Lecture 13: Multivariable Control of Robot Manipulators

- PD-Control

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- Feedback Linearization

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- Robust and Adaptive Motion Control

Improved Dynamical Model

Given a mechanical system with n-degrees of freedom

$$D(q)\ddot{q}+C(q,\dot{q})\dot{q}+g(q) = \tau, \; q = \left[q_1, \ldots, q_n\right]^T, \; \tau = \left[\tau_1, \ldots, \tau_n\right]^T$$

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We suppose that each of degrees of freedom is controlled by a geared DC-motor

$$J_{m_k}\ddot{\theta}_k + B_k\dot{\theta}_k = \frac{K_{m_k}}{R_k} V_k - \frac{1}{r_k}\tau_k, \quad k = 1, \ldots, n$$

where

- θ_k is the k^{th}-motor angle;
- r_k is a gear ration;
- J_{m_k}, B_k, K_{m_k}, R_k are parameters or computed from parameters of the k^{th} DC-motor

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and that the motor and the link angles are related by

$$\theta_{m_k} = r_k q_k, \quad k = 1, \ldots, n$$

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and that the motor and the link angles are related by

$$\theta_{m_k} = r_k q_k, \quad k = 1, \ldots, n$$

Then the actuator equations are

$$r_k^2 J_{m_k}\ddot{q}_k + r_k^2 B_k\dot{q}_k = r_k\frac{K_{m_k}}{R_k} V_k - \tau_k, \quad k = 1, \ldots, n$$

Improved Dynamical Model (Cont'd)

The dynamical systems

$$D(q)\ddot{q}+C(q,\dot{q})\dot{q}+g(q) = \tau,\ q = \left[q_1, \ldots, q_n\right]^T,\ \tau = \left[\tau_1, \ldots, \tau_n\right]^T$$

$$r_k^2 J_{m_k}\ddot{q}_k + r_k^2 B_k \dot{q}_k = r_k \frac{K_{m_k}}{R_k} V_k - \tau_k, \quad k = 1, \ldots, n$$

can be rewritten as one, if we exclude τ!

Improved Dynamical Model (Cont'd)

The dynamical systems

$$D(q)\ddot{q}+C(q,\dot{q})\dot{q}+g(q) = \tau,\ q = \left[q_1,\ldots,q_n\right]^T,\ \tau = \left[\tau_1,\ldots,\tau_n\right]^T$$

$$r_k^2 J_{m_k}\ddot{q}_k + r_k^2 B_k \dot{q}_k = r_k \frac{K_{m_k}}{R_k} V_k - \tau_k, \quad k = 1,\ldots,n$$

can be rewritten as one, if we exclude τ!

Indeed, it is

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + B\dot{q} + g(q) = u$$

where

- $M(q) = D(q) + J$ with $J = \mathrm{diag}\left\{r_1^2 J_{m_1},\ldots,r_n^2 J_{m_n}\right\}$
- $B = \left[r_1^2 B_1, r_2^2 B_2,\ldots,r_n^2 B_n\right]^T$
- $u = \left[u_1, u_2,\ldots,u_n\right]^T$ with $u_k = r_k \frac{K_{m_k}}{R_k} V_k$, $k = 1,\ldots,n$

PD-Controller Design

Given a mechanical system

$$\boldsymbol{M(q)\ddot{q} + C(q,\dot{q})\dot{q} + B\dot{q} + g(q) = u}$$

we are interested

- to design a controller to stabilize a particular configuration of the robot: $\boldsymbol{q = q_d}$
- to analyze the closed-loop system

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Assume that $B = 0$ and $g(p) = 0$

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we are interested

- to design a controller to stabilize a particular configuration of the robot: $q = q_d$
- to analyze the closed-loop system

Assume that $B = 0$ and $g(p) = 0$

The first controller to analyze is

$$u = -K_p(q - q_d) - K_d\dot{q}$$

with K_p and K_d are diagonal matrices with positive elements.

