Lecture 10: Dynamics: Euler-Lagrange Equations

- Examples

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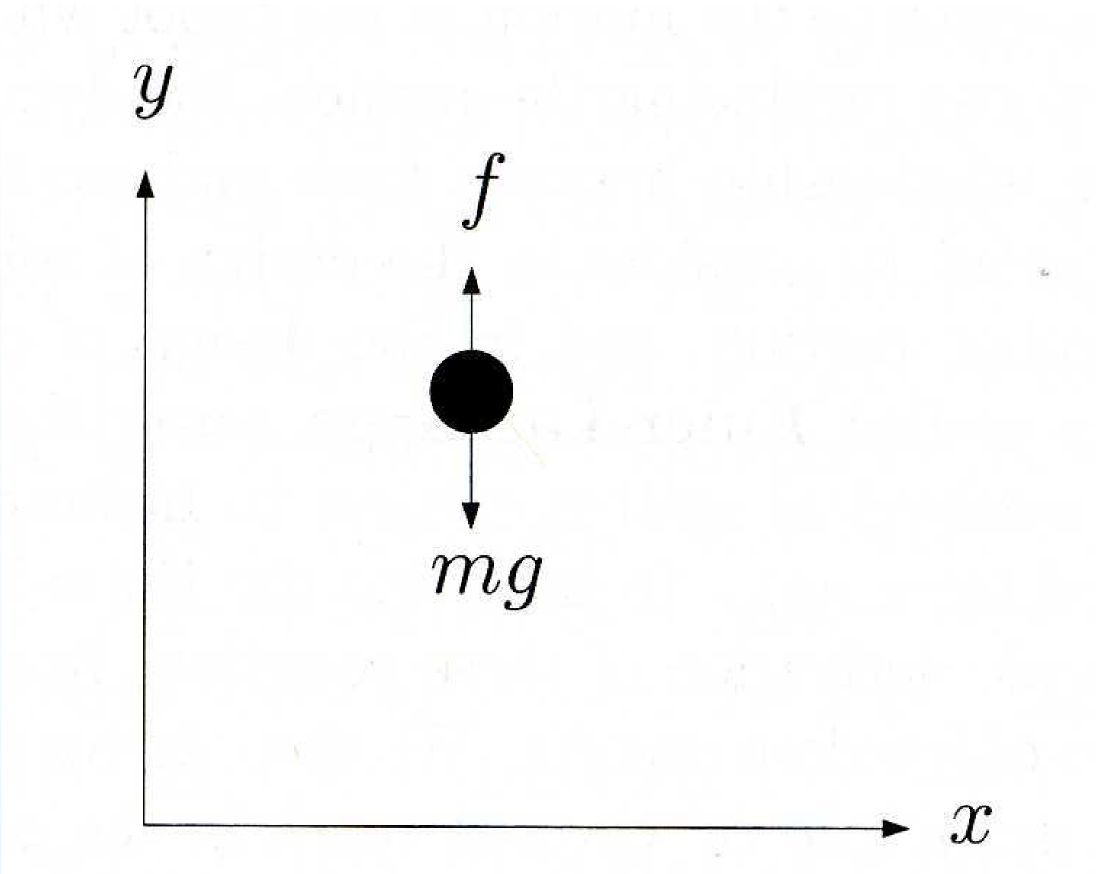
- Holonomic Constraints and Virtual Work

Example



The second Newton law says that the equation of motion of the particle is

$$m\frac{d^2}{dt^2}y = \sum_i F_i = f - mg$$

- f is an external force;
- mg is the force acting on the particle due to gravity.

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$$m\frac{d^2}{dt^2}y \;=\; \frac{d}{dt}\left(m\frac{d}{dt}y\right) = \frac{d}{dt}\left(m\frac{\partial}{\partial \dot{y}}\left[\frac{1}{2}\,\dot{y}^2\right]\right) = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{y}}\mathcal{K}\right)$$

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$$\begin{aligned} m\frac{d^2}{dt^2}y &= \frac{d}{dt}\left(m\frac{d}{dt}y\right) = \frac{d}{dt}\left(m\frac{\partial}{\partial \dot{y}}\left[\frac{1}{2}\,\dot{y}^2\right]\right) = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{y}}\mathcal{K}\right) \\ mg &= \frac{\partial}{\partial y}\left[mgy\right] = \frac{\partial}{\partial y}\mathcal{P} \end{aligned}$$

with kinetic/potential energies defined by $\mathcal{K} = \frac{1}{2}m\dot{y}^2$, $\mathcal{P} = mgy$

