Lecture 12: Dynamics: Euler-Lagrange Equations

- Examples

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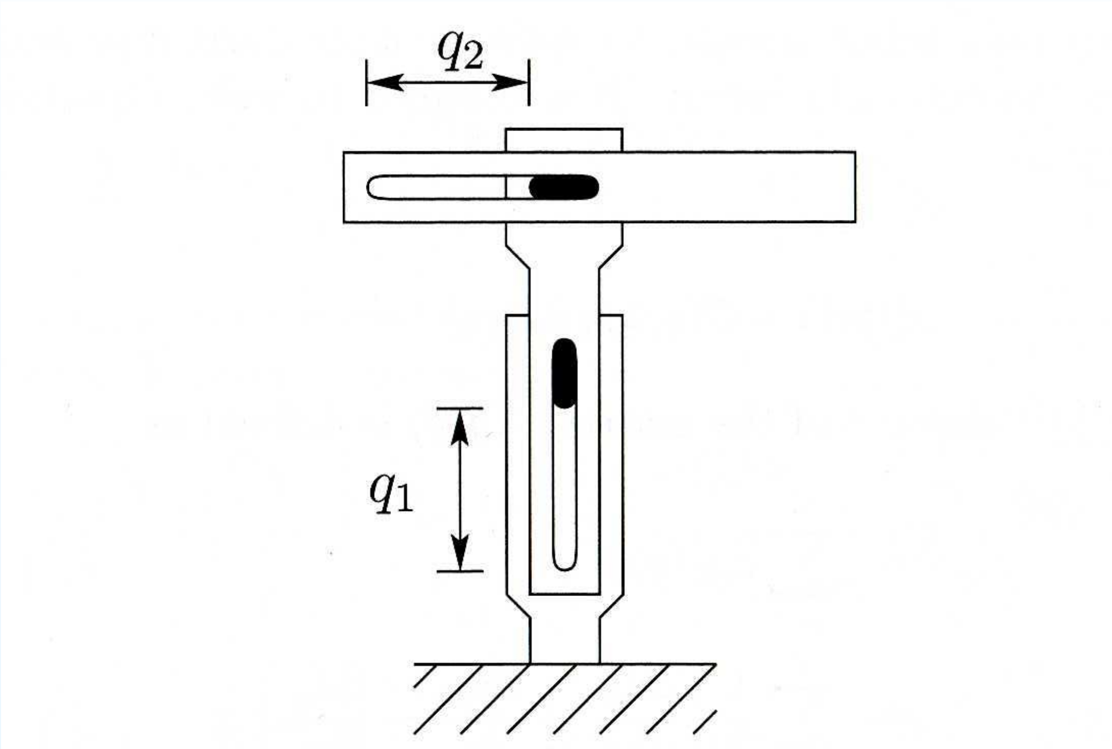
- Examples
- Properties of Equations of Motion

Example: Two-Link Cartesian Manipulator



For this system we need

- to solve forward kinematics problem;
- to compute manipulator Jacobian;
- to compute kinetic and potential energies and the Euler-Lagrange equations

Forward Kinematics and Jacobian

DH parameters for computing homogeneous transformations

$$\boldsymbol{T(q_i)} = \mathrm{Rot}_{\boldsymbol{z,\theta}} \cdot \mathrm{Trans}_{\boldsymbol{z,d}} \cdot \mathrm{Trans}_{\boldsymbol{x,a}} \cdot \mathrm{Rot}_{\boldsymbol{x,\alpha}}$$

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The kinetic energy of the system is

$$\mathcal{K} = \tfrac{1}{2}\left[m_1 v_{c1}^2 + \omega_1^T \mathcal{I}_1 \omega_1\right] + \tfrac{1}{2}\left[m_2 v_{c2}^2 + \omega_2^T \mathcal{I}_2 \omega_2\right]$$

and

$$\begin{aligned} v_{c1} &= \left[J_{v1}^{(1)}\ J_{v1}^{(2)}\right]\begin{bmatrix} \dot q_1 \\ \dot q_2 \end{bmatrix} = J_{v1}^{(1)}\dot q_1 + J_{v1}^{(2)}\dot q_2 \\ v_{c2} &= \left[J_{v2}^{(1)}\ J_{v2}^{(2)}\right]\begin{bmatrix} \dot q_1 \\ \dot q_2 \end{bmatrix} = J_{v2}^{(1)}\dot q_1 + J_{v2}^{(2)}\dot q_2 \end{aligned}$$