PD-Controller Design (Cont'd)

To analyze the behavior of the closed loop system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = u = -K_p\left(q - q_d\right) - K_d\dot{q}$$

consider a scalar function

$$V(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \frac{1}{2}\left(q - q_d\right)^T K_p\left(q - q_d\right)$$

PD-Controller Design (Cont'd)

To analyze the behavior of the closed loop system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = u = -K_p(q - q_d) - K_d\dot{q}$$

consider a scalar function

$$V(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \frac{1}{2}(q - q_d)^T K_p (q - q_d)$$

Its time-derivative along solutions of the closed-loop system is

$$\frac{d}{dt}V \quad = \quad \dot{q}^T M(q)\ddot{q} + \dot{q}^T \frac{d}{dt}\frac{1}{2}\left[M(q)\right]\dot{q} + \dot{q}^T K_p (q - q_d)$$

PD-Controller Design (Cont'd)

To analyze the behavior of the closed loop system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = u = -K_p(q - q_d) - K_d\dot{q}$$

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Its time-derivative along solutions of the closed-loop system is

$$\begin{aligned} \frac{d}{dt}V &= \dot{q}^T M(q)\ddot{q} + \dot{q}^T \frac{d}{dt}\frac{1}{2}\left[M(q)\right]\dot{q} + \dot{q}^T K_p (q - q_d) \\ &= \dot{q}^T \left[u - C(q,\dot{q})\dot{q}\right] + \dot{q}^T \frac{d}{dt}\frac{1}{2}\left[M(q)\right]\dot{q} + \dot{q}^T K_p (q - q_d) \end{aligned}$$

PD-Controller Design (Cont'd)

To analyze the behavior of the closed loop system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = u = -K_p(q - q_d) - K_d\dot{q}$$

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Its time-derivative along solutions of the closed-loop system is

$$\begin{aligned}
\frac{d}{dt}V &= \dot{q}^T M(q)\ddot{q} + \dot{q}^T \frac{d}{dt}\frac{1}{2}\left[M(q)\right]\dot{q} + \dot{q}^T K_p (q - q_d) \\
&= \dot{q}^T \left[u - C(q,\dot{q})\dot{q}\right] + \dot{q}^T \frac{d}{dt}\frac{1}{2}\left[M(q)\right]\dot{q} + \dot{q}^T K_p (q - q_d) \\
&= \dot{q}^T \left[u + K_p(q - q_d)\right] + \dot{q}^T \left\{\frac{d}{dt}\frac{1}{2}\left[M(q)\right] - C(q,\dot{q})\right\}\dot{q}
\end{aligned}$$

PD-Controller Design (Cont'd)

To analyze the behavior of the closed loop system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = u = -K_p(q - q_d) - K_d\dot{q}$$

consider a scalar function

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Its time-derivative along solutions of the closed-loop system is

$$\begin{aligned}
\frac{d}{dt}V &= \dot{q}^T M(q)\ddot{q} + \dot{q}^T \frac{d}{dt}\frac{1}{2}\left[M(q)\right]\dot{q} + \dot{q}^T K_p (q - q_d) \\
&= \dot{q}^T \left[u - C(q,\dot{q})\dot{q}\right] + \dot{q}^T \frac{d}{dt}\frac{1}{2}\left[M(q)\right]\dot{q} + \dot{q}^T K_p (q - q_d) \\
&= \dot{q}^T \left[u + K_p(q - q_d)\right] + \underbrace{\dot{q}^T \left\{\frac{d}{dt}\frac{1}{2}\left[D(q) + J\right] - C(q,\dot{q})\right\}\dot{q}}_{=0}
\end{aligned}$$

PD-Controller Design (Cont'd)

To analyze the behavior of the closed loop system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = u = -K_p(q - q_d) - K_d\dot{q}$$

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$$V(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \frac{1}{2}(q - q_d)^T K_p(q - q_d)$$