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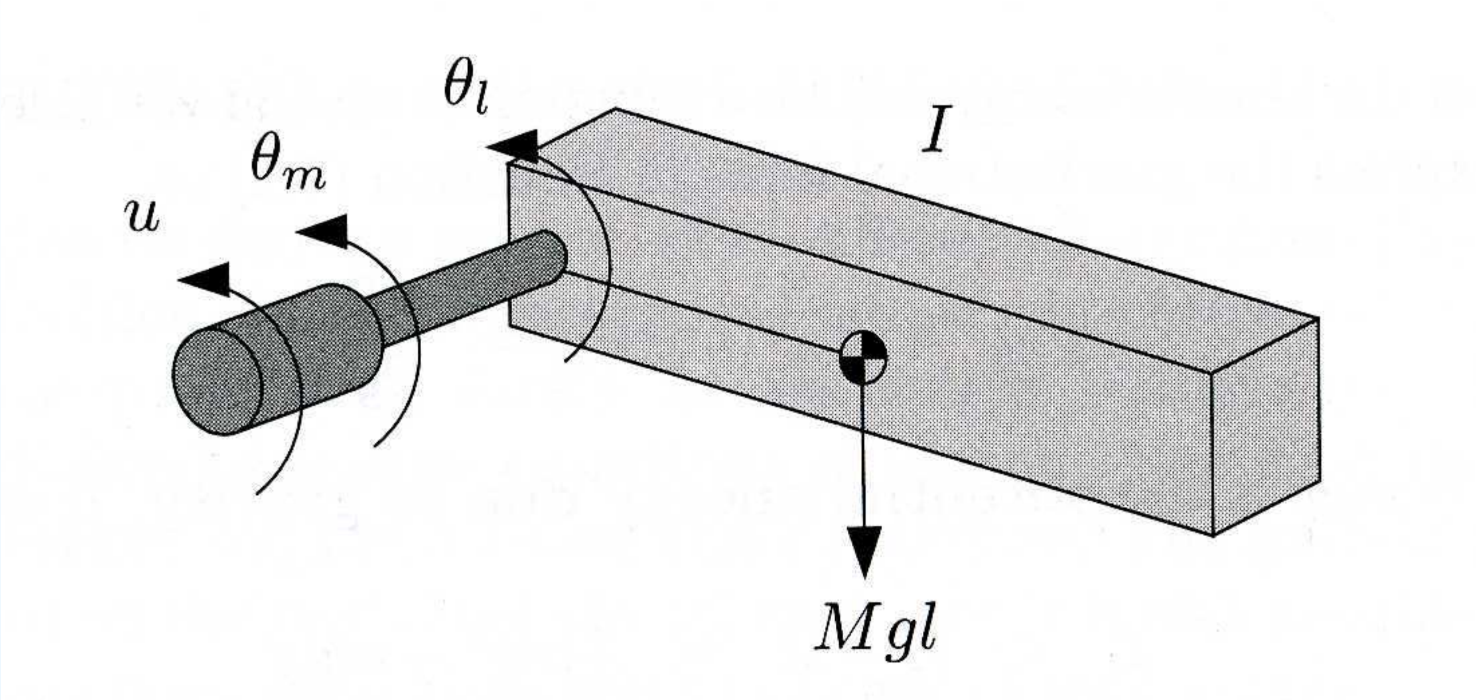
Then the second Newton law can be rewritten as

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{y}}\mathcal{L}\right) - \frac{\partial}{\partial y}\mathcal{L} = f \qquad \text{with} \qquad \mathcal{L} = \mathcal{K} - \mathcal{P}$$

where the function $\mathcal{L}(y, \dot{y})$ is called the Lagrangian.

