Lecture 6: Kinematics: Velocity Kinematics - the Jacobian

- Skew Symmetric Matrices

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Angular Velocity for Describing Rotation around Fixed Axis

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- As the body rotates a perpendicular from any point of the body to the axis sweeps the same an angle θ
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$$\vec{\omega} = \frac{d}{dt}\theta \cdot \vec{k}$$

- The linear velocity of any point of the body is then

$$\vec{v} = \vec{\omega} \times \vec{r}$$

where $\vec{r}$ is the vector from the origin, which is assumed to lie on the axis.

Skew Symmetric Matrices

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For example, if $n = 3$, then any $S \in so(3)$ has the form

$$S = \begin{bmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{bmatrix}$$

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For example, if $n = 3$, then any $S \in so(3)$ has the form

$$S = \begin{bmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{bmatrix} = \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix}$$

Skew Symmetric Matrices (Cont'd)

Given $\vec{a} = [a_x,\ a_y,\ a_z]^T$, define

$$S(\vec{a}) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

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Property:

$$S(\alpha\vec{a} + \beta\vec{c}) = \alpha S(\vec{a}) + \beta S(\vec{c})$$

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Property:

$$S(\vec{a})\,\vec{p} = \vec{a} \times \vec{p} \quad \text{for any } \vec{a},\ \vec{p} \in \mathbb{R}^3$$

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$$S(\vec{a})\,\vec{p} \;=\; \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} p_z a_y - p_y a_z \\ p_x a_z - p_z a_x \\ p_y a_x - p_x a_y \end{bmatrix}$$

Skew Symmetric Matrices (Cont'd)

Property:

$$\boldsymbol{R}\boldsymbol{S}\left(\vec{\boldsymbol{a}}\right)\boldsymbol{R}^{T} = \boldsymbol{S}\left(\boldsymbol{R}\vec{\boldsymbol{a}}\right) \quad \text{for any } \boldsymbol{R} \in \mathcal{SO}(3),\ \vec{\boldsymbol{a}} \in \mathbb{R}^{3}$$

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$$\boldsymbol{R}\left[\boldsymbol{S}\left(\vec{\boldsymbol{a}}\right)\boldsymbol{R}^{T}\vec{\boldsymbol{p}}\right] = \boldsymbol{R}\left[\vec{\boldsymbol{a}} \times \boldsymbol{R}^{T}\vec{\boldsymbol{p}}\right] = \boldsymbol{R}\left[\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}\right]$$

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Skew Symmetric Matrices (Cont'd)

Property:

$$RS(\vec{a})R^T = S\left(R\vec{a}\right) \quad \text{for any } R \in \mathcal{SO}(3),\ \vec{a} \in \mathbb{R}^3$$

$$\begin{aligned} R\left[S(\vec{a})R^T\vec{p}\right] &= R\left[\vec{a} \times R^T\vec{p}\right] = R\left[\vec{a} \times \vec{b}\right] \\ &= R\vec{a} \times R\vec{b} = R\vec{a} \times R\left(R^T\vec{p}\right) = R\vec{a} \times \vec{p} \end{aligned}$$

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\boldsymbol{R}\left[\boldsymbol{S}\left(\vec{a}\right)\boldsymbol{R}^{T}\vec{p}\right] &= \boldsymbol{R}\left[\vec{a} \times \boldsymbol{R}^{T}\vec{p}\right] = \boldsymbol{R}\left[\vec{a} \times \vec{b}\right] \\
&= \boldsymbol{R}\vec{a} \times \boldsymbol{R}\vec{b} = \boldsymbol{R}\vec{a} \times \boldsymbol{R}\left(\boldsymbol{R}^{T}\vec{p}\right) = \boldsymbol{R}\vec{a} \times \vec{p} \\
&= \boldsymbol{S}\left(\boldsymbol{R}\vec{a}\right)\vec{p}
\end{aligned}$$

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Property:

$$x^T S x = 0 \quad \text{for any } S \in so(3),\ x \in \mathbb{R}^3$$

Derivative of Rotation Matrix

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How to compute $\boldsymbol{\frac{d}{d\theta} R(\theta)}$?

