11 L2 Gain

In this section the notion of *L2 gain* is introduced for general input output system models. Examples of gain calculation are given, as well as general criteria for computing L2 gain of memoryless time invariant and linear time invariant models.

11.1 L2 Gain of General Models

L2 gain of an input-output system quantifies the maximal gain in "energy transmission" from input to output, where "energy" is of a signal g is understood as the integral of $|g(t)|^2$ over a time interval. In this respect, a system's L2 gain is similar to the gain of a function $\Delta : \mathbb{R}^m \mapsto \mathbb{R}^k$, defined as the minimal $\gamma \geq 0$ such that $\gamma^2|v|^2 \geq |\Delta(v)|^2$ for all $v \in \mathbb{R}^m$. There are, however, important differences. First, for a system, it is possible to have some energy accumulated in its initial state. Then the output energy is a combination of input energy and the energy of the initial condition. Naturally, the effect of initial conditions is to be separated from the energy transmission gain. Second, the total energy of many signals of interest in applications, such as sinusoids, ramps, etc., is infinite, which calls for introduction of a system gain defined it in terms of finite time interval beavior.

Definition 11.1 L2 gain of a CT i/o model $\mathcal{S} \subset \mathcal{L}_m \times \mathcal{L}_k$ is the maximal lower bound of those $\gamma > 0$ for which the inequality

$$\inf_{T>0} \int_0^T \{\gamma^2|f(t)|^2 - |y(t)|^2\}dt > -\infty \tag{11.1}$$

holds for all i/o pairs $(f, y) \in \mathcal{S}$.

For a given $T > 0$, the integral in (11.1) describes the difference between input energy (scaled by the factor γ^2 playing the role of an upper bound for the square of the L2 gain) and output energy. The difference is allowed to be finite, as this can be attributed to the energy stored in the initial state of the system. However, having an infinite difference would imply that the L2 gain is not smaller than γ.

For DT systems, the definition is the same except that (11.1) is replaced by its analog

$$\inf_{T>0} \sum_{t=0}^{T} \{\gamma^2|f(t)|^2 - |y(t)|^2\} > -\infty. \tag{11.2}$$

Proving that L2 gain of a system equals a given number $\gamma_0 > 0$ typically involves two steps. On one hand, one has to prove that for every $\gamma > \gamma_0$ condition (11.1) (or its DT equivalent (11.2)) is satisfied for *every* i/o pair. In addition, for every $\gamma < \gamma_0$ one has to generate an i/o pair such that the infimum in (11.1) is minus infinity. Indeed, only the first step has to be completed to prove that L2 gain equals zero, and only the second step is needed to prove that L2 gain is infinite.

Example 11.1 The system mapping every input $f \in \mathcal{L}$ to the same output $y(t) = e^{at}$ has infinite L2 gain for $a \geq 0$ and zero L2 gain for $a < 0$.

Indeed, for $a \geq 0$ and $f(t) \equiv 0$

$$\lim_{T\to\infty} \int_0^T \{\gamma^2|f(t)|^2 - |y(t)|^2\}dt = -\infty.$$

On the other hand, for $a < 0$

$$\int_0^T \{\gamma^2|f(t)|^2 - |y(t)|^2\}dt \geq -\int_0^T |e^{at}|^2 dt \geq \frac{1}{2a}$$

for all inputs f and all $\gamma \geq 0$.

Example 11.2 The "moving integration and hold" system $\mathcal{S}_\tau$ defined for $\tau > 0$ as the set of pairs $(f, y) \in \mathcal{L} \times \mathcal{L}$ such that

$$y(t) = \begin{cases} \int_{k\tau}^{k\tau+\tau} f(r)dr, & k\tau + \tau \leq t < k\tau + 2\tau, \ k \in \mathbb{Z}_+, \\ 0, & 0 \leq t < \tau, \end{cases}$$

has L2 gain $\gamma_\tau = \tau$.