Example:



A rigid link (θ_l) coupled through a gear to DC motor ($\theta_m = r\theta_l$):

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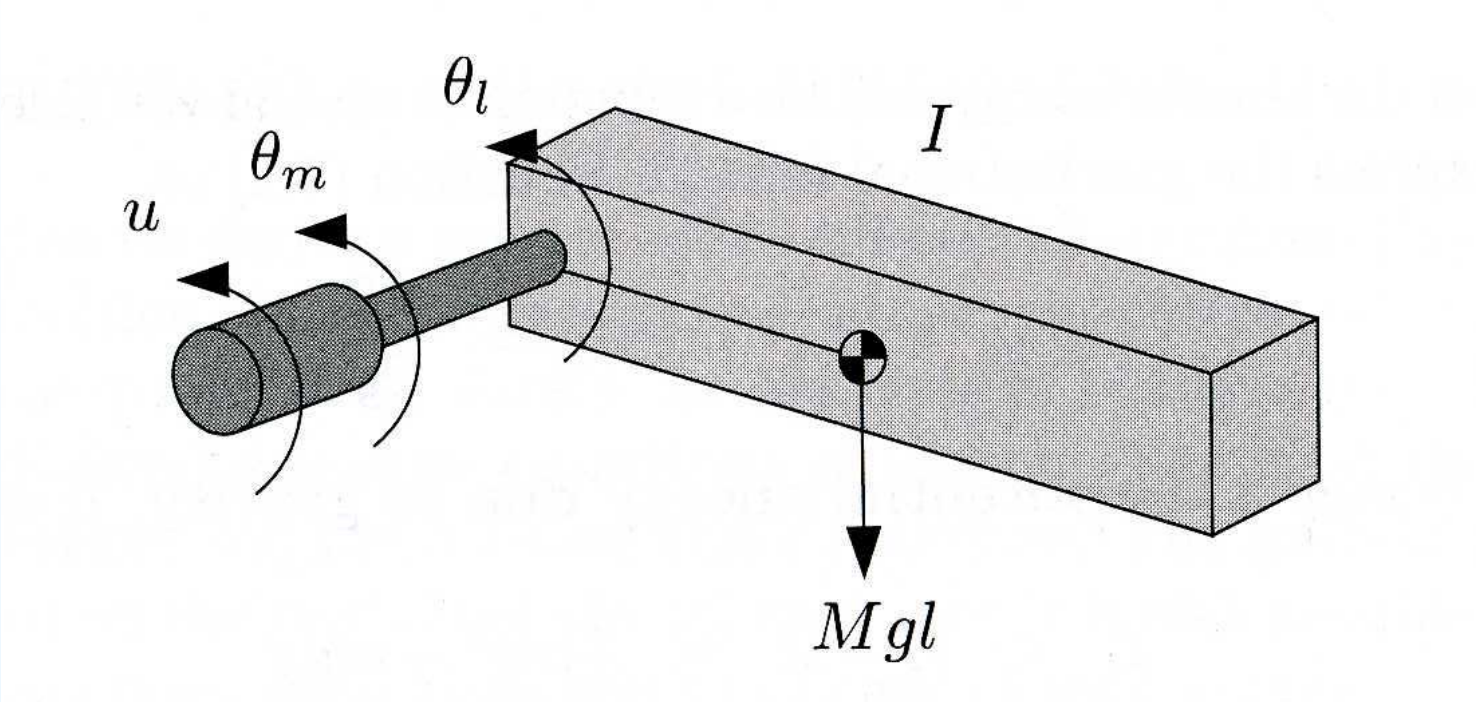

A rigid link (θ_l) coupled through a gear to DC motor ($\theta_m = r\theta_l$):

- Kinetic energy: $\mathcal{K} = \frac{1}{2} J_m \dot{\theta}_m^2 + \frac{1}{2} J_l \dot{\theta}_l^2 = \frac{1}{2} \left(r^2 J_m + J_l \right) \dot{\theta}_l^2$

