Lecture 2: Kinematics:

Rigid Motions and Homogeneous Transformations

- Frames, Points and Vectors

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- Frames, Points and Vectors
- Rotations:
  - In 2 and 3 Dimensions
  - Transformations by Rotation

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- Composition of Rotations:
  - Rotations with Respect to the Current Frame
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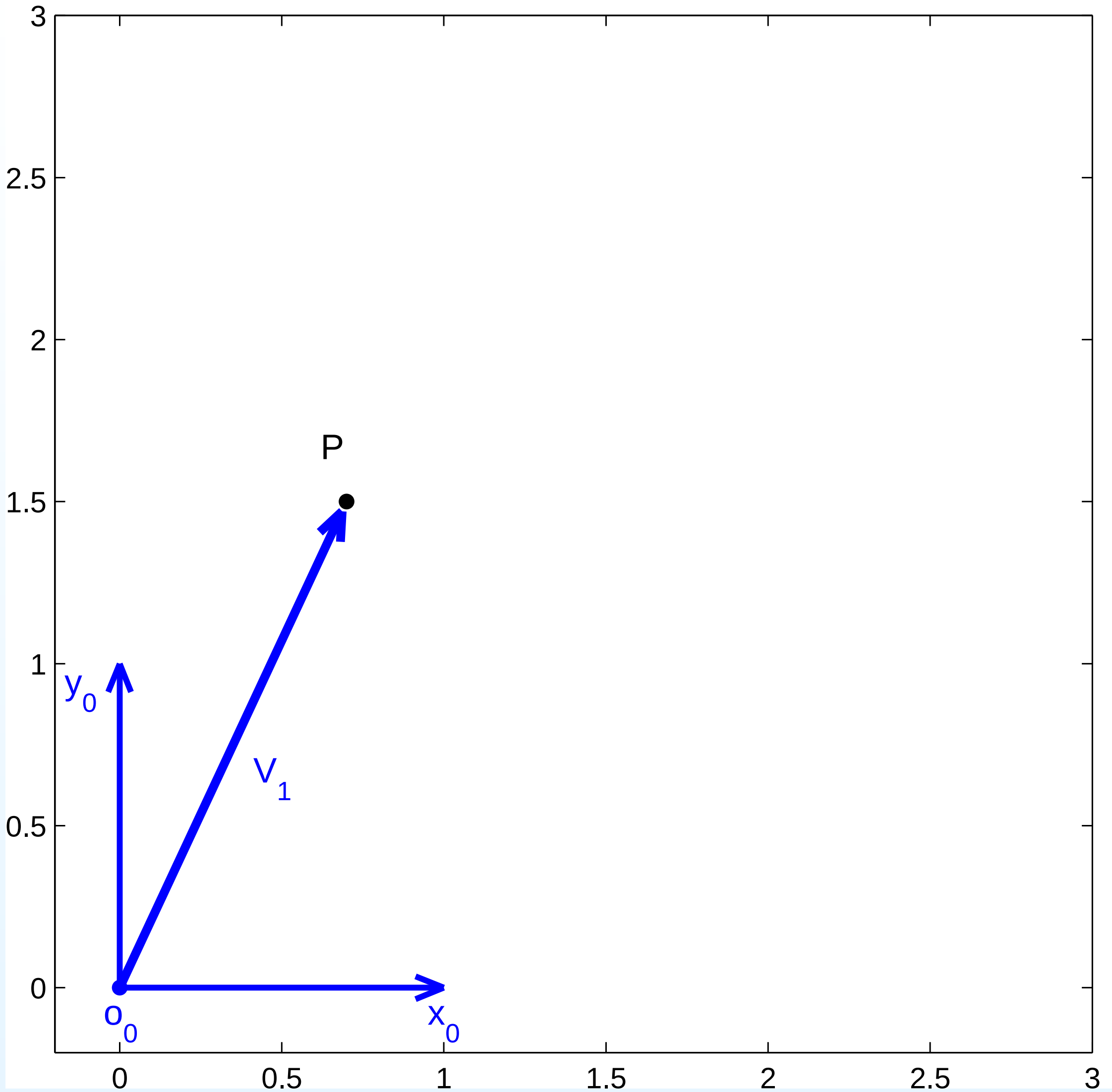
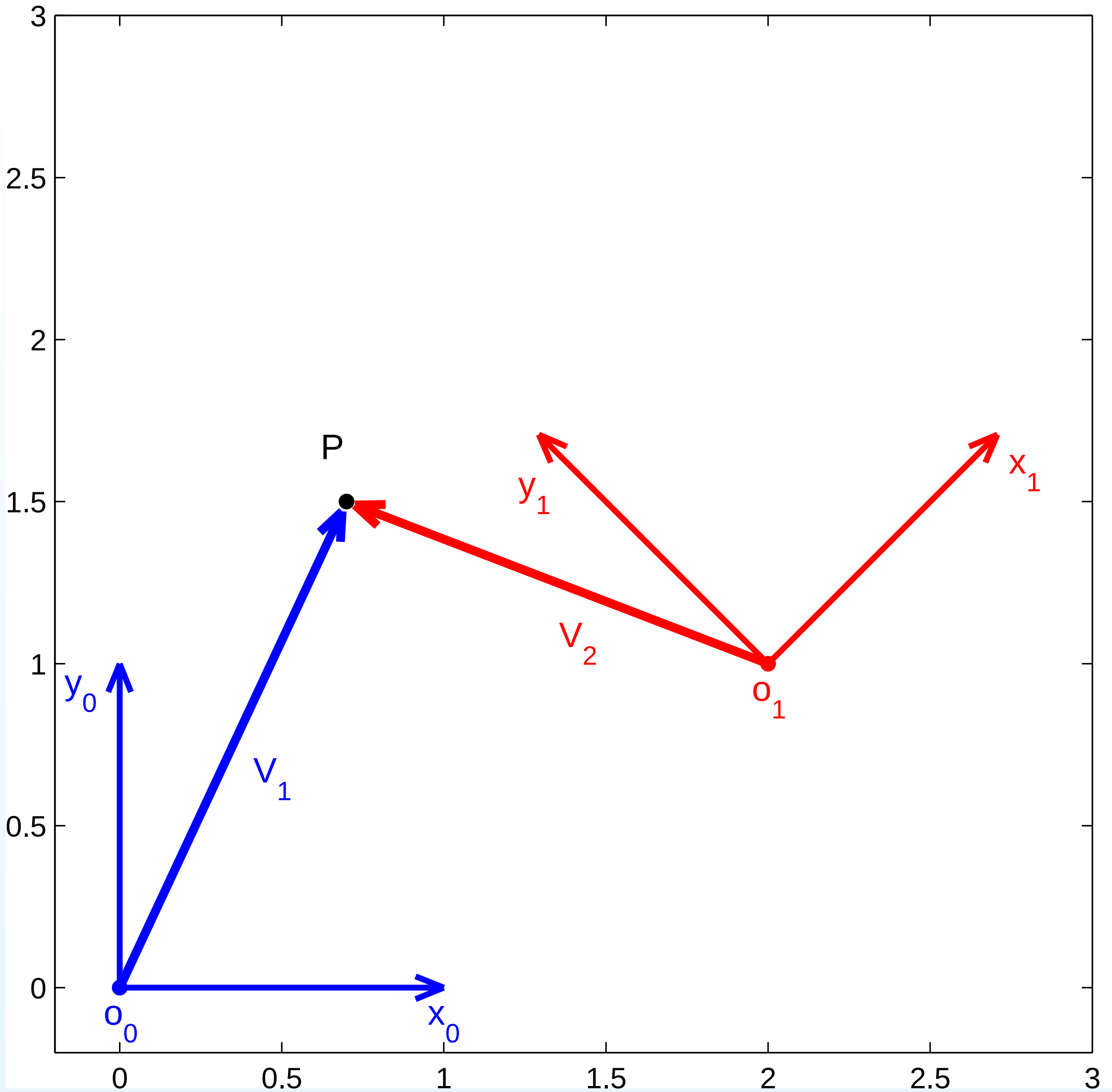


A coordinate frame in R^2: the point $P = [x_p^0, y_p^0]$ can be associated with the vector $\vec{V}_1$. Here x_p^0 denotes x-coordinate of point P in (x_0, o_0, y_0)-frame, i.e. in the 0-frame.



Two coordinate frames in $\boldsymbol{R^2}$: the point $\boldsymbol{P = [x_p^0, y_p^0] = [x_p^1, y_p^1]}$ can be associated with vectors $\boldsymbol{\vec{V}_1}$ and $\boldsymbol{\vec{V}_2}$. Here $\boldsymbol{x_p^1}$ denotes $\boldsymbol{x}$-coordinate of point $\boldsymbol{P}$ in the $\boldsymbol{1}$-frame.

Frames, Points and Vectors

- A point corresponds a particular location in the space.