For any $\boldsymbol{\theta}$ the matrix $\boldsymbol{R(\theta)}$ is a rotation, then

$$\boldsymbol{R(\theta) R^T(\theta) = R^T(\theta) R(\theta) = I}$$

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Therefore

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$$\Rightarrow \qquad \boxed{\frac{d}{dt}\left[R(\theta)\right] = S R(\theta), \quad S \in so(3)}$$

Derivative of Rotation Matrix (Example 4.2)

If $R(\theta) = R_{z,\theta}$, that is the basic rotation around the axis z, then

$$S = \frac{d}{d\theta}\left[R_{z,\theta}\right] R^T_{z,\theta} = \frac{d}{d\theta}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^T$$

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$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} = S(\vec{a})$$

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$$= S(\vec{k}), \qquad \vec{k} = [0,\ 0,\ 1]^{T}$$

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Angular Velocity

If $R(t) \in \mathcal{SO}(3)$ is time-varying and has a time-derivative, then

$$\frac{d}{dt}R(t) = S(t)R(t) = S(\omega(t))R(t), \quad S(\cdot) \in so(3)$$

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Consider a point p rigidly attached to a moving frame, then

$$p^0(t) = R_1^0(t)\, p^1$$

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Differentiating this expression we obtain

$$\begin{aligned} \frac{d}{dt}\left[p^0(t)\right] &= \frac{d}{dt}\left[R_1^0(t)\right] p^1 = S(\omega(t))R_1^0(t)p^1 = \omega(t) \times R_1^0(t)p^1 \\ &= \omega(t) \times p^0(t) \end{aligned}$$

Addition of Angular Velocities

An angular velocity is a free vector

$$\omega_{i,j}^{k}(t)$$

it corresponds to $\left\{ \frac{d}{dt} R_j^i(t) \right\}$ expressed in k-coordinate frame

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Consider two moving frames and the fixed one all with the common origin

$$R_2^0(t) = R_1^0(t)R_2^1(t)$$

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$$R_2^0(t) = R_1^0(t) R_2^1(t)$$

Taking time derivative of the left and right sides we have

$$\frac{d}{dt} R_2^0(t) \quad = \quad \frac{d}{dt} \left[R_1^0(t) \right] R_2^1(t) + R_1^0(t) \frac{d}{dt} \left[R_2^1(t) \right]$$

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&= S(\omega_{0,2}^0(t))R_2^0 \\
&= \Big[S(\omega_{0,1}^0(t))R_1^0(t)\Big]R_2^1(t) + R_1^0(t)\Big[S(\omega_{1,2}^1(t))R_2^1(t)\Big]
\end{aligned}$$

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Addition of Angular Velocities

An angular velocity is a free vector

it corresponds to $\underbrace{\left\{\frac{d}{dt}R^i_j(t)\right\}}^{\omega^k_{i,j}(t)}$ expressed in k-coordinate frame

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Consider two moving frames and the fixed one all with the common origin

$$R^0_2(t) = R^0_1(t)R^1_2(t)$$

Taking time derivative of the left and right sides we have

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\frac{d}{dt}R^0_2(t) &= \frac{d}{dt}\left[R^0_1(t)\right]R^1_2(t) + R^0_1(t)\frac{d}{dt}\left[R^1_2(t)\right] \\
&= S(\omega^0_{0,2}(t))R^0_2 \\
&= S(\omega^0_{0,1}(t))R^0_2(t) + R^0_1(t)S(\omega^1_{1,2}(t))\underbrace{R^0_1(t)^T R^0_1(t)}_{=I}R^1_2(t)
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&= S(\omega_{0,2}^0(t))R_2^0 \\
&= S(\omega_{0,1}^0(t))R_2^0(t) + \underbrace{R_1^0(t)S(\omega_{1,2}^1(t))R_1^0(t)^T}_{= S(R_1^0(t)\omega_{1,2}^1(t))} R_1^0(t)R_2^1(t)
\end{aligned}$$

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An angular velocity is a free vector

it corresponds to $\left\{\frac{d}{dt}R_j^i(t)\right\}$ expressed in k-coordinate frame: $\omega_{i,j}^k(t)$