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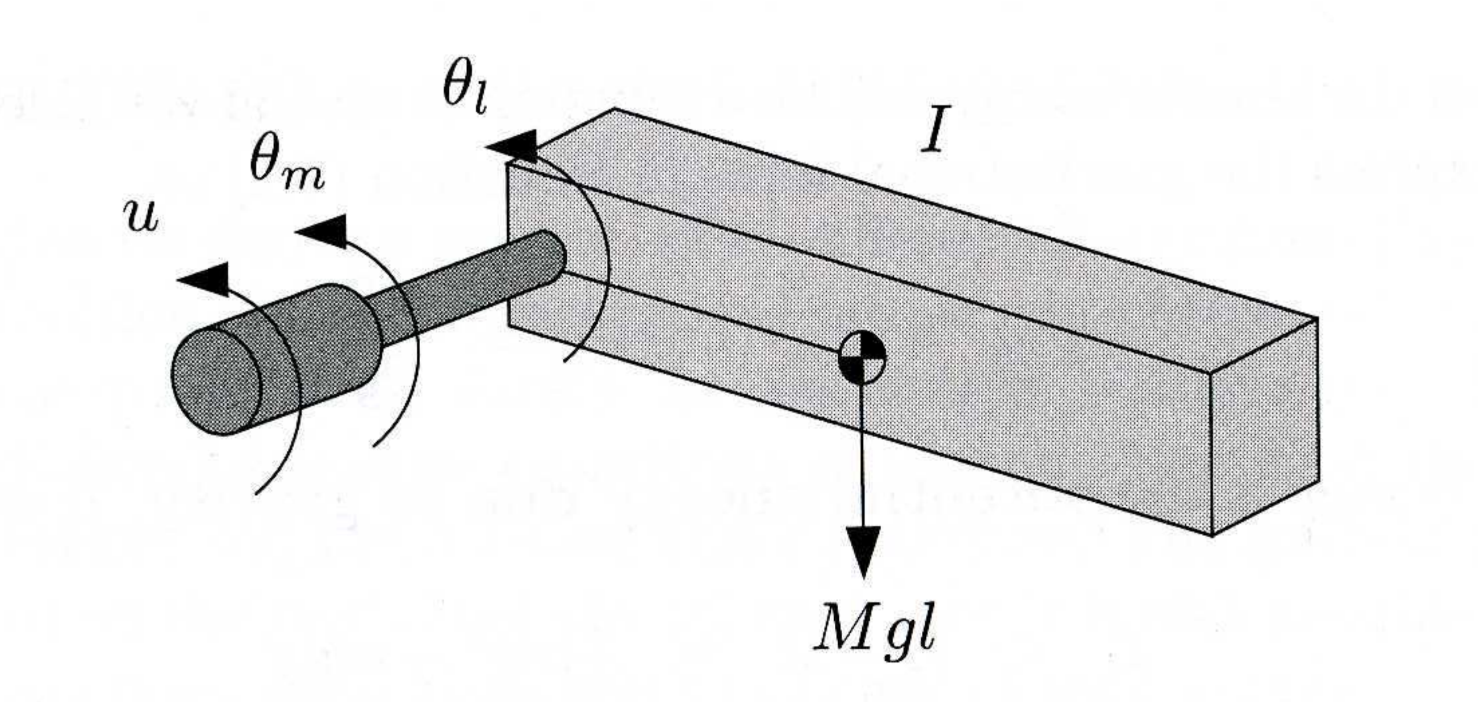

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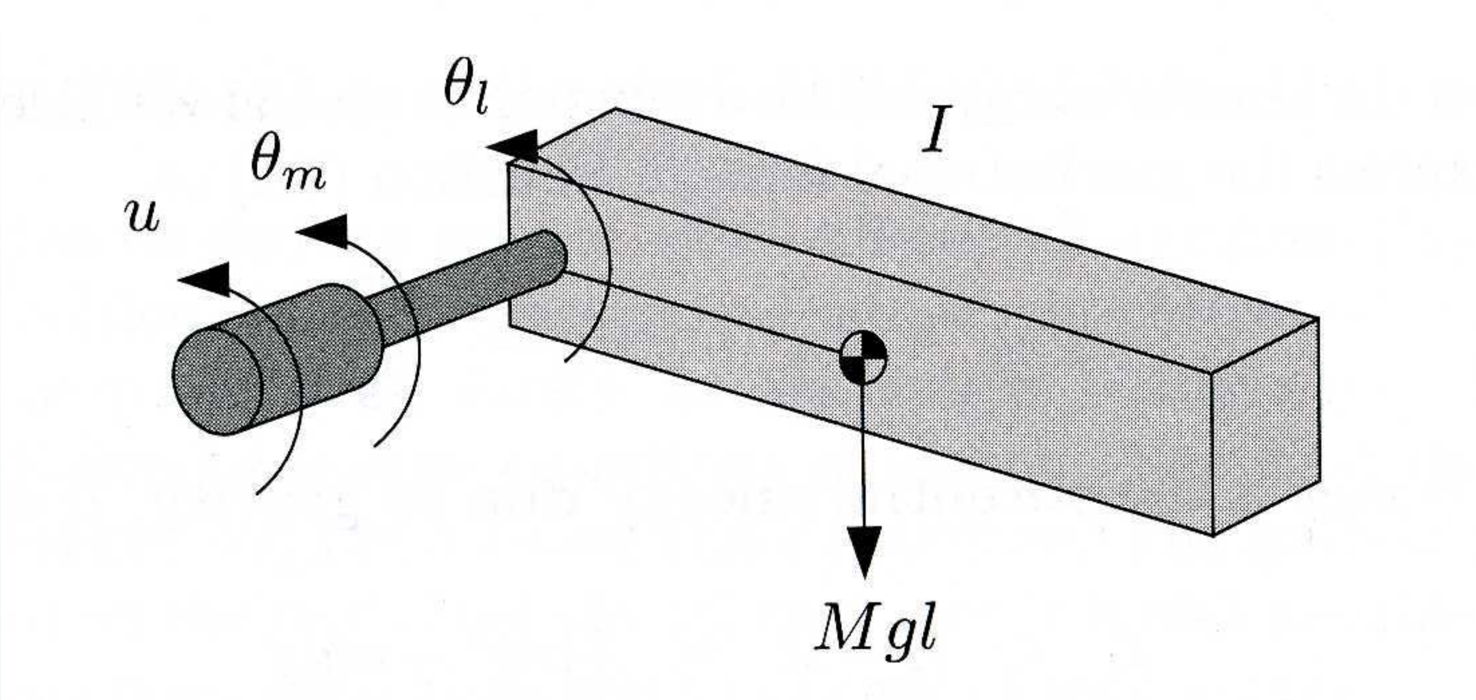