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To compute the Jacobian we can use the DH-frames, i.e

$$J_v^{(i)} = \begin{cases} z_{i-1}^0, & \text{for prismatic joint} \\ z_{i-1}^0 \times \left[o_c^0 - o_{i-1}^0 \right], & \text{for revolute joint} \end{cases}$$

$$J_\omega^{(i)} = \begin{cases} 0, & \text{for prismatic joint} \\ z_{i-1}^0, & \text{for revolute joint} \end{cases}$$

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$$\Rightarrow J_{v1} = [\vec{z}_0^0, 0] = \left[\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right], \quad J_{v2} = [\vec{z}_0^0, \vec{z}_1^0] = \left[\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right]$$

Forward Kinematics and Jacobian (Cont'd)

To sum up:

- Angular velocities ω_1 and ω_2 of both links are zeros

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To sum up:

- Angular velocities ω_1 and ω_2 of both links are zeros
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$$
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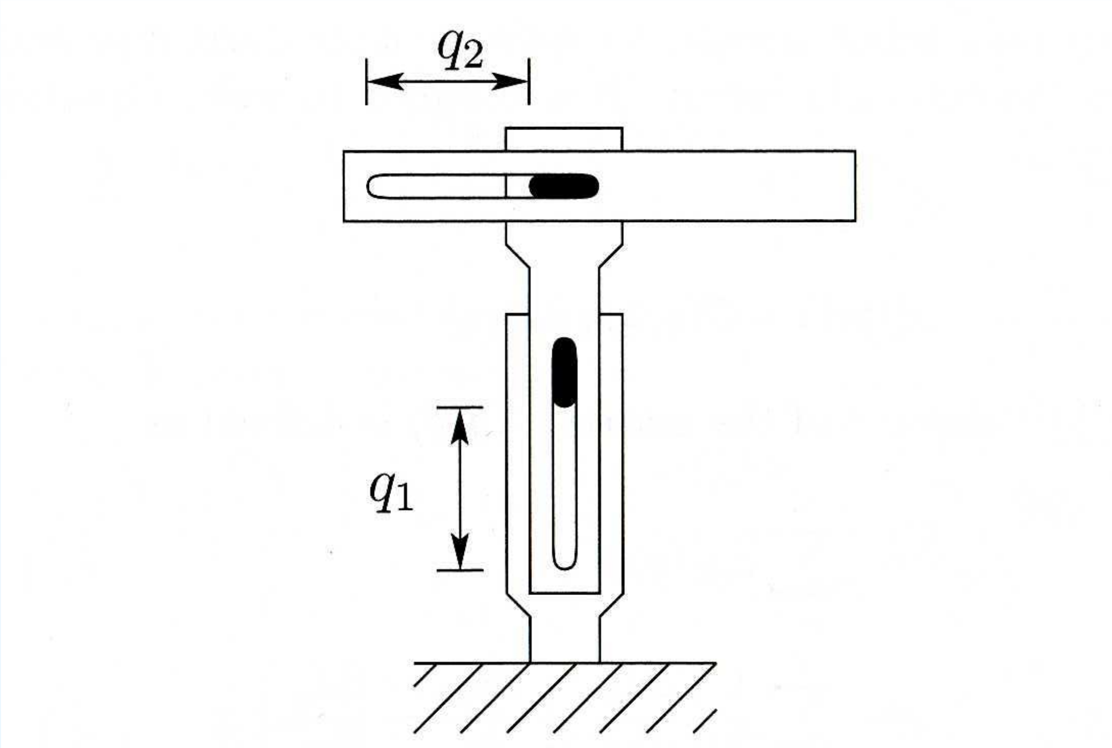
\end{aligned}
$$

