Lecture 11: Dynamics: Euler-Lagrange Equations

- D'Alembert Principle

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- Computing Kinetic and Potential Energies

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Principle of Virtual Work

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$$\sum_i \left(f_i^c + f_i^e\right) = 0$$

Then the work done by all forces applied to i^{th}-particle along each set of virtual displacement is zero, i.e.

$$\boxed{0} = \sum_i \left(f_i^c + f_i^e\right) \delta r_i = \underbrace{\sum_i f_i^c \delta r_i}_{=0} + \boxed{\sum_i f_i^e \delta r_i}$$

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Then the work done by all forces applied to i^{th}-particle along each set of virtual displacement is zero if we add the inertia forces

$$\boldsymbol{0} = \sum_i \left(f_i^e - \frac{d}{dt} \left[m_i \dot{r}_i \right] \right) \delta r_i$$

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$$0 = \sum_i \left(f_i^e - \frac{d}{dt}\left[m_i \dot{r}_i\right] \right) \delta r_i$$

Steps to be done

- Rewrite $\sum_i f_i^e \delta r_i$ as function of generalized coordinates q;
- Rewrite $\sum_i \frac{d}{dt}\left[m_i \dot{r}_i\right] \delta r_i$ as function of generalized coordinates q

D'Alembert Principle

Virtual displacements are computed as

$$\boldsymbol{\delta r_i} = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial \boldsymbol{q}_j} \boldsymbol{\delta q_i}, \quad i = 1, \ldots, k$$

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$$\delta r_i = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \delta q_i, \quad i = 1, \ldots, k$$

Then

$$\sum_{i=1}^{k} f_i^e \delta r_i = \sum_{i=1}^{k} f_i^e \left(\sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \delta q_j \right) = \sum_{j=1}^{n} \left(\sum_{i=1}^{k} f_i^e \frac{\partial r_i}{\partial q_j} \right) \delta q_j$$

$$= \sum_{j=1}^{n} \psi_j \delta q_j$$

The functions ψ_j are called generalized forces

D'Alembert Principle

The second term can be rewritten as

$$\sum_{i=1}^{k} \frac{d}{dt}\left[m_i \dot{r}_i\right] \boldsymbol{\delta r_i} \;=\; \sum_{i=1}^{k} m_i \ddot{r}_i \boldsymbol{\delta r_i} = \sum_{i=1}^{k} m_i \ddot{r}_i \left(\sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \boldsymbol{\delta q_j} \right)$$

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$$= \sum_{i=1}^{k} \sum_{j=1}^{n} m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j} \delta q_j$$

$$\frac{d}{dt}\left[m_i \dot{r}_i \frac{\partial r_i}{\partial q_j}\right] = m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j} + m_i \dot{r}_i \frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j}\right]$$

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$$\sum_{i=1}^{k} \tfrac{d}{dt}\left[m_i \dot{r}_i\right] \delta r_i \quad = \quad \sum_{i=1}^{k} m_i \ddot{r}_i \delta r_i = \sum_{i=1}^{k} m_i \ddot{r}_i \left(\sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \delta q_j \right)$$

$$= \quad \sum_{i=1}^{k} \sum_{j=1}^{n} m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j} \delta q_j$$

$$\frac{d}{dt}\left[m_i \dot{r}_i \frac{\partial r_i}{\partial q_j} \right] = m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j} + m_i \dot{r}_i \frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j} \right]$$

$$\Rightarrow \quad \sum_{i=1}^{k} m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j} = \sum_{i=1}^{k} \left\{ \frac{d}{dt}\left[m_i \dot{r}_i \frac{\partial r_i}{\partial q_j} \right] - m_i \dot{r}_i \frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j} \right] \right\}$$

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$$v_i = \dot{r}_i = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \dot{q}_j \quad \Rightarrow \quad \frac{\partial v_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j}$$

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$$\frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j}\right] = \sum_{l=1}^{n} \frac{\partial^2 r_i}{\partial q_j \partial q_l} \dot{q}_l = \frac{\partial}{\partial q_j}\left[\sum_{l=1}^{n} \frac{\partial r_i}{\partial q_l} \dot{q}_l \right] = \frac{\partial v}{\partial q_j}$$

