation of the induced subgraphs of a fixed choice architecture. Clearly, IIP*-(i) is the natural extension of IIP, while IIP*-(ii) rules out a specific situation in which some alternatives are eliminated from a choice architecture, with it being chosen from every possible member of induced partition, only if the architecture is presented as a whole.

The following theorem and proposition provide two equivalent characterizations of choice correspondence from architectures with different primitives. In Theorem 2, a unique pair of transitive binary relations $(\succsim_{\mathbf{0}}, \succsim_{\mathbf{1}})$ and a unique filtration mapping γ characterize the choice correspondence. Meanwhile, in Proposition 3, the choice correspondence is characterized by the unique filtration mapping γ and a unique choice profile $\mathit{\Delta}$ that assigns the chosen alternatives directly to every potential edge.

Given D, $\mathcal{Y}(D) \subseteq [V(D)]^2$ denotes the set of all pairs of vertices contained in the same cycles in D. That is, $\mathcal{Y}(D) := \{\{u, v\} \mid \exists C \in \mathcal{C}(D), u, v \in V(C)\}$. Let $(\succsim_{\mathbf{0}}, \succsim_{\mathbf{1}})$ be a pair of partial orders on X. Denote by $Z_{\succsim_\delta} : \mathcal{S} \twoheadrightarrow X$ the choice correspondence that picks, from each string, all the vertices which are not dominated under $\succsim_{\mathbf{0}}$ from the front, nor under $\succsim_{\mathbf{1}}$ from behind. Formally, for any $S \in \mathcal{S}$,

$$Z_{\succsim_\delta}(S) := \left\{ v \in V(S) \,\middle|\, \forall u \in V(S), \left\{ \begin{array}{l} uPv \subseteq S \Rightarrow \neg(u \succsim_{\mathbf{0}} v) \wedge \\ vPu \subseteq S \Rightarrow \neg(u \succsim_{\mathbf{1}} v) \end{array} \right\} \right\}.$$

Theorem 2. *A choice correspondence $Z : \mathcal{D} \twoheadrightarrow X$ satisfies DAC*, DAD, and IIP* if and only if there exist a connex quasi-transitive binary relation $\mathcal{R} = \succsim_{\mathbf{0}} \cup \succsim_{\mathbf{1}} \cup \sim^*$ and a unique mapping $\gamma : [X]^2 \to [X]^1 \cup \{\emptyset\}$ such that for all $D \in \mathcal{D}$,*

$$Z(D) = \left(\bigcup_{\varphi \in \Phi(\mathcal{A} \circ TD)} Z_{\succsim_\delta} (\varphi \circ \mathcal{A} \circ TD) \right) \cap \Gamma(D);$$

$$\Gamma(D) := V(D) \setminus \left(\bigcup_{\{u,v\} \in \mathcal{Y}(D)} \gamma(\{u, v\}) \right).$$

Moreover, $\succsim_{\mathbf{0}}, \succsim_{\mathbf{1}}$ are transitive, and given $(\succsim_{\mathbf{0}}, \succsim_{\mathbf{1}})$, the expression of Z is unique.

Proof. See Appendix A.3.

Remark 2. In the theorem, binary relations $(\succsim_{\mathbf{0}}, \succsim_{\mathbf{1}})$ and $\sim^*$ are specified as follows:

$$\succsim_{\mathbf{0}} := \{(u, v) \mid v \notin Z((u, v))\}; \quad \succsim_{\mathbf{1}} := \{(u, v) \mid v \notin Z((v, u))\};$$
$$\sim^* := \{(u, v) \mid \{u, v\} = Z((u, v)) = Z((v, u))\}.$$

Then $\sim^*$ is symmetric yet not necessarily transitive, while $\succsim_{\mathbf{0}}, \succsim_{\mathbf{1}}$ are transitive. When $\succ^* := (\succsim_{\mathbf{0}} \cap \succsim_{\mathbf{1}}) \neq \emptyset$, it characterizes the *unconditional choice* in the sense that, if $u \succ^* v$, then v would never be chosen whenever u is available.

The theorem suggests that "selection" and "elimination" are behaviorally compatible, when we consider the cumulative choice data. That is, given a choice architecture, the potential choice can be described by the collective outcome of a specific elimination. From every sorted list that obeys the order induced by the given digraph, the DM rejects an alternative if another alternative is listed in front (or listed behind) that weakly dominates it under $\succsim_{\mathbf{0}}$ (or $\succsim_{\mathbf{1}}$), or it is excluded by $\Gamma(D)$.

Note that the mapping $\gamma : [X]^2 \to [X]^1 \cup \{\emptyset\}$ assigns to each choice architecture a set of alternatives that would never be chosen from the architecture even if some of them might survive the elimination. That is, for particular choice architectures, γ systematically rules out some alternatives, as if the DM ignores them from those choice architectures. This observation is compatible with the notions of limited attention and limited consideration (e.g., Manzini and Mariotti (2014); Lleras, Masatlioglu, Nakajima, and Ozbay (2017)).

Let $\mathcal{E} := (X \times X) \setminus \{(x, x) \mid x \in X\}$ be the set of all potential edges on X. Let $\Delta : \mathcal{E} \twoheadrightarrow \{\mathbf{0}, \mathbf{1}\}$ be a nonempty correspondence. Denote by $Z_\Delta : \mathcal{S} \twoheadrightarrow X$ the choice correspondence that picks, from every string, the vertices which are neither the initials of edges with $\Delta(e) = \{\mathbf{1}\}$ nor the terminals of edges with $\Delta(e) = \{\mathbf{0}\}$ in TS. Formally, for any $S \in \mathcal{S}$,

$$Z_\Delta(S) := \left\{ v \in V(S) \,\middle|\, \forall e \in E(TS), \left\{ \begin{array}{l} \iota(e) = v \Rightarrow \Delta(e) \neq \{\mathbf{1}\} \wedge \\ \tau(e) = v \Rightarrow \Delta(e) \neq \{\mathbf{0}\} \end{array} \right\} \right\}.$$

Proposition 3. *A choice correspondence $Z : \mathcal{D} \twoheadrightarrow X$ satisfies DAC*, DAD, and IIP* if and only if there exist a unique nonempty correspondence $\Delta : \mathcal{E} \twoheadrightarrow \{\mathbf{0}, \mathbf{1}\}$ and a unique mapping $\gamma : [X]^2 \to [X]^1 \cup \{\emptyset\}$ such that*

$$Z(D) = \left(\bigcup_{\varphi \in \Phi(\mathcal{A} \circ TD)} Z_\Delta (\varphi \circ \mathcal{A} \circ TD) \right) \cap \Gamma(D);$$

$$\Gamma(D) := V(D) \setminus \left(\bigcup_{\{u,v\} \in \mathcal{Y}(D)} \gamma(\{u, v\}) \right)$$

for all $D \in \mathcal{D}$. Moreover, given $\Delta : \mathcal{E} \twoheadrightarrow \{\mathbf{0}, \mathbf{1}\}$, the expression of Z is unique.

Proof. See Appendix A.4.1.

The following example gives a demonstration of these equivalent choice procedures.

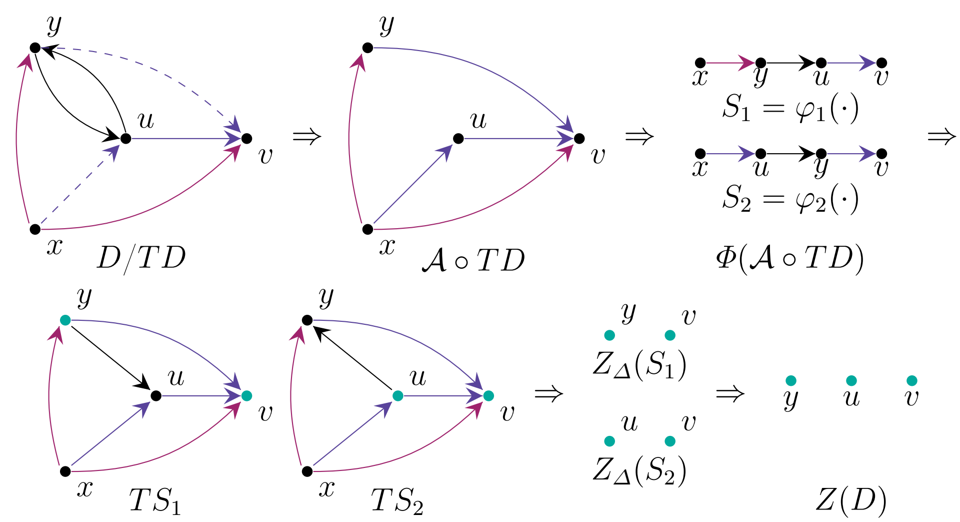

FIGURE 4 – Choice Correspondence from Architectures

Example 3 (Choice Correspondence)**.** Consider D given in Figure 4, where $\Gamma(D) = V(D) = \{x, y, u, v\}$. (i) Suppose $\Delta((x, u)) = \Delta((u, v)) = \Delta((y, v)) = \{\mathbf{0}, \mathbf{1}\}$ (blue edges), $\Delta((u, y)) = \Delta((y, u)) = \{\mathbf{0}\}$ (black edges) and $\Delta((x, y)) = \Delta((x, v)) = \{\mathbf{1}\}$

(red edges). Clearly, $\mathcal{A} \circ TD$ has topological sorting S_1, S_2. The DM eliminates x, u from S_1 since x is the initial of the edges labeled $\Delta(\cdot) = \{\mathbf{1}\}$ $((x, y), (x, v))$, while u is the terminal of the edge labeled $\Delta(\cdot) = \{\mathbf{0}\}$ $((y, u))$ in TS_1. Analogously, x, y are eliminated from S_2. As a result, $Z(D) = \{y, u, v\}$. (ii) Suppose $v \succsim_{\mathbf{1}} x, v \sim^* u, v \sim^* y, y \sim_{\mathbf{0}} u$ and $u \succsim_{\mathbf{0}} x$ in Figure 4. Again, the DM rejects x, u from S_1 as v is listed after x, satisfying $v \succsim_{\mathbf{1}} x$, while y appears before u, satisfying $y \succsim_{\mathbf{0}} u$. Similarly, the DM rejects x, y from S_2. Hence, $Z(D) = \{y, u, v\}$.

In what follows, we establish a connection between the properties of choice from the architectures and those of choice from sets. To this end, we explicitly distinguish a nonempty subset $V \subseteq X$ from the digraph $(V, \emptyset)$ and consider the induced choice correspondence $Z^*(V) := Z((V, \emptyset))$ on nonempty subsets $V \subseteq X$. Then, $Z^* : 2^X \setminus \{\emptyset\} \twoheadrightarrow X$ represents the choice from sets of alternatives without explicit structures. The next corollary shows that, under DAC*, DAD, and IIP*, Z^* is rationalized by a transitive binary relation.

Corollary 1. *If a choice correspondence $Z : \mathcal{D} \twoheadrightarrow X$ satisfies DAC*, DAD and IIP*, then a unique transitive binary relation $\succ^* \subset X \times X$ exists such that $Z^*(V) = \{v \in V \mid \nexists u \in V, u \succ^* v\}$ for all $V \in 2^X \setminus \{\emptyset\}$.*

Proof. See Appendix A.5.

Note that the rationale $\succ^* \subset X \times X$ might lack explanatory power or normative implication as a $v \in X$ might exists such that $\neg(u \succ^* v)$ and $\neg(v \succ^* u)$ for all $u \in X$ even when $\# \succ^*$ is sufficiently large. The following property provides a sufficient condition under which every alternative is related to some alternatives regarding the rationale. We show that, under the following property, Z^* can be described by the maximization of a semiorder.[5]

Relevance— *For any $u, v \in X$, $Z((u, v)) = \{u, v\}$ implies $Z((v, u)) \neq \{u, v\}$.*

Relevance requires that, in binomial choices, the DM should be sensitive or sophisticated to either the alternatives or architectures, such that there is a simple architecture (in this case, a digraph on two vertices with a single edge) under which he/she consistently prefers one alternative over another. In terms of preference, the implication of Relevance is restrictive in the sense that, even if some alternatives are indifferent or incomparable, ties must be broken under some interventions.

[5] See Luce (1956); Sen (1971); Jamison and Lau (1973).

Corollary 2. *If a choice correspondence $Z : \mathcal{D} \twoheadrightarrow X$ satisfies DAC*, DAD, IIP*, and Relevance, then there is a unique semiorder $\succ^S \subset X \times X$ such that $Z^*(V) = \{v \in V \mid \nexists u \in V, u \succ^S v\}$ for all $V \in 2^X \setminus \{\emptyset\}$.*

Proof. See Appendix A.5.

We now establish a connection between the properties of choice from architectures and WARP of classical choice correspondences. In the context of choice from sets, a choice correspondence $Z : 2^X \setminus \{\emptyset\} \twoheadrightarrow X$ satisfies WARP if for any $A, B \subseteq X$ and any $x, y \in A \cap B, x \in Z(A)$ and $y \in Z(B)$ imply $x \in Z(B)$. Similarly, we consider a sufficient condition. The following property states that every binomial choice should consistently yield a single alternative whenever they are intervened by simple choice architectures (connected by a single edge).

Strong Relevance— *$Z((u, v)), Z((v, u))$ are singletons for any $u, v \in X$.*

Proposition 4. *If a choice correspondence $Z : \mathcal{D} \twoheadrightarrow X$ satisfies DAC*, DAD, IIP*, and Strong Relevance then $Z^* : 2^X \setminus \{\emptyset\} \twoheadrightarrow X$ satisfies WARP.*

Proof. See Appendix A.4.2.

5. Discussion

5.1. More on Axioms

The axioms imposed on $z : \mathcal{D} \to X$ and $Z : \mathcal{D} \twoheadrightarrow X$ are the UNCAF type proposed by Chambers, Echenique, and Shmaya (2014). In addition, all of these except IIP* become *vacuous* when we focus on the choice from sets ($z^* : 2^X \setminus \{\emptyset\} \to X$ and $Z^* : 2^X \setminus \{\emptyset\} \twoheadrightarrow X$), meaning that choice data from sets might not be sufficient to falsify properties of choice behavior. IIP* induces the following property on the restricted domain. Being close to the IIA axiom, it states that eliminating some alternatives that are rejected from disjoint submenus does not alter the choice from the given menu.

IIP* on Sets— *For any $V \in 2^X \setminus \{\emptyset\}$, it holds that (i) $Z^*(V) = Z^*(\bigcup_{V_j \in \{V_i\}_i} Z^*(V_j))$ for any partition $\{V_i\}_i$ of V, and (ii) if $\bigcup_{V_j \in \{V_i\}_i} Z^*(V_j) = V$ for every non-trivial partition $\{V_i\}_i$, then $Z^*(V) = V$.*

In the characterization of $Z : \mathcal{D} \twoheadrightarrow X$, DAD acts as a normative imposition rather than a necessary condition. It is imposed since Theorem 2 has a strong normative implication for the rationalization of $Z^* : 2^X \setminus \{\emptyset\} \twoheadrightarrow X$ (the induced choice correspondence from sets), when we consider the connection between Z and the choice from the sets. We present a characterization of Z without DAD.

For every $D \in \mathcal{D}$, let $\mathcal{H}(D) \subseteq [V(D)]^2$ be the collection of 2-element sets of isolated vertices in D. Formally, $\mathcal{H}(D) := \{\{u,v\} \in [V(D)]^2 \mid TD[\{u,v\}] = \{u,v\}\}$.

Proposition 5. *A choice correspondence $Z : \mathcal{D} \twoheadrightarrow X$ satisfies DAC* and IIP* if and only if there exist a unique nonempty correspondence $\Delta : \mathcal{E} \twoheadrightarrow \{\mathbf{0},\mathbf{1}\}$ and a unique pair of mappings $\gamma_C : [X]^2 \to [X]^1 \cup \{\emptyset\}$, $\gamma_I : [X]^2 \to [X]^1 \cup \{\emptyset\}$ such that*

$$Z(D) = \left(\bigcup_{\varphi \in \Phi(\mathcal{A} \circ TD)} Z_\Delta \left(\varphi \circ \mathcal{A} \circ TD\right) \right) \cap \Gamma^*(D);$$

$$\Gamma^*(D) := V(D) \setminus \left(\left(\bigcup_{\{u,v\} \in \mathcal{Y}(D)} \gamma_C(\{u,v\}) \right) \cup \left(\bigcup_{\{u,v\} \in \mathcal{H}(D)} \gamma_I(\{u,v\}) \right) \right)$$

for any $D \in \mathcal{D}$. Moreover, given $\Delta, \gamma_C, \gamma_I$, the expression of Z is unique.

Proof. See Appendix A.4.3.

5.2. Procedural Invariant and Sampling

Here, we study how our characterization of choice translates between the choice function (the non-repeated experiment that forces single choices) and the choice correspondence (sufficiently accumulated choice data). The following corollaries show that the selection $z_{\succsim,\delta} : \mathcal{S} \to X$ and the elimination $Z_{\succsim_\delta} : \mathcal{S} \twoheadrightarrow X$ are compatible, and every singleton sample of Z reveals the unconditional choice observed in Z.[6]

Corollary 3. *Let $\succsim := \succ \cup \sim_{\mathbf{0}} \cup \sim_{\mathbf{1}}$ be the preference obtained in Theorem 1. Then a unique pair of partial orders $Q_{\mathbf{0}}, Q_{\mathbf{1}} \subset X \times X$ exists such that (i) for every $u, v \in X$, $u \succsim v$ if and only if $uQ_{\mathbf{0}}v$ or $uQ_{\mathbf{1}}v$, and (ii) $Z_{Q_\delta}(S) = \{z_{\succsim,\delta}(S)\}$ for all $S \in \mathcal{S}$.*

Proof. See Appendix A.5.