- The kinetic energy is

$$
\mathcal{K} = \tfrac{1}{2} m_1 v_{c1}^2 + \tfrac{1}{2} m_2 v_{c2}^2 = \tfrac{1}{2} \begin{bmatrix} \dot q_1 \\ \dot q_2 \end{bmatrix}^T \begin{bmatrix} m_1 + m_2 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \dot q_1 \\ \dot q_2 \end{bmatrix}
$$

Potential Energy (PE) for Two-Link Cartesian Manipulator



Observations

- PE is independent of the second link position;
- It depends on the height of center of mass of robot;
- $\boldsymbol{\mathcal{P} = g \cdot (m_1 + m_2) \cdot q_1 + Const}$

Euler-Lagrange Equations for 2-Link Cartesian Manipulator

Given the kinetic $\mathcal{K}$ and potential $\mathcal{P}$ energies, the dynamics are

$$\frac{d}{dt}\left[\frac{\partial(\mathcal{K}-\mathcal{P})}{\partial \dot{q}}\right] - \frac{\partial(\mathcal{K}-\mathcal{P})}{\partial q} = \tau$$

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With kinetic and potential energies

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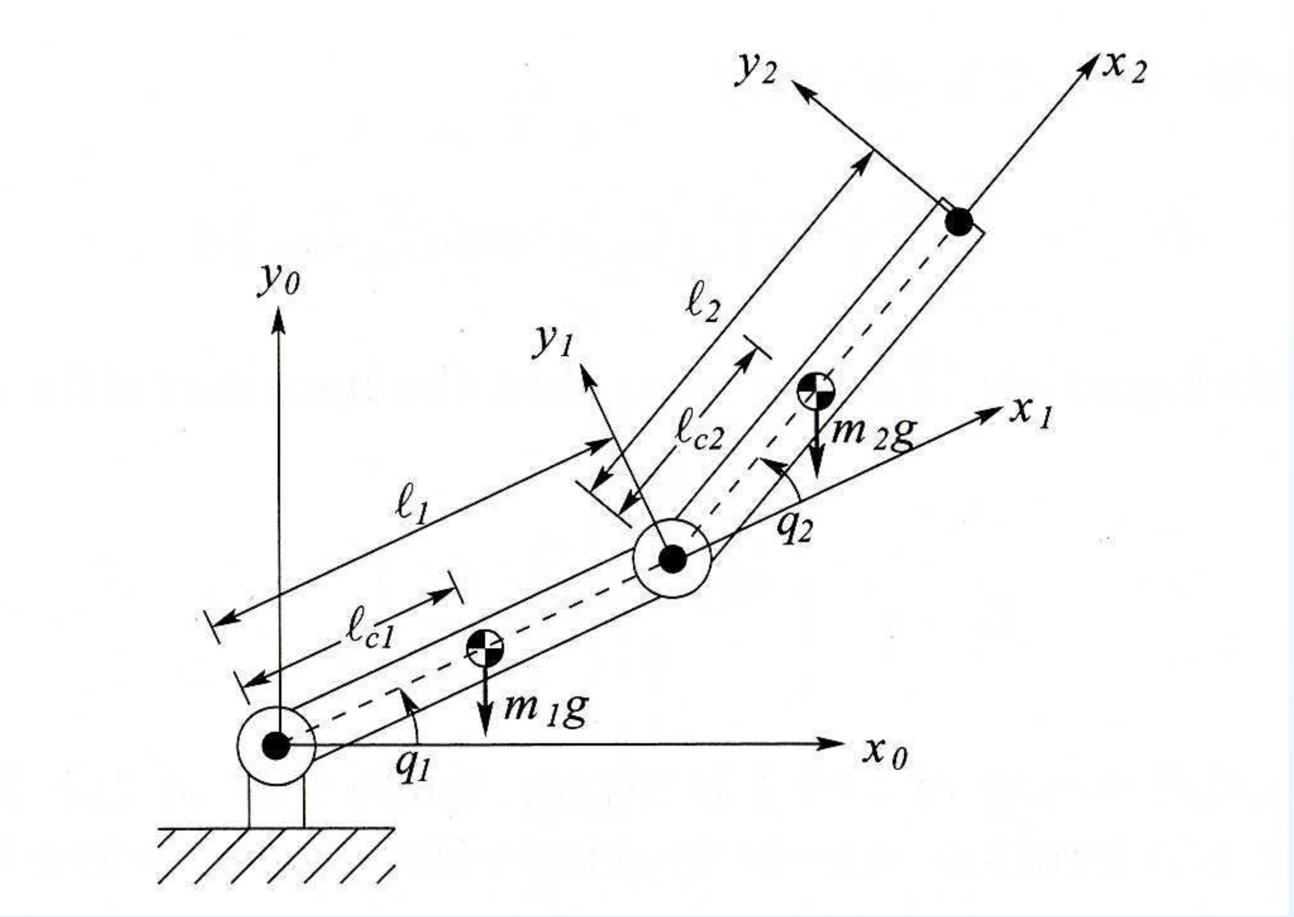
the Euler-Lagrange equations are