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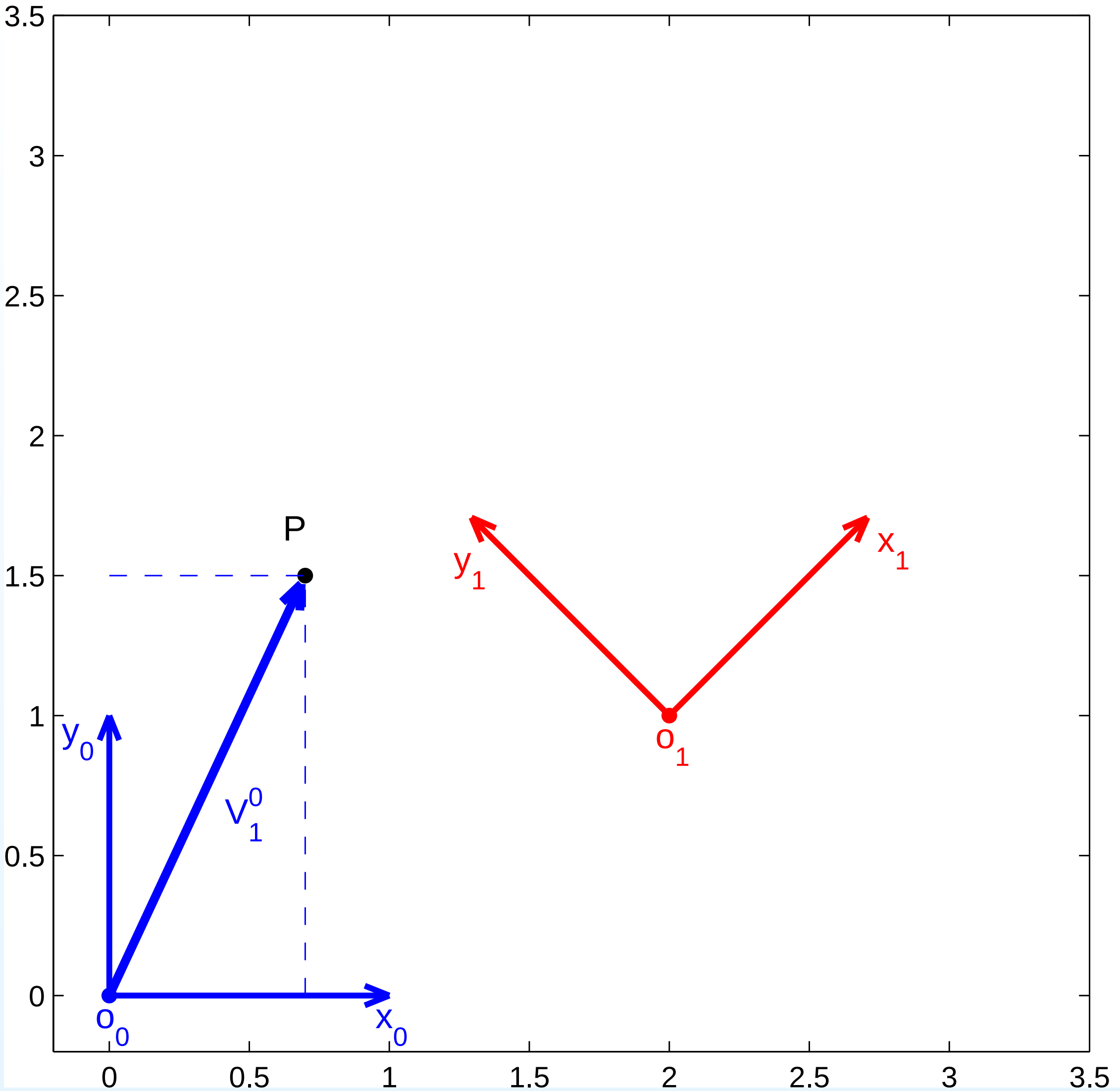
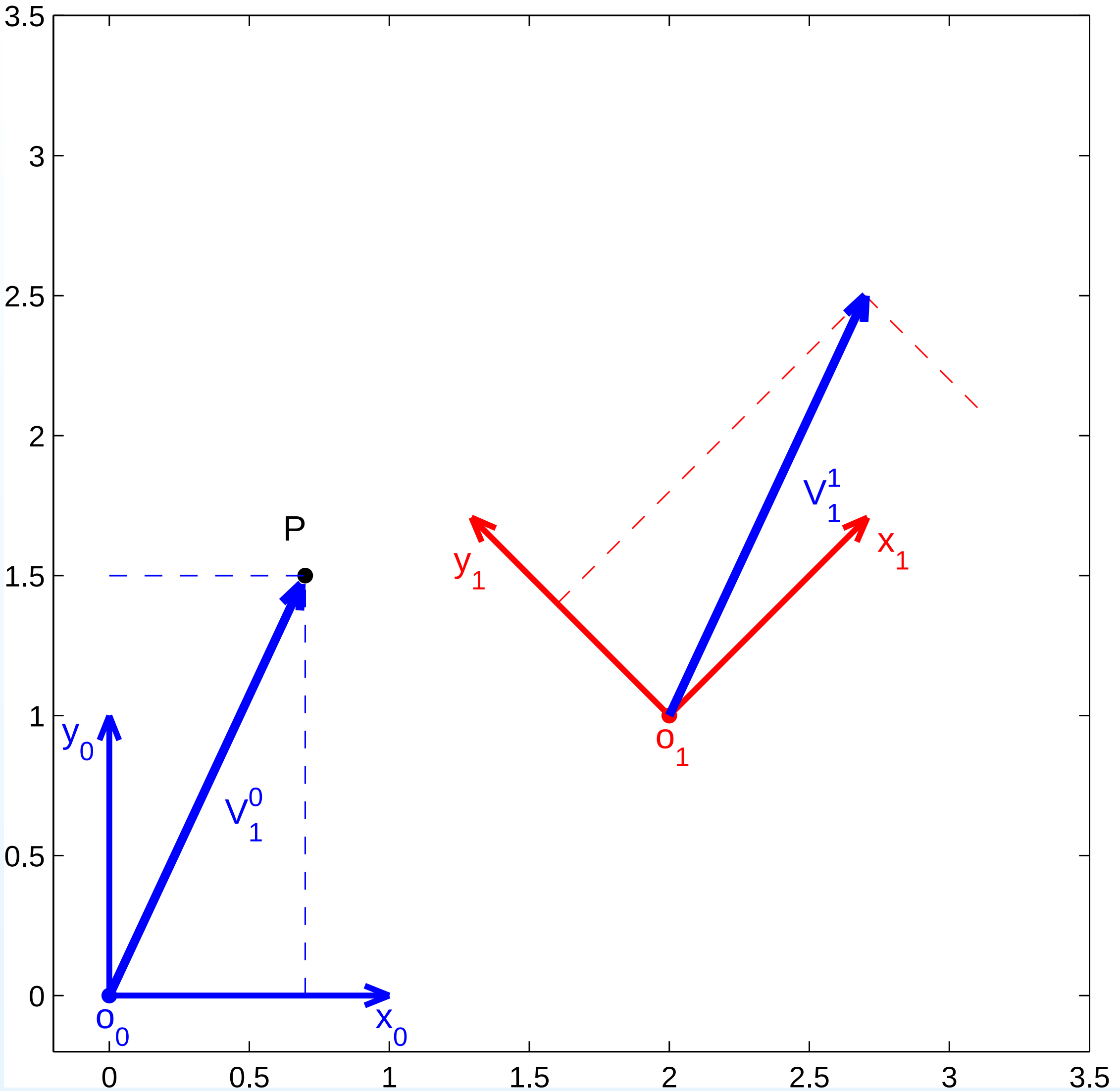
- A point has different representation (coordinates) in different frames.
- A vector is defined by direction and magnitude.
- Vectors with the same direction and magnitude are the same.



The coordinates of the vector $\vec{V}_1$ in the $\mathbf{0}$-frame $[x_p^0, y_p^0]$.

What would be coordinates of $\vec{V}_1$ in the $\mathbf{1}$-frame?



We need to consider the vector $\vec{V}_1^1$ of the same direction and magnitude as $\vec{V}_1$ but with the origin in o_1. Conclusion: we can sum vectors only if they are expressed in parallel frames.

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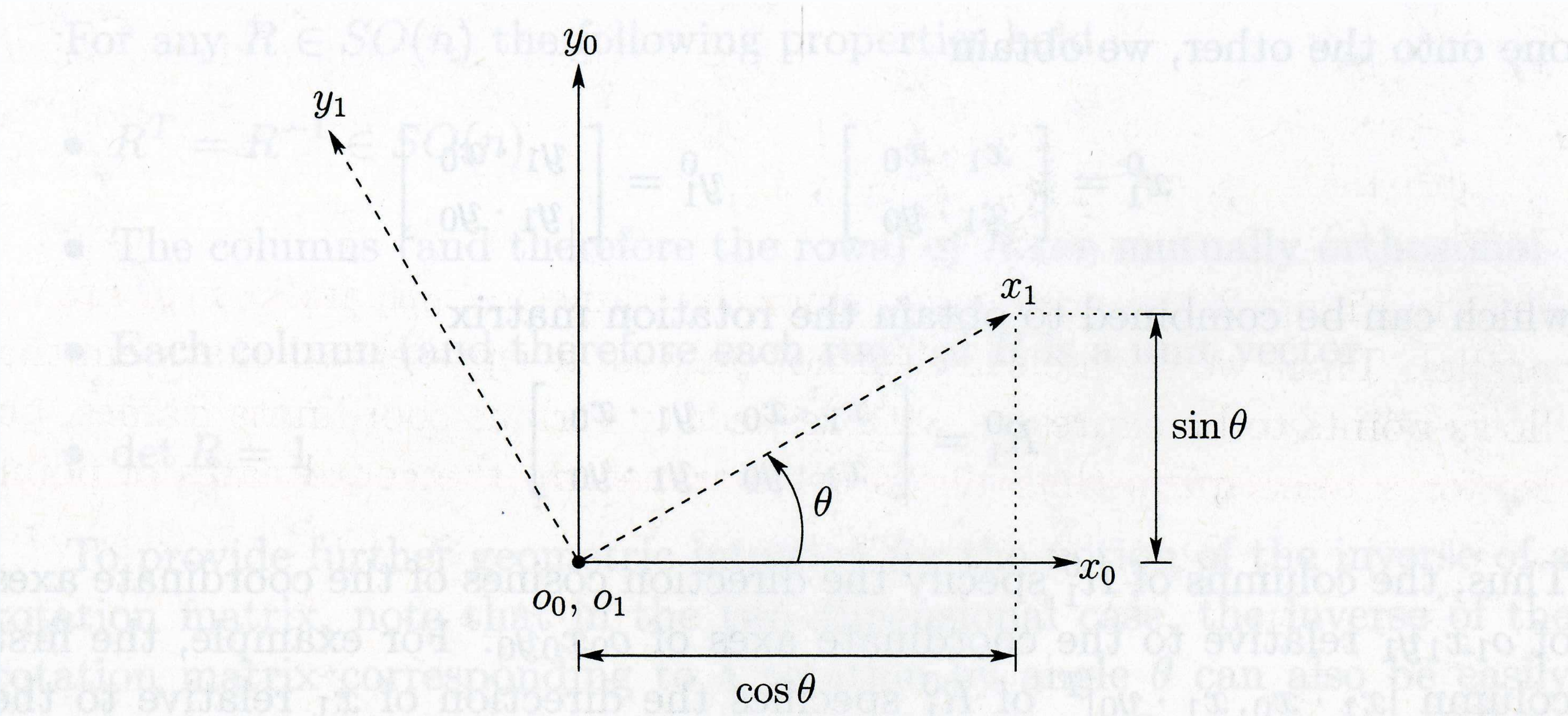
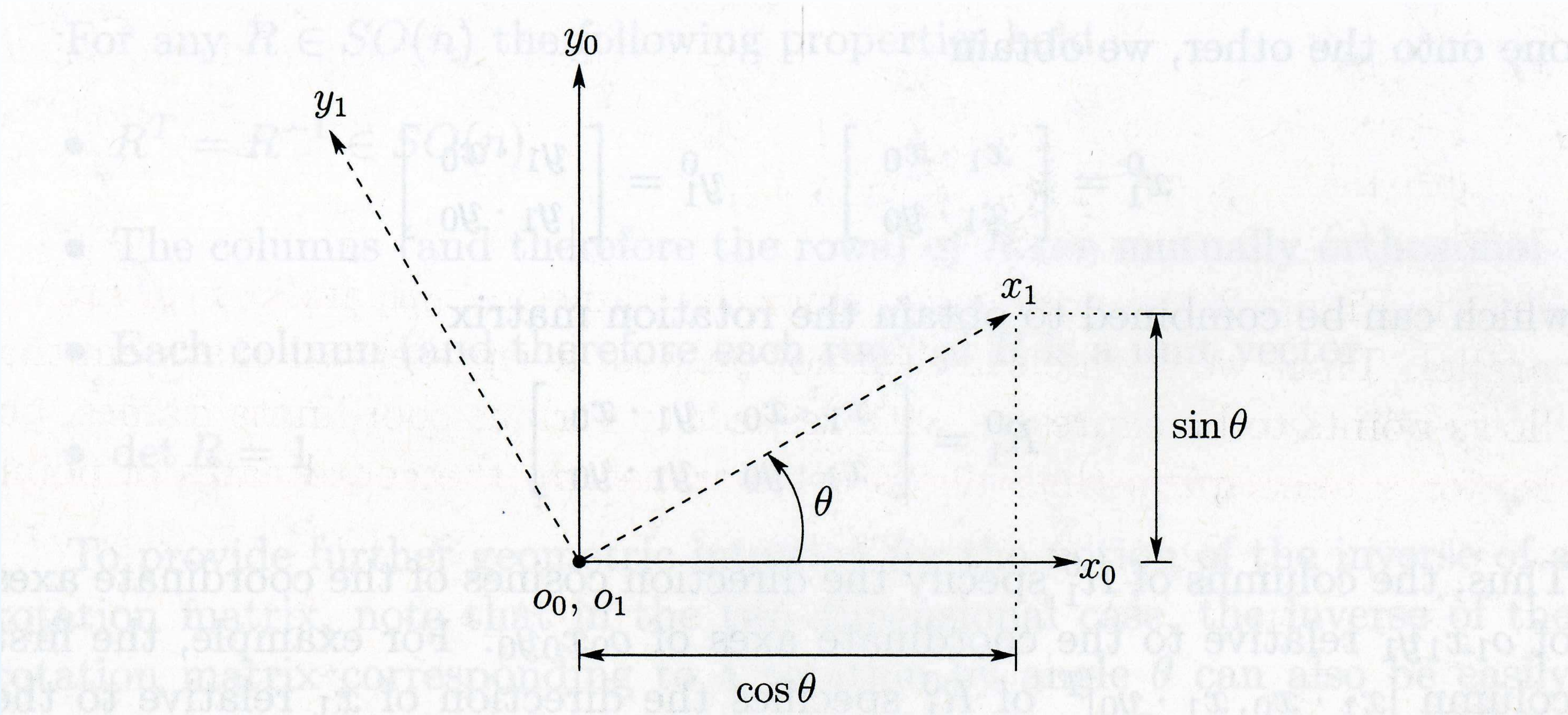
Rotations in $\mathbf{2}$-D