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$$= \sum_{j=1}^{n} \left[\sum_{i=1}^{k} \left\{ \frac{d}{dt} \left[m_i v_i \frac{\partial v_i}{\partial \dot{q}_j} \right] - m_i v_i \frac{\partial v_i}{\partial q_j} \right\} \right] \delta q_j$$

$$= \sum_{j=1}^{n} \left[\frac{d}{dt} \frac{\partial \mathcal{K}}{\partial \dot{q}_j} - \frac{\partial \mathcal{K}}{\partial q_j} \right] \delta q_j$$

where
$$\mathcal{K} = \sum_{i=1}^{k} \frac{1}{2} m_i \left| v_i \right|^2$$

D'Alembert Principle

To summarize, the equation

$$\mathbf{0} = \sum_{i} \left(\boldsymbol{f}_i^e - \frac{d}{dt}\left[\boldsymbol{m}_i \dot{\boldsymbol{r}}_i \right] \right) \boldsymbol{\delta r_i}$$

with

$$\sum_{i=1}^{k} \frac{d}{dt}\left[\boldsymbol{m}_i \dot{\boldsymbol{r}}_i \right] \boldsymbol{\delta r_i} = \sum_{j=1}^{n} \left[\frac{d}{dt} \frac{\partial \mathcal{K}}{\partial \dot{q}_j} - \frac{\partial \mathcal{K}}{\partial q_j} \right] \boldsymbol{\delta q_j}, \quad \sum_{i=1}^{k} \boldsymbol{f}_i^e \boldsymbol{\delta r_i} = \sum_{j=1}^{n} \psi_j \boldsymbol{\delta q_j}$$

is

$$\sum_{j=1}^{n} \left\{ \frac{d}{dt} \frac{\partial \mathcal{K}}{\partial \dot{q}_j} - \frac{\partial \mathcal{K}}{\partial q_j} - \psi_j \right\} \boldsymbol{\delta q_j} = \mathbf{0}$$

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To summarize, the equation

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is

$$\sum_{j=1}^{n} \left\{ \frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{q}_j} - \frac{\partial \mathcal{K}}{\partial q_j} - \psi_j \right\} \delta q_j = 0$$

If δq_j are independent then we obtain equations

$$\frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{q}_j} - \frac{\partial \mathcal{K}}{\partial q_j} = \psi_j, \quad j = 1, \ldots n$$

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To summarize, the equation

$$0 = \sum_i \left(f_i^e - \tfrac{d}{dt}\left[m_i \dot{r}_i\right] \right) \delta r_i$$

with

$$\sum_{i=1}^{k} \tfrac{d}{dt}\left[m_i \dot{r}_i\right] \delta r_i = \sum_{j=1}^{n} \left[\frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{q}_j} - \frac{\partial \mathcal{K}}{\partial q_j} \right] \delta q_j, \quad \sum_{i=1}^{k} f_i^e \delta r_i = \sum_{j=1}^{n} \psi_j \delta q_j$$

is

$$\sum_{j=1}^{n} \left\{ \frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{q}_j} - \frac{\partial \mathcal{K}}{\partial q_j} - \psi_j \right\} \delta q_j = 0$$