To prove that $\gamma_\tau \leq \tau$, note that

$$\left|\int_{k\tau}^{k\tau+\tau} f(r)dr\right|^2 \leq \left(\int_{k\tau}^{k\tau+\tau} |f(r)|^2 dr\right) \cdot \left(\int_{k\tau}^{k\tau+\tau} dr\right) \leq \tau \int_{k\tau}^{k\tau+\tau} |f(r)|^2 dr.$$

Hence

$$\int_{k\tau}^{k\tau+\tau} |y(r)|^2 dr \leq \tau \left|\int_{k\tau-\tau}^{k\tau} f(r)dr\right|^2 \leq \tau^2 \int_{k\tau-\tau}^{k\tau} |f(r)|^2 dr.$$

Therefore, for $k\tau \leq T < k\tau + \tau$,

$$\int_0^T |y(t)|^2 dt \leq \int_0^{k\tau+\tau} |y(t)|^2 dt \leq \tau^2 \int_0^{k\tau} |f(t)|^2 dt \leq \tau^2 \int_0^T |f(t)|^2 dt,$$

which proves that $\gamma_0 \leq \tau$.

To prove that $\gamma_\tau \geq \tau$ consider the input $f(t) \equiv 1$, and the corresponding output

$$y(t) = \begin{cases} \tau, & t \geq \tau, \\ 0, & \text{otherwise.} \end{cases}$$

Since the integrals

$$\int_0^T \{\gamma^2|f(t)|^2 - |y(t)|^2\}dt \leq \int_\tau^T \{\gamma^2 - \tau^2\}dt = (T - \tau)(\gamma^2 - \tau^2)$$

converge to $-\infty$ when $0 < \gamma < \tau$, it follows that $\gamma_0 \geq \tau$.

11.2 L2 Gains Of Memoryless Systems

Every continuous function $\phi: \mathbb{R}^m \mapsto \mathbb{R}^k$ defines a linear memoryless system $\mathcal{S}_\phi \subset \mathcal{L}_m \times \mathcal{L}_k$ with input f and output y according to

$$y(t) = \phi(f(t)). \tag{11.3}$$

The following statement offers an easy way of computing L2 gain of $\mathcal{S}_\phi$.

Theorem 11.1 *L2 gain of system (11.3) equals the gain of function ϕ, i.e. is given by*

$$\gamma_0 = \inf\{\gamma > 0 : \ \gamma|v| \geq |\phi(v)| \ \forall \ v \in \mathbb{R}^m\}. \tag{11.4}$$

Proof. If $\gamma > \gamma_0$ then $\gamma|f(t)| \geq |y(t)|$ for all t, hence

$$\int_0^T \{\gamma^2|f(t)|^2 - |y(t)|^2\}dt \geq 0$$

is bounded from below.

If $\gamma < \gamma_0$ then $\gamma|v_0| < |\phi(v_0)|$ for some $v_0 \in \mathbb{R}^m$. Hence for the input $f(t) \equiv v_0$ and the corresponding output $y(t) \equiv \phi(v_0)$ the integrals

$$\int_0^T \{\gamma^2|f(t)|^2 - |y(t)|^2\}dt = T(\gamma^2|v_0|^2 - |\phi(v_0)|^2)$$

converge to minus infinity as $T \to +\infty$. ∎

Example 11.3 L2 gain of system mapping $f \in \mathcal{L}$ to $y \in \mathcal{L}$ defined by $y(t) = \sin(f(t))$ equals 1.

Example 11.4 L2 gain of system mapping $f \in \mathcal{L}$ to $y \in \mathcal{L}$ defined by $y(t) = f(t)^{1/3}$ equals $+\infty$.