To find an appropriate way to parametrize rotations in $\mathbf{2}$-D, let us track vectors $x_1^0(\cdot)$, $y_1^0(\cdot)$ as θ varies, i.e.

Rotations in $\mathbf{2}$-D



To find an appropriate way to parametrize rotations in $\mathbf{2}$-D, let us track vectors $x_1^0(\cdot)$, $y_1^0(\cdot)$ as θ varies, i.e.

$$x_1^0(\theta) = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}, \; y_1^0(\theta) = x_1^0\left(\theta + \frac{\pi}{2}\right) = \begin{bmatrix} \cos\left(\theta + \frac{\pi}{2}\right) \\ \sin\left(\theta + \frac{\pi}{2}\right) \end{bmatrix} = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

Rotations in 2-D

The matrix

$$\boldsymbol{R(\theta) = \left[x_1^0(\theta) \,|\, y_1^0(\theta)\right] = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}$$

is called the rotation matrix.

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is called the rotation matrix.

It has a number of interesting properties:

- $\det \boldsymbol{R(\theta)} = \cos^2(\theta) + \sin^2(\theta) = 1$
- $\boldsymbol{R(\theta)^{-1}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{\det \boldsymbol{R(\theta)}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

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- $R(\theta)^{-1} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = R^T(\theta)$

A $n \times n$-matrix X that satisfies the property, $X^{-1} = X^T$

$$\Rightarrow \{XX^T = I_n, \ \det(XX^T) = 1 = \det(X)\det(X^T) = \det(X)^2\}$$

is called orthogonal, $X \in \mathcal{O}(n)$. If $\det X = 1 \ \Rightarrow \ X \in \mathcal{SO}(n)$

Rotations in $\mathbf{2}$-D

Let us consider another way for computing

$$\boldsymbol{R(\theta)} = \left[\boldsymbol{x_1^0(\theta)} \,|\, \boldsymbol{y_1^0(\theta)}\right] = \begin{bmatrix} \mathbf{cos}(\boldsymbol{\theta}) & -\,\mathbf{sin}(\boldsymbol{\theta}) \\ \mathbf{sin}(\boldsymbol{\theta}) & \mathbf{cos}(\boldsymbol{\theta}) \end{bmatrix}$$

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As known, scalar product between two vectors is

$$\boldsymbol{\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = |\vec{a}| \cdot |\vec{b}| \cdot \cos\left(\widehat{\vec{a}, \vec{b}}\right)}$$

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All vectors of coordinate frames of magnitude 1, therefore

$$x_1^0(\theta) = \begin{bmatrix} x_1^0(\theta) \cdot x_0 \\ x_1^0(\theta) \cdot y_0 \end{bmatrix}, \qquad y_1^0(\theta) = \begin{bmatrix} y_1^0(\theta) \cdot x_0 \\ y_1^0(\theta) \cdot y_0 \end{bmatrix}$$

Rotations in 2-D

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$$\Rightarrow \boxed{R(\theta) = \left[x_1^0(\theta) \middle| \, y_1^0(\theta)\right] = \left[\begin{array}{c|c} x_1^0(\theta) \cdot x_0 & y_1^0(\theta) \cdot x_0 \\ x_1^0(\theta) \cdot y_0 & y_1^0(\theta) \cdot y_0 \end{array}\right]}$$

Rotations in $\mathbf{3}$-D

Rotation matrix for $\mathbf{3}$-dimensions is then

$$
\begin{aligned}
\boldsymbol{R_1^0(\theta)} &= \left[\boldsymbol{x_1^0(\theta)} \,|\, \boldsymbol{y_1^0(\theta)} \,|\, \boldsymbol{z_1^0(\theta)} \right] \\
&= \left[\begin{array}{c|c|c}
\boldsymbol{x_1^0(\theta) \cdot x_0} & \boldsymbol{y_1^0(\theta) \cdot x_0} & \boldsymbol{z_1^0(\theta) \cdot x_0} \\
\boldsymbol{x_1^0(\theta) \cdot y_0} & \boldsymbol{y_1^0(\theta) \cdot y_0} & \boldsymbol{z_1^0(\theta) \cdot y_0} \\
\boldsymbol{x_1^0(\theta) \cdot z_0} & \boldsymbol{y_1^0(\theta) \cdot z_0} & \boldsymbol{z_1^0(\theta) \cdot z_0}
\end{array} \right]
\end{aligned}
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Rotations in $\mathbf{3}$-D

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Properties to check:

- Columns of $R_1^0(\cdot)$ are mutually orthogonal, for instance

$$\left[x_1^0(\theta)x_0, x_1^0(\theta)y_0, x_1^0(\theta)z_0\right] \left[y_1^0(\theta)x_0, y_1^0(\theta)y_0, y_1^0(\theta)z_0\right]^T =$$

$$= \underbrace{x_1^0(\theta)\cdot y_1^0(\theta)}_{=0} \cdot |x_0|^2 + \underbrace{x_1^0(\theta)\cdot y_1^0(\theta)}_{=0} \cdot |y_0|^2 + \underbrace{x_1^0(\theta)\cdot y_1^0(\theta)}_{=0} \cdot |z_0|^2$$

Rotations in $\mathbf{3}$-D

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\begin{aligned}
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&= \left[\begin{array}{c|c|c} x_1^0(\theta)\cdot x_0 & y_1^0(\theta)\cdot x_0 & z_1^0(\theta)\cdot x_0 \\ x_1^0(\theta)\cdot y_0 & y_1^0(\theta)\cdot y_0 & z_1^0(\theta)\cdot y_0 \\ x_1^0(\theta)\cdot z_0 & y_1^0(\theta)\cdot z_0 & z_1^0(\theta)\cdot z_0 \end{array} \right]
\end{aligned}
$$

Properties to check:

- Columns of $R_1^0(\cdot)$ are mutually orthogonal;
- $R_1^0(\theta) \left[R_1^0(\theta) \right]^T = I_3$

Rotations in $\mathbf{3}$-D

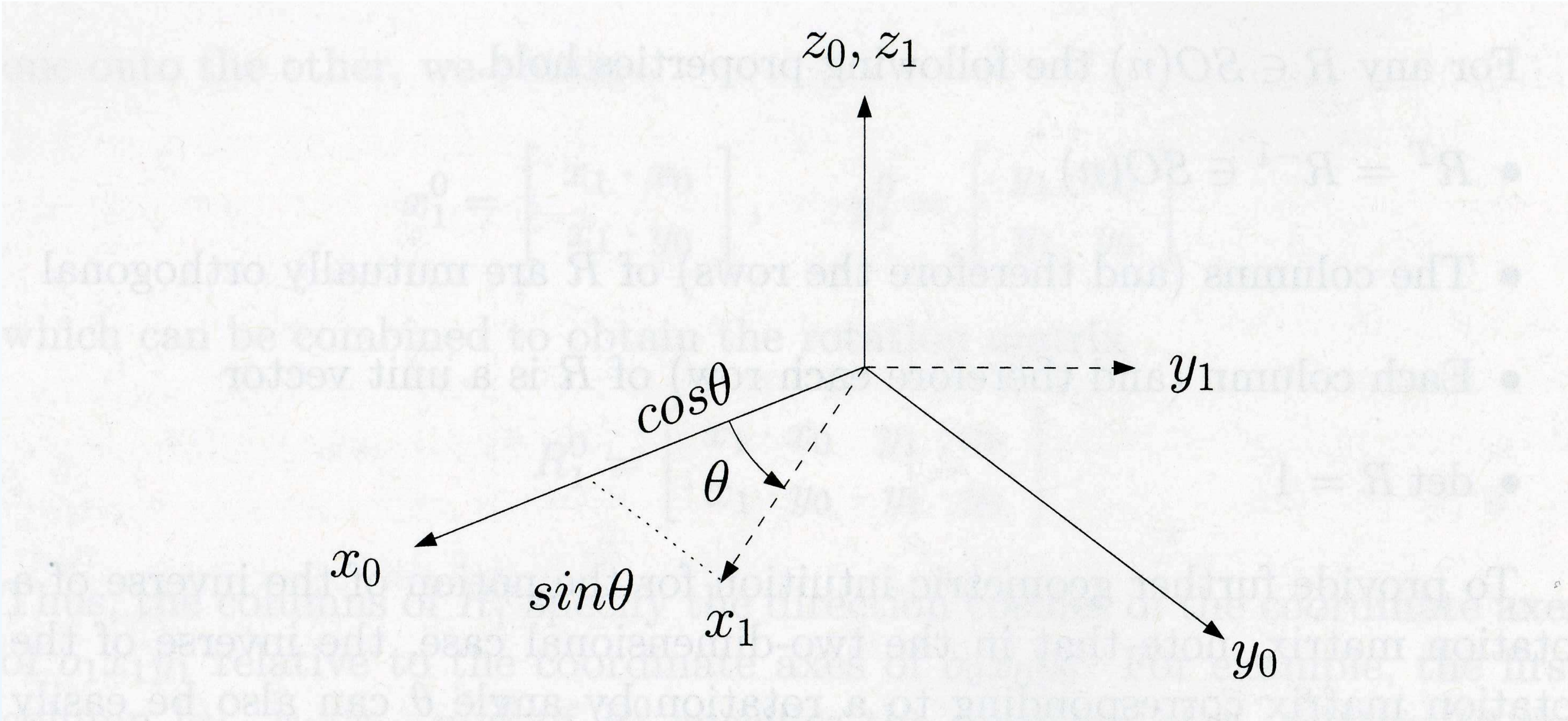
Rotation matrix for $\mathbf{3}$-dimensions is then