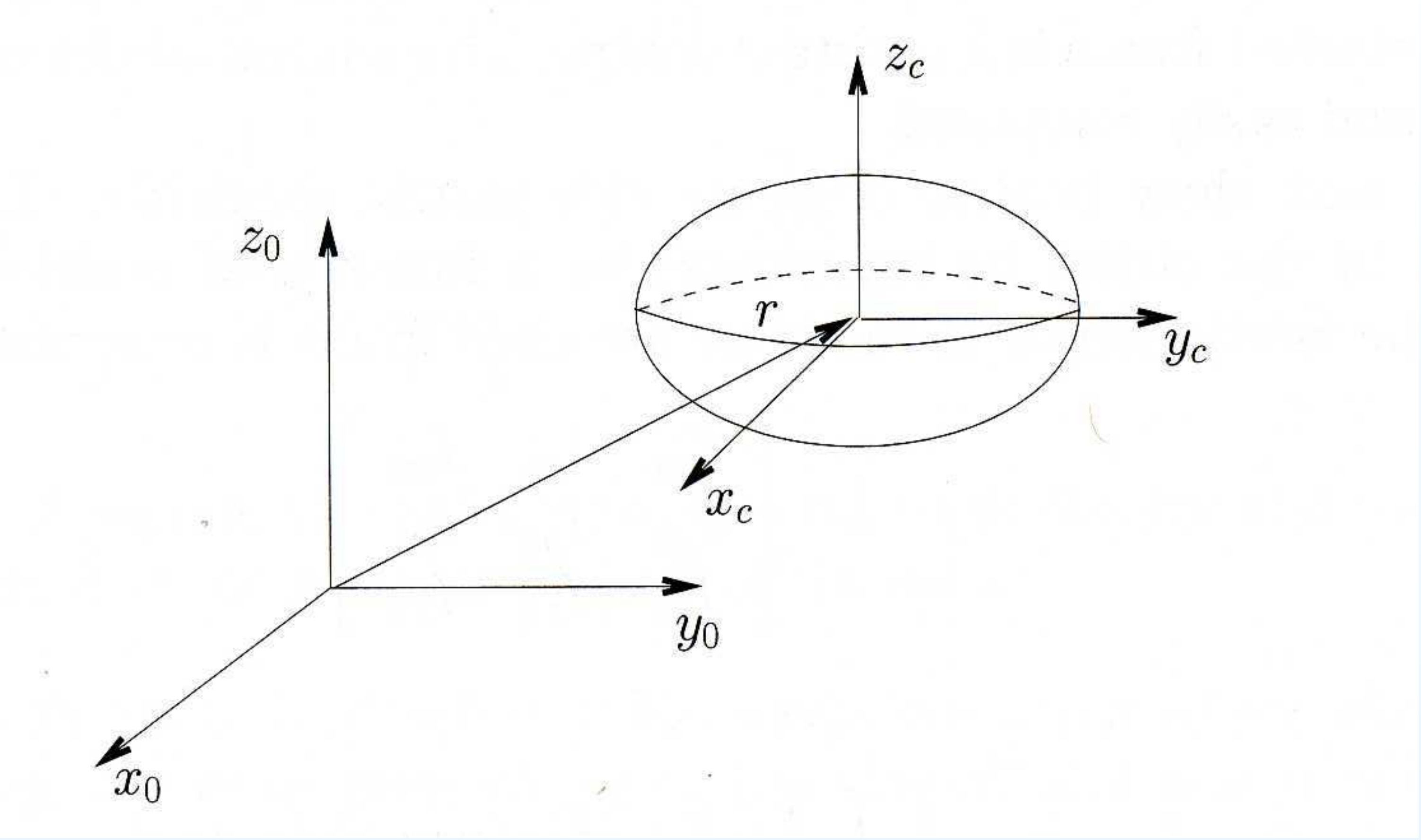
If ψ_j functions are particular form then the equations are

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = \tau_j, \quad \psi_j = -\frac{\partial \mathcal{P}}{\partial q_j} + \tau_j, \quad \mathcal{L} = \mathcal{K} - \mathcal{P}$$

Lecture 11: Dynamics: Euler-Lagrange Equations

- D'Alembert Principle
- Computing Kinetic and Potential Energies
- Equations of Motion

Computing Kinetic Energy



Rigid body has 6 degrees of freedom. Its kinetic energy consists of kinetic energy of rotation and kinetic energy of translation

$$\mathcal{K} = \frac{1}{2} m |v|^2 + \frac{1}{2} \omega^T \mathcal{I} \omega$$

Computing Kinetic Energy

We know how to compute the angular velocity

$$S(\omega) = \frac{d}{dt} R(t) R^T(t) \quad \to \quad \omega$$

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$$S(\omega) = \frac{d}{dt}R(t)R^T(t) \quad \rightarrow \quad \omega$$

The 3×3 matrix $\mathcal{I}$ is called the tensor of inertia

In the body frame it is constant $I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$ and

computed as

$$\begin{aligned}
I_{xx} &= \int\int\int (y^2 + z^2)\rho(x, y, z)dxdydz \\
I_{yy} &= \int\int\int (x^2 + z^2)\rho(x, y, z)dxdydz \\
I_{zz} &= \int\int\int (y^2 + x^2)\rho(x, y, z)dxdydz
\end{aligned}$$

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computed as

$$\begin{aligned} I_{xy} = I_{yx} &= -\int\int\int xy\rho(x,y,z)dxdydz \\ I_{xz} = I_{zx} &= -\int\int\int xz\rho(x,y,z)dxdydz \\ I_{yz} = I_{zy} &= -\int\int\int yz\rho(x,y,z)dxdydz \end{aligned}$$