Every piecewise continuous k-by-m real matrix valued function $h: \mathbb{R}_+ \mapsto \mathbb{R}^{k\times m}$ defines a linear memoryless system $\mathcal{S}_h \subset \mathcal{L}_m \times \mathcal{L}_k$ with input f and output y according to

$$y(t) = h(t)f(t). \tag{11.5}$$

The following statement offers an easy way of computing L2 gain of $\mathcal{S}_h$.

Theorem 11.2 *L2 gain of system (11.5) equals*

$$\gamma_0 = \lim_{T\to\infty} \sup_{t>T} \|h(t)\|. \tag{11.6}$$

Proof. For every $\gamma > \gamma_0$ there exists $\tau = \tau(\gamma)$ such that $\|h(t)\| \leq \gamma$ (and hence $|y(t)| \leq \gamma|f(t)|$) for all $t \geq \tau(\gamma)$. Hence for $T > \tau(\gamma)$

$$\int_0^T \{\gamma^2|f(t)|^2 - |y(t)|^2\}dt \geq \int_0^{\tau(\gamma)} \{\gamma^2|f(t)|^2 - |y(t)|^2\}dt$$

has a finite lower bound which is independent of T.

If $\gamma < \gamma_0$ then for every $T > 0$ there exist $\tau = \tau(T) > T + 1$, $v = v(T) \in \mathbb{R}^m$, and $\epsilon = \epsilon(T) > 0$ such that

$$\gamma^2|v(T)|^2 - |h(t)v(T)|^2 \leq -1/\epsilon \quad \forall\ t \in [\tau(T), \tau(T) + \epsilon].$$

Define $T_k, v_k m \epsilon_k$ for $k = 0, 1, 2, \dots$ according to

$$T_0 = \tau(0),\ v_0 = v(0),\ \epsilon_0 = \epsilon(0),\quad T_{k+1} = \tau(T_k + \epsilon_k),\ v_{k+1} = v(T_k + \epsilon_k),\ \epsilon_{k+1} = \epsilon(T_k + \epsilon_k).$$

Define $f \in \mathcal{L}_m$ by

$$f(t) = \begin{cases} v_k, & T_k \leq t < T_k + \epsilon_k,\ k \in \mathbb{Z}_+ \\ 0,\ \text{otherwise.} & \end{cases}$$

Then

$$\int_{T_k}^{T_k+\epsilon} \{\gamma^2|y(t)|^2 - |f(t)|^2\}dt \leq -1$$

for all $k \in \mathbb{Z}_+$, where $y(t) = h(t)f(t)$ is the output corresponding to input f, and $|f(t)| = |y(t)| = 0$ outside the union of intervals $[T_k, T_k + \epsilon_k]$. Hence

$$\lim_{T\to\infty} \int_0^T \{\gamma^2|f(t)|^2 - |y(t)|^2\}dt = -\infty.$$

■

Example 11.5 System

$$\mathcal{S} = \{(f, y) \in \mathcal{L} \times \mathcal{L} :\ y(t) = (1 + t)^{-1} f(t)\ \forall\ t \geq 0\}$$

has zero L2 gain.

Example 11.6 The harmonic modulation system

$$\mathcal{S} = \{(f, y) \in \mathcal{L} \times \mathcal{L} :\ y(t) = \sin(t) f(t)\ \forall\ t\}$$

has L2 gain $\gamma_0 = 1$.

11.3 L2 Gain Of Transfer Matrix Models

A k-by-m transfer matrix G defines the LTI system $\mathcal{S}_{lti}[G]$, as well as the "multiplication in the Laplace transform domain" system $\mathcal{S}_0[G]$, describing zero-state response of $\mathcal{S}_{lti}[G]$. The following very important statement provides a relatively straightforward way of computing L2 gains of both models.