Its time-derivative along solutions of the closed-loop system is

$$\begin{aligned}
\frac{d}{dt}V &= \dot{q}^T M(q)\ddot{q} + \dot{q}^T \frac{d}{dt}\frac{1}{2}\left[M(q)\right]\dot{q} + \dot{q}^T K_p(q - q_d) \\
&= \dot{q}^T \left[u - C(q,\dot{q})\dot{q}\right] + \dot{q}^T \frac{d}{dt}\frac{1}{2}\left[M(q)\right]\dot{q} + \dot{q}^T K_p(q - q_d) \\
&= \dot{q}^T \left[u + K_p(q - q_d)\right] \\
&= \dot{q}^T \left[-K_p(q - q_d) - K_d\dot{q} + K_p(q - q_d)\right] = -\dot{q}^T K_d\dot{q}
\end{aligned}$$

PD-Controller Design (Cont'd)

To analyze the behavior of the closed loop system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = u = -K_p(q - q_d) - K_d\dot{q}$$

consider a scalar function

$$V(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \frac{1}{2}(q - q_d)^T K_p (q - q_d)$$

Its time-derivative along solutions of the closed-loop system is

$$\boxed{\frac{d}{dt}V = -\dot{q}^T K_d \dot{q} \leq 0}$$

Therefore

- V is positive definite, $V(q,\dot{q}) = 0 \Rightarrow \{q = q_d, \dot{q} = 0\}$
- $V(q(t),\dot{q}(t))$ is monotonically decreasing!

$\Rightarrow$ $\boxed{\exists \lim_{t\to+\infty} V(q(t),\dot{q}(t)) = V_\infty}$ and $\boxed{\exists \lim_{t\to+\infty} \dot{q}(t) = \dot{q}_\infty(t) = 0}$

PD-Controller Design (Cont'd)

To analyze the behavior of the closed loop system

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} = u = -K_p\,(q - q_d) - K_d\dot{q}$$

consider a scalar function

$$V(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \frac{1}{2}\,(q - q_d)^T\,K_p\,(q - q_d)$$

If we substitute this limit trajectory into the dynamics, we obtain

$$M(q_\infty)\,\underbrace{\ddot{q}_\infty}_{=0} + C(q_\infty, \dot{q}_\infty)\,\underbrace{\dot{q}_\infty}_{=0} = -K_p\,(q_\infty - q_d) - K_d\,\underbrace{\dot{q}_\infty}_{=0}$$

that is

$$0 = -K_p\,(q_\infty - q_d)$$

PD-Controller Design (Cont'd)

To analyze the behavior of the closed loop system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = u = -K_p\,(q - q_d) - K_d\dot{q}$$

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$$M(q_\infty)\,\underbrace{\ddot{q}_\infty}_{=\,0} + C(q_\infty,\dot{q}_\infty)\,\underbrace{\dot{q}_\infty}_{=\,0} = -K_p\,(q_\infty - q_d) - K_d\,\underbrace{\dot{q}_\infty}_{=\,0}$$

that is

$$0 = -K_p\,(q_\infty - q_d)$$

$$K_p = \text{diag}\,\{K_{p1}, K_{p2}, \ldots, K_{pn}\} > 0 \;\Rightarrow\; \boxed{q_\infty = q_d}$$

PD-Controller Design with Gravity Compensation

Given a mechanical system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + B\dot{q} + g(q) = u$$

How to modify the controller

$$u = -K_p\,(q - q_d) - K_d\dot{q}$$

if $B \neq 0$ and $g(p) \neq 0$?

PD-Controller Design with Gravity Compensation

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How to modify the controller

$$u = -K_p(q - q_d) - K_d\dot{q}$$

if $B \neq 0$ and $g(p) \neq 0$?