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- the Lagrangian is $\mathcal{L} = \mathcal{K} - \mathcal{P}$ and the dynamics are

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{\theta}_l}\mathcal{L}\right) - \frac{\partial}{\partial \theta_l}\mathcal{L} = ru$$

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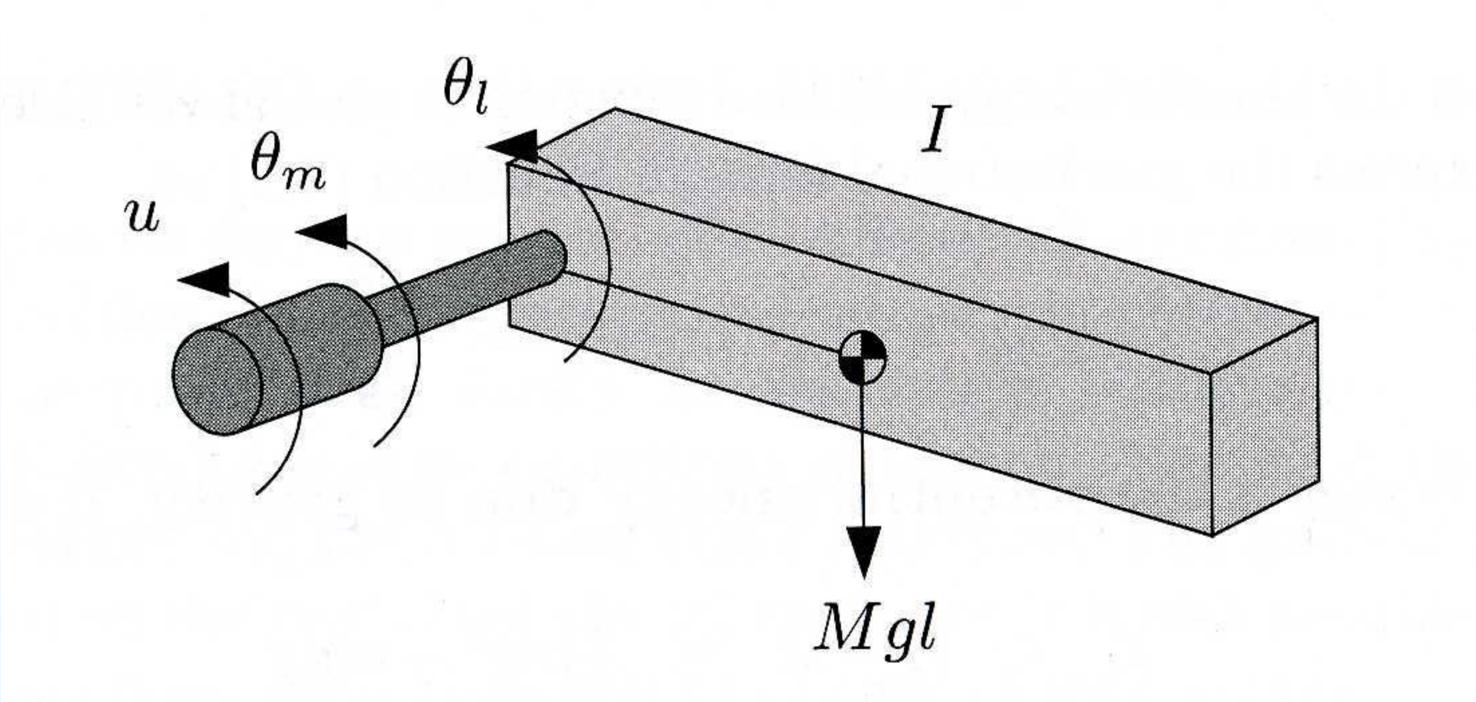