Computing Kinetic Energy

We know how to compute the angular velocity

$$\boldsymbol{S(\omega) = \frac{d}{dt}R(t)R^{T}(t) \quad \longrightarrow \quad \omega}$$

The $\mathbf{3} \times \mathbf{3}$ matrix $\boldsymbol{\mathcal{I}}$ is called the tensor of inertia

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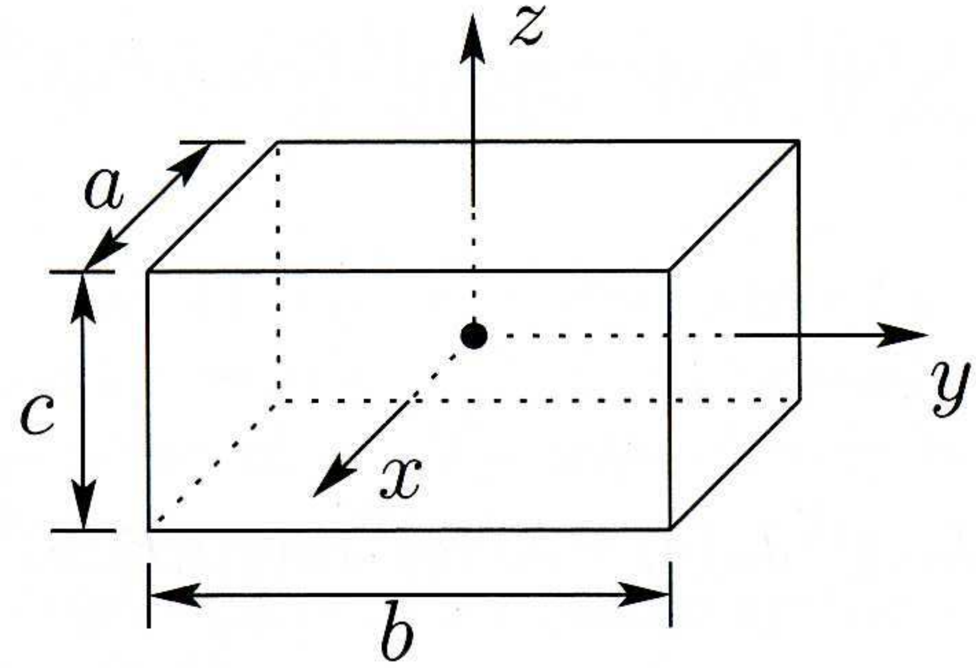
To compute the tensor of inertia in the inertia frame, we can use the formula