Corollary 4. *Suppose that choice correspondence $Z : \mathcal{D} \twoheadrightarrow X$ and binary relation $\mathcal{R}$ follow the statement in Theorem 2. Let $\hat{z} : \mathcal{D} \to X$ be a sample of Z such that $\hat{z}(D) \in Z(D)$ for every $D \in \mathcal{D}$. Denote by $\hat{\succsim} := \hat{\succ} \cup \hat{\sim}_{\mathbf{0}} \cup \hat{\sim}_{\mathbf{1}}$ the binary relation*

[6] Note that $\hat{z} : \mathcal{D} \to X$ in Corollary 4 does not necessarily satisfy DAC and IIP. The corollary only requires those binary relations to be defined in the same way as in Theorem 1, according to the data given by $\hat{z}$.

defined by $\hat{z} : \mathcal{D} \to X$ following Remark 1. Then, (i) $(\succsim_{\mathbf{0}} \cap \succsim_{\mathbf{1}}) \subseteq \hat{\succ}$, and (ii) for any $u, v \in X$, if $u \succsim_{\mathbf{0}} v$ or $u \succsim_{\mathbf{1}} v$, then $u \hat{\succsim} v$.

Proof. See Appendix A.5.

Regarding the translation of choice behavior, the followings are invariant between the choice function and correspondence: (i) the sorting of choice architectures and the selection/elimination procedure from each sorted list, and (ii) the unconditional choice ($\succ^* := \succsim_{\mathbf{0}} \cap \succsim_{\mathbf{1}}$) observed in data of sufficient size. Moreover, the unconditional choice cannot be underestimated theoretically in choice functions. Hence, when $\Gamma(D) = V(D)$ for all $D \in \mathcal{D}$, the choice correspondence $Z : \mathcal{D} \twoheadrightarrow X$ is a *twofold* extension of $z : \mathcal{D} \to X$. That is, all possible sorted lists (i.e., $\varphi \in \Phi(\mathcal{A} \circ TD)$) are considered in the cumulative observation, with the extended description of preference.

5.3. Consideration Set Formation and Source of Stochasticity

In recent papers, consideration sets have been explicitly derived as the result of models (Masatlioglu and Nakajima (2013); Caplin, Dean, and Leahy (2018)).[7] Some choice-theoretical studies also give insight into the formation of consideration sets.[8] As with these papers, our model has implications for the formation of consideration sets. For every $D \in \mathcal{D}$, Theorem 1 implies that $z(D) \in \{z_{\succsim,\delta}(\varphi_D \circ \mathcal{A} \circ TD) \mid \varphi_D \in \Phi(\mathcal{A} \circ TD)\}$ and does not falsify any outcome from the selection of topological sorting $\varphi_D \in \Phi(\cdot)$. Given the preference $\succsim := \succ \cup \sim_{\mathbf{0}} \cup \sim_{\mathbf{1}}$, the realization of φ_D uniquely determines the choice from D. Since the set $\Phi(\cdot)$ depends on D, $z_{\succsim,\delta}(\varphi_D(\cdot)) = z_{\succsim,\delta}(\varphi'_D(\cdot))$ might hold for some $\varphi_D, \varphi'_D \in \Phi(\cdot)$, or there might be a $\succsim$-maximal alternative $v \in V(D)$ such that $v \neq z_{\succsim,\delta}(\varphi_D(\cdot))$ for all $\varphi_D \in \Phi(\cdot)$. In addition, for some $D, \tilde{D} \in \mathcal{D}$ with $V(D) \subseteq V(\tilde{D})$, it is possible that $z(D) \neq z_{\succsim,\delta}(\varphi_{\tilde{D}} \circ \mathcal{A} \circ T\tilde{D})$ for all $\varphi_{\tilde{D}} \in \Phi(\mathcal{A} \circ T\tilde{D})$ and there is a $\varphi'_{\tilde{D}} \in \Phi(\mathcal{A} \circ T\tilde{D})$ such that $z_{\succsim,\delta}(\varphi'_{\tilde{D}} \circ \mathcal{A} \circ T\tilde{D}) \in V(D)$, as if $z(D)$ is ignored from $\tilde{D}$. These observations support the notion of consideration sets. Consequently, the set $K(D) := \{z_{\succsim,\delta}(\varphi_D \circ \mathcal{A} \circ TD) \mid \varphi_D \in \Phi(\mathcal{A} \circ TD)\}$ can be identified as the consideration set of a given $D \in \mathcal{D}$. Note that, (i) the set $K(D)$ is obtained as the result of preference maximization and summarizes different resolutions

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[7] Caplin et al. (2018) captured the formation of consideration sets by optimizing the Shannon model, whereas in Masatlioglu and Nakajima (2013), consideration sets are obtained by searching process. For other econometric and experimental approaches, see Mehta, Rajiv, and Srinivasan (2003); Caplin, Dean, and Martin (2011).

[8] For instance, *shortlisting* in Manzini and Mariotti (2007) and *categorizing* in Manzini and Mariotti (2012a).

of indifference, whereas the consideration set studied in the literature serves as the subdomain of preference maximization; (ii) our model identifies the consideration sets by isolating its formation mechanism, which is often sidestepped in the literature.

Manzini and Mariotti (2014) proposed a model that links the preference maximization to the stochastic choice data, where the source of stochasticity is given by a probabilistic membership of consideration sets. That is, each alternative is considered with a fixed probability. In an analogy, our model signifies the realization frequency of topological sorting as a source of stochasticity. Given a choice architecture D, Proposition 3 implies that each topological sorting $\varphi_D \in \Phi(\mathcal{A} \circ TD)$ uniquely determines a subset $\nu(\varphi_D) := Z_\Delta(\varphi_D(\cdot)) \subseteq V(D)$, where, for every $e \in E(T \circ \varphi_D \circ \mathcal{A} \circ TD[\nu(\varphi_D)])$, we have $\Delta(e) = \{\mathbf{0}, \mathbf{1}\}$. If we suppose the cumulative observation $Z : \mathcal{D} \twoheadrightarrow X$ is endowed with choice frequency data, then our model is related to the choice frequency in the following manner. Upon generating the data, each $\varphi_D \in \Phi(\cdot)$ might be realized with a probability $p(\varphi_D; D)$, and for every $(u, v) \in \{(u, v) \in \mathcal{E} \mid \Delta((u, v)) = \{\mathbf{0}, \mathbf{1}\}\}$, the image of $\Delta((u, v))$ might collapse to $\{\mathbf{0}\}, \{\mathbf{1}\}$ or $\{\mathbf{0}, \mathbf{1}\}$ with probability $q(\{\mathbf{0}\}; (u, v)), q(\{\mathbf{1}\}; (u, v))$ and $q(\{\mathbf{0}, \mathbf{1}\}; (u, v))$, respectively. Then, under suitable numerical extensions of our axioms, one might characterize the choice frequency by those probabilities, or represent all the frequencies as a function of alternatives (e.g., Luce (1959)). Notably, since $\nu(\varphi_D)$ only contains $\succ^*$-maximal alternatives for every $D \in \mathcal{D}$ and any $\varphi_D \in \Phi(\cdot)$, our model conjectures that the choice reveals to be stochastic because of the random tie-breaking. This implication is in analogy to Aguiar, Boccardi, and Dean (2016), where the choice is described by satisficing with fixed preference and random search orders.

6. Applications

In the abstract, our model can be interpreted as choice under exogenous irreflexive orders, which are identified as digraph architectures. The architecture dependency of choice can be exercised to a wide spectrum of choice-relevant studies due to the abundance of real-world objects, rules, and information that can induce orders over the alternatives in question. In what follows, we discuss two strands of applications in terms of demand and equilibrium.

6.1. Choice Leading and Manipulation

One major strand of application is that, with the architecture dependency being unveiled, interested parties can utilize the real-world objects or information (e.g., hyperlink connections, rules, user ratings) to lead the choice intentionally by shaping the choice architectures.[9] Here, we focus our argument on frequent choice situations, hence on the choice correspondence $Z : \mathcal{D} \twoheadrightarrow X$. The following result shows the possibility of such choice leading.

Proposition 6. *Suppose $Z : \mathcal{D} \twoheadrightarrow X$ satisfies DAC*, DAD, and IIP*. Then, for every nonempty $V \in 2^X$, there exist unique subsets $Core(V), Cl(V) \subseteq V$ such that (i) $Core(V) \subseteq Z(D) \subseteq Cl(V)$ for any $D(V,E) \in \mathcal{D}$ and (ii) for every $V^* \subseteq Cl(V)$, a connected DAG $D^*(V,E^*) \in \mathcal{D}$ exists such that $(Core(V) \cup V^*) \subseteq Z(D^*)$. Moreover, if $(Core(V) \cup V^*) \subset Cl(V)$, there is a $D_*(V,E_*) \in \mathcal{D}$, in which $\{e \in E_* \mid \iota(e) = v \vee \tau(e) = v\} \neq \emptyset$ for all $v \in V$, such that $(Core(V) \cup V^*) \subseteq Z(D_*) \subset Cl(V)$.*

Proof. See Appendix A.4.4.

In the proposition, for a given menu V, (i) clarifies the spectrum of manipulation in the sense that the alternatives in $V \setminus Cl(V)$ or in $Core(V)$ are rejected or selected by the DM solely according to his/her preference; hence, the variation of architecture cannot alter the choice involving those alternatives. Probing further to (ii), given any submenu in that spectrum, a connected DAG is sufficient to lead the choice to include the submenu, while the latter part of (ii) reveals that, by a certain class of architectures, one can also lead the DM *not to choose* particular alternatives.

Incorporating the discussion on the source of stochasticity, architecture shaping might also be related to leading the choice frequency. We now demonstrate both the leading choice and, under some simplifying assumptions, the leading choice frequency.

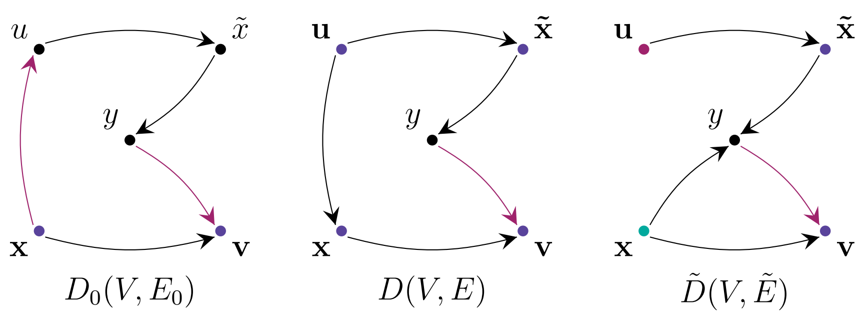

Figure 5 – Leading Choice (Frequency)

[9] Manzini, Mariotti, and Tyson (2011) studied the choice manipulation problem in two-stage threshold framework. In their paper, the manipulation is specified by varying one psychological variable with the other variables being fixed; hence, they are implicit on how the manipulation is exercised. For the two-stage threshold model, see also Manzini, Mariotti, and Tyson (2013).

Example 4 (Leading Choice)**.** Let a DM's choice behavior $Z : \mathcal{D} \twoheadrightarrow X$ satisfy DAC*, DAD, and IIP* in cumulative choice data. Consider the menu $V = \{u, v, x, \tilde{x}, y\} \subset X$, where it reveals $xR^I_{\mathbf{0}}\tilde{x}$, $vR^P_{\mathbf{0}}u$, $vR^P_{\mathbf{0}}x(\tilde{x})$, $vR^P_{\mathbf{1}}y$, $x(\tilde{x})R^P_{\mathbf{0}}u$, $y \sim^* x$ and $y \sim^* u$. By default, V is endowed with the architecture $D_0(V, E_0)$ given in Figure 5. Since D_0 has a unique topological sorting $\varphi_0(D_0) = \overrightarrow{xu\tilde{x}yv}$, we have $Z(D_0) = Z_\Delta(\varphi_0(D_0)) = \{x, v\}$. Suppose the interested party intends to ensure $u, \tilde{x}$ *would be chosen as well*. This could be managed by shaping the architecture to $D(V, E)$ in the figure. In fact, D has topological sorting $\varphi_1(D) = \overrightarrow{ux\tilde{x}yv}$, $\varphi_2(D) = \overrightarrow{u\tilde{x}xyv}$, and $\varphi_3(D) = \overrightarrow{u\tilde{x}yxv}$, which yield $Z_\Delta(\varphi_1(D)) = \{u, x, v\}$, $Z_\Delta(\varphi_2(D)) = Z_\Delta(\varphi_3(D)) = \{u, \tilde{x}, v\}$. As a result, $Z(D) = \bigcup_{i\in\{1,2,3\}} Z_\Delta(\varphi_i(D)) = \{u, v, x, \tilde{x}\}$.

Example 5 (Leading Frequency)**.** To simplify the argument, assume that in every i.i.d. choice instance, for every architecture $\hat{D}$, (i) the DM implements a topological sorting $\varphi \in \Phi(\mathcal{A} \circ T\hat{D})$ with equal probability and picks an alternative w uniformly from $Z_\Delta(\varphi(\cdot))$, and (ii) the realization of φ and the selection of w are i.i.d. Let $p(w, \hat{D})$ be the probability of w being chosen from $\hat{D}$, then

$$p(w, \hat{D}) := \sum_{\varphi_i \in \left\{\varphi | w \in Z_\Delta(\varphi \circ \mathcal{A} \circ T\hat{D}))\right\}} \frac{1}{\left(\#\Phi(\mathcal{A} \circ T\hat{D})\right)\left(\#Z_\Delta(\varphi_i \circ \mathcal{A} \circ T\hat{D})\right)}.$$

Hence, for $D(V, E)$ in Figure 5, we have $p(x, D) = 1/9$ and $p(u, D) = 1/3$. In the figure, $\tilde{D}(V, \tilde{E})$ has topological sorting $\tilde{\varphi}_1(\tilde{D}) = \overrightarrow{xu\tilde{x}yv}$, $\tilde{\varphi}_2(\tilde{D}) = \overrightarrow{ux\tilde{x}yv}$, and $\tilde{\varphi}_3(\tilde{D}) = \overrightarrow{u\tilde{x}xyv}$, where we have $Z_\Delta(\tilde{\varphi}_1(\tilde{D})) = \{x, v\}$, $Z_\Delta(\tilde{\varphi}_2(\tilde{D})) = \{u, x, v\}$, and $Z_\Delta(\tilde{\varphi}_3(\tilde{D})) = \{u, \tilde{x}, v\}$. Hence, we have $1/9 = p(x, D) < p(x, \tilde{D}) = 5/18$ and $1/3 = p(u, D) > p(u, \tilde{D}) = 2/9$. As a result, the interested party can *stimulate the frequency* of x and *reduce* that of u intentionally by shaping the architecture to $\tilde{D}(V, \tilde{E})$.

6.2. Choice Architectures as Games

Chambers et al. (2017) studied a sufficient condition under which the revealed preference formalization can reveal strategic group behavior, such as the Nash equilibrium. Similarly, in the following, we apply choice architectures as an alternative language of games with discrete strategies and present how our model reveals the pure-strategy Nash equilibrium and the competitive equilibrium.

Let A, B be two DMs and $V = \{u, v, x, y\}$ be the set of allocations. Suppose the DMs' preferences are given by the quasi-transitive binary relations $\succsim^A, \succsim^B \subset V \times V$. Given $V, \succsim^A, \succsim^B$, for each DM i, define a correspondences $\Delta^i : V \times V \twoheadrightarrow \{\mathbf{0}, \mathbf{1}\}$ and a mapping $\gamma^i : [V]^2 \to [V]^1 \cup \{\emptyset\}$ by

$$(*)\qquad \begin{aligned} w \succ^i \tilde{w} &\Leftrightarrow \left\{ \begin{array}{c} \Delta^i((w,\tilde{w})) = \{\mathbf{0},\mathbf{1}\} \wedge \Delta^i((\tilde{w},w)) = \{\mathbf{1}\} \\ \wedge\, \gamma^i(\{w,\tilde{w}\}) = \{\tilde{w}\} \end{array} \right\}; \\ w \sim^i \tilde{w} &\Leftrightarrow \left\{ \begin{array}{c} \Delta^i((w,\tilde{w})) = \{\mathbf{0},\mathbf{1}\} \wedge \Delta^i((\tilde{w},w)) = \{\mathbf{0},\mathbf{1}\} \\ \wedge\, \gamma^i(\{w,\tilde{w}\}) = \{\emptyset\} \end{array} \right\}. \end{aligned}$$

Let $D_A(V, E_A), D_B(V, E_B)$ be the digraphs given in Figure 6.

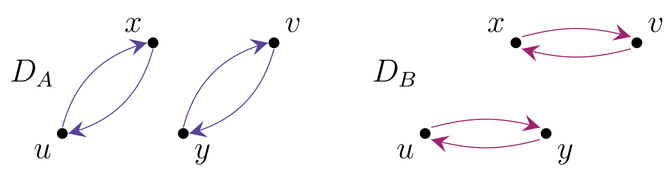

FIGURE 6 – 2×2-Games

Now, we consider the set of allocations $G(\{D_i\}_i, \{\succsim^i\}_i) := Z^A(D_A) \cap Z^B(D_B)$, where, for $i = A, B$, $Z^i : \mathcal{D} \twoheadrightarrow V$ denotes the choice correspondence given in Proposition 3 with respect to (Δ^i, γ^i) defined in $(*)$.