$$\begin{aligned} \boldsymbol{R_1^0(\theta)} &= \left[\boldsymbol{x_1^0(\theta)} \,|\, \boldsymbol{y_1^0(\theta)} \,|\, \boldsymbol{z_1^0(\theta)}\right] \\ &= \left[\begin{array}{c|c|c} \boldsymbol{x_1^0(\theta)\cdot x_0} & \boldsymbol{y_1^0(\theta)\cdot x_0} & \boldsymbol{z_1^0(\theta)\cdot x_0} \\ \boldsymbol{x_1^0(\theta)\cdot y_0} & \boldsymbol{y_1^0(\theta)\cdot y_0} & \boldsymbol{z_1^0(\theta)\cdot y_0} \\ \boldsymbol{x_1^0(\theta)\cdot z_0} & \boldsymbol{y_1^0(\theta)\cdot z_0} & \boldsymbol{z_1^0(\theta)\cdot z_0} \end{array}\right] \end{aligned}$$

Properties to check:

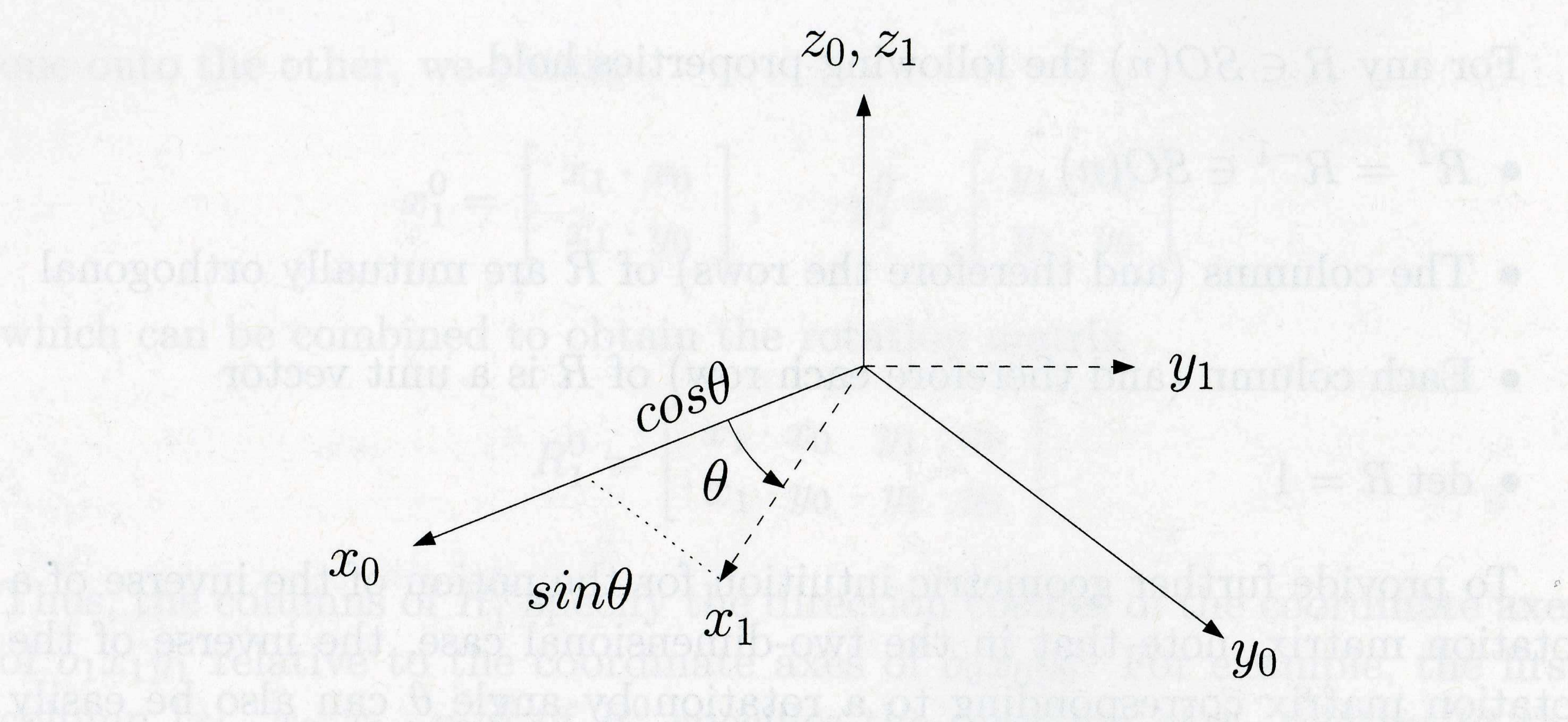
- Columns of $\boldsymbol{R_1^0(\cdot)}$ are mutually orthogonal;
- $\boldsymbol{R_1^0(\theta)\left[R_1^0(\theta)\right]^T = I_3}$
- $\mathbf{det}\, \boldsymbol{R_1^0(\theta) = 1}$

Example of Rotation in $\mathbf{3}$-D



$\mathbf{1}$-Frame is rotated through $\boldsymbol{\theta}$-angle around $\boldsymbol{z_0}$-axis.

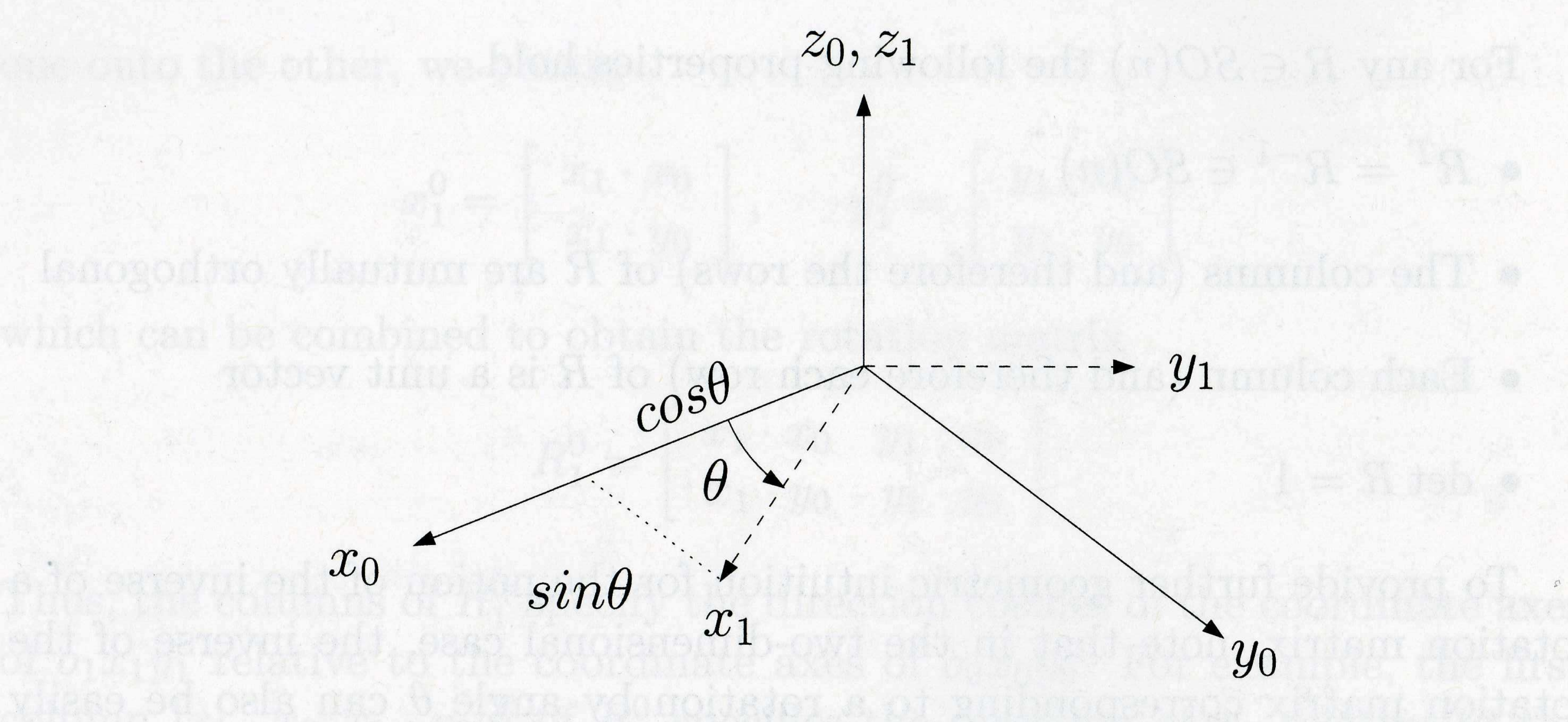
Example of Rotation in $\mathbf{3}$-D



1-Frame is rotated through θ-angle around z_0-axis.