$$\begin{aligned}(m_1+m_2)\ddot{q}_1 + g(m_1+m_2) &= \tau_1\\ m_2\ddot{q}_2 &= \tau_2\end{aligned}$$

Example: Planar Elbow Manipulator



For this system we need

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Forward Kinematics and Jacobian (Cont'd)

The formula

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gives

$$J_{v1}^{(1)} = \vec{z}_0 \times (\vec{o}_{c_1} - \vec{o}_0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} l_{c1} \cos q_1 \\ l_{c1} \sin q_1 \\ 0 \end{bmatrix}$$

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$$\begin{aligned} J_{v2}^{(1)} &= \vec{z}_0 \times (\vec{o}_{c2} - \vec{o}_0) \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \left(\begin{bmatrix} l_1 \cos q_1 \\ l_1 \sin q_1 \\ 0 \end{bmatrix} + \begin{bmatrix} l_{c_2} \cos(q_1 + q_2) \\ l_{c_2} \sin(q_1 + q_2) \\ 0 \end{bmatrix} \right) \end{aligned}$$

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Forward Kinematics and Jacobian (Cont'd)

The formula

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Forward Kinematics and Jacobian (Cont'd)

The formula

$$\boldsymbol{J_\omega^{(i)}} = \begin{cases} \mathbf{0}, & \text{for prismatic joint} \\ z_{i-1}^0, & \text{for revolute joint} \end{cases}$$

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$$J_{\omega_2}^{(2)} = \vec{z}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Forward Kinematics and Jacobian (Cont'd)

To sum up, the kinetic energy $\mathcal{K}$ is

$$
\begin{aligned}
\mathcal{K} &= \frac{1}{2}\left[m_1 v_{c1}^2 + \omega_1^T \mathcal{I}_1 \omega_1\right] + \frac{1}{2}\left[m_2 v_{c2}^2 + \omega_2^T \mathcal{I}_2 \omega_2\right] \\
&= \frac{1}{2}\left[m_1 \left(J_{v_1}^{(1)} \dot q_1\right)^2 + I_1 \left(J_{\omega_1}^{(1)} \dot q_1\right)^2\right] + \\
&\quad + \frac{1}{2}\left[m_2 \left(J_{v_2}^{(1)} \dot q_1 + J_{v_2}^{(2)} \dot q_2\right)^2 + I_2 \left(J_{\omega_2}^{(1)} \dot q_1 + J_{\omega_2}^{(2)} \dot q_2\right)^2\right]
\end{aligned}
$$