Example 6. (i) Suppose $x \succ^A u$, $u \succ^A v$, $v \succ^A y$, and $y \succ^B u$, $u \succ^B v$, $v \succ^B x$. Then, by $(*)$, $Z^A(D_A) = \{x, v\}$, $Z^B(D_B) = \{y, v\}$. Hence, $G(\{D_A, D_B\}, \{\succ^A, \succ^B\}) = \{v\}$. (ii) Suppose $u \succ^A v, v \succ^A x, x \succsim^A y$ and $v \succ^A u, u \succ^A x, x \succsim^A y$ with $\succsim^A, \succsim^B$ being transitive. Then, we have $G(\cdot) = Z^A(D_A) = Z^B(D_B) = \{u, v\}$. (iii) Suppose $u \sim^A v, v \succ^A x, x \sim^A y$ and $x \sim^A y, y \succ^A u, u \sim^A v$ with $\succsim^A, \succsim^B$ being transitive. Then, $Z^A(D_A) = \{u, v\}$, $Z^B(D_B) = \{x, y\}$, and hence $G(\cdot) = \emptyset$.

Under D_A and D_B, the preference profiles $\succ^A, \succ^B$ given in (i), (ii), and (iii) represent the *Prisoner's Dilemma*, *Battle of Genders*, and *Matching Pennies*, respectively. In each case, $G(\cdot)$ captures the pure-strategy Nash equilibria. Regarding this formalization, the following intrinsic points should be noted.

Remark 3. From each DM's perspective, the choice problem is not strategic per se, as it is degenerated to the choice from architectures. The strategic interaction, however, is implicitly captured by the followings.

(a) The choice objects are given as allocations of payoffs. Hence, for each i, the strict part of $\succsim^i$ might differ from i's strict preference over payoffs (e.g., $\succsim^i$ might incorporate envy).
(b) For each i, the intuition of (Δ^i, γ^i) in $(*)$ is that, given $u \succ^i v$, if the game allows someone (including i) to alter the allocation from v to u, or it is free

of alteration, then since i can always alter, u is the only possible allocation ($\mathit{\Delta}((v,u)) = \{\mathbf{1}\}, \gamma^i(\{u,v\}) = \{v\}$). If someone (including i) can commit the alteration (u,v), then i must admit both u, v ($\mathit{\Delta}((u,v)) = \{\mathbf{0}, \mathbf{1}\}$), as other players might make the alteration even if i would not.

(c) In Figure 6, each $D_i(V, E_i)$ describes the possible alterations of i's that are allowed by the game. For example, $(x,y),(y,x) \notin E_A$ as these alterations require B to change his/her playing strategy.

Since the size of players is irrelevant to Remark 3-(a)(b)(c), the formalization and the property of $G(\cdot)$ can be naturally extended to any finite set of players N.

Let V be the finite set of allocations in question. The set of players is denoted by N, and each $i \in N$ is described by the quasi-transitive preference $\succsim^i \subset V \times V$. For each $i \in N$, define a correspondences $\mathit{\Delta}^i : V \times V \twoheadrightarrow \{\mathbf{0}, \mathbf{1}\}$ and a mapping $\gamma^i : [V]^2 \to [V]^1 \cup \{\emptyset\}$ following $(*)$, and let D_i be the digraph on V with $E(D_i)$ containing all the edges on the allocations between which i can make the alteration. We consider the game $\mathcal{G} = (V, N, \{(\mathit{\Delta}^i, \gamma^i)\}_{i\in N}, \{D_i\}_{i\in N})$ and the set $G(\mathcal{G}) := \bigcap_{i\in N} Z^i(D_i)$, where each $Z^i : \mathcal{D} \twoheadrightarrow V$ is the correspondence given in Proposition 3 with respect to $(\mathit{\Delta}^i, \gamma^i)$.

Proposition 7. *$G(\mathcal{G})$ is the set of all pure-strategy Nash equilibria of $\mathcal{G}$.*

Proof. See Appendix A.4.5.

Suppose D_i is the complete digraph for each $i \in N$, then by $(*)$, each $Z^i(D_i)$ coincides with the maximization of $\succsim^i$. Hence, this formalization can also accommodate the individual choice with $G(\cdot) := \bigcap_{i\in N} Z^i(D_i)$ revealing the competitive equilibrium.

Moreover, given a set of allocations V and the preference profile $\{\succsim^i\}_{i\in N}$, Proposition 6 implies that, if $\bigcap_{i\in N} Cl^i(V) \neq \emptyset$, then there exists a game $\{D_i\}_{i\in N}$ that has a pure-strategy Nash equilibrium. Similarly, if $\bigcap_{i\in N} Core^i(V) \neq \emptyset$, then both types of equilibrium always exist for any form of game.

Integrating the leading choice with the above discussion, given a discrete set of allocations and the preference profile of N players, a social planner can lead a desirable Nash equilibrium to exist (e.g., mechanism design or implementation) by designing specific rules D_i as choice architectures for each $i \in N$. The related axiomatization then becomes equivalent to such an architecture design. A straightforward example is the rights structures proposed by Koray and Yildiz (2018).

6.3. Price as A Signal

Price is also an example of the information that induces orders (not necessarily linear) over the alternatives as choice architectures.[10] Price is often incorporated into the budget constraint in the standard revealed preference theory. Theorem 1, however, can accommodate the preference maximization in a discrete choice with the *signalling effect* of price (Milgrom and Roberts (1986)).

In this case, $u \sim_{\mathbf{0}} v$ indicates that the DM is indifferent to u, v and thus prefers the one with the lower price, while $u \sim_{\mathbf{1}} v$ suggests he/she is unable to compare u, v and is convinced that the higher price reflects the higher quality such that he/she tends to compromise on price. Consequently, $\succsim$ divides X into two categories, where every $\sim_{\mathbf{0}}$-indifferent set does not overlap with any $\sim_{\mathbf{1}}$-indifferent set (Claim 3), meaning that the DM always prefers one alternative over the other between two categories. Given a choice architecture ($D \in \mathcal{D}$) and the induced price perception as being a linear order over the underlying menu (φ_D), the procedure $z_{\succsim,\delta} : \mathcal{S} \to X$ maximizes the preference $\succsim$ and resolves $\sim_{\mathbf{0}}, \sim_{\mathbf{1}}$ by minimizing the price (expenditure) and maximizing the quality signalling, respectively.[11]

7. Related Literature

The study of choice with explicit information was pioneered by Rubinstein and Salant (2006, 2012); Salant and Rubinstein (2008). Their earlier paper considered choice with linear orders over alternatives (hence, from lists). Our model is a generalization in the sense that digraphs represent the orders that are not necessarily linear, or even cyclic. For the choice function, our selection procedure from sorted lists coincides with their characterization, which can also accommodate Simon (1955). Their later paper studied choice with choice-relevant yet alternative-irrelevant properties in terms of frames. The major difference between frames and architectures is that we specify architectures in a homogeneous way as digraphs to allow our model to be more translatable to various sources of architectures. Rubinstein and Salant (2012) proposed a general model in which a DM reveals different preferences under different

[10] For instance, given a menu V, let $(u, v) \in E(D)$ only if the hundreds digit of the price of v exceeds that of u. Then, the digraph D represents a nonlinear order on V. For the theoretical foundation of such a specification, see Matějka and McKay (2015).

[11] In marketing literature, a similar interpretation is reported, e.g., in Wathieu and Bertini (2007). For a more stylised analysis, see Erdem, Keane, and Sun (2008); Schmidbauer and Stock (2018).

frames. In our model, the description of preference is consistent throughout, and the realization of topological sorting explains the revealed inconsistency of choice.

Another strand of literature related to our model is the searching approach (Caplin et al. (2011); Masatlioglu and Nakajima (2013)). In Masatlioglu and Nakajima (2013), iterative searching serves as a formation mechanism of consideration sets. Neither their choice function nor ours is informative about the identification of a search path or the realization of topological sorting. In our model, however, for every given choice architecture, the potential topological sorting $\varPhi(\mathcal{A} \circ TD)$ is predetermined by D, meaning that one can predict the candidates of choice $\bigcup_{\varphi \in \varPhi(\cdot)} z_{\succsim,\delta}(\varphi(\cdot))$. If each $z_{\succsim,\delta}(\varphi(\cdot))$ varies sufficiently, one might infer the realized φ_D from $z(D) = z_{\succsim,\delta}(\varphi_D(\cdot))$. A limitation of our model is that we implicitly assume the DM would inspect the entire given architecture.[12] Hence, our model is in an analogy of the hybrid of the fixed-sample -size search and marginal search. los Santos, Hortaçsu, and Wildenbeest (2012), for example, provided an empirical study on these two types of searching.

Probing further, Manzini and Mariotti (2007, 2012a,b) are seminal in incorporating choice procedures into the preference maximization. In Manzini and Mariotti (2007) (RSM), the choice is explained by the sequential maximization of asymmetric binary relations. Our choice function is related to the canonical RSM in the sense that the DM maximizes his/her preference $\succsim = \succ \cup \sim_{\mathbf{0}} \cup \sim_{\mathbf{1}}$ in the first stage and resolves the second stage among $\sim_{\mathbf{0}}$-indifferent (resp., $\sim_{\mathbf{1}}$-indifferent) alternatives. By definition of $\sim_{\mathbf{0}}$ and $\sim_{\mathbf{1}}$, the second stage can also be considered as an order maximization with respect to the topological sorting.

A. Proofs

A.1. Primitives

Given a $D \in \mathcal{D}$, let $V_C(D)$ be the set of all vertices contained in cycles in D. Formally, $V_C(D) := \{v \in V(D) \mid \exists C \in \mathcal{C}(D), v \in V(C)\}$. We say a set of vertices $A \subseteq V(D)$ satisfies property $\mathcal{L}$ in D, denoted by $A \vDash \mathcal{L}(D)$, if for any $u, v \in A$,

$$\exists P \in \mathcal{P}(D),\, uPv \subset D \iff \nexists P' \in \mathcal{P}(D),\, vP'u \subset D.$$
$$((u,v) \in E(TD) \iff (v,u) \notin E(TD))$$

[12] In some situations, this might be resolved practically. For instance, the server of an online shopping site (e.g., Amazon) records the browsing history of each DM. According to the data, one can construct an actual architecture by taking the induced subgraph over the reviewed pages and including all the one-step-ahead pages.

Claim 1. *For any $D \in \mathcal{D}$, if $V(D) \vDash \mathcal{L}(D)$, then D is a DAG and has a unique topological sorting $\varphi \in \Phi(D)$.*

Proof. Let D be an arbitrary digraph that satisfies $V(D) \vDash \mathcal{L}(D)$. Suppose $\mathcal{C}(D) \neq \emptyset$, and fix an arbitrary $C \in \mathcal{C}(D)$. Then, for any $u, v \in V(C)$, it holds that

$$\exists P \in \mathcal{P}(C), uPv \subset C \iff \exists P' \in \mathcal{P}(C), vP'u \subset C.$$

This contradicts to $V(D) \vDash \mathcal{L}(D)$. Hence, D is a DAG as $\mathcal{C}(D) = \emptyset$.

Since D is a DAG under $V(D) \vDash \mathcal{L}(D)$, it follows that $\Phi(D) \neq \emptyset$.[13] Suppose $\varphi, \tilde{\varphi} \in \Phi(D)$ and $\varphi \neq \tilde{\varphi}$. Then, there must be $u, v \in V(D)$ such that $(u, v) \in T\varphi(D)$ and $(v, u) \in T\tilde{\varphi}(D)$. Meanwhile, $V(D) \vDash \mathcal{L}(D)$ implies that $(u, v) \in E(TD)$ or $(v, u) \in E(TD)$ holds exclusively in TD. As a result, either $E(T\varphi(D)) = E(T\tilde{\varphi}(D))$, or one of $\varphi, \tilde{\varphi}$ is not a topological sorting of D. By contradictions, $\varphi, \tilde{\varphi} \in \Phi(D)$ implies $\varphi = \tilde{\varphi}$. Hence, D has a unique topological sorting. Q.E.D.

A.2. Theorem 1

Since it is easy to verify that $z(D) = z_{\succsim,\delta}(\varphi \circ \mathcal{A} \circ TD)$ satisfies DAC and IIP, we show the sufficiency part. The proof is conducted by the following series of claims.

For every $u, v \in X$, define the following binary relations.

$$\begin{aligned}
\succ &:= \{(u, v) \mid z((u, v)) = z((v, u)) = u\}; \\
\sim_{\mathbf{0}} &:= \{(u, v) \mid z((u, v)) = u \wedge z((v, u)) = v\}; \\
\sim_{\mathbf{1}} &:= \{(u, v) \mid z((u, v)) = v \wedge z((v, u)) = u\}.
\end{aligned}$$

Define accordingly $\succsim := \succ \cup \sim_{\mathbf{0}} \cup \sim_{\mathbf{1}}$, then $\succsim \subset X \times X$ is connex on X.

Claim 2. *$\succ$, $\sim_{\mathbf{0}}$, and $\sim_{\mathbf{1}}$ are transitive.*

Proof. Let $u, v, w \in X$ be arbitrary and denote by $\overrightarrow{uvw}$ the string, say S, with $V(S) = \{u, v, w\}$, $E(S) = \{(u, v), (v, w)\}$.

(a) Suppose $u \succ v, v \succ w$, and consider $S = \overrightarrow{uvw}, S' = \overrightarrow{wvu}$. Then, we have $z((u, v)) = u, z((v, w)) = v$. Since $\{(u, v), \{w\}\}$ and $\{(v, w), \{x\}\}$ define two induced partitions of S, by IIP, we have $z(TS[\{u, w\}]) = z(S) = z(TS[\{u, v\}])$. As a result, $z((u, w)) = z((u, v)) = u$. Similarly, it follows that $z(TS'[\{v, u\}]) = z(S') = z(TS'[\{w, u\}])$. Hence, $z((w, u)) = z((v, u)) = u$. That is, $u \succ w$.

(b) Suppose $u \sim_{\mathbf{0}} v, v \sim_{\mathbf{0}} w$. By IIP, we have $z(TS[\{u, w\}]) = z(TS[\{v, u\}]) = Z(S)$. This implies $z((u, w)) = z((u, v)) = u$ by definition of $\sim_{\mathbf{0}}$. Analogously, it holds

[13] By Szpilrajn extension theorem, it is a well-know result that every DAG has topological sorting.

that $z(TS'[\{w,u\}]) = z(S') = z(TS'[\{w,v\}])$, which implies $z((w,u)) = z((w,v)) = w$. Hence, $u \sim_{\mathbf{0}} w$. (c) The transitivity of $\sim_{\mathbf{1}}$ follows similarly. Q.E.D.

Claim 3. *For every $u, v, w \in X$, it is impossible to have $u \sim_{\mathbf{0}} v$ and $v \sim_{\mathbf{1}} w$.*

Proof. Assume that $u \sim_{\mathbf{0}} v$ and $v \sim_{\mathbf{1}} w$. Let $S = \overrightarrow{uwv}$ and $S' = \overrightarrow{vuw}$. For S, consider induced partitions $\{(u,v), \{w\}\}$ and $\{(v,w), \{u\}\}$. By IIP, it follows that $z(TS[\{u,w\}]) = z(S) = z(TS[\{u,v\}])$. Hence, $z((u,w)) = z((u,v)) = u$. Meantime, for S', consider induced partitions $\{(v,w), \{u\}\}$ and $\{(u,v), \{w\}\}$. Then, we have $z(TS'[\{u,w\}]) = z(S') = z(TS'[\{v,w\}])$. As a result, $z((u,w)) = z((v,w)) = w$. This contradicts to $z((u,w)) = u$ obtained in $S = \overrightarrow{uwv}$. Q.E.D.

Under Claim 2, $\sim_{\mathbf{0}}$ and $\sim_{\mathbf{1}}$ are symmetric and transitive. Since $\succsim \subset X \times X$ is connex on X, it follows Claim 3 that, for every $\succsim$-indifferent set A, if there exist $u, v \in A$ with $u \sim_{\mathbf{0}} v$, then $x \sim_{\mathbf{0}} y$ for all $x, y \in A$. Consequently, define a bivalent function $\delta : X \to \{\mathbf{0}, \mathbf{1}\}$ by the following, then $\delta : X \to \{\mathbf{0}, \mathbf{1}\}$ is well-defined.

$$\delta(v) := \begin{cases} \mathbf{1}, & \exists u \in X, \{u \neq v \wedge u \sim_{\mathbf{1}} v\} \\ \mathbf{0}, & \text{otherwise} \end{cases} . \qquad \forall v \in X$$

For a given $\succsim$-indifferent set A, simply write $A \in X/{\sim_{\mathbf{0}}}$ (resp., $A \in X/{\sim_{\mathbf{1}}}$) if $\delta(v) = \mathbf{0}$ (resp., $\mathbf{1}$) for any $v \in A$.

Claim 4. *$z(D)$ is $\succsim$-maximal in $V(D)$ for any $D \in \mathcal{D}$.*

Proof. Let $x, y \in X$ be arbitrary and suppose $x \succ y$. By definition of $\succsim$, $z((x,y)) = z((y,x)) = x$. Note that the possible connected DAG on $\{x,y\}$ is (x,y) or (y,x). Hence by DAC, $z(D) \neq y$ for any D with $x, y \in V(D)$. That is, for any $D \in \mathcal{D}$ and any $v \in V(D)$, if there is a $u \in V(D)$ such that $u \succ v$, then $v \neq z(D)$. Q.E.D.

Given a $D \in \mathcal{D}$, let $M(D)$ be the set of all $\succsim$-maximal vertices in $V(D)$. Formally,

$$M(D) := \{v \in V(D) \mid \nexists u \in V(D),\, u \succ v\}.$$

Then, for every $D \in \mathcal{D}$, it follows that $\delta(v) = \mathbf{0}$ for any $v \in M(D)$ ($M(D) \in X/{\sim_{\mathbf{0}}}$), or $\delta(v) = \mathbf{1}$ for every $v \in M(D)$ ($M(D) \in X/{\sim_{\mathbf{1}}}$), exclusively.