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$$\boxed{\left(r^2 J_m + J_l\right)\ddot{\theta}_l + Mgl\sin\theta_l = ru}$$

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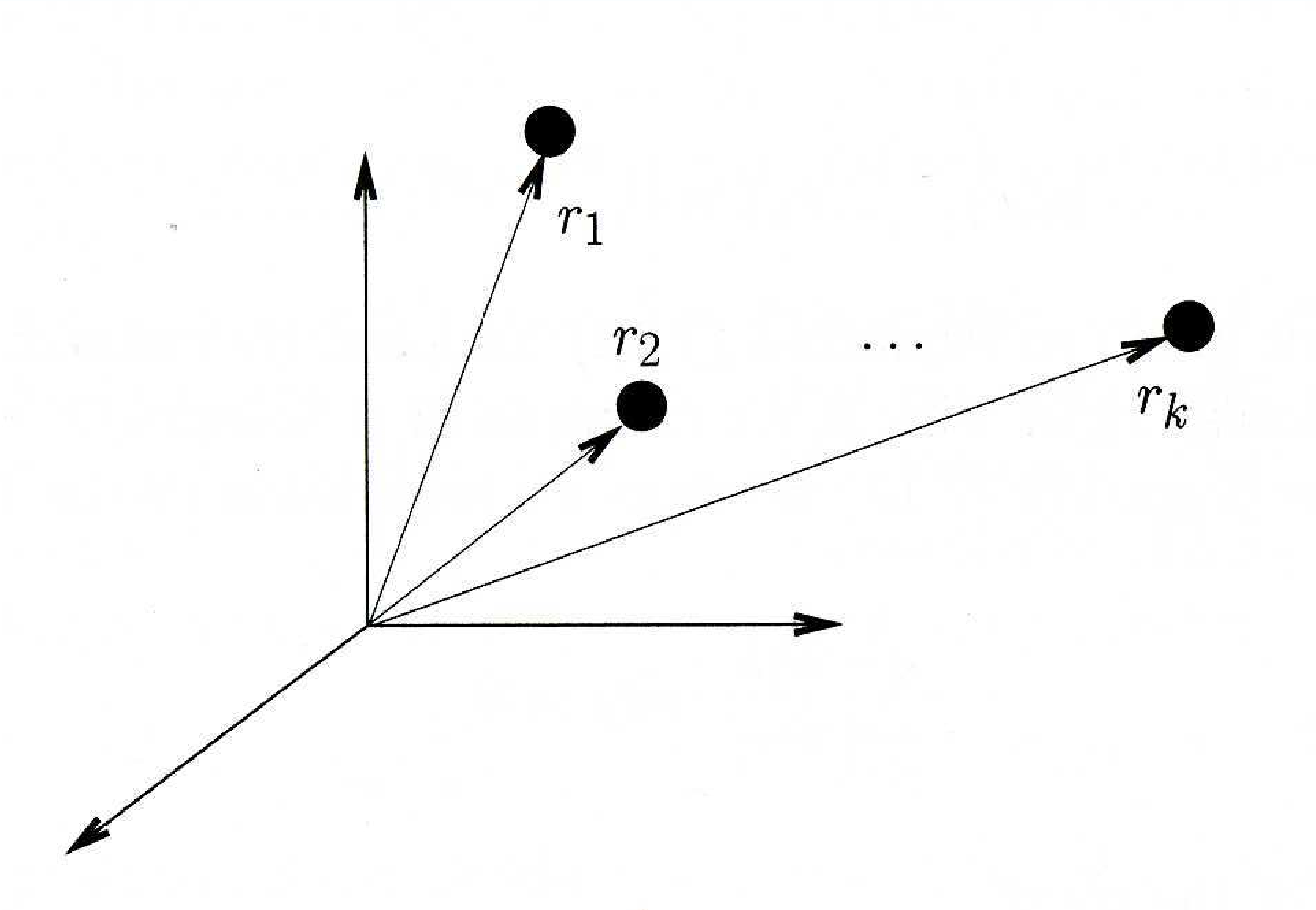
- Holonomic Constraints and Virtual Work