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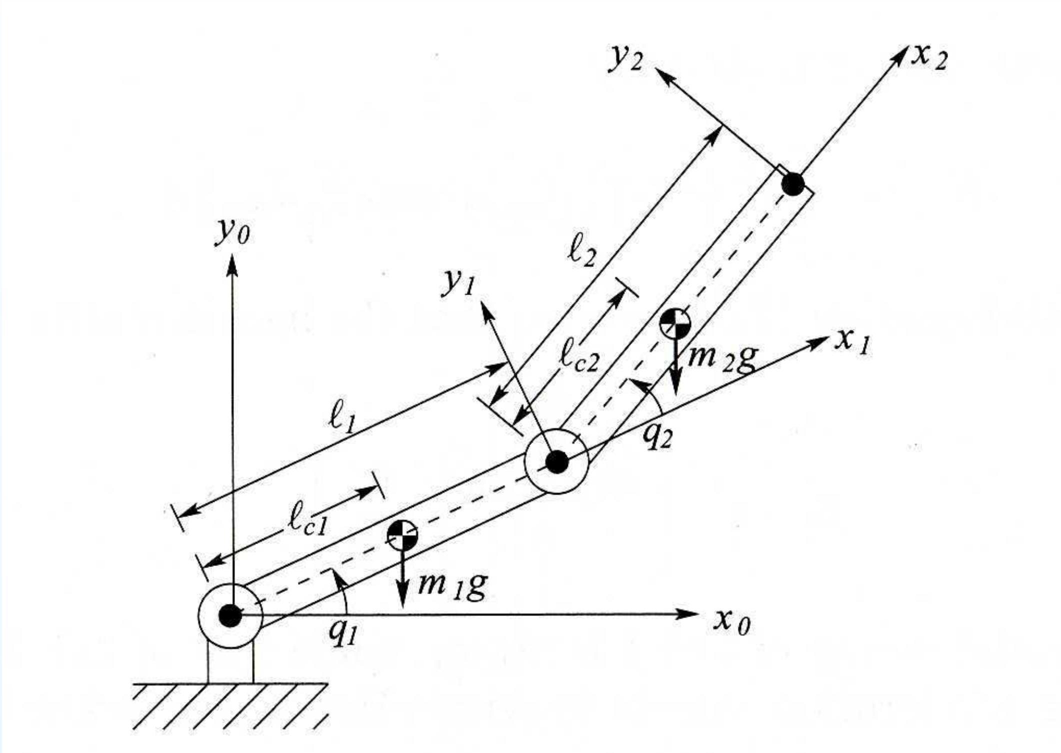
&= \tfrac{1}{2}\left[m_1\left(J_{v_1}^{(1)}\dot q_1\right)^2 + I_1\left(J_{\omega_1}^{(1)}\dot q_1\right)^2\right] + \\
&\quad + \tfrac{1}{2}\left[m_2\left(J_{v_2}^{(1)}\dot q_1 + J_{v_2}^{(2)}\dot q_2\right)^2 + I_2\left(J_{\omega_2}^{(1)}\dot q_1 + J_{\omega_2}^{(2)}\dot q_2\right)^2\right] \\
&= \tfrac{1}{2}\begin{bmatrix}\dot q_1 \\ \dot q_2\end{bmatrix}^T \begin{bmatrix} d_{11} & d_{12} \\ d_{12} & d_{22}\end{bmatrix}\begin{bmatrix}\dot q_1 \\ \dot q_2\end{bmatrix}
\end{aligned}
$$

with

$$
\begin{aligned}
d_{11} &= m_1 l_{c_1}^2 + m_2\left(l_1^2 + l_{c_2}^2 + 2 l_1 l_{c_2}\cos q_2\right) + I_1 + I_2 \\
d_{12} &= m_2\left(l_{c_2}^2 + l_1 l_{c_2}\cos q_2\right) + I_2 \\
d_{22} &= m_2 l_{c_2}^2 + I_2
\end{aligned}
$$

Potential Energy (PE) for Two-Link Elbow Manipulator



- PE of the 1st link is $\mathcal{P}_1 = m_1 g y_{c_1} = m_1 g l_{c_1} \sin q_1$
- PE of the 2nd link is
  $\mathcal{P}_2 = m_1 g y_{c_2} = m_2 g \left(l_1 \sin q_1 + l_{c_2} \sin(q_1 + q_2) \right)$
- Total PE is $\mathcal{P}_1 + \mathcal{P}_2$