Claim 5. *The followings hold for all $D \in \mathcal{D}$:*
(a) $M(D) \in X/{\sim_{\mathbf{0}}} \implies \{e \in E(\mathcal{A} \circ TD[M(D)]) \mid \tau(e) = z(D)\} = \emptyset$;
(b) $M(D) \in X/{\sim_{\mathbf{1}}} \implies \{e \in E(\mathcal{A} \circ TD[M(D)]) \mid \iota(e) = z(D)\} = \emptyset$.

Proof. Fix an arbitrary $D \in \mathcal{D}$ and let $z(D) = v$.

(a) Suppose a $u \in M(D)$ exists with satisfying $TD[\{u, v\}] = (u, v)$. Let $\{V_i\}_i$ be an arbitrary partition of $V(D)$ such that $V_j \cap M(D) \neq \emptyset$ for all $V_j \in \{V_i\}_i$, and a $V_k \in \{V_i\}_i$ exists with satisfying $V_k = \{u, v\}$. Fix this k, and for each i, let $D_i := TD[V_i]$, then $\{D_i\}_i$ forms an induced partition of D. By Claim 4, $z(D_i) \in M(D)$ for all i, and $z(D_k) = u$, since $u \sim_{\mathbf{0}} v$ with $TD[\{u, v\}] = (u, v)$. That is, $v \notin \{z(D_i)\}_i$. Hence, we have $z(D) = v \neq z(TD[\{z(D_i)\}_i])$ and $v \in V(D_k) \setminus \{z(D_k)\}$. By IIP-(ii), $(u, v) \in E(TD)$ if and only if $(v, u) \in E(TD)$, which contradicts to $TD[\{u, v\}] = (u, v)$. Thus, by Claim 4, $M(D) \in X/{\sim_{\mathbf{0}}}$ implies $TD[\{u, v\}] \neq (u, v)$ for any $u \in M(D)$, meaning that $\{e \in E(\mathcal{A} \circ TD[M(D)]) \mid \tau(e) = v\} = \emptyset$.

(b) Suppose there exists a $u \in M(D)$ such that $TD[\{u, v\}] = (v, u)$. Let $\{D_i\}_i$ be the induced partition of D defined in (a). By Claim 4, $\{z(D_i)\}_i \subset M(D)$. Since $u \sim_{\mathbf{1}} v$ and $TD[\{u, v\}] = (v, u)$, we have $z(D_k) = u$. Thus, it yields $z(D) = v \neq z(TD[\{z(D_i)\}_i])$ and $v \in V(D_k) \setminus \{z(D_k)\}$. In analogy to (a), a contradiction occurs against $TD[\{u, v\}] = (v, u)$ under IIP-(ii). Hence, $M(D) \in X/{\sim_{\mathbf{1}}}$ implies that $TD[\{u, v\}] \neq (v, u)$ for all $u \in M(D)$. The statement follows. Q.E.D.

Claim 6. *For any $D \in \mathcal{D}$, $M(D) \vDash \mathcal{L}(D)$ implies $z(D) = z(TD[M(D)])$.*

Proof. Fix an arbitrary $D \in \mathcal{D}$ that satisfies $M(D) \vDash \mathcal{L}(D)$.

Let $\{V_i\}_i$ be an arbitrary partition of $V(D)$ such that $\#(V_j \cap M(D)) = 1$ for all $V_j \in \{V_i\}_i$. Define a digraph $D_j := TD[V_j]$ for each $V_j \in \{V_i\}_i$, then $\{D_i\}_i$ forms an induced partition of D. By Claim 4 and $M(D) \vDash \mathcal{L}(D)$, we have $z(D) \in M(D) = \{z(D_i)\}_i$ and $\{z(D_i)\}_i \vDash \mathcal{L}(D)$. That is, IIP-(i)(ii) are falsified for $\{D_i\}_i$. Hence, by IIP, we have $z(D) = z(TD[\{z(D_i)\}_i]) = z(TD[M(D)])$. Q.E.D.

Claim 7. *For any $D \in \mathcal{D}$, if $M(D) \vDash \mathcal{L}(D)$, then $z(D) = z_{\succsim,\delta}(\varphi \circ \mathcal{A} \circ TD)$ for all $\varphi \in \Phi(\mathcal{A} \circ TD)$.*

Proof. Let $D \in \mathcal{D}$ be an arbitrary digraph that satisfies $M(D) \vDash \mathcal{L}(D)$. Under Claim 6, it suffices to show $z(TD[M(D)]) = z_{\succsim,\delta}(\varphi \circ \mathcal{A} \circ TD)$ for all $\varphi \in \Phi(\mathcal{A} \circ TD)$.

Note that $V(S) \vDash \mathcal{L}(S)$ for any $S \in \mathcal{S}$. Hence, by Claim 6, if there is a $S \in \mathcal{S}$ such that $V(S) = M(D)$ and $TS = TD[M(D)]$, then we have $z(TD[M(D)]) = z(S)$. Since $M(D) \vDash \mathcal{L}(D)$ implies $M(D) \vDash \mathcal{L}(TD[M(D)])$, by Claim 1, $TD[M(D)]$ has a unique topological sorting. Denote by φ_M this topological sorting, then we have $T\varphi_M(TD[M(D)]) = TD[M(D)]$. Moreover, by Claim 1, we have $TD[M(D)] = \mathcal{A} \circ TD[M(D)]$. As a result, $z(TD[M(D)]) = z(\varphi_M(\mathcal{A} \circ TD[M(D)]))$.

Suppose that $M(D) \in X/{\sim_{\mathbf{0}}}$. Then, by Claim 5-(a), if $(u,v) \in E(T\varphi_M(\mathcal{A} \circ TD[M(D)]))$, then $z(\varphi_M(\mathcal{A}\circ TD[M(D)])) \neq v$. Since $M(D) \vDash \mathcal{L}(\varphi_M(\mathcal{A}\circ TD[M(D)])$, $z(\varphi_M(\mathcal{A}\circ TD[M(D)])) = u$ implies that $(u,v) \in E(T\varphi_M(\mathcal{A}\circ TD[M(D)]))$ for all $v \in M(D)$. Suppose now $M(D) \in X/{\sim_{\mathbf{1}}}$. Then, by Claim 5-(b), $z(\varphi_M(\mathcal{A}\circ TD[M(D)])) = u$ implies that $(v,u) \in E(T\varphi_M(\mathcal{A}\circ TD[M(D)]))$ for all $v \in M(D)$. Hence, $z(\varphi_M(\mathcal{A}\circ TD[M(D)])) = z_{\succsim,\delta}(\varphi_M(\mathcal{A}\circ TD[M(D)]))$.

Given $\mathcal{A}\circ TD$ being a DAG, let $\varphi \in \Phi(\mathcal{A}\circ TD)$ be arbitrary. Then, we have $M(D) \vDash \mathcal{L}(\varphi_M(\mathcal{A}\circ TD[M(D)]))$ and $V(D) \vDash \mathcal{L}(\varphi(\mathcal{A}\circ TD))$. Since φ_M is the unique topological sorting of $\mathcal{A}\circ TD[M(D)]$, it follows that $T\varphi(\mathcal{A}\circ TD)[M(D)] = \varphi_M(\mathcal{A}\circ TD[M(D)])$. Hence, by Claim 4, we have $z_{\succsim,\delta}(\varphi\circ\mathcal{A}\circ TD) = z_{\succsim,\delta}(\varphi_M(\mathcal{A}\circ TD[M(D)]))$ for all $\varphi \in \Phi(\mathcal{A}\circ TD)$. The claim follows. Q.E.D.

Completion of Sufficiency. By the definition of topological sorting, for any DAG $D \in \mathcal{D}$ and any $u \in V(D)$, it follows that

$$\begin{aligned} &\left\{\begin{array}{l} \forall w \in V(D), \\ (u,w) \notin E(TD) \end{array}\right\} \implies \left\{\begin{array}{r} \exists \varphi \in \Phi(D), \forall w \in V(D), \\ (w,u) \in E(T\varphi(D)) \end{array}\right\}; \\ &\left\{\begin{array}{l} \forall w \in V(D), \\ (w,u) \notin E(TD) \end{array}\right\} \implies \left\{\begin{array}{r} \exists \varphi \in \Phi(D), \forall w \in V(D), \\ (u,w) \in E(T\varphi(D)) \end{array}\right\}. \end{aligned} \tag{1}$$

Let $D \in \mathcal{D}$ be an arbitrary digraph, and let $z(D) = v$. By Claim 5, we have (i) if $M(D) \in X/{\sim_{\mathbf{0}}}$, then $(u,v) \notin E(\mathcal{A}\circ TD)$ for all $u \in M(D)$, and (ii) $(v,u) \notin E(\mathcal{A}\circ TD)$ for any $u \in M(D)$ when $M(D) \in X/{\sim_{\mathbf{1}}}$. Since $\mathcal{A}\circ TD$ is a DAG, by (1),

$$\begin{aligned} \{\forall u \in M(D), \delta(u) = \mathbf{0}\} &\implies \left\{\begin{array}{c} \exists \varphi_D \in \Phi(\mathcal{A}\circ TD), \forall u \in M(D), \\ (z(D),u) \in E(T\varphi_D(\mathcal{A}\circ TD)) \end{array}\right\}; \\ \{\forall u \in M(D), \delta(u) = \mathbf{1}\} &\implies \left\{\begin{array}{c} \exists \varphi_D \in \Phi(\mathcal{A}\circ TD), \forall u \in M(D), \\ (u,z(D)) \in E(T\varphi_D(\mathcal{A}\circ TD)) \end{array}\right\}. \end{aligned}$$

Thus, it yields $z(D) = z_{\succsim,\delta}(\varphi_D\circ\mathcal{A}\circ TD)$.

Now we show the uniqueness. Let $D, \tilde{D} \in \mathcal{D}$ be arbitrary digraphs such that $M(D)\cap M(\tilde{D}) \neq \emptyset$. Clearly, we have $\delta(u) = \delta(\tilde{u})$ for any $u \in M(D), \tilde{u} \in M(\tilde{D})$. Suppose that $M(D) \vDash \mathcal{L}(D), M(\tilde{D}) \vDash \mathcal{L}(\tilde{D})$, and without loss of generality, assume $\delta(u) = \mathbf{0}$ for all $u \in M(D)$. Then, it follows Claim 1 that every $\varphi_D \in \Phi(\mathcal{A}\circ TD)$ yields the same permutation on $M(D)$, and so does every $\varphi_{\tilde{D}} \in \Phi(\mathcal{A}\circ T\tilde{D})$ on $M(\tilde{D})$. Hence, by Claim 7, if a $v \in M(D)\cap M(\tilde{D})$ exists such that $(v,u) \in E(TD)$ and $(v,\tilde{u}) \in E(T\tilde{D})$ for all $u \in M(D)$ and any $\tilde{u} \in M(\tilde{D})$, then

$$z(D) = z_{\succsim,\delta}(\varphi_D\circ\mathcal{A}\circ TD) = v = z_{\succsim,\delta}(\varphi_{\tilde{D}}\circ\mathcal{A}\circ T\tilde{D}) = z(\tilde{D}),$$

for all $\varphi_D \in \Phi(\mathcal{A} \circ TD)$ and any $\varphi_{\tilde{D}} \in \Phi(\mathcal{A} \circ T\tilde{D})$. That is, for any $D, \tilde{D} \in \mathcal{D}$, $z(D) = z(\tilde{D})$ whenever every topological sorting of $\mathcal{A} \circ TD$ and $\mathcal{A} \circ T\tilde{D}$ yields the same first (or the same last) $\succsim$-maximal vertex. The uniqueness follows. Q.E.D.

A.3. Theorem 2

We show only the sufficiency part via the following series of claims.

Define the following binary relations on X.

$$
\begin{aligned}
\succ^* &:= \{(x, y) \mid Z((x, y)) = Z((y, x)) = \{x\}\}\,;\\
\sim^* &:= \{(x, y) \mid Z((x, y)) = Z((y, x)) = \{x, y\}\}\,;\\
R^P_{\mathbf{0}} &:= \{(x, y) \mid Z((x, y)) = \{x\} \wedge Z((y, x)) = \{x, y\}\}\,;\\
R^I_{\mathbf{0}} &:= \{(x, y) \mid Z((x, y)) = \{x\} \wedge Z((y, x)) = \{y\}\}\,;\\
R^P_{\mathbf{1}} &:= \{(x, y) \mid Z((x, y)) = \{x, y\} \wedge Z((y, x)) = \{x\}\}\,;\\
R^I_{\mathbf{1}} &:= \{(x, y) \mid Z((x, y)) = \{y\} \wedge Z((y, x)) = \{x\}\}\,.
\end{aligned}
$$

Define $\mathcal{R} := \succ^* \cup \sim^* \cup R^P_\delta \cup R^I_\delta$, where $\delta = \mathbf{0}, \mathbf{1}$, then $\mathcal{R}$ is connex on X.

Claim 8. *$\succ^*$, $\left(\succ^* \cup R^P_\delta\right)$, and R^I_δ are transitive for $\delta = \mathbf{0}, \mathbf{1}$.*

Proof. Let $u, v, w \in X$ be arbitrary, and without loss of generality, let $\delta = \mathbf{0}$. Denote by $\overrightarrow{uvw}$ the string S such that $V(S) = \{u, v, w\}$ and $E(S) = \{(u, v), (v, w)\}$.

Suppose $u \succ^* v$ and $v \succ^* w$ and consider $\overrightarrow{uvw}$. By IIP*, $Z\left(\overrightarrow{uvw}\right) = Z((u, v)) = \{u\}$. Also IIP* implies that $Z((u, w)) = Z\left(\overrightarrow{uvw}\right)$, hence $Z((u, w)) = \{u\}$. Similarly for $\overrightarrow{wvu}$, it follows that $Z((w, u)) = Z(\overrightarrow{wvu}) = Z((v, u)) = \{u\}$. Thus $u \succ^* w$.

Now suppose $u(\succ^* \cup R^P_{\mathbf{0}})v$ and $v(\succ^* \cup R^P_{\mathbf{0}})w$. Since $\succ^*$ is transitive, it suffices to show: (i) $uR^P_{\mathbf{0}}v$ and $vR^P_{\mathbf{0}}w$ implies $u(\succ^* \cup R^P_{\mathbf{0}})w$; (ii) $u \succ^* v$ and $vR^P_{\mathbf{0}}w$ implies $u(\succ^* \cup R^P_{\mathbf{0}})w$. (i) Assume $uR^P_{\mathbf{0}}v$, $vR^P_{\mathbf{0}}w$ and consider $\overrightarrow{uvw}, \overrightarrow{wuv}$. By IIP* and the definition of $R^P_{\mathbf{0}}$, $Z((u, v)) = Z(\overrightarrow{uvw}) = Z((u, w))$. Hence, we have $Z((u, w)) = Z((u, v)) = \{u\}$. Again, by IIP*, $Z((w, u)) = Z(\overrightarrow{wuv}) = Z(T\overrightarrow{wuv}[Z(w, u) \cup \{v\}])$ for $\overrightarrow{wuv}$. If $Z((w, u)) = \{w\}$, it follows that $Z(\overrightarrow{wuv}) = Z(T\overrightarrow{wuv}[Z(w, u) \cup \{v\}]) = Z((w, v)) = \{v, w\}$. Since this contradicts to $Z(\overrightarrow{wuv}) = Z((w, u)) = \{w\}$, it yields that $Z((w, u)) \neq \{w\}$. Thus, we have $u\left(\succ^* \cup R^P_{\mathbf{0}}\right)w$. (ii) Assume $u \succ^* v$, $vR^P_{\mathbf{0}}w$ and consider $\overrightarrow{uvw}, \overrightarrow{vwu}$. By IIP*, $Z((u, w)) = Z(\overrightarrow{uvw}) = Z((u, v)) = \{u\}$ and $Z((w, u)) = Z(\overrightarrow{vwu}) = Z((v, u)) = \{u\}$. Thus, $u \succ^* v$ and $vR^P_{\mathbf{0}}w$ imply $u \succ^* w$. Therefore, $(\succ^* \cup R^P_{\mathbf{0}}) \subset X \times X$ is transitive.

Suppose $uR^I_{\mathbf{0}}v$, $vR^I_{\mathbf{0}}w$ and consider $\overrightarrow{uvw}, \overrightarrow{wvu}$. Then by IIP* we have $Z((u, w)) = Z(\overrightarrow{uvw}) = Z((u, v)) = \{u\}$ holds for $\overrightarrow{uvw}$; while $Z((w, u)) = Z(\overrightarrow{wvu}) = Z((w, v)) = \{w\}$ for $\overrightarrow{wvu}$. Hence, it yields that $uR^I_{\mathbf{0}}w$. Q.E.D.

Claim 9. *For any $u, v, w \in X$, it is impossible to have*

(a) $uR^I_{\mathbf{0}}v \wedge vR^I_{\mathbf{1}}w$;

(b) $uR^I_{\mathbf{0}}v \wedge vR^P_{\mathbf{1}}w$ *(resp.,* $uR^P_{\mathbf{1}}v \wedge vR^I_{\mathbf{0}}w$*);*

(c) $uR^I_{\mathbf{1}}v \wedge vR^P_{\mathbf{0}}w$ *(resp.,* $uR^P_{\mathbf{0}}v \wedge vR^I_{\mathbf{1}}w$*).*

Proof. (a) Assume that there exist $u, v, w \in X$ such that $uR^I_{\mathbf{0}}v$ and $vR^I_{\mathbf{1}}w$. For $\overrightarrow{uwv}$, IIP* implies that $Z((u,w)) = Z(\overrightarrow{uwv}) = Z((u,v)) = \{u\}$, while it follows that $Z((u,w)) = Z(\overrightarrow{vuw}) = Z((v,w)) = \{w\}$ for $\overrightarrow{vuw}$. A contradiction occurs.