The controller

$$u = -K_p(q - q_d) - K_d\dot{q} + g(q)$$

is stabilizing, if K_p and K_d are diagonal matrices such that

$$K_p > 0 \qquad K_d - B > 0$$

Lecture 13: Multivariable Control of Robot Manipulators

- PD-Control
- Feedback Linearization
- Robust and Adaptive Motion Control

Feedback Linearization

Given a mechanical system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + B\dot{q} + g(q) = u$$

How to define a control variable

$$u = \alpha(q,\dot{q}) + \beta(q,\dot{q})v$$

so that the closed loop system is:

- linear? I.e. is equivalent to

$$\dot{x} = Ax + Bv$$

- linear and stabilizable? I.e. the pair (A, B) is stabilizable

Feedback Linearization

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---

What if

$$\boldsymbol{u = M(q)v + C(q,\dot{q})\dot{q} + B\dot{q} + g(q)?}$$

Feedback Linearization

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Then

$$M(q)\ddot{q} = M(q)v \quad \Rightarrow \quad \ddot{q} = v$$

Feedback Linearization

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How to define a control variable

$$\boldsymbol{u = \alpha(q,\dot{q}) + \beta(q,\dot{q})v}$$

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---

Then

$$\boldsymbol{\ddot{q} = v} \quad \Rightarrow \quad \boxed{\boldsymbol{\dot{x} = \begin{bmatrix} 0_n & I_n \\ 0_n & 0_n \end{bmatrix} x + \begin{bmatrix} 0_n \\ I_n \end{bmatrix} v, \quad x = [q^T, \dot{q}^T]^T}}$$

Feedback Linearization

Given a mechanical system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + B\dot{q} + g(q) = u$$

and the desired trajectory $q_d = q_d(t)$, introduce the controller

$$\begin{aligned} u &= M(q)v + C(q,\dot{q})\dot{q} + B\dot{q} + g(q) \\ v &= \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t)) \end{aligned}$$

Feedback Linearization

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Then the closed loop system is

$$\ddot{q} = v = \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t))$$

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Then the closed loop system is

$$\ddot{q} = v = \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t))$$

It can be rewritten in error-variables as

$$\boxed{\ddot{e} + K_d\dot{e} + K_p e = 0, \qquad e = q - q_d(t)}$$

Lecture 13: Multivariable Control of Robot Manipulators

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Robust and Adaptive Motion Control

Given a mechanical system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u$$

and the desired trajectory $q_d = q_d(t)$, the controller

$$\begin{aligned} u &= \boxed{M(q)v + C(q,\dot{q})\dot{q} + g(q)} \\ v &= \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t)) \end{aligned}$$

cannot be implemented!

Robust and Adaptive Motion Control

Given a mechanical system

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and the desired trajectory $q_d = q_d(t)$, the controller

$$\begin{aligned} u &= \boxed{M(q)v + C(q,\dot{q})\dot{q} + g(q)} \\ v &= \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t)) \end{aligned}$$

cannot be implemented!

The approximation might be used

$$\begin{aligned} u &= \boxed{[M(q) + \triangle M]\, v + [C(q,\dot{q}) + \triangle C]\,\dot{q} + [g(q) + \triangle g]} \\ v &= \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t)) \end{aligned}$$

Robust and Adaptive Motion Control

The uncertainty in parameters of controller

$$
\begin{aligned}
\boldsymbol{u} &= \boxed{[M(q) + \triangle M]\, \boldsymbol{v} + [C(q,\dot{q}) + \triangle C]\, \dot{q} + [g(q) + \triangle g]} \\
\boldsymbol{v} &= \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t))
\end{aligned}
$$

motivates two approaches

- Robust Control:
  Design K_p, K_d and $q_d(t)$ such that the error signal

$$
e(t) = q(t) - q_d(t) \approx 0 \qquad \forall \{\triangle M, \triangle C, \triangle g\} \in \mathcal{W}
$$