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&= S(\omega_{0,2}^0(t))R_2^0 \\
&= S(\omega_{0,1}^0(t))R_2^0(t) + S(R_1^0(t)\omega_{1,2}^1(t))\underbrace{R_1^0(t)R_2^1(t)}_{=R_2^0(t)}
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$$\omega_{i,j}^k(t)$$

Consider two moving frames and the fixed one all with the common origin

$$R_2^0(t) = R_1^0(t)R_2^1(t)$$

Taking time derivative of the left and right sides we have

$$\begin{aligned}
\frac{d}{dt}R_2^0(t) &= \frac{d}{dt}\left[R_1^0(t)\right]R_2^1(t) + R_1^0(t)\frac{d}{dt}\left[R_2^1(t)\right] \\
&= S(\omega_{0,2}^0(t))R_2^0 \\
&= \left[S(\omega_{0,1}^0(t)) + S(R_1^0(t)\omega_{1,2}^1(t))\right]R_2^0(t)
\end{aligned}$$

Addition of Angular Velocities

An angular velocity is a free vector

$$\omega_{i,j}^{k}(t)$$

it corresponds to $\left\{ \frac{d}{dt} R_j^i(t) \right\}$ expressed in k-coordinate frame

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$$\Rightarrow \qquad \boxed{S(\omega_{0,2}^0(t)) = S(\omega_{0,1}^0(t)) + S(R_1^0(t)\omega_{1,2}^1(t))}$$

Addition of Angular Velocities (Cont'd)

The relation

$$S(\omega_{0,2}^{0}(t)) = S(\omega_{0,1}^{0}(t)) + S(R_{1}^{0}(t)\omega_{1,2}^{1}(t))$$

together with the property: $S(a) + S(c) = S(a + c)$, imply that the angular velocity can be computed as

$$\boxed{\omega_{0,2}^{0}(t) = \omega_{0,1}^{0}(t) + R_{1}^{0}(t)\omega_{1,2}^{1}(t) = \omega_{0,1}^{0}(t) + \omega_{1,2}^{0}(t)}$$

Addition of Angular Velocities (Cont'd)

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Given n-moving frames with the same origins as for fixed one

$$R_n^0(t) = R_1^0(t)R_2^1(t)\cdots R_n^{n-1}(t) \;\Rightarrow\; \frac{d}{dt}R_n^0(t) = S(\omega_{0,n}^0(t))R_n^0(t)$$

$$\begin{aligned}
\omega_{0,n}^0(t) &= \omega_{0,1}^0(t) + \omega_{1,2}^0(t) + \omega_{2,3}^0(t) + \cdots + \omega_{n-1,n}^0(t) \\
&= \omega_{0,1}^0 + R_1^0\omega_{1,2}^1 + R_2^0\omega_{2,3}^2 + \cdots + R_{n-1}^0\omega_{n-1,n}^{n-1}
\end{aligned}$$

Computing a Linear Velocity of a Point

Given moving and fixed frames related by a homogeneous transform

$$\boldsymbol{H}_1^0(t) = \begin{bmatrix} \boldsymbol{R}_1^0(t) & \boldsymbol{o}_1^0(t) \\ \boldsymbol{0} & \boldsymbol{1} \end{bmatrix}$$

that is, coordinates of each point of moving frame are

$$\boldsymbol{p}^0(t) = \boldsymbol{R}_1^0(t)\boldsymbol{p}^1 + \boldsymbol{o}_1^0(t)$$

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Hence

$$\frac{d}{dt}p^0(t) \quad = \quad \frac{d}{dt}\left[R_1^0(t)\right]p^1 + \frac{d}{dt}\left[o_1^0(t)\right]$$

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Hence

$$\begin{aligned} \frac{d}{dt}p^0(t) &= \frac{d}{dt}\left[R_1^0(t)\right]p^1 + \frac{d}{dt}\left[o_1^0(t)\right] \\ &= S(\omega_1^0(t))R_1^0 p^1 + \frac{d}{dt}\left[o_1^0(t)\right] \\ &= \omega_1^0(t) \times (R_1^0 p^1) + \frac{d}{dt}\left[o_1^0(t)\right] \end{aligned}$$