(b) Suppose there exist $u, v, w \in X$ that satisfy $uR^I_{\mathbf{0}}v$ and $vR^P_{\mathbf{1}}w$. Consider $\overrightarrow{uwv}$ and $\overrightarrow{vuw}$. By IIP* and definition of $R^I_{\mathbf{0}}, R^P_{\mathbf{1}}$, we have $Z((u,w)) = Z(\overrightarrow{uwv}) = Z((u,v)) = \{u\}$. Since $Z((u,w)) = \{u\}$, it again follows IIP* that $Z(\overrightarrow{vuw}) = Z((v,w)) = \{v,w\}$ and $Z(\overrightarrow{vuw}) = Z((v,u)) = \{v\}$, which contradict to each other. The case $uR^P_{\mathbf{1}}v$ and $vR^I_{\mathbf{0}}w$ follows similarly by considering the same pair of digraphs.

(c) Let $uR^I_{\mathbf{1}}v$ and $vR^P_{\mathbf{0}}w$. Then, by IIP*, $Z(T\overrightarrow{uwv}[Z((u,w)) \cup \{v\}]) = Z(\overrightarrow{uwv}) = Z((w,v)) = \{w,v\}$ for $\overrightarrow{uwv}$. Hence, $w \in Z((u,w))$. Meanwhile, for $\overrightarrow{vuw}$, we have $Z((u,w)) = Z(\overrightarrow{vuw}) = Z((v,u)) = \{u\}$, which contradicts to $w \in Z((u,w))$. Q.E.D.

Claim 10. *For any $u, v, w \in X$,*

(a) $\{uR^P_{\mathbf{0}}v \wedge vR^P_{\mathbf{1}}w\} \implies u \succ^* w$; *(b)* $\{uR^P_{\mathbf{1}}v \wedge vR^P_{\mathbf{0}}w\} \implies u \succ^* w$;

(c) $\{uR^P_{\mathbf{0}}v \wedge wR^P_{\mathbf{1}}v\} \implies u \sim^* w$; *(d)* $\{uR^P_{\mathbf{0}}v \wedge uR^P_{\mathbf{1}}w\} \implies v \sim^* w$;

(e) $\{uR^P_{\mathbf{0}}v \wedge vR^I_{\mathbf{0}}w\} \implies uR^P_{\mathbf{0}}w$; $\{uR^P_{\mathbf{1}}v \wedge vR^I_{\mathbf{1}}w\} \implies uR^P_{\mathbf{1}}w$;

(f) $\{uR^I_{\mathbf{0}}v \wedge vR^P_{\mathbf{0}}w\} \implies uR^P_{\mathbf{0}}w$; $\{uR^I_{\mathbf{1}}v \wedge vR^P_{\mathbf{1}}w\} \implies uR^P_{\mathbf{1}}w$;

(g) $\{u \succ^* v \wedge vR^I_{\delta}w\} \implies u \succ^* w$, $\delta = \mathbf{0}, \mathbf{1}$;

(h) $\{uR^I_{\delta}v \wedge v \succ^* w\} \implies u \succ^* w$, $\delta = \mathbf{0}, \mathbf{1}$;

(i) $\{u \sim^* v \wedge vR^I_{\delta}w\} \implies u \sim^* w$, $\delta = \mathbf{0}, \mathbf{1}$.

Proof. We show (a), (c), (e), (g) and (i) since the rest can be shown similarly.

(a) Suppose $uR^P_{\mathbf{0}}v$ and $vR^P_{\mathbf{1}}w$. For $\overrightarrow{uwv}$, IIP* implies that $Z((u,w)) = Z(\overrightarrow{uwv}) = Z((u,v)) = \{u\}$. Similarly for $\overrightarrow{wuv}$, it holds that $Z((w,u)) = Z(\overrightarrow{wuv}) = Z((u,v)) = \{u\}$. Hence, $u \succ^* w$.

(c) Suppose $uR^P_{\mathbf{0}}v$ and $wR^P_{\mathbf{1}}v$. For $S = \overrightarrow{uwv}$, it holds that $Z(S) = Z((u,w)) = Z(TS[Z((u,w)) \cup \{v\}])$. If $Z((u,w)) = \{w\}$ then it yields that $Z(S) = Z((u,w)) = \{v,w\}$, which suggests a contradiction. Hence, $u \in Z((u,w))$. For $S' = \overrightarrow{vuw}$, we have $Z(S') = Z((u,w)) = Z(TS'[Z((u,w)) \cup \{v\}])$, which implies $w \in Z((u,w))$. Consider $\tilde{S} = \overrightarrow{vwu}$ and $\tilde{S}' = \overrightarrow{wuv}$. By IIP*, $Z(\tilde{S}) = Z(T\tilde{S}[Z((w,u)) \cup \{v\}]) = Z((w,u)) = Z(T\tilde{S}'[Z((w,u)) \cup \{v\}]) = Z(\tilde{S}')$. If $Z((w,u)) = \{u\}$ then $Z(\tilde{S}) = Z(T\tilde{S}[\{u,v\}]) =$

$\{u,v\} \neq Z((w,u))$. If $Z((w,u)) = \{w\}$, then we have $Z(\tilde{S}') = Z(T\tilde{S}'[\{w,v\}]) = \{w,v\} \neq Z((w,u))$. Thus, $Z((w,u)) = Z((u,w)) = \{u,w\}$.

(e) Suppose $uR^P_{\mathbf{0}}v, vR^I_{\mathbf{0}}w$. Let $S = \overrightarrow{uvw}, S' = \overrightarrow{vwu}$, and $S'' = \overrightarrow{wvu}$. Then, IIP* yields $Z(S) = Z((u,w)) = Z((u,v)) = \{u\}$, $Z(S') = Z(TS'[Z((w,u)) \cup \{v\}]) = Z((v,u)) = \{u,v\}$, and $Z(S'') = Z(TS''[Z((w,u)) \cup \{v\}]) = Z((w,u))$. If $u \notin Z((w,u))$ then $Z(TS'[Z((w,u)) \cup \{v\}]) = Z((w,v)) = \{w\}$, which contradicts to $Z(S') = \{u,v\}$. If $w \notin Z((w,u))$ then it yields $Z(TS''[Z((w,u)) \cup \{v\}]) = Z((v,u)) = \{u,v\}$. This contradicts to $Z(S'') = Z((w,u))$, as $v \notin Z((w,u))$. Hence, $Z((u,w)) = \{u\}$ and $Z((w,u)) = \{u,w\}$. The case $uR^P_{\mathbf{1}}v, vR^I_{\mathbf{1}}w$ follows analogously.

(g) Let $u \succ^* v, vR^I_{\mathbf{0}}w$ and $S = \overrightarrow{uvw}, S' = \overrightarrow{vwu}$. By IIP*, $Z(S) = Z((u,w)) = Z((u,v)) = \{u\}$ and $Z(S') = Z((w,u)) = Z((v,u)) = \{u\}$. Thus, $Z((u,w)) = Z((w,u)) = \{u\}$, that is, $u \succ^* w$. The case $\delta = \mathbf{1}$ can be shown similarly.

(i) Suppose $u \sim^* v, vR^I_{\mathbf{0}}w$, and consider $S_1 = \overrightarrow{uvw}, S_2 = \overrightarrow{uwv}, S_3 = \overrightarrow{wuv}, S_4 = \overrightarrow{vwu}$. By IIP*, it follows that $Z(TS_1[Z((u,w)) \cup \{v\}]) = Z(S_1) = Z(TS_1[Z((v,w)) \cup \{u\}]) = Z((u,v)) = \{u,v\}$, which implies $u \in Z((u,w))$. For S_2, IIP* implies that $Z(TS_2[Z((u,w)) \cup \{v\}]) = Z(S_2) = Z(TS_2[Z((w,v)) \cup \{u\}]) = Z((u,w))$. If $Z((u,w)) = \{u\}$, it yields the contradiction $\{u\} = Z((u,w)) = Z(S_2) = Z((u,v)) = \{u,v\}$. Hence, $Z((u,w)) = \{u,w\}$. For S_3, it holds that $Z(TS_3[Z((w,u)) \cup \{v\}]) = Z(S_3) = Z(TS_3[Z((w,v)) \cup \{u\}]) = Z((w,u))$. Similarly, $Z((w,u)) = \{u\}$ yields the contradiction $\{u,v\} = Z((u,v)) = Z(S_3) = Z((w,u)) = \{u\}$, meaning that $w \in Z((w,u))$. For S_4, we have $Z(TS_4[Z((w,u)) \cup \{v\}]) = Z(S_4) = Z(TS_4[Z((v,w)) \cup \{u\}]) = Z((v,u)) = \{u,v\}$, which implies $u \in Z((w,u))$. As a result, $Z((u,w)) = Z((w,u)) = \{u,w\}$. That is, $u \sim^* w$. The case $\delta = \mathbf{1}$ follows similarly. Q.E.D.

Let $\succsim_\delta := \succ^* \cup R_\delta$ and $R_\delta := R^P_\delta \cup R^I_\delta$, where $\delta = \mathbf{0}, \mathbf{1}$.

For a given $D \in \mathcal{D}$, denote by $M^*(D) \subseteq V(D)$ the set of all $\succ^*$-maximal elements in $V(D)$. Formally, $M^*(D) := \{v \in V(D) \mid \nexists u \in V(D), u \succ^* v\}$. Denote by $C(\{u,v\})$ the 2-cycle on $\{u,v\}$. Let $\gamma : [X]^2 \to [X]^1 \cup \{\emptyset\}$ be the mapping given by

$$\gamma(\{u,v\}) := \begin{cases} \{u\}, & Z(C(\{u,v\})) = \{v\} \\ \{v\}, & Z(C(\{u,v\})) = \{u\} \\ \emptyset, & \text{otherwise} \end{cases}, \quad \forall \{u,v\} \in [X]^2.$$

Accordingly, define a correspondence $\Gamma : \mathcal{D} \twoheadrightarrow X$ by

$$\Gamma(D) := V(D) \setminus \left(\bigcup_{\{u,v\} \in \mathcal{Y}(D)} \gamma(\{u,v\}) \right), \quad \forall D \in \mathcal{D}.$$

Claim 11. *Let $M^*_\Gamma(D) := M^*(D) \cap \Gamma(D)$, then $Z(D) \subseteq M^*_\Gamma(D)$ for all $D \in \mathcal{D}$.*

Proof. For any $D \in \mathcal{D}$, if there is a $C \in \mathcal{C}(D)$ with $\#V(C) = n$, then given any $V \in [V(C)]^k$ with $2 \leq k \leq n$, there is a $\tilde{C} \in \mathcal{C}(TD)$ such that $V = V(\tilde{C})$. In particular, $TD[\{u,v\}] = C(\{u,v\})$ holds for any $\{u,v\} \in \mathcal{Y}(D)$. The claim is trivial if $\mathcal{C}(D) = \emptyset$. Suppose $\mathcal{C}(D) \neq \emptyset$, and let $v \in \bigcup_{\{x,y\} \in \mathcal{Y}(D)} \gamma(\{x,y\})$ be arbitrary. Then, a $u \in V_C(D)$ exists such that $\{u,v\} \in \mathcal{Y}(D)$, and thus $TD[\{u,v\}] = C(\{u,v\})$. By definition of γ, $\gamma(\{u,v\}) = \{v\}$ implies $Z(C(\{u,v\})) = \{u\}$. Hence, $v \notin Z(C(\{u,v\})) = Z(TD[\{u,v\}])$, which yields $v \notin Z(D)$ by IIP*. As a result, $v \in \bigcup_{\{x,y\} \in \mathcal{Y}(D)} \gamma(\{x,y\})$ implies $v \notin Z(D)$ for any $v \in V(D)$, meaning that $Z(D) \subseteq \Gamma(D)$.

Fix an arbitrary $D \in \mathcal{D}$. Let $u, v \in V(D)$ and suppose that $u \succ^* v$. For the set $\{u,v\}$, the possible connected DAGs are (u,v) and (v,u). Also, $u \succ^* v$ implies $v \notin Z((u,v))$ and $v \notin Z((v,u))$. Hence, by DAC*, we have $v \notin Z(D)$, as $\{u,v\} \subseteq V(D)$. Since $u, v \in V(D)$ are arbitrary, it yields that, for any $D \in \mathcal{D}$ and any $v \in V(D)$, $v \notin Z(D)$ if there is a $u \in V(D)$ such that $u \succ^* v$. That is, $Z(D) \subseteq M^*(D)$. Q.E.D.

Claim 12. *For any $D \in \mathcal{D}$, any $u \in V(D)$, and any $v \in Z(D)$,*

(a) $uR_{\mathbf{0}}v \implies TD[\{u,v\}] \neq (u,v)$;

(b) $uR_{\mathbf{1}}v \implies TD[\{u,v\}] \neq (v,u)$.

Proof. Let $D \in \mathcal{D}$, $u \in V(D)$, and $v \in Z(D)$ be arbitrary. Set $V = \{u,v\}$ and $V' = V(D) \setminus V$, then $\{V, V'\}$ defines a partition of $V(D)$. Thus, by IIP*,

$$Z(D) = Z\left(TD\left[Z(TD[V']) \cup Z(TD[V])\right]\right). \tag{2}$$

(a) Suppose that $uR_{\mathbf{0}}v$ and $TD[V] = (u,v)$. Then, $Z(TD[V]) = \{u\}$. Since $v \notin Z(TD[V'])$, by (2), $v \notin Z(D) = Z\left(TD\left[Z(TD[V']) \cup \{u\}\right]\right)$. This contradicts to $v \in Z(D)$. As a result, $uR_{\mathbf{0}}v$ implies $TD[V] \neq (u,v)$.

(b) Let $uR_{\mathbf{1}}v$ and $TD[V] = (v,u)$. Then, $Z(TD[V]) = \{u\}$. By (2), we have $Z(D) = Z\left(TD\left[Z(TD[V']) \cup \{u\}\right]\right)$, which implies $v \notin Z(D)$ since $v \notin Z(TD[V'])$. Hence, by contradiction, $uR_{\mathbf{1}}v$ implies $TD[V] \neq (v,u)$. The claim follows. Q.E.D.

Claim 13. *For any $D \in \mathcal{D}$, if $V(D) \vDash \mathcal{L}(D)$ then $Z(D) = Z_{\succsim_\delta}(\varphi \circ \mathcal{A} \circ TD)$ for the unique $\varphi \in \Phi(\mathcal{A} \circ TD)$.*

Proof. Fix an arbitrary such $D \in \mathcal{D}$. Then, by Claim 1, D is a DAG and $\#\Phi(D) = 1$. Thus, $\Gamma(D) = V(D)$ and $\Phi(D) = \Phi(\mathcal{A} \circ TD)$. Let $\Phi(\mathcal{A} \circ TD) = \{\varphi\}$.

Suppose $v \in Z(D)$. By Claim 11, $Z(D) \subseteq M^*(D)$. Since D is a DAG, by Claim 12, for any $u \in V(D)$, $(u,v) \in E(TD)$ implies $\neg(uR_{\mathbf{0}}v)$, and $\neg(uR_{\mathbf{0}}v)$ if $(v,u) \in E(TD)$. Hence, $Z(D) \subseteq Z_{\succsim_\delta}(\varphi \circ \mathcal{A} \circ TD)$.

Now suppose $v \notin Z(D)$. Then, by IIP*, a $V_1 \subset V(D)$ exists such that $v \in V_1$ and $v \notin Z(TD[V_1])$. Fix this V_1. Then, it follows IIP* that there exists a $V_2 \subset V_1$ that satisfies $v \in V_2$ and $v \notin Z(TD[V_2])$. Following the iterative manner, a $u \in V(D)$ exists such that $v \notin Z(TD[\{u,v\}])$. Since D is a DAG, it yields that

$$TD[\{u,v\}] = (u,v) \implies \{u \succ^* v \vee uR_{\mathbf{0}}v\}\,;$$
$$TD[\{u,v\}] = (v,u) \implies \{u \succ^* v \vee uR_{\mathbf{1}}v\}\,.$$

Thus, we have $v \notin Z_{\succsim_\delta}(\varphi \circ \mathcal{A} \circ TD)$, which implies $Z_{\succsim_\delta}(\varphi \circ \mathcal{A} \circ TD) \subseteq Z(D)$. Q.E.D.

Completion of Sufficiency. Fix an arbitrary $D \in \mathcal{D}$.

Suppose $v \in Z(D)$. By Claim 12, for all $u \in V(D)$, it holds that

$$uR_{\mathbf{0}}v \implies \mathcal{A} \circ TD[\{u,v\}] \neq (u,v);$$
$$uR_{\mathbf{1}}v \implies \mathcal{A} \circ TD[\{u,v\}] \neq (v,u). \tag{3}$$

Hence, a $\varphi \in \Phi(\mathcal{A} \circ TD)$ exists such that (i) $T\varphi(\mathcal{A} \circ TD)[\{u,v\}] = (v,u)$ for all $u \in \{u \in V(D) \mid uR_{\mathbf{0}}v\}$, and (ii) $T\varphi(\mathcal{A} \circ TD)[\{u,v\}] = (u,v)$ for all $u \in \{u \in V(D) \mid uR_{\mathbf{1}}v\}$. Moreover, by Claim 11, $v \in M^*_\Gamma(D)$. As a result, it follows that $v \in Z_{\succsim_\delta}(\varphi \circ \mathcal{A} \circ TD)$ for some $\varphi \in \Phi(\mathcal{A} \circ TD)$, and $v \in \Gamma(D)$. Consequently, we have $Z(D) \subseteq \bigcup_{\varphi \in \Phi(\mathcal{A} \circ TD)} (Z_{\succsim_\delta}(\varphi \circ \mathcal{A} \circ TD) \cap \Gamma(D))$.