Robust and Adaptive Motion Control

The uncertainty in parameters of controller

$$u = \left[M(q) + \triangle M\right] v + \left[C(q,\dot q) + \triangle C\right]\dot q + \left[g(q) + \triangle g\right]$$

$$v = \ddot q_d(t) - K_p(q - q_d(t)) - K_d(\dot q - \dot q_d(t))$$

motivates two approaches

- Robust Control:
  Design K_p, K_d and $q_d(t)$ such that the error signal

  $$e(t) = q(t) - q_d(t) \approx 0 \qquad \forall \{\triangle M, \triangle C, \triangle g\} \in \mathcal{W}$$

- Adaptive Control: Improve estimates for

  $$M(q), \quad C(q,\dot q), \quad g(q)$$

  in the course of regulating the system

Robust Motion Control Based on Feedback Linearization

Given a trajectory $q = q_d(t)$, consider the closed loop system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u$$

$$\begin{aligned} u &= [M(q) + \triangle M]\, v + [C(q,\dot{q}) + \triangle C]\, \dot{q} + [g(q) + \triangle g] \\ v &= \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t)) \end{aligned}$$

Robust Motion Control Based on Feedback Linearization

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$$\begin{aligned} u &= [M(q) + \triangle M]\, v + [C(q,\dot{q}) + \triangle C]\, \dot{q} + [g(q) + \triangle g] \\ v &= \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t)) \end{aligned}$$

$$\begin{aligned} M(q)\ddot{q}+C(q,\dot{q})\dot{q}+g(q) &= \\ &= [M(q) + \triangle M]\, v + [C(q,\dot{q}) + \triangle C]\, \dot{q} + [g(q) + \triangle g] \end{aligned}$$

Robust Motion Control Based on Feedback Linearization

Given a trajectory $q = q_d(t)$, consider the closed loop system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u$$

$$\begin{aligned} u &= [M(q) + \triangle M]\, v + [C(q,\dot{q}) + \triangle C]\,\dot{q} + [g(q) + \triangle g] \\ v &= \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t)) \end{aligned}$$

$$\begin{aligned} M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) &= \\ &= [M(q) + \triangle M]\, v + [C(q,\dot{q}) + \triangle C]\,\dot{q} + [g(q) + \triangle g] \end{aligned}$$

$$\Rightarrow \quad M(q)\ddot{q} = [M(q) + \triangle M]\, v + \triangle C\dot{q} + \triangle g$$

Robust Motion Control Based on Feedback Linearization

Given a trajectory $q = q_d(t)$, consider the closed loop system

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = u$$

$$\begin{aligned} u &= [M(q) + \triangle M]\, v + [C(q, \dot{q}) + \triangle C]\, \dot{q} + [g(q) + \triangle g] \\ v &= \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t)) \end{aligned}$$

$$\begin{aligned} M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) &= \\ &= [M(q) + \triangle M]\, v + [C(q, \dot{q}) + \triangle C]\, \dot{q} + [g(q) + \triangle g] \end{aligned}$$

$$\Rightarrow \quad M(q)\ddot{q} = [M(q) + \triangle M]\, v + \triangle C\dot{q} + \triangle g$$

$$\Rightarrow \quad \ddot{q} = M(q)^{-1}\, [M(q) + \triangle M]\, v + M(q)^{-1}\, [\triangle C\dot{q} + \triangle g]$$

Robust Motion Control Based on Feedback Linearization

Given a trajectory $q = q_d(t)$, consider the closed loop system

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$$\ddot{q} = M^{-1}[M + \triangle M]\, v + M^{-1}[\triangle C\dot{q} + \triangle g] \pm \boxed{M^{-1} M v}$$

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$$\Rightarrow \quad \ddot{q} = v + M^{-1}(q)\, [\triangle M(q) v + \triangle C(q,\dot{q})\dot{q} + \triangle g(q)]$$