Lecture 12: Dynamics: Euler-Lagrange Equations

- Examples
- Properties of Equations of Motion

Passivity Relation

Given a mechanical system

$$\boxed{\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] - \frac{\partial \mathcal{L}}{\partial q} = \tau} \quad \Leftrightarrow \quad \boxed{D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau}$$

with

$$\mathcal{L} = \frac{1}{2}\dot{q}^T D(q)\dot{q} - P(q)$$

Passivity Relation

Given a mechanical system

$$\boxed{\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] - \frac{\partial \mathcal{L}}{\partial q} = \tau} \Leftrightarrow \boxed{D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau}$$

with

$$\mathcal{L} = \frac{1}{2}\dot{q}^T D(q)\dot{q} - P(q)$$

Its energy is given by

$$\mathcal{H} = \frac{1}{2}\dot{q}^T D(q)\dot{q} + P(q)$$

Passivity Relation

Given a mechanical system

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] - \frac{\partial \mathcal{L}}{\partial q} = \tau \quad \Leftrightarrow \quad D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

with

$$\mathcal{L} = \tfrac{1}{2}\dot{q}^T D(q)\dot{q} - P(q)$$

Its energy is given by

$$\mathcal{H} = \tfrac{1}{2}\dot{q}^T D(q)\dot{q} + P(q)$$

What will happen with $\frac{d}{dt}\mathcal{H}$?

Passivity Relation (Cont'd)

Differentiating $\mathcal{H}$ along a solution of the system, we have

$$\begin{aligned} \frac{d}{dt}\mathcal{H} &= \frac{d}{dt}\left[\frac{1}{2}\dot{q}^T D(q)\dot{q} + P(q)\right] \\ &= \frac{1}{2}\ddot{q}^T D(q)\dot{q} + \frac{1}{2}\dot{q}^T D(q)\ddot{q} + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \end{aligned}$$

Passivity Relation (Cont'd)

Differentiating $\mathcal{H}$ along a solution of the system, we have

$$\begin{aligned}
\frac{d}{dt}\mathcal{H} &= \frac{d}{dt}\left[\frac{1}{2}\dot{q}^T D(q)\dot{q} + P(q)\right] \\
&= \frac{1}{2}\ddot{q}^T D(q)\dot{q} + \frac{1}{2}\dot{q}^T D(q)\ddot{q} + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T D(q)\ddot{q} + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q}
\end{aligned}$$

Passivity Relation (Cont'd)

Differentiating $\mathcal{H}$ along a solution of the system, we have

$$
\begin{aligned}
\frac{d}{dt}\mathcal{H} &= \frac{d}{dt}\left[\frac{1}{2}\dot{q}^T D(q)\dot{q} + P(q)\right] \\
&= \frac{1}{2}\ddot{q}^T D(q)\dot{q} + \frac{1}{2}\dot{q}^T D(q)\ddot{q} + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T D(q)\ddot{q} + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T \left[\tau - C(q,\dot{q})\dot{q} - g(q)\right] + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q}
\end{aligned}
$$

Here we use the Euler-Lagrange equations

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

Passivity Relation (Cont'd)

Differentiating $\mathcal{H}$ along a solution of the system, we have

$$\begin{aligned}
\frac{d}{dt}\mathcal{H} &= \frac{d}{dt}\left[\frac{1}{2}\dot{q}^T D(q)\dot{q} + P(q)\right] \\
&= \frac{1}{2}\ddot{q}^T D(q)\dot{q} + \frac{1}{2}\dot{q}^T D(q)\ddot{q} + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T D(q)\ddot{q} + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T\left[\tau - C(q,\dot{q})\dot{q} - g(q)\right] + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T\tau + \dot{q}^T\left(\frac{1}{2}\frac{d}{dt}\left[D(q)\right] - C(q,\dot{q})\right)\dot{q} + \dot{q}^T\left(\frac{\partial \mathcal{P}}{\partial q} - g(q)\right)
\end{aligned}$$

Passivity Relation (Cont'd)

Differentiating $\mathcal{H}$ along a solution of the system, we have

$$
\begin{aligned}
\frac{d}{dt}\mathcal{H} &= \frac{d}{dt}\left[\frac{1}{2}\dot{q}^T D(q)\dot{q} + P(q)\right] \\
&= \frac{1}{2}\ddot{q}^T D(q)\dot{q} + \frac{1}{2}\dot{q}^T D(q)\ddot{q} + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T D(q)\ddot{q} + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T\left[\tau - C(q,\dot{q})\dot{q} - g(q)\right] + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T\tau + \dot{q}^T\left(\frac{1}{2}\frac{d}{dt}\left[D(q)\right] - C(q,\dot{q})\right)\dot{q} + \dot{q}^T \underbrace{\left(\frac{\partial \mathcal{P}}{\partial q} - g(q)\right)}_{=0}
\end{aligned}
$$