$$
\begin{aligned}
x_1^0(\theta)x_0 &= \cos\theta & y_1^0(\theta)x_0 &= -\sin\theta & z_1^0(\theta)x_0 &= 0\\
x_1^0(\theta)y_0 &= \sin\theta & y_1^0(\theta)y_0 &= \cos\theta & z_1^0(\theta)y_0 &= 0\\
x_1^0(\theta)z_0 &= 0 & y_1^0(\theta)z_0 &= 0 & z_1^0(\theta)z_0 &= 1
\end{aligned}
$$

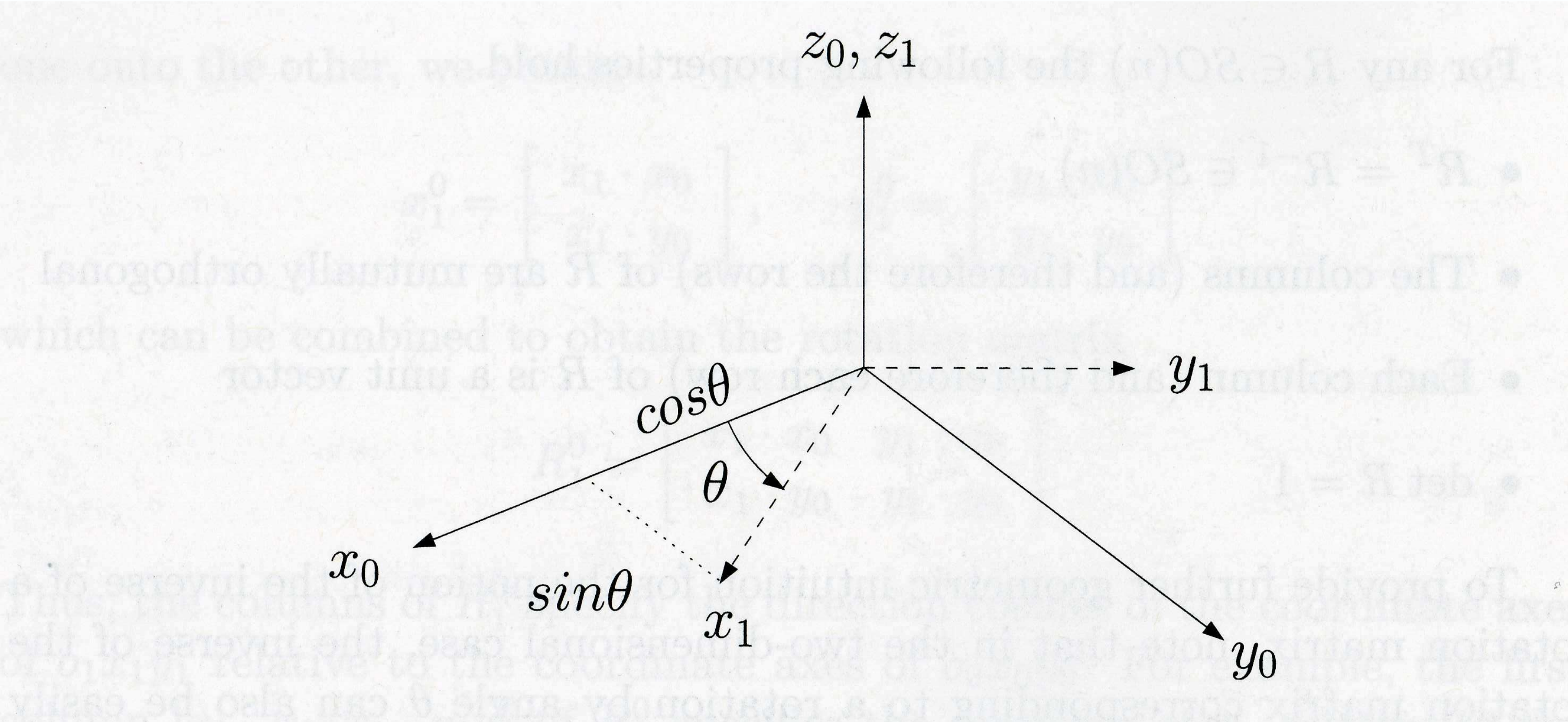
Example of Rotation in $\mathbf{3}$-D



1-Frame is rotated through θ-angle around z_0-axis.

$$R_1^0(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \{=: R_{z,\theta}\}$$

Example of Rotation in $\mathbf{3}$-D



$\mathbf{1}$-Frame is rotated through θ-angle around z_0-axis.

This basic rotation matrix clearly satisfies the properties

$$\boldsymbol{R_{z,0} = I_3}, \qquad \boldsymbol{R_{z,\theta} R_{z,\phi} = R_{z,\theta+\phi}}, \qquad \boldsymbol{[R_{z,\theta}]^{-1} = R_{z,-\theta}}$$

Basic Rotations in $\mathbf{3}$-D

In the way we have introduced the basic rotation matrix

$$\boldsymbol{R_{z,\theta}} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we can introduce basic rotation matrices

$$\boldsymbol{R_{x,\theta}}, \quad \boldsymbol{R_{y,\theta}}$$

Basic Rotations in 3-D

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we can introduce basic rotation matrices

$$R_{x,\theta}, \quad R_{y,\theta}$$

They are

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}, \qquad R_{y,\theta} = \begin{bmatrix} * & 0 & * \\ 0 & 1 & 0 \\ * & 0 & * \end{bmatrix}$$

Basic Rotations in $\mathbf{3}$-D

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$$\boldsymbol{R_{z,\theta}} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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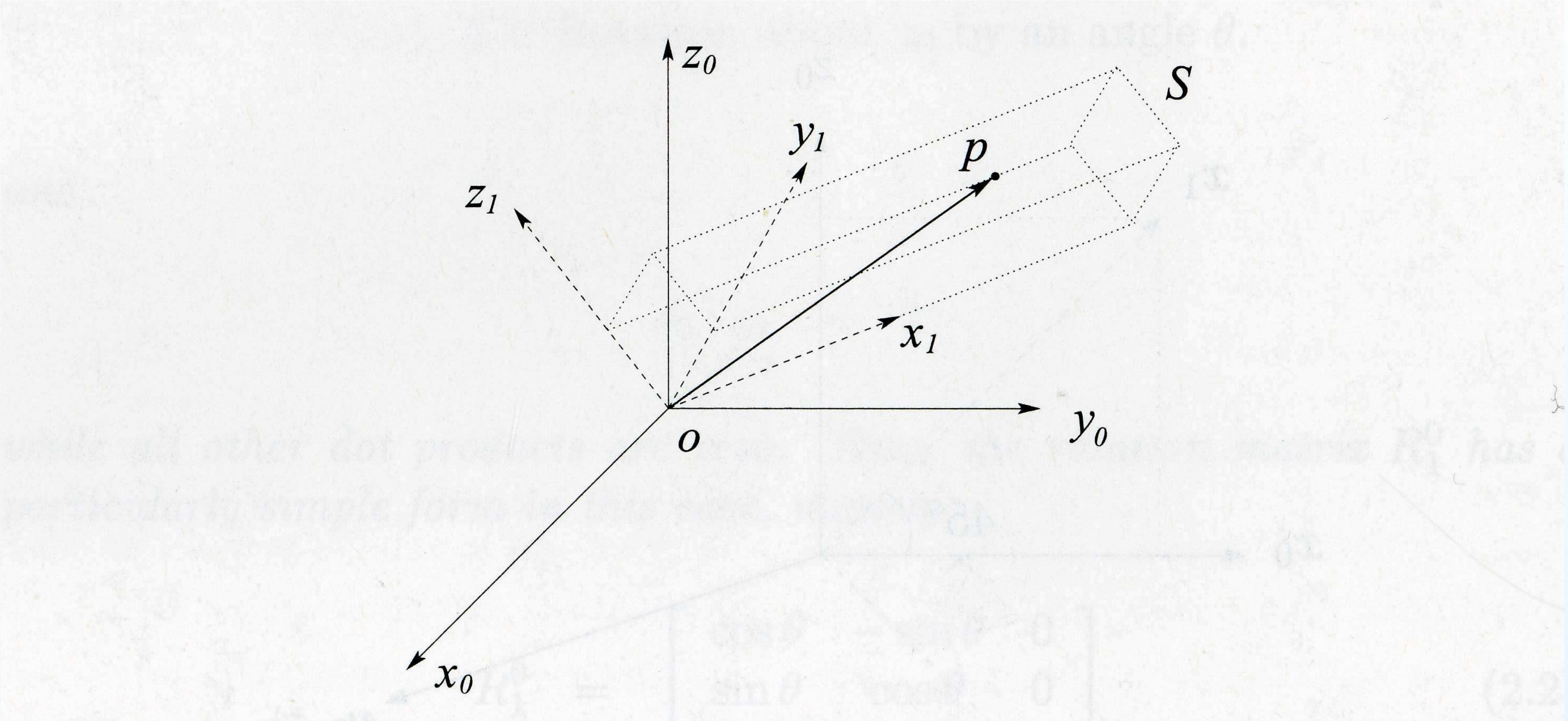
$$\boldsymbol{R_{x,\theta}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \qquad \boldsymbol{R_{y,\theta}} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