Suppose there is a $v \in V(D)$ and a $\varphi \in \Phi(\mathcal{A} \circ TD)$ such that $v \in Z_{\succsim_\delta}(\varphi \circ \mathcal{A} \circ TD) \cap \Gamma(D)$. Then, by definition of $Z_{\succsim_\delta} : \mathcal{S} \twoheadrightarrow X$, we have (i), (ii) and $\neg(u \succ^* v)$ held for all $u \in V(D)$. Hence, (3) holds for all $u \in V(D)$. By DAD, $v \in Z(\{u,v\})$ as $v \in M^*(D)$ and $Z(\{u,v\}) = Z((u,v)) \cup Z((v,u))$. Moreover, if $v \in V_C(D)$, then a $u \in V_C(D)$ exists such that $\{u,v\} \in \mathcal{Y}(D)$ and $TD[\{u,v\}] = C(\{u,v\})$. Since $v \in \Gamma(D)$, (3) implies that $v \in Z(TD[\{u,v\}])$ whenever $E(TD[\{u,v\}]) \neq \emptyset$. As a result, $v \in Z(TD[\{u,v\}])$ for all $u \in V(D)$. By IIP*, it follows that $v \in Z(TD[V])$ for any $V \in [V(D)]^3$ such that $v \in V$. Following the iterative manner, $v \in Z(TD[V_{n-1}])$ for all $V_{n-1} \in [V(D)]^{n-1}$ with $v \in V_{n-1}$, where $n = \#V(D)$. Hence, it yields $v \in Z(D)$ when there is a $\varphi \in \Phi(\mathcal{A} \circ TD)$ such that $v \in (Z_{\succsim_\delta}(\varphi \circ \mathcal{A} \circ TD) \cap \Gamma(D))$. That is, $\bigcup_{\varphi \in \Phi(\mathcal{A} \circ TD)} (Z_{\succsim_\delta}(\varphi \circ \mathcal{A} \circ TD) \cap \Gamma(D)) \subseteq Z(D)$. Since $D \in \mathcal{D}$ is arbitrary,

$$Z(D) = \bigcup_{\varphi \in \Phi(\mathcal{A} \circ TD)} \big(Z_{\succsim_\delta}(\varphi \circ \mathcal{A} \circ TD) \cap \Gamma(D)\big), \quad \forall D \in \mathcal{D}.$$

Now we show that the binary relation $\mathcal{R} \subset X \times X$ given by $\mathcal{R} := \succsim_{\mathbf{0}} \cup \succsim_{\mathbf{1}} \cup \sim^*$ is quasi-transitive. Let $u, v, w \in X$ be arbitrary. Denote by $\mathcal{R}^P, \mathcal{R}^I$ the asymmetric and symmetric part of $\mathcal{R}$, respectively. Then, $u\mathcal{R}v$ if and only if $(u \succ^* v) \vee (uR^P_{\mathbf{0}}v) \vee (uR^P_{\mathbf{1}}v) \vee (uR^I_{\mathbf{0}}v) \vee (uR^I_{\mathbf{1}}v) \vee (u \sim^* v)$, and $v\mathcal{R}u$ if and only if $(v \succ^* u) \vee (vR^P_{\mathbf{0}}u) \vee (vR^P_{\mathbf{1}}u) \vee (uR^I_{\mathbf{0}}v) \vee (uR^I_{\mathbf{1}}v) \vee (u \sim^* v)$. Hence, we have $\mathcal{R}^P = (u\mathcal{R}v) \wedge \neg(v\mathcal{R}u) = \succ^*$

$\cup R^P_{\mathbf{0}} \cup R^P_{\mathbf{1}}$, and $\mathcal{R}^I = (u\mathcal{R}v) \wedge (v\mathcal{R}u) = \sim^* \cup R^I_{\mathbf{0}} \cup R^I_{\mathbf{1}}$. Since $\mathcal{R}$ is quasi-transitive if and only if $\mathcal{R}^P$ is transitive, it suffices to show $\mathcal{R}^P$ is transitive. By Claim 8, $(u \succ^* v) \wedge (v \succ^* w) \Rightarrow (u \succ^* w)$, $(u \succ^* v) \wedge (vR^P_\delta w) \Rightarrow (u \succ^* w)$ for $\delta = \mathbf{0}, \mathbf{1}$, and $(uR^P_\delta v) \wedge (vR^P_\delta w) \Rightarrow (u \succ^* w) \vee (uR^P_\delta w)$ for $\delta = \mathbf{0}, \mathbf{1}$. Moreover, by Claim 10-(a)(b), we have $(uR^P_{\mathbf{0}}v) \wedge (vR^P_{\mathbf{1}}w) \Rightarrow (u \succ^* w)$ and $(uR^P_{\mathbf{1}}v) \wedge (vR^P_{\mathbf{0}}w) \Rightarrow (u \succ^* w)$. Therefore, $\mathcal{R}^P \subset X \times X$ is transitive, equivalently, $\mathcal{R} \subset X \times X$ is quasi-transitive.

The transitivity of $\mathcal{R}^P$ and Claim 10-(e)(f)(g)(h) imply the transitivity of $\succsim_{\mathbf{0}} = \succ^* \cup R^P_{\mathbf{0}} \cup R^I_{\mathbf{0}}$ and $\succsim_{\mathbf{1}} = \succ^* \cup R^P_{\mathbf{1}} \cup R^I_{\mathbf{1}}$. Since $\Phi(\mathcal{A} \circ TD)$ is singleton for those $D \in \mathcal{D}$ with $V(D) \vDash \mathcal{L}(D)$, by Claim 13, $Z(D) = Z_{\succsim_\delta}(\varphi \circ TD)$ is uniquely determined for all such $D \in \mathcal{D}$. Hence, given $(\succsim_{\mathbf{0}}, \succsim_{\mathbf{1}})$, the choice procedure is unique. Q.E.D.

A.4. Propositions

A.4.1. Proposition 3

Proof. Define a nonempty correspondence $\Delta : \mathcal{E} \twoheadrightarrow \{\mathbf{0}, \mathbf{1}\}$ by

$$\Delta((u,v)) := \begin{cases} \{\mathbf{0}\}, & Z((u,v)) = \{u\} \\ \{\mathbf{1}\}, & Z((u,v)) = \{v\} \\ \{\mathbf{0},\mathbf{1}\}, & Z((u,v)) = \{u,v\} \end{cases}, \qquad \forall (u,v) \in \mathcal{E}.$$

Then, by the definitions of $\succ^*, R^P_\delta, R^I_\delta$, it follows that

$$\begin{aligned} \neg(u \succ^* v) &\iff \{\Delta((u,v)) \neq \{\mathbf{0}\} \vee \Delta((v,u)) \neq \{\mathbf{1}\}\}; \\ \neg(uR^I_{\mathbf{0}}v) &\iff \{\Delta((u,v)) \neq \{\mathbf{0}\} \vee \Delta((v,u)) \neq \{\mathbf{0}\}\}; \\ \neg(uR^I_{\mathbf{1}}v) &\iff \{\Delta((u,v)) \neq \{\mathbf{1}\} \vee \Delta((v,u)) \neq \{\mathbf{1}\}\}; \\ \neg(uR^P_{\mathbf{0}}v) &\iff \{\Delta((u,v)) \neq \{\mathbf{0}\} \vee \Delta((v,u)) \neq \{\mathbf{0},\mathbf{1}\}\}; \\ \neg(uR^P_{\mathbf{1}}v) &\iff \{\Delta((u,v)) \neq \{\mathbf{0},\mathbf{1}\} \vee \Delta((v,u)) \neq \{\mathbf{1}\}\}. \end{aligned}$$

Fix an arbitrary $S \in \mathcal{S}$. By Theorem 2, it suffices to show $Z_{\succsim_\delta}(S) = Z_\Delta(S)$. Suppose $v \in Z_{\succsim_\delta}(S)$. By the definition of $Z_{\succsim_\delta}$, for any $u \in V(S)$, we have

$$\begin{aligned} TS[\{u,v\}] = (u,v) &\Rightarrow \left\{\neg(u \succ^* v) \wedge \neg(uR^P_{\mathbf{0}}v) \wedge \neg(uR^I_{\mathbf{0}}v)\right\} \\ &\Leftrightarrow \Delta((u,v)) \neq \{\mathbf{0}\}; \\ TS[\{u,v\}] = (v,u) &\Rightarrow \left\{\neg(u \succ^* v) \wedge \neg(uR^P_{\mathbf{1}}v) \wedge \neg(uR^I_{\mathbf{1}}v)\right\} \\ &\Leftrightarrow \Delta((v,u)) \neq \{\mathbf{1}\}. \end{aligned} \tag{4}$$

Since $V(S) \vDash \mathcal{L}(S)$, $\#E(TS[V]) = 1$ for all $V \in [V(S)]^2$. As a result, $v \in Z_{\succsim_\delta}(S)$ implies $\Delta(e) \neq \{\mathbf{0}\}$ for all $e \in \{e \in E(TS) \mid \tau(e) = v\}$; and $\Delta(\tilde{e}) \neq \{\mathbf{1}\}$ for all $\tilde{e} \in \{e \in E(TS) \mid \iota(e) = v\}$. Hence, $Z_{\succsim_\delta}(S) \subseteq Z_\Delta(S)$. Suppose $v \in Z_\Delta(S)$. By the definition of Z_Δ and (4), we have $\neg(u \succsim_{\mathbf{0}} v)$ when $TS[\{u,v\}] = (u,v)$; and $\neg(u \succsim_{\mathbf{1}} v)$ when $TS[\{v,u\}] = (v,u)$. That is, $v \in Z_{\succsim_\delta}(S)$. Therefore, $Z_\Delta(S) \subseteq Z_{\succsim_\delta}(S)$. Q.E.D.

A.4.2. Proposition 4

Since $Z : \mathcal{D} \twoheadrightarrow X$ satisfies DAC*, DAD and IIP*, Theorem 2 holds as a primitive.

Under Strong Relevance, the binary relation $Q := \succ^* \cup R^I_{\mathbf{0}} \cup R^I_{\mathbf{1}}$ is connex on X. By Claim 9, $uR^I_{\mathbf{0}}v$ and $vR^I_{\mathbf{1}}w$ cannot hold simultaneously for any $u, v, w \in X$. Define an indicator function $\delta : X \to \{\mathbf{0}, \mathbf{1}\}$ by

$$\delta(v) = \begin{cases} \mathbf{1} & \exists u \in X, uR^I_{\mathbf{1}}v \\ \mathbf{0} & \text{otherwise} \end{cases}.$$

Then, $\delta : X \to \{\mathbf{0}, \mathbf{1}\}$ is well-defined, and it holds that $\delta(u) = \delta(v)$ whenever $\neg(u \succ^* v)$ and $\neg(v \succ^* u)$. To simplify the statements, given a nonempty $A \subseteq X$, write $A \in X/R^I_{\mathbf{0}}$ (resp., $A \in X/R^I_{\mathbf{1}}$) if $uR^I_{\mathbf{0}}v$ (resp., $uR^I_{\mathbf{1}}v$) for any $u, v \in A$. Let $z_{Q,\delta} : \mathcal{S} \to X$ be the choice function defined in the same way in Theorem 1, with respect to the binary relation Q and the indicator function δ.

Claim 14. *$Z(D) = \bigcup_{\varphi \in \varPhi(\mathcal{A} \circ TD)} (\{z_{Q,\delta}(\varphi \circ \mathcal{A} \circ TD)\} \cap \Gamma(D))$ for any $D \in \mathcal{D}$.*

Proof. Fix an arbitrary $D \in \mathcal{D}$.

Without loss of generality, suppose $M^*(D) \in X/R^I_{\mathbf{0}}$.

Assume $v \in Z(D)$. Then, by Claim 11 and 12, (i) $v \in M^*_\Gamma(D)$, and (ii) $(u,v) \in E(TD)$ only if $(v,u) \in E(TD)$ for any $u \in M^*(D)$. That is, $v \in V_C(TD[M^*(D)])$, or $\{e \in E(TD[M^*(D)]) \mid \tau(e) = v\} = \emptyset$. As a result, it follows that $\{e \in E(\mathcal{A} \circ TD[M^*_\Gamma(D)]) \mid \tau(e) = v\} = \emptyset$. Hence, a $\varphi \in \varPhi(\mathcal{A} \circ TD)$ exists such that, for any $u \in M^*_\Gamma(D)$, there exists a $P \in \mathcal{P}(\varphi \circ \mathcal{A} \circ TD)$ that satisfies $vPu \subset \varphi(\mathcal{A} \circ TD)$. Thus, it yields $Z(D) \subseteq \bigcup_{\varphi \in \varPhi(\mathcal{A} \circ TD)} (\{z_{Q,\delta}(\varphi \circ \mathcal{A} \circ TD)\} \cap \Gamma(D))$.

Now suppose $v \in \left(\bigcup_{\varphi \in \varPhi(\mathcal{A} \circ TD)} (\{z_{Q,\delta}(\varphi \circ \mathcal{A} \circ TD)\} \cap \Gamma(D))\right) \setminus Z(D)$. Then, we have $\{e \in E(T\varphi(\mathcal{A} \circ TD)[M^*_\Gamma(D)]) \mid \tau(e) = v\} = \emptyset$ for some $\varphi \in \varPhi(\mathcal{A} \circ TD)$. By IIP*, a $V \subset M^*_\Gamma(D)$ exists such that $v \in V$ and $v \notin Z(TD[V])$. Following the iterative manner, a $u \in M^*_\Gamma(D)$ exists such that $v \notin Z(TD[\{u,v\}])$. Since $M^*_\Gamma(D) \subseteq M^*(D) \in X/R^I_{\mathbf{0}}$ and $\{e \in E(T\varphi(\mathcal{A} \circ TD)[M^*_\Gamma(D)]) \mid \tau(e) = v\} = \emptyset$ for some $\varphi \in \varPhi(\mathcal{A} \circ TD)$, $v \notin Z(TD[\{u,v\}])$ implies that $TD[\{u,v\}] = \{u,v\}$, or $TD[\{u,v\}] = C(\{u,v\})$. By DAD, we have $Z(\{u,v\}) = \{u,v\}$, as $\{u,v\} = (u,v) \cap (v,u)$. Hence, $TD[\{u,v\}] = C(\{u,v\})$ and $Z(C(\{u,v\})) = \{u\}$. This contradicts to $v \in M^*_\Gamma(D)$. Consequently, $v \in \bigcup_{\varphi \in \varPhi(\mathcal{A} \circ TD)} (\{z_{Q,\delta}(\varphi \circ \mathcal{A} \circ TD)\} \cap \Gamma(D))$ implies $v \in Z(D)$, and the claim holds for all $D \in \mathcal{D}$ such that $M^*(D) \in X/R^I_{\mathbf{0}}$. Q.E.D.

Completion. Let $V \in 2^X \setminus \{\emptyset\}$ be arbitrary. Denote by $\omega(V)$ the string that links every consecutive elements following a given permutation of V. Let $\Omega(V)$ be the set of all such strings on V. Clearly, $\varPhi(\mathcal{A} \circ T((V, \emptyset))) = \Omega(V)$ and $\Gamma((V, \emptyset)) = V$. Hence, by Claim 14, $Z^*(V) = Z((V, \emptyset)) = \{z_{Q,\delta}(\omega(V)) \mid \omega \in \Omega(V)\}$. Note that, given $\delta : X \to \{\mathbf{0}, \mathbf{1}\}$, each $\omega \in \Omega(V)$ represents a particular linear order on V, and $Q \subset X \times X$ is connex and transitive. Therefore, by Rubinstein and Salant (2006, Proposition 3, .pp 10-11), $Z^* : 2^X \setminus \{\emptyset\} \twoheadrightarrow X$ satisfies WARP. Q.E.D.

A.4.3. Proposition 5

Proof. Let $\succ^*, R_{\mathbf{0}}, R_{\mathbf{1}}, \sim^* \subset X \times X$ and $\varDelta : \mathcal{E} \twoheadrightarrow \{\mathbf{0}, \mathbf{1}\}$ follow the definitions given in Theorem 2 and Proposition 3. Since $Z : \mathcal{D} \twoheadrightarrow X$ satisfies DAC* and IIP*, Claim 8-Claim 12 hold as primitives.

Define a mapping $\gamma_I : [X]^2 \to [X]^1 \cup \{\emptyset\}$ by

$$\gamma_I(\{u, v\}) := \begin{cases} \{u\}, & Z(\{u, v\}) = \{v\}; \\ \{v\}, & Z(\{v, u\}) = \{u\}; \\ \emptyset, & \text{otherwise} \end{cases}.$$

Moreover, for every $\{u, v\} \in [X]^2$, let $\gamma_C(\{u, v\}) = \gamma(\{u, v\})$. Then, by IIP*, for every $D \in \mathcal{D}$, the followings hold for every $u, v \in V(D)$.

$$\{TD[\{u, v\}] = \{u, v\} \wedge \gamma_I(\{u, v\}) = \{v\}\} \implies v \notin Z(D);$$
$$\{TD[\{u, v\}] = C(\{u, v\}) \wedge \gamma_C(\{u, v\}) = \{v\}\} \implies v \notin Z(D).$$

Hence, for every $D \in \mathcal{D}$, we have

$$Z(D) \subseteq \Gamma^*(D) := V(D) \setminus \left(\left(\bigcup_{\{u,v\} \in \mathcal{Y}(D)} \gamma_C(\{u, v\}) \right) \cup \left(\bigcup_{\{u,v\} \in \mathcal{H}(D)} \gamma_I(\{u, v\}) \right) \right).$$

Fix an arbitrary $D \in \mathcal{D}$.