Passivity Relation (Cont'd)

Differentiating $\mathcal{H}$ along a solution of the system, we have

$$\begin{aligned}
\frac{d}{dt}\mathcal{H} &= \frac{d}{dt}\left[\frac{1}{2}\dot{q}^T D(q)\dot{q} + P(q)\right] \\
&= \frac{1}{2}\ddot{q}^T D(q)\dot{q} + \frac{1}{2}\dot{q}^T D(q)\ddot{q} + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T D(q)\ddot{q} + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T\left[\tau - C(q,\dot{q})\dot{q} - g(q)\right] + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T\tau + \dot{q}^T \underbrace{\left(\frac{1}{2}\frac{d}{dt}\left[D(q)\right] - C(q,\dot{q})\right)}_{=0}\dot{q}
\end{aligned}$$

Passivity Relation (Cont'd)

Differentiating $\mathcal{H}$ along a solution of the system, we have

$$\begin{aligned}
\frac{d}{dt}\mathcal{H} &= \frac{d}{dt}\left[\frac{1}{2}\dot{q}^T D(q)\dot{q} + P(q)\right] \\
&= \frac{1}{2}\ddot{q}^T D(q)\dot{q} + \frac{1}{2}\dot{q}^T D(q)\ddot{q} + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T D(q)\ddot{q} + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T\left[\tau - C(q,\dot{q})\dot{q} - g(q)\right] + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T\tau + \dot{q}^T \underbrace{\left(\frac{1}{2}\frac{d}{dt}\left[D(q)\right] - C(q,\dot{q})\right)}_{=0}\dot{q} \\
&= \dot{q}^T\tau
\end{aligned}$$

Passivity Relation (Cont'd)

The differential relation

$$\frac{d}{dt}\mathcal{H} = \dot{q}^T\tau$$

can be integrated, so that

$$\begin{aligned} \int_0^T \frac{d}{dt}\mathcal{H}(q(t),\dot{q}(t))dt &= \mathcal{H}(q(T),\dot{q}(T)) - \mathcal{H}(q(0),\dot{q}(0)) \\ &= \int_0^T \dot{q}(t)^T\tau(t)dt \end{aligned}$$

Passivity Relation (Cont'd)

The differential relation

$$\frac{d}{dt}\mathcal{H} \;=\; \dot{q}^{T}\tau$$

can be integrated, so that

$$\begin{aligned}\int_0^T \frac{d}{dt}\mathcal{H}(q(t),\dot{q}(t))dt &= \mathcal{H}(q(T),\dot{q}(T)) - \mathcal{H}(q(0),\dot{q}(0)) \\ &= \int_0^T \dot{q}(t)^{T}\tau(t)dt\end{aligned}$$

$$\boxed{\Rightarrow\ \int_0^T \dot{q}(t)\tau(t)dt \geq -\mathcal{H}(q(0),\dot{q}(0))}$$

Passivity Relation (Cont'd)

The differential relation

$$\frac{d}{dt}\mathcal{H} = \dot{q}^T \tau$$

can be integrated, so that

$$\int_0^T \frac{d}{dt}\mathcal{H}(q(t),\dot{q}(t))dt = \mathcal{H}(q(T),\dot{q}(T)) - \mathcal{H}(q(0),\dot{q}(0))$$

$$= \int_0^T \dot{q}(t)^T \tau(t) dt$$

$$\Rightarrow \int_0^T \dot{q}(t)\tau(t)dt \geq -\mathcal{H}(q(0),\dot{q}(0))$$

These relations are called

- passivity (dissipativity) relation
- passivity (dissipativity) relation in the integral form

Skew Symmetry of $\dot{D}(q) - C(q, \dot{q})$

To check that

$$N = \frac{d}{dt}\left[D(q)\right] - 2C(q, \dot{q}), \quad N^T = -N$$

look at $(k, j)^{th}$-component

$$\frac{d}{dt} d_{kj} - 2c_{kj} \;=\; \sum_{i=1}^{n} \frac{\partial d_{kj}}{\partial q_i}\dot{q}_i - \sum_{i=1}^{n}\left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right]\dot{q}_i$$