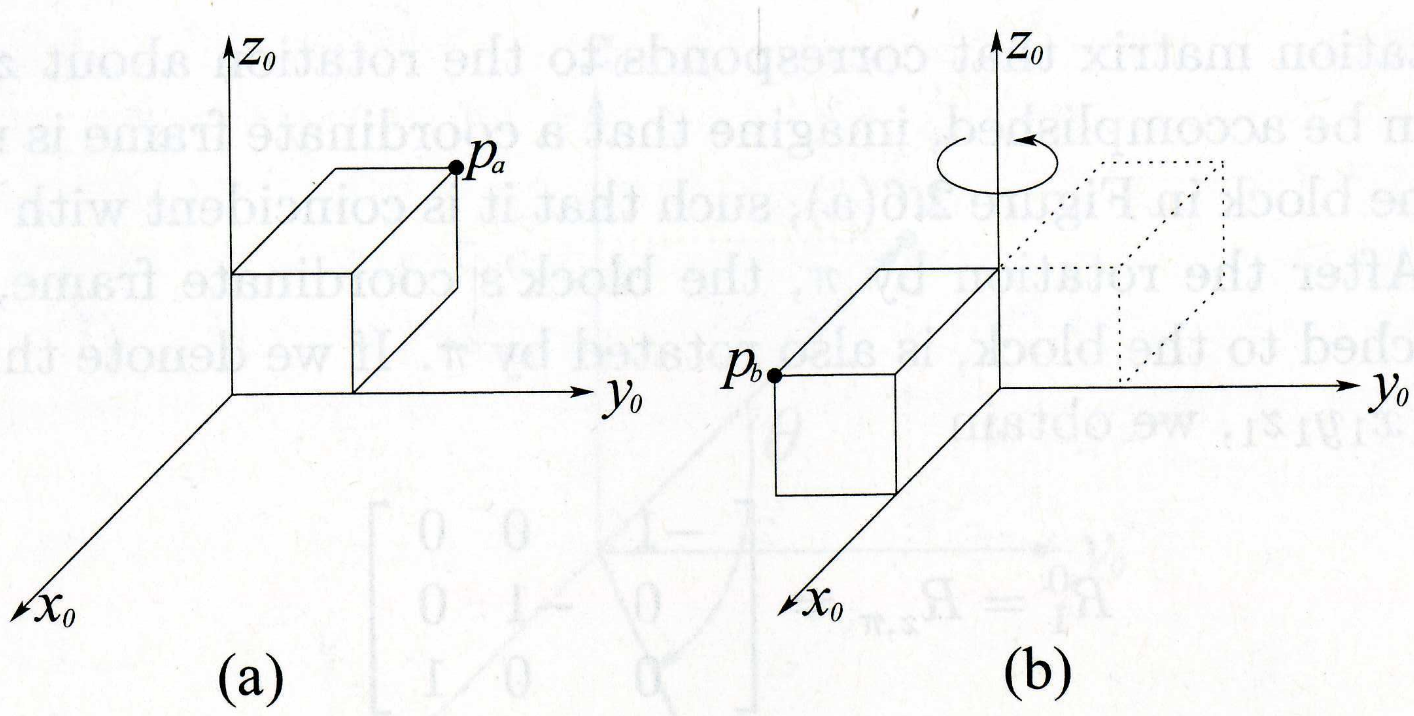
Transformations by Rotations in 3-D



The 0-frame is our world, the 1-frame is fixed to a rigid body.

What will happen with points of body (let say p)
if we rotate the body, i.e. the 1-frame?

Transformations by Rotations in $\mathbf{3}$-D



How to trace the change of position of the point in the $\mathbf{0}$-frame?

Let say, we are interested in coordinates of the point $\boldsymbol{p}$, in the $\mathbf{1}$-frame they are constant, but the $\mathbf{0}$-frame they are changed!

Transformations by Rotations in 3-D

The coordinates of point p in the 1-frame is $p^1 = [u, v, w]^T$, i.e.

$$p^1 = u \cdot \vec{x}_1 + v \cdot \vec{y}_1 + w \cdot \vec{z}_1$$

and the coordinates u, v, w do not change when the 1-frame is rotated. We need to find the coordinates of p in the 0-frame.

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Clearly, with the rotation the basis of the 1-frame is changing and we know how it does that!

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Clearly, with the rotation the basis of the 1-frame is changing and we know how it does that!

For instance,

$$x_1^0 = \begin{bmatrix} x_1 \cdot x_0 \\ x_1 \cdot y_0 \\ x_1 \cdot z_0 \end{bmatrix} = \underbrace{\left[\begin{array}{c|c|c} x_1^0 \cdot x_0 & y_1^0 \cdot x_0 & z_1^0 \cdot x_0 \\ x_1^0 \cdot y_0 & y_1^0 \cdot y_0 & z_1^0 \cdot y_0 \\ x_1^0 \cdot z_0 & y_1^0 \cdot z_0 & z_1^0 \cdot z_0 \end{array}\right]}_{= R} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Transformations by Rotations in 3-D

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For instance,

$$x_1^0 = R \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad y_1^0 = R \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad z_1^0 = R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Transformations by Rotations in 3-D

The coordinates of point p in the 1-frame is $p^1 = [u, v, w]^T$, i.e.

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$$p^0 = u \cdot x_1^0 + v \cdot y_1^0 + w \cdot z_1^0 = R \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix} + R \begin{bmatrix} 0 \\ v \\ 0 \end{bmatrix} + R \begin{bmatrix} 0 \\ 0 \\ w \end{bmatrix}$$

Transformations by Rotations in 3-D

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$$p^0 = u \cdot x_1^0 + v \cdot y_1^0 + w \cdot z_1^0 = R \begin{bmatrix} u \\ v \\ w \end{bmatrix} = R_1^0 p^1$$

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Rotation with Respect to the Current Frame:

Suppose that we have 3 frames:

$$(o_0, x_0, y_0, z_0), \quad (o_1, x_1, y_1, z_1), \quad (o_2, x_2, y_2, z_2).$$

Any point p will have three representations:

$$p^0 = [u_0, v_0, w_0]^T, \quad p^1 = [u_1, v_1, w_1]^T, \quad p^2 = [u_2, v_2, w_2]^T$$

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We know that

$$p^0 = R_1^0 p^1, \quad p^1 = R_2^1 p^2, \quad p^0 = R_2^0 p^2$$

How are the matrices R_1^0, R_2^1 and R_2^0 related?

Rotation with Respect to the Current Frame:

Suppose that we have 3 frames:

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We know that

$$p^0 = R_1^0 \, p^1, \quad p^1 = R_2^1 \, p^2, \quad p^0 = R_2^0 \, p^2$$

How are the matrices R_1^0, R_2^1 and R_2^0 related?

We can compute p^0 in two different ways

$$p^0 = R_1^0 \, p^1 = R_1^0 \, R_2^1 \, p^2, \qquad p^0 = R_2^0 \, p^2$$

Rotation with Respect to the Current Frame:

Suppose that we have 3 frames:

$$(o_0, x_0, y_0, z_0), \quad (o_1, x_1, y_1, z_1), \quad (o_2, x_2, y_2, z_2).$$

Any point p will have three representations:

$$p^0 = [u_0, v_0, w_0]^T, \quad p^1 = [u_1, v_1, w_1]^T, \quad p^2 = [u_2, v_2, w_2]^T$$

We know that

$$p^0 = R_1^0\, p^1, \quad p^1 = R_2^1\, p^2, \quad p^0 = R_2^0\, p^2$$

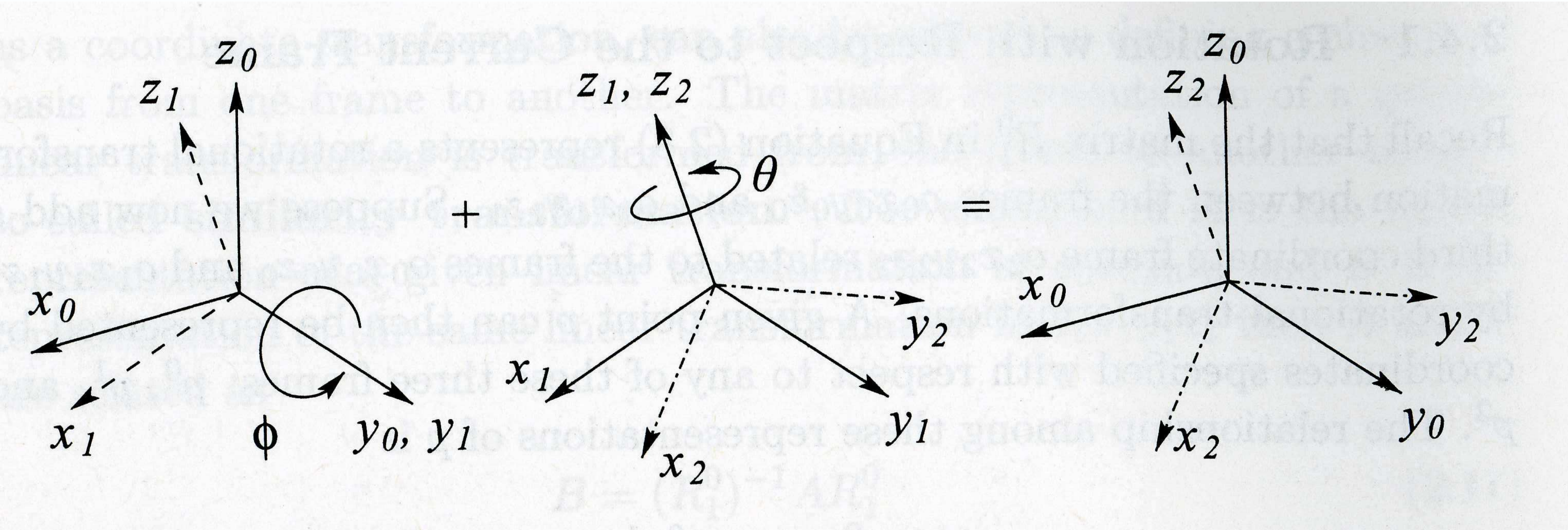
How are the matrices R_1^0, R_2^1 and R_2^0 related?

We can compute p^0 in two different ways

$$p^0 = R_1^0\, p^1 = R_1^0\, R_2^1\, p^2, \qquad p^0 = R_2^0\, p^2$$

$$\Rightarrow R_1^0\, R_2^1 \equiv R_2^0$$

Example 2.5:

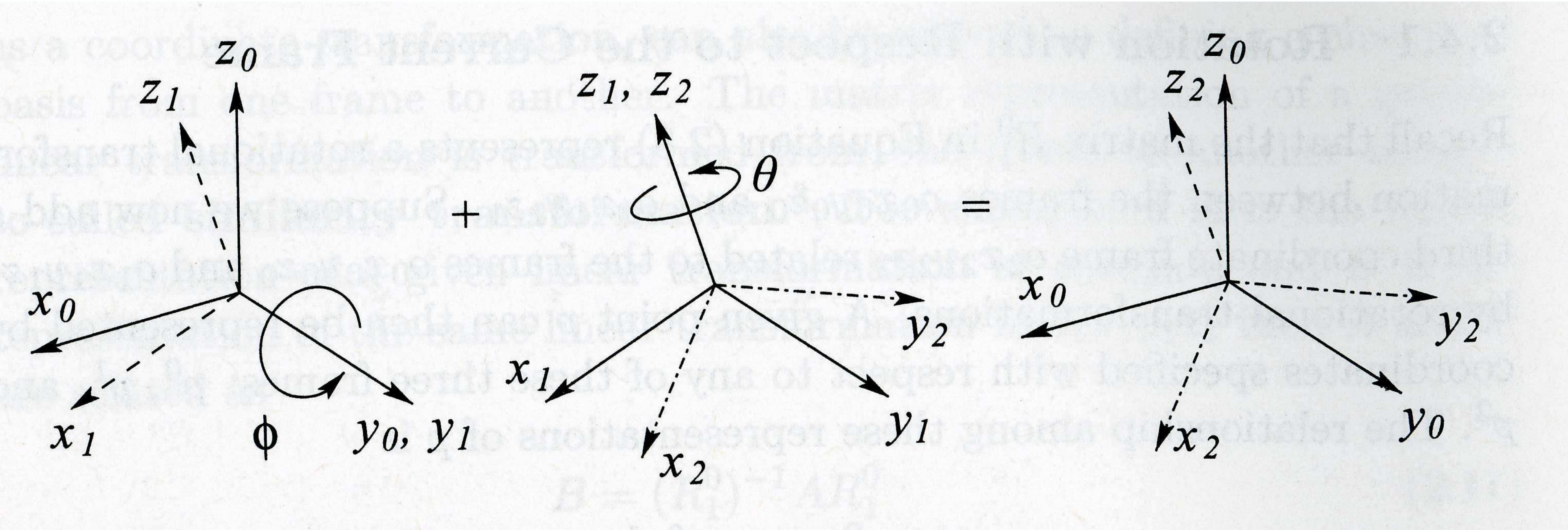


Suppose we rotate

- first the frame by angle ϕ around current y-axis,
- then rotate by angle θ around the current z-axis.