Suppose there exist a $v \in V(D)$ and a $\varphi_D \in \varPhi(\mathcal{A} \circ TD)$ such that $v \in Z_\varDelta(\varphi_D \circ \mathcal{A} \circ TD) \cap \Gamma^*(D)$. By the definitions of $Z_\varDelta$ and Γ^*, for every $u \in V(D)$, it holds that

$$TD[\{u, v\}] = (u, v) \implies \varDelta((u, v)) \neq \{\mathbf{0}\} \iff v \in Z((u, v));$$
$$TD[\{u, v\}] = (v, u) \implies \varDelta((v, u)) \neq \{\mathbf{1}\} \iff v \in Z((v, u));$$
$$TD[\{u, v\}] = \{u, v\} \implies \gamma_I(\{u, v\}) \neq \{v\} \iff v \in Z(\{u, v\});$$
$$TD[\{u, v\}] = C(\{u, v\}) \implies \gamma_C(\{u, v\}) \neq \{v\} \iff v \in Z(C(\{u, v\})).$$

Hence, by IIP*, $v \in Z(TD[V_3])$ for any $V_3 \in [V(D)]^3$ with $v \in V_3$. By iteration, we have $v \in Z(TD[V_{n-1}])$ for any $V_{n-1} \in [V(D)]^{n-1}$, where $\#V(D) = n$. As a result,

$v \in Z(TD[V])$ for every $V \subset V(D)$ that contains v. If an induced partition $\{D_i\}_i$ of D exists such that $V = \bigcup_{D_j \in \{D_i\}_i} Z(D_j) \subset V(D)$, then by IIP*-(i), $v \in Z(TD[V]) = Z(D)$. Oppositely, if $\bigcup_{D_j \in \{D_i\}_i} Z(D_j) = V(D)$ for all induced partition $\{D_i\}_i$, then by IIP*-(ii), $v \in V(D) = Z(D)$. Thus, it follows that $\bigcup_{\varphi_D \in \Phi(\mathcal{A} \circ TD)} (Z_\Delta(\varphi_D \circ \mathcal{A} \circ TD) \cap \Gamma^*(D)) \subseteq Z(D)$.

Suppose $v \in Z(D)$. Then, by Claim 12, for any $u \in V(D)$, $\mathcal{A} \circ TD[\{u, v\}] \neq (u, v)$ if $uR_{\mathbf{0}}v$, and $\mathcal{A} \circ TD[\{u, v\}] \neq (v, u)$ when $uR_{\mathbf{1}}v$. In addition, by Claim 11, we have $\neg(u \succ^* v)$ for all $u \in V(D)$. Under the definition of $\Delta : \mathcal{E} \twoheadrightarrow \{\mathbf{0}, \mathbf{1}\}$, for every $e \in E(\mathcal{A} \circ TD)$, $\Delta(e) \neq \{\mathbf{1}\}$ if $\iota(e) = v$, and $\Delta(e) \neq \{\mathbf{0}\}$ when $\tau(e) = v$. Hence, a $\varphi_D \in \Phi(\mathcal{A} \circ TD)$ exists such that $v \in Z_\Delta(\varphi_D \circ \mathcal{A} \circ TD)$. Since $v \in Z(D) \subseteq \Gamma^*(D)$, it yields $Z(D) \subseteq \left(\bigcup_{\varphi_D \in \Phi(\mathcal{A} \circ TD)} Z_\Delta(\varphi_D \circ \mathcal{A} \circ TD)\right) \cap \Gamma^*(D)$. The proposition follows. Q.E.D.

A.4.4. Proposition 6

Proof. Since $Z : \mathcal{D} \twoheadrightarrow X$ satisfies DAC*, DAD, and IIP*, Theorem 2 and Proposition 3 hold. Fix an arbitrary nonempty $V \in 2^X$.

(i) For any $D, \tilde{D} \in \mathcal{D}$ with $V(D) = V(\tilde{D}) = V$, it follows that $M^*(D) = M^*(\tilde{D})$. By Claim 11, $Z(D) \subseteq M^*(D) \cap \Gamma(D)$ for all $D \in \mathcal{D}$. Hence, let

$$Cl(V) := \{v \in V \mid \forall u \in V, \neg(u \succ^* v)\},$$

then it holds that $Z(D) \subseteq Cl(V)$ for all $D(V, E) \in \mathcal{D}$. By Claim 12, if a $v \in Cl(V)$ exists such that $v \in Z(D)$ for all $D(V, E) \in \mathcal{D}$, then $v \succ^* u$ or $v \sim^* u$, and $\gamma(\{u, v\}) \neq \{v\}$, for all $u \in V$. Define $Core(V) \subseteq V$ by

$$Core(V) := \left\{ v \in V \;\middle|\; \forall u \in V, \left\{ \begin{array}{c} \neg(v \succ^* u) \Rightarrow v \left(R_{\mathbf{0}}^P \cup R_{\mathbf{1}}^P \cup \sim^*\right) u \\ \wedge \quad \gamma(\{u, v\}) \neq \{v\} \end{array} \right\} \right\}.$$

Then, it follows that $Core(V) \subseteq Z_{\succsim_\delta}(S)$ for all $S(V, E) \in \mathcal{S}$, and $Core(V) \subseteq \Gamma(D)$ for all $D(V, E) \in \mathcal{D}$. That is, for any $D(V, E) \in \mathcal{D}$, $Core(V)$ would be selected from every $\varphi \in \Phi(\mathcal{A} \circ TD)$. Hence, $Core(V) \subseteq Z(D) \subseteq Cl(V)$ for any $D(V, E) \in \mathcal{D}$.

(ii) Let $\hat{V} := V \setminus Cl(V)$. By definition, for any $u \in \hat{V}$, there is a $v \in Cl(V)$ such that $v \succ^* u$. Thus, by Claim 8 and Claim 10-(g)(h), we have $\neg(uR_{\mathbf{0}}v \vee uR_{\mathbf{1}}v)$ for any $u \in \hat{V}, v \in Cl(V)$.[14] Let $\hat{E} \subset Cl(V) \times \hat{V}$ be the set that satisfies $v \succ^* u$ for any $(v, u) \in \hat{E}$, and $\#\{e \in \hat{E} \mid \tau(e) = u\} = 1$ for all $u \in \hat{V}$.

Claim 8 implies that $R_{\mathbf{0}}^P, R_{\mathbf{1}}^P$ are transitive on $Cl(V)$, and by DAD, $Z(\{u, v\}) = \{u, v\}$ for every $u, v \in Cl(V)$. Let $V_{\mathbf{0}}^1 \subseteq Cl(V)$ be the largest $R_{\mathbf{0}}^I$-indifferent subset of

[14] It is shown in the proof of Claim 8 that for any $u, v, w \in X$, $u \succ^* v, vR_\delta^P w$ imply $u \succ^* w$.

$Cl(V)$, and let $V_{\mathbf{0}}^2$ be the largest such subset of $Cl(V) \setminus V_{\mathbf{0}}^1$, and so on. Then, it yields a collection of $R_{\mathbf{0}}^I$-indifferent subsets $\{V_{\mathbf{0}}^1, \ldots, V_{\mathbf{0}}^k\}$. Similarly, let $\{V_{\mathbf{1}}^1, \ldots, V_{\mathbf{1}}^l\}$ be the collection of $R_{\mathbf{1}}^I$-indifferent subsets. Denote by V_0 the set of remaining elements of $Cl(V)$, and let $\{V^1, \ldots, V^m\}$ be the collection of all singleton subsets of V_0. Then, by Claim 9, $\mathcal{V}(V) := \{V^1, \ldots, V^m, V_{\mathbf{0}}^1, \ldots, V_{\mathbf{0}}^k, V_{\mathbf{1}}^1, \ldots, V_{\mathbf{1}}^l\}$ defines a partition of $Cl(V)$. By construction, $R_{\mathbf{0}}^P \cup R_{\mathbf{1}}^P \cup \sim^*$ is connex on V_0, and Claim 9, Claim 10-(a)(b) jointly imply that $v^0 \sim^* v^1$ for any $v^0 \in V_{\mathbf{0}}^i, v^1 \in V_{\mathbf{1}}^j$ with $1 \leq i \leq k, 1 \leq j \leq l$. Moreover, by Claim 10 and the transitivity of $R_{\mathbf{0}}^P, R_{\mathbf{1}}^P$, it follows that:

a1) $\forall u, v, w \in Cl(V), \neg \left(uR_{\mathbf{0}}^P v \wedge vR_{\mathbf{1}}^P w\right) \wedge \neg \left(uR_{\mathbf{1}}^P v \wedge vR_{\mathbf{0}}^P w\right)$;

a2) $\forall V^i, V^j \in \mathcal{V}(V), \forall Q \in \{R_{\mathbf{0}}^P, R_{\mathbf{1}}^P, \sim^*\}$,

$$\exists u \in V^i, \exists v \in V^j, uQv \Longrightarrow \forall v^i, \forall v^j, v^i Q v^j;$$

a3) $\forall V^i, V^j, V^h \in \mathcal{V}(V)$,

$$\begin{aligned} &\left\{ \begin{array}{r} \forall u \in V^i, \forall v \in V^j, \forall w \in V^h, \\ uR_{\mathbf{0}}^P w \ \wedge \ vR_{\mathbf{1}}^P w \end{array} \right\} \Rightarrow \left\{ \begin{array}{r} \forall V^r \in \mathcal{V}(V), \forall \tilde{v} \in V^r, \\ \tilde{v}\left(R_{\mathbf{0}}^P \cup R_{\mathbf{1}}^P \cup \sim^*\right) w \end{array} \right\}; \\ &\left\{ \begin{array}{r} \forall u \in V^i, \forall v \in V^j, \forall w \in V^h, \\ wR_{\mathbf{0}}^P u \ \wedge \ wR_{\mathbf{1}}^P v \end{array} \right\} \Rightarrow \left\{ \begin{array}{r} \forall V^r \in \mathcal{V}(V), \forall \tilde{v} \in V^r, \\ w\left(R_{\mathbf{0}}^P \cup R_{\mathbf{1}}^P \cup \sim^*\right) \tilde{v} \end{array} \right\}. \end{aligned} \tag{5}$$

Hence, under the transitivity of $R_{\mathbf{0}}^P, R_{\mathbf{1}}^P$, a permutation $(V_1, \ldots V_n)$ of $\mathcal{V}(V)$ exists such that, for any $V_i, V_j \in \{V_1, \ldots V_n\}$, $i < j$ implies

$$\forall v_i \in V_i, \forall v_j \in V_j, \ (v_j R_{\mathbf{0}}^P v_i) \vee (v_i R_{\mathbf{1}}^P v_j) \vee (v_j \sim^* v_i). \tag{6}$$

Fix this permutation. Since there is a $i \in \{1, \ldots, n\}$ such that $u \in V_i$ for every $v \in Cl(V)$, to simplify the statement, let v_i (resp., v_j, v_h, u_r, w_t) imply $v_i \in V_i$ (resp., V_j, V_h, V_r, V_t) in $(V_1, \ldots, V_n)$. Then, by (5) and (6), we have

$$\left\{ \begin{array}{l} \exists V_i, V_j, V_h \in \mathcal{V}(V), \\ v_i R_{\mathbf{0}}^P v_h \ \wedge \ v_i R_{\mathbf{1}}^P v_j \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \forall V_r, V_t \in \mathcal{V}(V) : r \geq i \geq t, \\ \forall v_r \in V_r, \forall v_t \in V_t, \forall v_i \in V_i, \\ v_i \left(R_{\mathbf{1}}^P \cup \sim^*\right) v_r \ \wedge \ v_i \left(R_{\mathbf{0}}^P \cup \sim^*\right) v_t \end{array} \right\}; \tag{7}$$

$$\left\{ \begin{array}{l} \exists V_i, V_j, V_h \in \mathcal{V}(V), \\ v_j R_{\mathbf{0}}^P v_i \ \wedge \ v_h R_{\mathbf{1}}^P v_i \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \forall V_r, V_t \in \mathcal{V}(V) : r \geq i \geq t, \\ \forall v_r \in V_r, \forall v_t \in V_t, \forall v_i \in V_i, \\ v_r \left(R_{\mathbf{0}}^P \cup \sim^*\right) v_i \ \wedge \ v_t \left(R_{\mathbf{1}}^P \cup \sim^*\right) v_i \end{array} \right\}. \tag{8}$$

Fix an arbitrary $V^* \subseteq Cl(V)$, and without loss of generality, suppose $Core(V) \subseteq V^*$. Define the following subsets of $\tilde{V} := Cl(V) \setminus V^*$:

$$
\begin{aligned}
\tilde{V}^I_\delta &:= \left\{ u \in \tilde{V} \mid \exists v \in Cl(V),\, vR^I_\delta u \right\}, \quad \delta = \mathbf{0}, \mathbf{1}; \\
\tilde{V}^P_\delta &:= \left\{ u \in \tilde{V} \mid \exists v \in Cl(V),\, vR^P_\delta u \wedge \nexists v' \in Cl(V),\, uR^I_\delta v' \right\}, \quad \delta = \mathbf{0}, \mathbf{1}; \\
\tilde{V}_+ &:= \left\{ u \in \tilde{V} \mid \forall v \in Cl(V), u \left(R^P_{\mathbf{0}} \cup R^P_{\mathbf{1}} \cup \sim^* \right) v \right\}.
\end{aligned}
$$

Then, by (a1), $\tilde{V} = \tilde{V}^P_{\mathbf{0}} \cup \tilde{V}^P_{\mathbf{1}} \cup \tilde{V}^I_{\mathbf{0}} \cup \tilde{V}^I_{\mathbf{1}} \cup \tilde{V}_+$, and every pair of these sets excepting $\tilde{V}^P_{\mathbf{0}}, \tilde{V}^P_{\mathbf{1}}$ are disjoint, where $\tilde{V}^P_{\mathbf{0}} \cap \tilde{V}^P_{\mathbf{1}}$ satisfies (8). Moreover, if $\tilde{V}_+ \neq \emptyset$, then, for any $u \in \tilde{V}_+$, a $v \in Cl(V)$ exists such that $\gamma(\{u, v\}) = \{u\}$.

The idea is that, if some of $\tilde{V}^I_\delta, \tilde{V}^P_\delta$ are nonempty, say $\tilde{V}^I_{\mathbf{0}} \neq \emptyset$, then construct a connected DAG $D = D^* = D_*$ that satisfies $Core(V) \cup V^* \subsetneq Z(D) \subset Cl(V)$ and $\tilde{V}^I_{\mathbf{0}} \setminus Z(D) \neq \emptyset$. Otherwise, define a $D_*(V, E_*)$ that satisfies $\tilde{V}_+ \cap Z(D_*) = \emptyset$, where $\{e \in E_* \mid \iota(e) = v \vee \tau(e) = v\} \neq \emptyset$ for all $v \in V$.