$$\boldsymbol{\mathcal{I} = R(t) I R^{T}(t)}$$

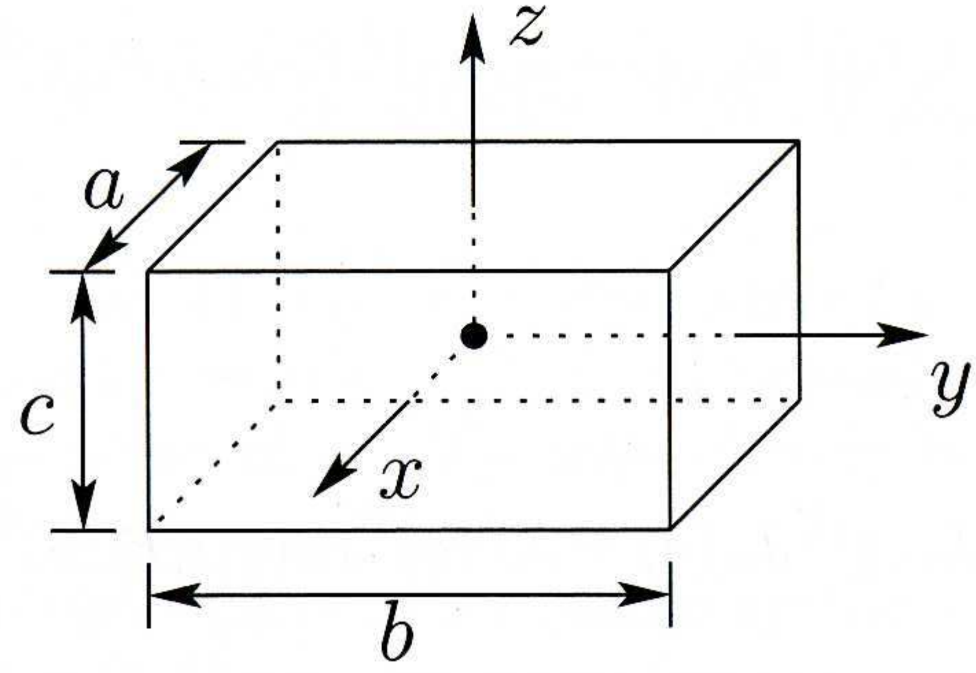
Computing Kinetic Energy



Rectangular brick with uniform mass density. Let us compute

$$I_{xx} = \int_{-c/2}^{c/2} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} (y^2 + z^2)\rho(x, y, z) dx dy dz = ???$$

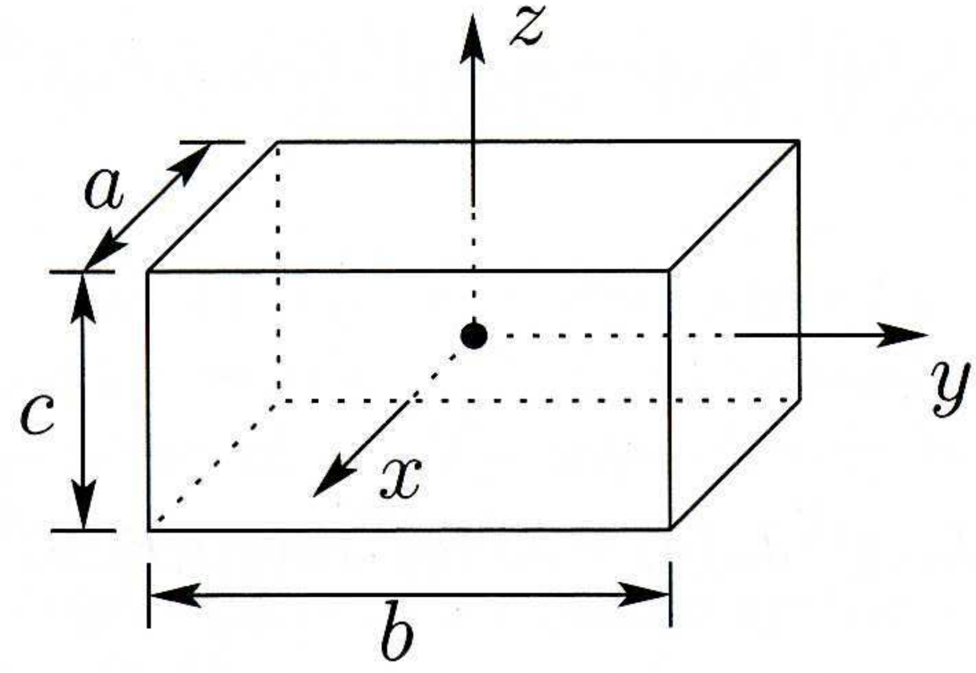
Computing Kinetic Energy



Rectangular brick with uniform mass density. Let us compute

$$
\begin{aligned}
I_{xx} &= \int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}(y^2+z^2)\rho(x,y,z)dxdydz \\
&= \rho\frac{abc}{12}(b^2+c^2) = \frac{m}{12}(b^2+c^2)
\end{aligned}
$$

Computing Kinetic Energy



Rectangular solid brick with uniform mass density. In the same way

$$I_{yy} = \frac{m}{12}(a^2+c^2), \quad I_{zz} = \frac{m}{12}(a^2+b^2), \quad I_{xy} = I_{xz} = I_{yz} = 0$$

Computing Kinetic Energy for n-Link Robot

To use the formula

$$\mathcal{K} = \frac{1}{2} m |v|^2 + \frac{1}{2} \omega^T \mathcal{I} \omega$$

we need to express

- $v = \dot{r}$ as function of generalized coordinates q and velocities $\dot{q}$;
- ω as function of generalized coordinates q and velocities $\dot{q}$;

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These relations are given by Jacobian matrices

$$v_i = J_{v_i}(q)\dot{q}, \quad \omega_i = J_{\omega_i}(q)\dot{q}$$

Computing Kinetic Energy for n-Link Robot

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The final form of kinetic energy is

$$\mathcal{K} \;=\; \frac{1}{2}\dot{q}^T \left[\sum_{i=1}^{k} m_i J_{v_i}(q)^T J_{v_i}(q) + J_{\omega_i}(q)^T R_i(q) I R_i(q)^T J_{\omega_i}(q) \right] \dot{q}$$