Theorem 11.3 *L2 gains of systems $\mathcal{S}_{lti}[G]$ and $\mathcal{S}_0[G]$ equal the H-Infinity norm of G, defined by*

$$\|G\|_\infty = \sup_{s\in\Omega} \|G(s)\|, \tag{11.7}$$

where Ω is the right half plane $\Omega = \mathbb{C}_0$ for the CT case, and the exterior of the unit disc $\Omega = \mathbf{D}_1$ in the DT case. Moreover, for rational transfer matrices with no poles in Ω, the supremum can be calculated on the boundary of Ω (the imaginary axis in the CT case and the unit circle in the DT case).

Proof. The proof will be given for continuous time systems (the DT case is similar).

First, note that L2 gains of $\mathcal{S}_0[G]$ and $\mathcal{S}_{lti}[G]$ must be equal. Indeed, since $\mathcal{S}_0[G] \subset \mathcal{S}_{lti}[G]$, the L2 gain of $\mathcal{S}_{lti}[G]$ is not smaller than L2 gain of $\mathcal{S}_0[G]$. On the other hand, every i/o pair in $\mathcal{S}_{lti}[G]$ can be represented in the form $(f + f_0, y + y_0)$, where $(f, y) \in \mathcal{S}_0[G]$ and f_0, y_0 have finite energy. Since for every $\epsilon > 0$ and every pair of vectors $a, b \in \mathbb{C}^n$ the inequalities

$$(1-\epsilon)|a|^2 + (1-\epsilon^{-1})|b|^2 \le |a+b|^2 \le (1+\epsilon)|a|^2 + (1+\epsilon^{-1})|b|^2$$

take place, one can conclude that

$$\int_0^T \{\gamma^2|f+f_0|^2 - |y+y_0|^2\}dt \ge \tag{11.8}$$

$$\ge (1+\epsilon)\int_0^T \left\{\frac{1-\epsilon}{1+\epsilon}\gamma^2|f|^2 - |y|^2\right\} dt + (1+\epsilon^{-1})\int_0^T \left\{\frac{1-\epsilon^{-1}}{1+\epsilon^{-1}}\gamma^2|f_0|^2 - |y_0|^2\right\} dt$$

for all $\gamma, T \ge 0$. If γ is larger than the L2 gain of $\mathcal{S}_0[G]$, the first integral on the right side of (11.8) has a finite lower bound for sufficiently small $\epsilon > 0$. On he other hand, the second integral on the right side of (11.8) has a finite lower bound because f_0 and y_0 have finite energy. Hence L2 gain of $\mathcal{S}_0[G]$ is not smaller than L2 gain of $\mathcal{S}_{lti}[G]$.

Let us show that L2 gain of $\mathcal{S}_0[G]$ is not larger than $\|G\|_\infty$. Indeed, assume that $\gamma > \|G\|_\infty$ and let (f, y) be an i/o pair in $\mathcal{S}_0[G]$. Using the notation from the definition of transfer matrix models,

$$|G(s)F_T(s)| \le \gamma|F_T(s)| \quad \forall\ s \in \mathbb{C}_0.$$

Hence for every $h > 0$

$$
\begin{aligned}
\int_0^T |e^{-ht} y(t)|^2 dt &\leq \int_0^\infty |e^{-ht} y_T(t)|^2 dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} |Y_T(j\omega + h)|^2 d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(j\omega + h) F_T(j\omega + h)|^2 d\omega \\
&\leq \frac{\gamma^2}{2\pi} \int_{-\infty}^{\infty} |F_T(j\omega + h)|^2 d\omega \\
&= \gamma^2 \int_0^\infty |e^{-ht} f_T(t)|^2 dt \\
&= \gamma^2 \int_0^T |e^{-ht} f(t)|^2 dt.
\end{aligned}
$$

As $h \to 0$, this implies

$$
\int_0^T \{\gamma^2 |f(t)|^2 - |y(t)|^2\} dt \geq 0.
$$

Now let us show that L2 gain γ_0 of $\mathcal{S}_{lti}[G]$ is not smaller than $\|G\|_\infty$. There is nothing to prove when $\gamma_0 = +\infty$, hence assume that $\gamma_0 < \infty$.