Find the combined rotation

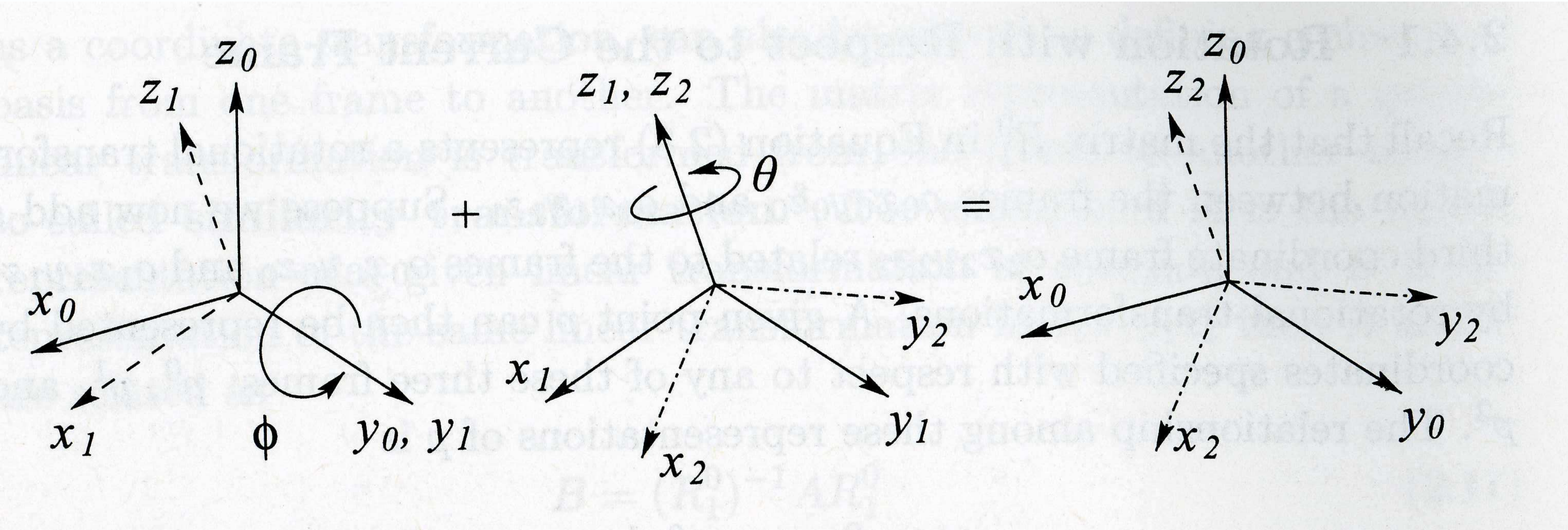
Example 2.5:



The rotations around y- and z-axis are basic rotations

$$\boldsymbol{R_{y,\phi}} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}, \qquad \boldsymbol{R_{z,\theta}} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

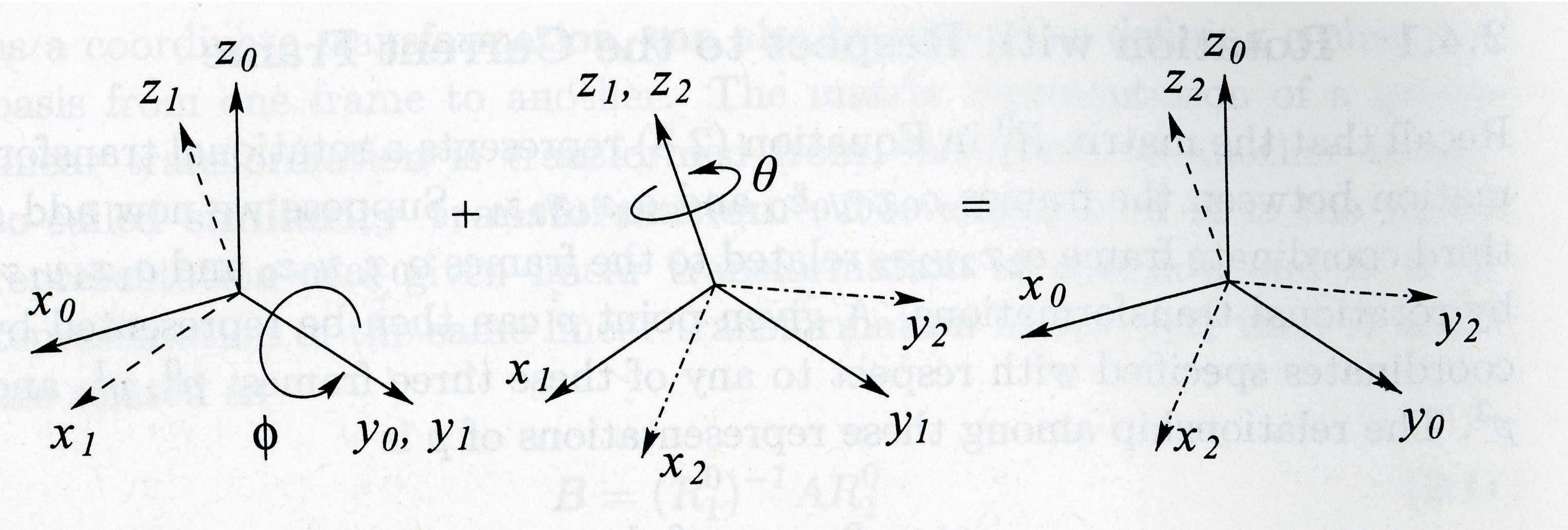
Example 2.5:



Therefore the overall rotation is

$$R = R_{y,\phi} R_{z,\theta} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 2.5:



Therefore the overall rotation is

$$R = R_{y,\phi} R_{z,\theta} = \begin{bmatrix} c_\phi c_\theta & -c_\phi s_\theta & s_\phi \\ s_\theta & c_\theta & 0 \\ -s_\phi c_\theta & s_\phi s_\theta & c_\phi \end{bmatrix}, \qquad \{\Rightarrow \; p^0 = R\,p^2\}$$

Example 2.5:

Important Observation: Rotations do not commute

$$R_{y,\phi} R_{z,\theta} \neq R_{z,\theta} R_{y,\phi}$$

So that the order of rotations is important!

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So that the order of rotations is important!

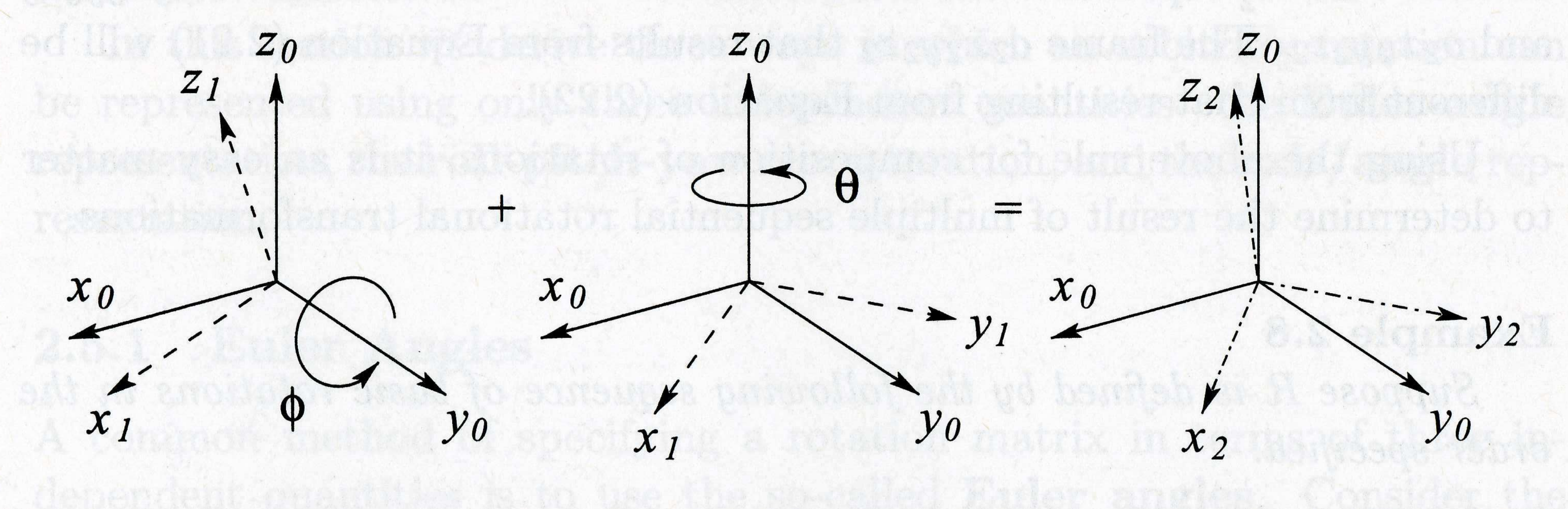
Indeed

$$R_{y,\phi}R_{z,\theta} = \begin{bmatrix} c_\phi c_\theta & -c_\phi s_\theta & s_\phi \\ s_\theta & c_\theta & 0 \\ -s_\phi c_\theta & s_\phi s_\theta & c_\phi \end{bmatrix}$$

and

$$R_{z,\theta}R_{y,\phi} = \begin{bmatrix} c_\phi c_\theta & -s_\theta & c_\theta s_\phi \\ s_\theta c_\phi & c_\theta & s_\phi s_\theta \\ -s_\phi & 0 & c_\phi \end{bmatrix}$$

Rotation with Respect to the Fixed Frame:



How to compute the rotation if the basic rotations are done with respect to fixed frames?

Rotation with Respect to the Fixed Frame:

Given two frames and the rotation:

$$(o_0, x_0, y_0, z_0), \quad (o_1, x_1, y_1, z_1), \quad p^0 = R_1^0 p^1$$

Rotation with Respect to the Fixed Frame:

Given two frames and the rotation:

$$(o_0, x_0, y_0, z_0), \quad (o_1, x_1, y_1, z_1), \quad p^0 = R_1^0 p^1$$

Given a linear transform A, for which we know how it acts on vectors of the 0-frame

$$\vec{a}^0 = A\vec{b}^0$$

How it acts on the vectors of the 1-frame?