Computing Potential Energy for n-Link Robot

Potential energy of i^{th}-link is

$$\mathcal{P}_i = m_i g^T r_{ci}$$

where r_{ci} is the position of its center of mass

Computing Potential Energy for n-Link Robot

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where r_{ci} is the position of its center of mass

The total potential energy of the robot is then

$$\mathcal{P} = \sum_{i=1}^{k} \mathcal{P}_i = \sum_{i=1}^{k} m_i g^T r_{ci}$$

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Equations of Motion

We have seen that

- In general, kinetic energy is

$$\mathcal{K} = \tfrac{1}{2}\dot{q}^T \left[\sum_{i=1}^{k} m_i J_{v_i}(q)^T J_{v_i}(q) + J_{\omega_i}(q)^T R_i(q) I R_i(q)^T J_{\omega_i}(q)\right] \dot{q}$$

$$= \frac{1}{2}\dot{q}^T D(q)\dot{q} = \sum_{i,j} d_{ij}(q)\dot{q}_i\dot{q}_j$$

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- If generalized forces are potential, then $\psi_j = -\frac{\partial \mathcal{P}}{\partial q_j} + \tau_j$

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$$= \frac{1}{2}\dot{q}^T D(q)\dot{q} = \sum_{i,j} d_{ij}(q)\dot{q}_i \dot{q}_j$$

- If generalized forces are potential, then $\psi_j = -\frac{\partial \mathcal{P}}{\partial q_j} + \tau_j$
- We can introduce a scalar function $\mathcal{L} = \mathcal{K} - \mathcal{P}$ and write the equation of motion in compact form

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0$$

Equations of Motion

We have seen

- In general, kinetic energy is

$$\mathcal{K} = \frac{1}{2}\dot{q}^T \left[\sum_{i=1}^{k} m_i J_{v_i}(q)^T J_{v_i}(q) + J_{\omega_i}(q)^T R_i(q) I R_i(q)^T J_{\omega_i}(q)\right] \dot{q}$$

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- If generalized forces are potential, then $\psi_j = -\frac{\partial \mathcal{P}}{\partial q_j} + \tau_j$
- We can introduce a scalar function $\mathcal{L} = \mathcal{K} - \mathcal{P}$ and write the equation of motion in compact form

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = \tau_j = \frac{d}{dt}\frac{\partial(\mathcal{K} - \mathcal{P})}{\partial \dot{q}_j} - \frac{\partial(\mathcal{K} - \mathcal{P})}{\partial q_j}$$

Equations of Motion

We have seen

- In general, kinetic energy is

$$\mathcal{K} = \frac{1}{2}\dot{q}^T \left[\sum_{i=1}^{k} m_i J_{v_i}(q)^T J_{v_i}(q) + J_{\omega_i}(q)^T R_i(q) I R_i(q)^T J_{\omega_i}(q) \right] \dot{q}$$

$$= \frac{1}{2}\dot{q}^T D(q)\dot{q} = \sum_{i,j} d_{ij}(q)\dot{q}_i\dot{q}_j$$

- If generalized forces are potential, then $\psi_j = -\frac{\partial \mathcal{P}}{\partial q_j} + \tau_j$
- We can introduce a scalar function $\mathcal{L} = \mathcal{K} - \mathcal{P}$ and write the equation of motion in compact form

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = \tau_j = \frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{q}_j} - \frac{\partial (\mathcal{K} - \mathcal{P})}{\partial q_j}$$

Equations of Motion

The equations of motion have a particular structure

$$\frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{q}_k} - \frac{\partial(\mathcal{K} - \mathcal{P})}{\partial q_k} = \tau_k, \quad k = 1, \ldots, n$$

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Indeed

$$\frac{\partial \mathcal{K}}{\partial \dot q_k} = \frac{\partial}{\partial \dot q_k}\left[\frac{1}{2}\dot q^T D(q)\dot q\right] = \sum_{j=1}^{n} d_{kj}\dot q_j \Rightarrow \frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot q_k} = \frac{d}{dt}\left[\sum_{j=1}^{n} d_{kj}\dot q_j\right]$$