Let $\tilde{E} \subset V \times V$ and $J \subset \{1, \ldots, n\}$ be the sets given by the following criteria:

b1) for any $u_i \in \tilde{V}^I_{\mathbf{0}}$ and any $u_s \in \tilde{V}^I_{\mathbf{1}}$,

$$
\begin{aligned}
&\{V_i \cap V^* \neq \emptyset\} \Rightarrow \left\{ \forall v \in V_i \cap V^*,\, (v, u_i) \in \tilde{E} \right\}; \\
&\left\{ \begin{array}{l} V_i \cap V^* = \emptyset \quad \wedge \\ \exists k \geq i, v_k R^P_{\mathbf{0}} u_i \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} i \in J \ \wedge \ \forall v_j \in V_j,\, (v_j, u_i) \in \tilde{E} \\ (j = \min_{k \geq i} k \ \text{ s.t. } \ v_k R^P_{\mathbf{0}} u_i) \end{array} \right\}; \\
&\{V_s \cap V^* \neq \emptyset\} \Rightarrow \left\{ \forall v \in V_s \cap V^*,\, (u_s, v) \in \tilde{E} \right\}; \\
&\left\{ \begin{array}{l} V_s \cap V^* = \emptyset \quad \wedge \\ \exists l \leq s, v_l R^P_{\mathbf{1}} u_s \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} s \in J \ \wedge \ \forall v_r \in V_r,\, (u_s, v_r) \in \tilde{E} \\ (r = \max_{l \leq s} l \ \text{ s.t. } \ v_l R^P_{\mathbf{1}} u_s) \end{array} \right\};
\end{aligned}
$$

b2) for any $V_i \in \mathcal{V}(V)$ that satisfies $V_i \subseteq \tilde{V}^I_{\mathbf{0}}$ and $\neg(v_j R^P_{\mathbf{0}} u_i)$ for all $j \geq i$, we have $i \in J$, and a unique $v^0_i \in V_i$ exists such that

$$
\begin{aligned}
&\forall u \in V_i \setminus \{v^0_i\},\, (v^0_i, u) \in \tilde{E}; \\
&\left\{ \begin{array}{l} i \neq 1 \wedge \\ \exists k \leq i,\, V_k \cap V^* \neq \emptyset \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \forall v_h \in V_h \cap V^*,\, (v_h, v^0_i) \in \tilde{E} \\ (h = \max_{k \leq i} k \ \text{ s.t. } \ V_k \cap V^* \neq \emptyset) \end{array} \right\}; \\
&\left\{ \begin{array}{l} i \neq n \wedge \\ \exists k \geq i,\, V_k \cap V^* \neq \emptyset \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \forall v_j \in V_j \cap V^*, \forall u \in V_i \setminus \{v^0_i\}, \\ (u, v_j) \in \tilde{E} \\ (j = \min_{k \geq i} k \ \text{ s.t. } \ V_k \cap V^* \neq \emptyset) \end{array} \right\}.
\end{aligned}
\tag{9}
$$

b3) for any $V_s \in \mathcal{V}(V)$ which satisfies $V_s \subseteq \tilde{V}_{\mathbf{1}}^I$ and $\neg(v_r R_{\mathbf{1}}^P u_s)$ for all $r \leq s$, we have $s \in J$, and a unique $v_s^1 \in V_s$ exists such that

$$\forall u \in V_s \setminus \{v_s^1\},\ (u, v_s^1) \in \tilde{E};$$

$$\left\{\begin{array}{l} s \neq n \wedge \\ \exists l \geq s,\ V_l \cap V^* \neq \emptyset \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \forall v_t \in V_t \cap V^*,\ (v_s^1, v_t) \in \tilde{E} \\ (t = \min_{l \geq s} l \text{ s.t. } V_l \cap V^* \neq \emptyset) \end{array}\right\};$$

$$\left\{\begin{array}{l} s \neq 1 \wedge \\ \exists l \leq s,\ V_l \cap V^* \neq \emptyset \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \forall v_r \in V_r \cap V^*, \forall u \in V_s \setminus \{v_s^1\}, \\ (v_r, u) \in \tilde{E} \\ (r = \max_{l \leq s} l \text{ s.t. } V_l \cap V^* \neq \emptyset) \end{array}\right\}. \tag{10}$$

b4) for any $u_i \in \tilde{V}_{\mathbf{0}}^P$ and any $u_s \in \tilde{V}_{\mathbf{1}}^P \setminus \tilde{V}_{\mathbf{0}}^P$,

$$i \in J \ \wedge\ \forall v_j \in V_j : (j = \min_{k \geq i} k \text{ s.t. } v_k R_{\mathbf{0}}^P u_i),\ (v_j, u_i) \in \tilde{E};$$
$$s \in J \ \wedge\ \forall v_r \in V_r : (r = \max_{l \leq s} l \text{ s.t. } v_l R_{\mathbf{1}}^P u_s),\ (u_s, v_r) \in \tilde{E}.$$

Let $J^c := \{1, \ldots, n\} \setminus J$. Define a set $E_J \subset Cl(V) \times Cl(V)$ by

$$E_J := \left\{(v_i, v_j) \mid v_i \in V_i, v_j \in V_j \wedge i \in J^c \wedge j = \min_{k > i,\, k \in J^c} k\right\}.$$

Define $D^*(V, E^*)$ by $E := \tilde{E} \cup E_J \cup \hat{E}$. Then, by construction, D^* is a connected DAG. Moreover, by (b1)–(b4) and (6)(7)(8), for any $e \in E(TD^*)$, $\Delta(e) \neq \{\mathbf{1}\}$ when $\iota(e) \in V^*$, and $\Delta(e) \neq \{\mathbf{0}\}$ when $\tau(e) \in V^*$. Hence, for any $v \in V^*$, a $\varphi \in \Phi(D^*)$ exists such that $v \in Z_\Delta(\varphi(D^*))$.[15] Consequently, $(Core(V) \cup V^*) \subseteq Z(D^*)$.

Suppose $Core(V) \cup V^* \cup \tilde{V}_+ \subset Cl(V)$. Then, not all of (b1)–(b4) are vacuous. By (b4), for any $u \in \tilde{V}_{\mathbf{0}}^P \cup \tilde{V}_{\mathbf{1}}^P$, an $e \in E^*$ exists such that $u \notin Z(e)$. By (b1)–(b3), for every $u \in \tilde{V}_{\mathbf{0}}^I \cup \tilde{V}_{\mathbf{1}}^I$, if u is neither the vertex v_i^0 in (9) nor v_s^1 in (10), then an $e \in E^*$ exists such that $u \notin Z(e)$. Since D^* is a DAG, $TD^*[\{u, v\}]$ cannot be a cycle for any $u, v \in V$. Hence, by IIP*, $Core(V) \cup V^* \cup \tilde{V}_+ \subset Cl(V)$ implies $Z(D^*) \subset Cl(V)$.

Suppose $Cl(V) \setminus (Core(V) \cup V^*) = \tilde{V}_+$. In this case, $\tilde{E} = \emptyset$. Let $\tilde{E}_\gamma \subset Cl(V) \times Cl(V)$ be the set that satisfies:

c1) $\forall u, v \in Cl(V),\ (u, v) \in \tilde{E}_\gamma \iff (v, u) \in \tilde{E}_\gamma$;

c2) $\forall u \in \tilde{V}_+, \exists! v \in Cl(V), \left\{\gamma(\{u, v\}) = \{u\} \wedge (u, v), (v, u) \in \tilde{E}_\gamma\right\}$.

Let $V_\gamma := \{v \in Cl(V) \mid \exists u \in Cl(V), (u, v) \in \tilde{E}_\gamma\}$, and let $\tilde{J} := \{i \in \{1, \ldots, n\} \mid V_i \not\subseteq \tilde{V}_+\}$.[16] Define a set $E_{\tilde{J}} \subset V^* \times V^*$ accordingly by

$$E_{\tilde{J}} := \left\{(v_i, v_j) \mid v_i \in V_i \setminus V_\gamma, v_j \in V_j \setminus V_\gamma \ \wedge\ i \in \tilde{J} \ \wedge\ j = \min_{k > i,\, k \in \tilde{J}} k\right\}.$$

---

[15] See (4) ff. in Appendix A.4.1.

[16] Note that, for every $u_i \in \tilde{V}_+$, $V_i = \{u_i\}$ is singleton.

Define $D_*(V, E_*)$ by $E_* = \hat{E} \cup \tilde{E}_\gamma \cup E_{\tilde{J}}$. Then, for any $v \in V$, a $u \in V$ exists such that $(u, v) \in E_*$ or $(v, u) \in E_*$. By (6)(7)(8) and Theorem 2, $v \in Z(TD_*[\{u, v\}])$ for any $v \in Core(V) \cup V^*$ and any $u \in V$. In particular, $u \notin \Gamma(D_*)$ for any $u \in \tilde{V}_+$, since a $v \in Cl(V)$ exists such that $TD[\{u, v\}] = C(\{u, v\})$ with $\gamma(\{u, v\}) = \{u\}$. Hence, by Theorem 2 and IIP*, $(Core(V) \cup V^*) = Z(D_*) \subset Cl(V)$. Q.E.D.

A.4.5. Proposition 7

Proof. Given a $\mathcal{G} = \left(V, N, \{(\varDelta^i, \gamma^i)\}_{i\in N}, \{D_i\}_{i\in N}\right)$, denote by $\mathcal{N}(\mathcal{G})$ the set of Nash equilibria of $\mathcal{G}$. Let $v \in V$ be an arbitrary allocation.

Suppose $v \in G(\mathcal{G})$. By definition, $v \in Z^i(D_i)$ for every $i \in N$, meaning that, for every $i \in N$, (a) $\{v\} \neq \gamma^i(\{u, v\})$ for any $u \in V$ with $TD_i[\{u, v\}] = C(\{u, v\})$, and (b) a $\varphi_i \in \varPhi(\mathcal{A} \circ TD_i)$ exists such that $v \in Z_{\varDelta^i}(\varphi_i \circ \mathcal{A} \circ TD_i)$, under Proposition 3. Hence, it follows that $\varDelta^i((u, v)) \neq \{\mathbf{0}\}$ for every $u \in V$ that satisfies $T\varphi_i(\cdot)[\{u, v\}] = (u, v)$, and $\varDelta^i((v, u)) \neq \{\mathbf{1}\}$ for all $u \in V$ with $T\varphi_i(\cdot)[\{u, v\}] = (v, u)$. By ($*$), since $\varDelta^i((w, \tilde{w})) \neq \{\mathbf{0}\}$ for any $w, \tilde{w} \in V$, we have $TD_i[\{u, v\}] \neq (v, u)$ and $TD_i[\{u, v\}] \neq C(\{u, v\})$ for all $u \in V$ with $\varDelta^i((v, u)) = \{\mathbf{1}\}$. As a result, for every $i \in N$ and any $u \in V$, $u \succ^i v$ implies that $TD_i[\{u, v\}] = (u, v)$ or $TD_i[\{u, v\}] = \{u, v\}$. Thus, $G(\mathcal{G}) \subseteq \mathcal{N}(\mathcal{G})$ as $v \in \mathcal{N}(\mathcal{G})$.

Now suppose $v \in \mathcal{N}(\mathcal{G})$. Then, for every $i \in N$, we have $v \succsim^i u$ for any $u \in V$ with $(v, u) \in E(TD_i[\{u, v\}])$. By ($*$), it yields that, for every $i \in N$, $\varDelta^i((v, u)) = \{\mathbf{0}, \mathbf{1}\}$ for any $u \in V$ with $TD_i[\{u, v\}] = (v, u)$, and $\gamma^i(\{u, v\}) \neq \{v\}$ for any $u \in V$ with $TD_i[\{u, v\}] = C(\{u, v\})$. Moreover, $\varDelta^i((w, \tilde{w})) \neq \{\mathbf{0}\}$ for any $w, \tilde{w} \in V$. Hence, for every $i \in N$, a $\varphi_i \in \varPhi(\cdot)$ exists such that $v \in Z_{\varDelta^i}(\varphi_i(\cdot))$, meaning that $v \in Z^i(D_i)$ for all $i \in N$. Consequently, $\mathcal{N}(\mathcal{G}) \subseteq G(\mathcal{G})$ as $v \in G(\mathcal{G})$. Q.E.D.

A.5. Corollaries

Proof of Corollary 1. Let $\omega(V)$ be the string that links every consecutive elements following a given permutation of V. Denote by $\Omega(V)$ the set of all such strings on V. Then, for any $V \in 2^X \setminus \{\emptyset\}$, each $\omega \in \Omega(V)$ corresponds to a permutation of V. Clearly, $T((V, \emptyset)) = (V, \emptyset)$, $\mathcal{A}((V, \emptyset)) = (V, \emptyset)$, and $\Gamma((V, \emptyset)) = V$ for any $V \in 2^X \setminus \{\emptyset\}$. Thus, $\varPhi(\mathcal{A} \circ TD) = \Omega(V)$. Consequently, by Proposition 3, for any $V \in 2^X \setminus \{\emptyset\}$,

$$Z((V, \emptyset)) = \bigcup_{\omega\in\Omega(V)} Z_\varDelta(\omega(V)) = V \setminus \left(\bigcap_{\omega\in\Omega(V)} V \setminus Z_\varDelta(\omega(V)) \right).$$

Let $Q(V) := \bigcap_{\omega\in\Omega(V)} V \setminus Z_\varDelta(\omega(V))$. Fix a $V \in 2^X \setminus \{\emptyset\}$ and a $q \in Q(V)$ arbitrarily. Then, by the definition of $Z_\varDelta$, for any $\omega \in \Omega(V)$, a $v \in V$ exists such that $\varDelta((v, q)) =$

$\{\mathbf{0}\}$, or $\Delta((q,v)) = \{\mathbf{1}\}$. Let $\omega^+(V)$ denote the string in which $E(T\omega^+(V)[\{v,q\}]) = (v,q)$ for all v with $\Delta((q,v)) = \{\mathbf{1}\}$. Note that $q \in Q(V)$ implies $q \notin Z_\Delta(\omega^+(V))$. Hence, a $v \in V$ exists such that $\Delta((v,q)) = \{\mathbf{0}\}$ and $\Delta((q,v)) = \{\mathbf{1}\}$. Consequently, $Q(V) = \{q \in V \mid \exists v \in V, \{\Delta((v,q)) = \{\mathbf{0}\} \wedge \Delta((q,v)) = \{\mathbf{1}\}\}\}$. As a result, $Z^*(V) = Z((V,\emptyset)) = V \setminus Q(V) = \{v \in V \mid \nexists u \in V, u \succ^* v\}$ for all $V \in 2^X \setminus \{\emptyset\}$. By Claim 8, $\succ^* \subset X \times X$ is transitive. Q.E.D.

Proof of Corollary 2. Given Theorem 2 and Corollary 1, it suffices to show $\succ^*$ is a semiorder under Relevance. By Relevance, for any $u, v \in X$, it is impossible to have $Z((u,v)) = \{u,v\}$ and $Z((v,u)) = \{v,u\}$. That is, $\sim^* = \emptyset$. Hence, define a binary relation $\succsim^S := \succ^* \cup R^P_\delta \cup R^I_\delta$, then $\succsim^S \subset X \times X$ is connex, where $\delta = \{\mathbf{0},\mathbf{1}\}$. By Claim 10-(c)(d), for any $u, v, w \in X$, it is impossible to have $uR^P_{\mathbf{0}}v$ and $wR^P_{\mathbf{1}}v$. Let $\succ^S := \succ^*$ and $\sim^S := R^P_\delta \cup R^I_\delta$. Clearly, $\succ^S$ is asymmetric. Let $u, v, x, y \in X$ be arbitrary. Suppose $u \succ^* x$, $x \sim^* y$, and $y \succ^S v$. Then, by Claim 8 and Claim 10-(g)(h), $u \succ^S y$, and hence, $u \succ^S v$. Suppose $x \succ^S u$, $u \succ^S y$, and $u \sim^S v$. If $uR^I_\delta v$ for $\delta = \mathbf{0}, \mathbf{1}$, then by Claim 10-(g)(h), $x \succ^* v$ and $v \succ^* y$. If $uR^P_\delta v$ for $\delta = \mathbf{0}, \mathbf{1}$, then by Claim 8, $x \succ^S v$. If $vR^P_\delta u$ for $\delta = \mathbf{0}, \mathbf{1}$, then by Claim 8, $v \succ^S y$. Thus, $x \sim^S v$ and $v \sim^S y$ cannot be true simultaneously. Hence, $\succ^S \subset X \times X$ is a semiorder. Q.E.D.

Proof of Corollary 3. (i) By Theorem 1, $\succsim \subset X \times X$ has a unique decomposition $\succsim = \succ \cup \sim_{\mathbf{0}} \cup \sim_{\mathbf{1}}$. Let $Q_{\mathbf{0}} := \succ \cup \sim_{\mathbf{0}}$ and $Q_{\mathbf{1}} := \succ \cup \sim_{\mathbf{1}}$. Then, $(Q_{\mathbf{0}}, Q_{\mathbf{1}})$ is the pair that satisfies the statement (i).

(ii) Let $S \in \mathcal{S}$ be arbitrary. By the definition of Z_{Q_δ} and $(Q_{\mathbf{0}}, Q_{\mathbf{1}})$, we have $Z_{Q_\delta}(S) \subseteq M(S)$. Then, by Claim 3, $Z_{Q_\delta}(S) \in X/{\sim_{\mathbf{0}}}$ or $Z_{Q_\delta}(S) \in X/{\sim_{\mathbf{1}}}$. Suppose $Z_{Q_\delta}(S) \subseteq M(S) \in X/{\sim_{\mathbf{0}}}$. Then, for any $u \in M(S)$, if $\{e \in E(TS[M(S)]) \mid \tau(e) = u\} \neq \emptyset$, then $u \notin Z_{Q_\delta}(S)$. For S, a unique $v \in M(S)$ exists such that $\{e \in E(TS[M(S)]) \mid \tau(e) = v\} = \emptyset$, i.e., the first $\succsim$-maximal vertex. Suppose $Z_{Q_\delta}(S) \subseteq M(S) \in X/{\sim_{\mathbf{1}}}$, then for any $u \in M(S)$, $\{e \in E(TS[M(S)]) \mid \iota(e) = u\} \neq \emptyset$ implies $u \notin Z_{Q_\delta}(S)$. Similarly, there is a unique $v \in M(S)$ that satisfies $\{e \in E(TS[M(S)]) \mid \iota(e) = v\} = \emptyset$, the last $\succsim$-maximal vertex. Hence, $Z_{Q_\delta}(S)$ is singleton in both case, and by the definition of $z_{\succsim,\delta}$, we have $Z_{Q_\delta}(S) = \{z_{\succsim,\delta}(S)\}$. Q.E.D.

Proof of Corollary 4. (i) By Theorem 2, $(u,v) \in (\succsim_{\mathbf{0}} \cap \succsim_{\mathbf{1}})$ if and only if $Z((u,v)) = Z((v,u)) = \{u\}$. Hence, for all $(u,v) \in (\succsim_{\mathbf{0}} \cup \succsim_{\mathbf{1}})$, we have $\hat{z}((u,v) = \hat{z}((v,u)) = u$, as $\hat{z}(D) \in Z(D)$ for all $D \in \mathcal{D}$. Thus, it yields that $u \hat{\succ} v$ for all $(u,v) \in (\succsim_{\mathbf{0}} \cap \succsim_{\mathbf{1}})$.

(ii) Let $u, v \in X$ be arbitrary and suppose that $u \succsim_{\mathbf{0}} v$ or $u \succsim_{\mathbf{1}} v$. Then, by the definition of $\succsim_{\mathbf{0}}, \succsim_{\mathbf{1}}$, we have $v \notin Z((u,v))$ or $v \notin Z((v,u))$. Since $\hat{z}(D) \in$

$Z(D)$ for all $D \in \mathcal{D}$, it follows that $v \neq \hat{z}((u,v))$ or $v \neq \hat{z}((v,u))$. Hence, we have $(u\hat{\succ}v) \vee (u\hat{\sim}_{\mathbf{0}}v) \vee (u\hat{\sim}_{\mathbf{1}}v)$. That is, $u\hat{\succsim}v$. Q.E.D.

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