Concept of Holonomic Constraint



Unconstrained system of k particles has $3k$ degrees of freedom.
The number of Dof is less if the particles are constrained

Concept of Holonomic Constraint

A constraint imposed on k particles (with coordinates r_1, r_2, $\ldots$, $r_k \in \mathbb{R}^3$) is called holonomic, if it is of the form

$$g_i(r_1, r_2, \ldots, r_k) = 0, \qquad i = 1, 2, \ldots, l$$

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$$r_1, r_2 \in \mathbb{R}^3 : \; \left\|r_1 - r_2\right\|^2 = \left(r_1 - r_2\right)^T \left(r_1 - r_2\right) = l^2$$

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Presence of constraint implies presence a force

(called constraint force), that forces this constraint to hold.

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$$g_i(r_1, r_2, \ldots, r_k) = 0, \qquad i = 1, 2, \ldots, l$$

Differentiating the constraint function $g_i(\cdot)$ with respect to time, we obtain new constraint

$$\tfrac{d}{dt} g_i(r_1, r_2, \ldots, r_k) = \frac{\partial g_i}{\partial r_1} \tfrac{d}{dt} r_1 + \cdots + \frac{\partial g_i}{\partial r_k} \tfrac{d}{dt} r_k = 0$$

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The constraint of the form

$$\omega_1(r_1, \ldots r_k) dr_1 + \cdots + \omega_k(r_1, \ldots r_k) dr_k = 0$$

is called non-holonomic if it cannot be integrated back.

Concept of Generalized Coordinates

If the system is subject to holonomic constraint then

- If system consists of k particles, then it may be possible to express their coordinates as functions of fewer than $3k$ variables

$$r_1 = r_1(q_1, \ldots, q_n),\ r_2 = r_2(q_1, \ldots, q_n),\ \ldots,$$

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- The smallest set of variables is called generalized coordinates
- This smallest number is called a number of degrees of freedom
- If the system consists of an infinite number of particles, then it might have finite number of degrees of freedom

Concept of Virtual Displacement

Given a system of k-particles and a holonomic constraint

$$g_i(r_1, r_2, \ldots, r_k) = 0, \qquad i = 1, 2, \ldots, l$$

or the same

$$\frac{\partial g_i}{\partial r_1} dr_1 + \cdots + \frac{\partial g_i}{\partial r_k} dr_k = 0, \quad i = 1, 2, \ldots, l$$

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By definition a set of infinitesimal displacements

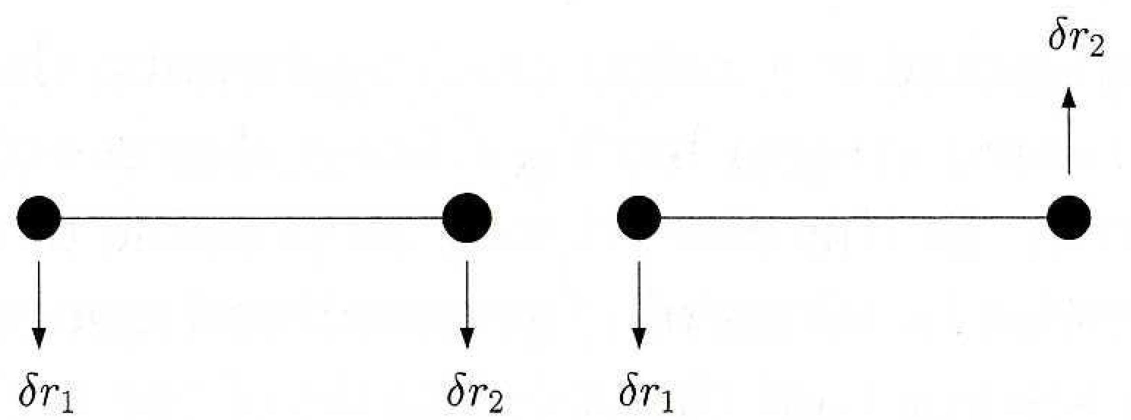
$$\delta r_1, \delta r_2, \ldots, \delta r_k$$

that are consistent with the constraint, i.e.