Rotation with Respect to the Fixed Frame:

Given two frames and the rotation:

$$(o_0, x_0, y_0, z_0), \quad (o_1, x_1, y_1, z_1), \quad p^0 = R_1^0 p^1$$

Given a linear transform A, for which we know how it acts on vectors of the 0-frame

$$\vec{a}^0 = A\vec{b}^0$$

How it acts on the vectors of the 1-frame?

To compute its action, we need to observe that vectors in both frames are in one-to-one correspondence, i.e

- given b^0, then $b^1 = \left[R_1^0\right]^{-1} b^0$
- given b^1, then $b^0 = R_1^0 b^1$

Rotation with Respect to the Fixed Frame:

Given two frames and the rotation:

$$(o_0, x_0, y_0, z_0), \quad (o_1, x_1, y_1, z_1), \quad p^0 = R_1^0 p^1$$

To compute its action, we need to observe that vectors in both frames are in one-to-one correspondence, i.e

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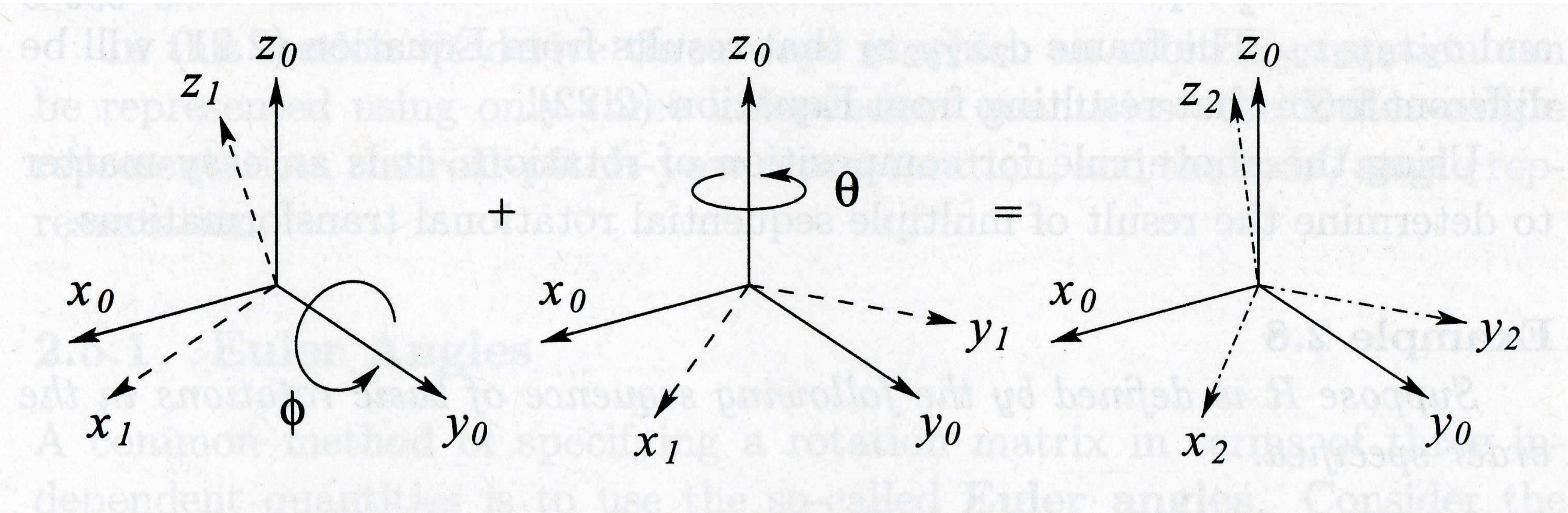
- given b^1, then $b^0 = R_1^0 b^1$

To define A acting in the 1-frame, use its definition in 0-frame

$$\underbrace{\left[R_1^0\right]^{-1} a^0}_{= a^1} = \left[R_1^0\right]^{-1} A b^0 = \underbrace{\left[R_1^0\right]^{-1} A R_1^0}_{B} b^1$$

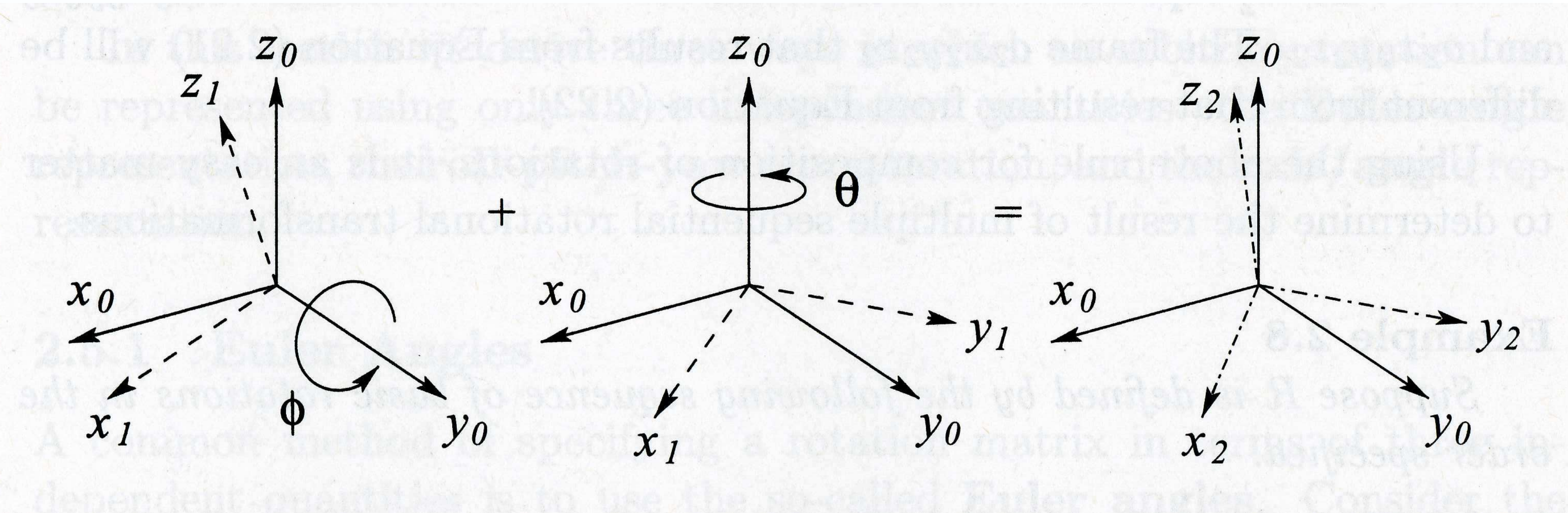
Rotation with Respect to the Fixed Frame:



We have two rotations

- the basic rotation $\boldsymbol{R_1^0 = R_{y_0,\phi}}$: by angle $\boldsymbol{\phi}$ around $\boldsymbol{y_0}$-axis
- the rotation $\boldsymbol{R_2^1}$ defined as the rotation by angle $\boldsymbol{\theta}$ around $\boldsymbol{z_0}$-axis (not $\boldsymbol{z_1}$-axis)

Rotation with Respect to the Fixed Frame:



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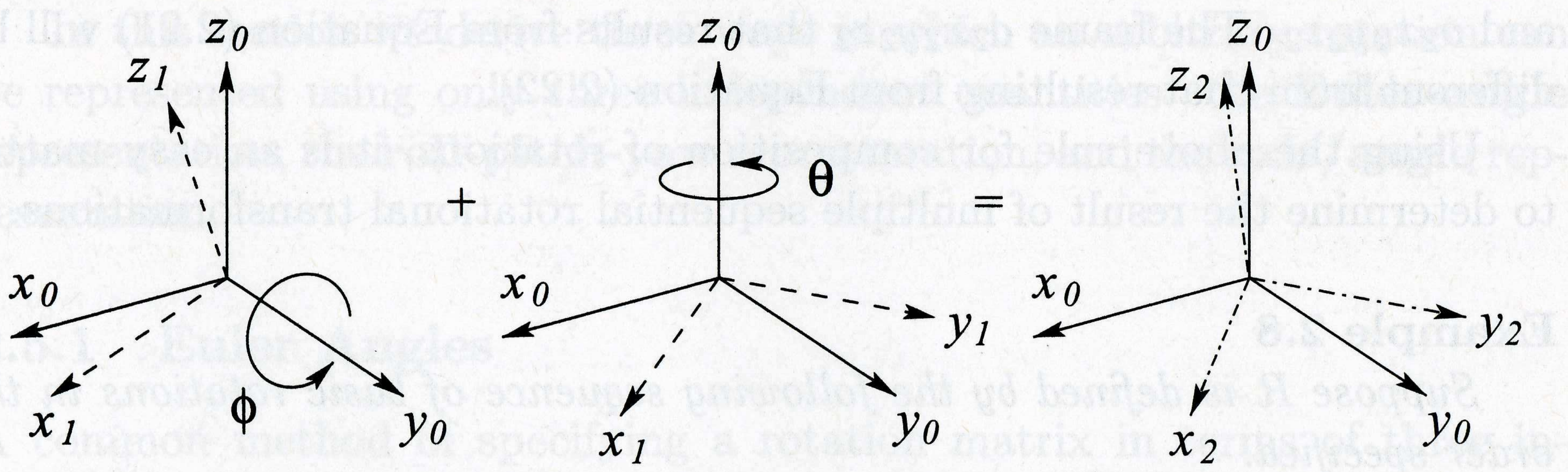
- the basic rotation $R_1^0 = R_{y_0,\phi}$: by angle ϕ around y_0-axis
- the rotation R_2^1 defined as the rotation by angle θ around z_0-axis (not z_1-axis)

The combined rotation will be

$$R_2^0 = R_1^0 R_2^1 = R_{y_0,\phi} R_2^1 = R_{y_0,\phi}\left[\{R_{y_0,\phi}\}^{-1} R_{z_0,\theta} R_{y_0,\phi}\right]$$

Rotation with Respect to the Fixed Frame:



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- the basic rotation $R_1^0 = R_{y_0,\phi}$: by angle ϕ around y_0-axis
- the rotation R_2^1 defined as the rotation by angle θ around z_0-axis (not z_1-axis)

The combined rotation will be

$$R_2^0 = R_{y_0,\phi}\left[\{R_{y_0,\phi}\}^{-1} R_{z_0,\theta} R_{y_0,\phi}\right] = R_{z_0,\theta} R_{y_0,\phi}$$