If L2 gain of system $\mathcal{S}_{lti}[G]$ is smaller than $\gamma < \infty$ then every response $y(t)$ to the finite energy input $f(t) = e^{-t} f_0$, where $f_0 \in \mathbb{R}^m$, has finite energy as well. Hence for every $f_0 \in \mathbb{R}^m$ the function $(s+1)^{-1} G(s) f_0$ is Laplace transform of a finite energy signal. Therefore for every $a \in \mathbb{C}_0$ and $v_0 \in \mathbb{C}^m$ the function

$$
\frac{a+1}{s-a} \left[\frac{G(s)}{s+1} - \frac{G(a)}{a+1} \right]
$$

is Laplace transform of a (complex valued) square integrable function $g : \mathbb{R}_+ \mapsto \mathbb{C}^m$. Since

$$
G(s) \left[\frac{1}{s-a} - \frac{1}{s+1} \right] = \frac{G(a)}{s-a} + \frac{a+1}{s-a} \left[\frac{G(s)}{s+1} - \frac{G(a)}{a+1} \right],
$$

once can conclude that if, for some $a \in \mathbb{C}_0$ and $v_0 \in \mathbb{C}^m$, the signals $f_1, f_2 \in \mathcal{L}_m^{2,loc}$ and $y_1, y_2 \in \mathcal{L}_k^{2,loc}$ are defined by

$$
e^{s_0 t} v_0 = f_1(t) + j f_2(t), \quad e^{s_0 t} G(s_0) v_0 = y_1(t) + j y_2(t), \quad (t \in \mathbb{R}_+)
$$

then $(f_k, y_k) \in \mathcal{S}_{lti}[G]$ are valid i/o pairs for $k = 1, 2$.

For every $\gamma < \|G\|_\infty$ there exists $v_0 \in \mathbb{C}^m$ and $s_0 \in \mathbb{C}_0$ such that $|G(s_0) v_0| > \gamma |v_0|$. The functions

$$
f_1(t) = \mathrm{Re}\left[e^{s_0 t} v_0 \right], \quad y_1(t) = \mathrm{Re}\left[e^{s_0 t} G(s_0) v_0 \right]
$$

define an i/o pair in $\mathcal{S}_{lti}[G]$, and so do the functions

$$
f_2(t) = \mathrm{Im}\left[e^{s_0 t} v_0 \right], \quad y_2(t) = \mathrm{Im}\left[e^{s_0 t} G(s_0) v_0 \right].
$$

Let $h > 0$ be the real part of s_0. Since

$$|f_1(t)|^2 + |f_2(t)|^2 = e^{2ht}|v_0|^2,$$

and

$$|y_1(t)|^2 + |y_2(t)|^2 = e^{2ht}|G(s_0)v_0|^2,$$

the integrals

$$\int_0^T \{\gamma^2|f_1|^2 + \gamma^2|f_2|^2 - |y_1|^2 - |y_2|^2\}dt$$

converge to minus infinity as $T \to +\infty$. Hence at least one of the two integrals

$$\int_0^T \{\gamma^2|f_k(t)|^2 - |y_k(t)|^2\}dt, \quad k \in \{1, 2\}$$

converge to minus infinity as $T \to +\infty$. ■

Example 11.7 L2 gain of the system $\mathcal{S} = \{f, y\} \subset \mathcal{L} \times \mathcal{L}$ described by the differential equation

$$\dot{y} = ay + f$$

equals $|a|^{-1}$ for $a < 0$ and $+\infty$ for $a \geq 0$.

Indeed, the transfer function equals $G(s) = (s - a)^{-1}$. For $a \geq 0$, $|G(s)|$ is unbounded in $\mathbb{C}_0$. For $a < 0$, G has no poles in $\mathbb{C}_0$, and the maximum of $|G(s)|$ on the imaginary axis is achieved at $s = 0$.