Skew Symmetry of $\dot{D}(q) - C(q,\dot{q})$

To check that

$$N = \frac{d}{dt}\left[D(q)\right] - 2C(q,\dot{q}), \quad N^T = -N$$

look at $(k,j)^{th}$-component

$$\begin{aligned} \frac{d}{dt}d_{kj} - 2c_{kj} &= \sum_{i=1}^{n} \frac{\partial d_{kj}}{\partial q_i}\dot{q}_i - \sum_{i=1}^{n}\left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right]\dot{q}_i \\ &= \sum_{i=1}^{n}\left\{\frac{\partial d_{kj}}{\partial q_i} - \left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right]\right\}\dot{q}_i \end{aligned}$$

Skew Symmetry of $\dot{D}(q) - C(q, \dot{q})$

To check that

$$N = \frac{d}{dt}\left[D(q)\right] - 2C(q, \dot{q}), \quad N^T = -N$$

look at $(k, j)^{th}$-component

$$\begin{aligned}
\frac{d}{dt} d_{kj} - 2c_{kj} &= \sum_{i=1}^{n} \frac{\partial d_{kj}}{\partial q_i} \dot{q}_i - \sum_{i=1}^{n} \left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right] \dot{q}_i \\
&= \sum_{i=1}^{n} \left\{ \frac{\partial d_{kj}}{\partial q_i} - \left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right] \right\} \dot{q}_i \\
&= \sum_{i=1}^{n} \left\{ \frac{\partial d_{ij}}{\partial q_k} - \frac{\partial d_{ki}}{\partial q_j} \right\} \dot{q}_i
\end{aligned}$$

Skew Symmetry of $\dot{D}(q) - C(q, \dot{q})$

To check that

$$N = \frac{d}{dt}[D(q)] - 2C(q, \dot{q}), \quad N^T = -N$$

look at $(k, j)^{th}$-component

$$\begin{aligned}
\frac{d}{dt} d_{kj} - 2c_{kj} &= \sum_{i=1}^{n} \frac{\partial d_{kj}}{\partial q_i}\dot{q}_i - \sum_{i=1}^{n}\left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right]\dot{q}_i \\
&= \sum_{i=1}^{n}\left\{\frac{\partial d_{kj}}{\partial q_i} - \left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right]\right\}\dot{q}_i \\
&= \sum_{i=1}^{n}\left\{\frac{\partial d_{ij}}{\partial q_k} - \frac{\partial d_{ki}}{\partial q_j}\right\}\dot{q}_i = \sum_{i=1}^{n}\left\{\frac{\partial d_{ji}}{\partial q_k} - \frac{\partial d_{ki}}{\partial q_j}\right\}\dot{q}_i
\end{aligned}$$

Skew Symmetry of $\dot{D}(q) - C(q, \dot{q})$

To check that

$$N = \frac{d}{dt}\left[D(q)\right] - 2C(q, \dot{q}), \quad N^T = -N$$

look at $(k, j)^{th}$-component

$$\begin{aligned}
\frac{d}{dt} d_{kj} - 2c_{kj} &= \sum_{i=1}^{n} \frac{\partial d_{kj}}{\partial q_i}\dot{q}_i - \sum_{i=1}^{n}\left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right]\dot{q}_i \\
&= \sum_{i=1}^{n}\left\{\frac{\partial d_{kj}}{\partial q_i} - \left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right]\right\}\dot{q}_i \\
&= \sum_{i=1}^{n}\left\{\frac{\partial d_{ij}}{\partial q_k} - \frac{\partial d_{ki}}{\partial q_j}\right\}\dot{q}_i = \sum_{i=1}^{n}\left\{\frac{\partial d_{ji}}{\partial q_k} - \frac{\partial d_{ki}}{\partial q_j}\right\}\dot{q}_i
\end{aligned}$$

$$\Rightarrow \qquad n_{kj} = -n_{jk}$$