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and

$$\frac{d}{dt}\left[\sum_{j=1}^{n} d_{kj}\dot q_j\right] = \sum_{j=1}^{n} d_{kj}\ddot q_j + \sum_{j=1}^{n} \tfrac{d}{dt}\left[d_{kj}(q)\right]\dot q_j$$

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and

$$\begin{aligned} \frac{d}{dt}\left[\sum_{j=1}^{n} d_{kj}\dot{q}_j\right] &= \sum_{j=1}^{n} d_{kj}\ddot{q}_j + \sum_{j=1}^{n} \tfrac{d}{dt}\left[d_{kj}(q)\right]\dot{q}_j \\ &= \sum_{j=1}^{n} d_{kj}\ddot{q}_j + \sum_{j=1}^{n}\left(\sum_{i=1}^{n}\frac{\partial d_{kj}(q)}{\partial q_i}\dot{q}_i\right)\dot{q}_j \end{aligned}$$

Equations of Motion

The equations of motion have a particular structure

$$\frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{q}_k} - \frac{\partial(\mathcal{K} - \mathcal{P})}{\partial q_k} = \tau_k, \quad k = 1, \ldots, n$$

Indeed

$$\frac{\partial \mathcal{K}}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k}\left[\frac{1}{2}\dot{q}^T D(q)\dot{q}\right] = \sum_{j=1}^{n} d_{kj}\dot{q}_j \Rightarrow \frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{q}_k} = \frac{d}{dt}\left[\sum_{j=1}^{n} d_{kj}\dot{q}_j\right]$$

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Equations of Motion

The second term of the equations of motion is equal to

$$\begin{aligned}
\frac{\partial(\mathcal{K}-\mathcal{P})}{\partial q_k} &= \frac{\partial}{\partial q_k}\left[\frac{1}{2}\dot{q}D(q)\dot{q}-\mathcal{P}\right] = \frac{1}{2}\dot{q}\left[\frac{\partial}{\partial q_k}D(q)\right]\dot{q}-\frac{\partial}{\partial q_k}\mathcal{P} \\
&= \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial d_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j-\frac{\partial}{\partial q_j}\mathcal{P}
\end{aligned}$$

Equations of Motion

The second term of the equations of motion is equal to

$$\frac{\partial(\mathcal{K}-\mathcal{P})}{\partial q_k} = \frac{\partial}{\partial q_k}\left[\frac{1}{2}\dot q D(q)\dot q - \mathcal{P}\right] = \frac{1}{2}\dot q\left[\frac{\partial}{\partial q_k}D(q)\right]\dot q - \frac{\partial}{\partial q_k}\mathcal{P}$$

$$= \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial d_{ij}}{\partial q_k}\dot q_i \dot q_j - \frac{\partial}{\partial q_j}\mathcal{P}$$

To sum up, the equations of motion are

$$\sum_{j=1}^{n} d_{kj}\ddot q_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\left(\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j}\right)\dot q_i \dot q_j -$$

$$-\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial d_{ij}}{\partial q_k}\dot q_i \dot q_j + \frac{\partial}{\partial q_k}\mathcal{P} = \tau_k$$

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$$= \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial d_{ij}}{\partial q_k}\dot q_i \dot q_j - \frac{\partial}{\partial q_j}\mathcal{P}$$

To sum up, the equations of motion are

$$\sum_{j=1}^{n} d_{kj}\ddot q_j + \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ijk}(q)\dot q_i \dot q_j + g_k(q) = \tau_k$$

with

$$c_{ijk}(q) = \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\left(\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_j}\right), \quad g_k(q) = \frac{\partial}{\partial q_k}\mathcal{P}$$

Equations of Motion

The second term of the equations of motion is equal to

$$\frac{\partial(\mathcal{K}-\mathcal{P})}{\partial q_k} = \frac{\partial}{\partial q_k}\left[\frac{1}{2}\dot{q}D(q)\dot{q} - \mathcal{P}\right] = \frac{1}{2}\dot{q}\left[\frac{\partial}{\partial q_k}D(q)\right]\dot{q} - \frac{\partial}{\partial q_k}\mathcal{P}$$

$$= \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial d_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j - \frac{\partial}{\partial q_j}\mathcal{P}$$

To sum up, the equations of motion

$$\sum_{j=1}^{n} d_{kj}\ddot{q}_j + \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ijk}(q)\dot{q}_i\dot{q}_j + g_k(q) = \tau_k$$

in vectorial form are

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$