$$\frac{\partial g_i}{\partial r_1} \delta r_1 + \cdots + \frac{\partial g_i}{\partial r_k} \delta r_k = 0, \quad i = 1, 2, \ldots, l$$

are called virtual displacements

Concept of Virtual Displacement



Virtual displacements of a rigid bar. Such infinitesimal motions do not destroy the constraint

$$(\boldsymbol{r_1} - \boldsymbol{r_2})^T (\boldsymbol{r_1} - \boldsymbol{r_2}) = \boldsymbol{l^2}$$

if $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ are perturbed

$$\boldsymbol{r_1} \longrightarrow (\boldsymbol{r_1} + \boldsymbol{\delta r_1}) \qquad \boldsymbol{r_2} \longrightarrow (\boldsymbol{r_2} + \boldsymbol{\delta r_2})$$

that is

$$\big((\boldsymbol{r_1} + \boldsymbol{\delta r_1}) - (\boldsymbol{r_2} + \boldsymbol{\delta r_2})\big)^T \big((\boldsymbol{r_1} + \boldsymbol{\delta r_1}) - (\boldsymbol{r_2} + \boldsymbol{\delta r_2})\big) = \boldsymbol{l^2}$$

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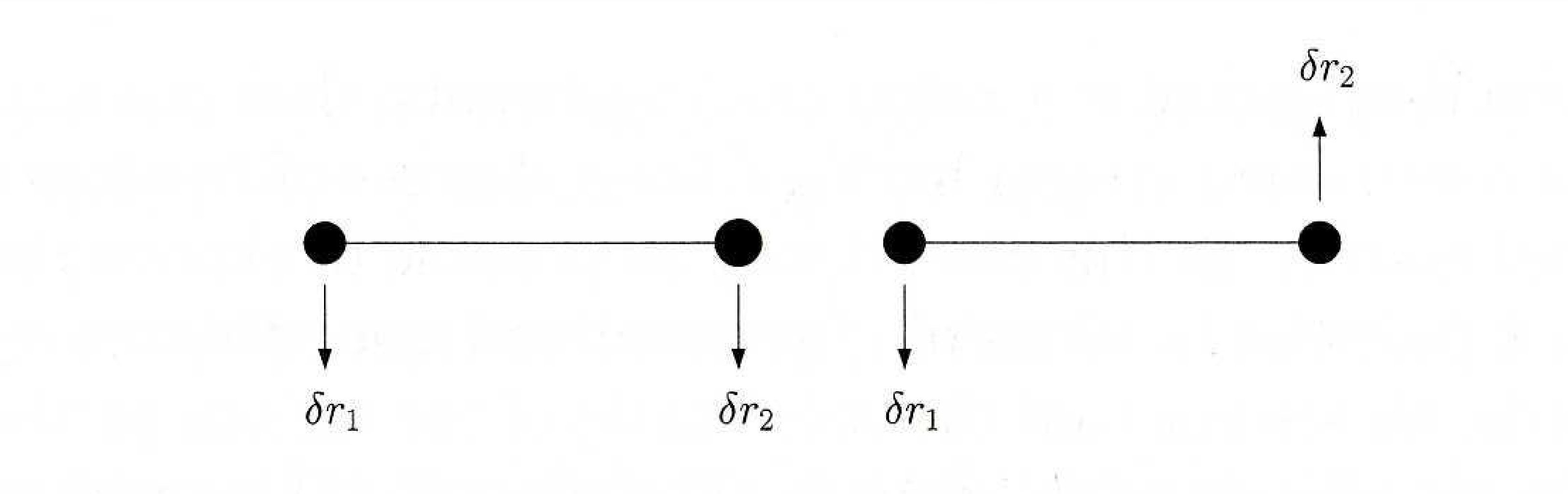

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Then the total sum of all forces applied to i^{th}-particle is zero

$$\sum_i \left(f_i^c + f_i^e\right) = 0$$

Then the work done by all forces applied to i^{th}-particle along each set of virtual displacement is zero, i.e.

$$\boxed{0} = \sum_i \left(f_i^c + f_i^e\right) \delta r_i = \underbrace{\sum_i f_i^c \delta r_i}_{=0} + \boxed{\sum_i f_i^e \delta r_i}$$