Robust Motion Control Based on Feedback Linearization

Given a trajectory $q = q_d(t)$, consider the closed loop system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \color{red}{u}$$

$$\begin{aligned} \color{red}{u} &= [M(q) + \triangle M]\,\color{red}{v} + [C(q,\dot{q}) + \triangle C]\,\dot{q} + [g(q) + \triangle g] \\ \color{red}{v} &= \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t)) \end{aligned}$$

$$M(q)\ddot{q}+C(q,\dot{q})\dot{q}+g(q) =$$

$$= [M(q) + \triangle M]\,\color{red}{v} + [C(q,\dot{q}) + \triangle C]\,\dot{q} + [g(q) + \triangle g]$$

$$\Rightarrow \quad M(q)\ddot{q} = [M(q) + \triangle M]\,\color{red}{v} + \triangle C\dot{q} + \triangle g$$

$$\ddot{q} = M^{-1}\,[M + \triangle M]\,\color{red}{v} + M^{-1}\,[\triangle C\dot{q} + \triangle g] \pm \boxed{M^{-1}M\color{red}{v}}$$

$$\boxed{\ddot{q} = \color{red}{v} + \eta(q,\dot{q},\color{red}{v})}$$

Robust Motion Control Based on Feedback Linearization

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It can be rewritten as

$$\ddot{q} = v + \eta(q, \dot{q}, v)$$

or

$$(\ddot{q} - \ddot{q}_d) + K_d\,(\dot{q} - \dot{q}_d) + K_p\,(q - q_d) = w + \eta(q, \dot{q}, v)$$

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or

$$\frac{d}{dt}e = \begin{bmatrix} 0_n & I_n \\ -K_p & -K_d \end{bmatrix} e + \begin{bmatrix} 0_n \\ I_n \end{bmatrix} (w + \eta)\,, \quad e = \begin{bmatrix} (q - q_d) \\ (\dot{q} - \dot{q}_d) \end{bmatrix}$$

Robust Motion Control Based on Feedback Linearization

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or

$$\frac{d}{dt}e = Ae + B\,(w + \eta)\,, \quad e = \begin{bmatrix} (q - q_d) \\ (\dot{q} - \dot{q}_d) \end{bmatrix}$$

Robust Motion Control Based on Feedback Linearization

To continue with design of $\boldsymbol{w}$ for the system

$$\frac{d}{dt}e = Ae + B\left[w + \eta(e, w)\right]$$

we need to impose some assumptions on $\eta(\cdot)$, namely

$$\|\eta(e, w)\| \leq \alpha\, \|w\| + \gamma_1\, \|e\| + \gamma_2\, \|e\|^2 + \gamma_3, \quad \alpha < 1$$

Robust Motion Control Based on Feedback Linearization

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Matrix A is stable, therefore $\forall\, Q > 0$, $\exists\, P = P^T > 0$ such that

$$A^T P + PA = -Q$$

Robust Motion Control Based on Feedback Linearization

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Matrix A is stable, therefore $\forall\, Q > 0$, $\exists\, P = P^T > 0$ such that

$$A^T P + PA = -Q$$

Consider a Lyapunov function candidate as $V = e^T P e$, then

$$\begin{aligned}
\frac{d}{dt}V &= \frac{d}{dt}e^T Pe + e^T P\frac{d}{dt}e \\
&= e^T(A^T P + PA)e + 2e^T PB\left[w + \eta(e, w)\right]
\end{aligned}$$

Robust Motion Control Based on Feedback Linearization

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$$A^T P + PA = -Q$$

Consider a Lyapunov function candidate as $V = e^T Pe$, then

$$\begin{aligned} \frac{d}{dt}V &= -e^T Qe + 2e^T PB\left[w + \eta(e, w)\right] \\ &\leq 0 \quad \leftarrow \quad \text{How to achieve this by choosing } w\text{?} \end{aligned}$$

Robust Motion Control Based on Feedback Linearization

To continue with design of w for the system

$$\frac{d}{dt}e = Ae + B\left[w + \eta(e, w)\right]$$

we need to impose some assumptions on $\eta(\cdot)$, namely

$$\|\eta(e, w)\| \leq \alpha\,\|w\| + \gamma_1\,\|e\| + \gamma_2\,\|e\|^2 + \gamma_3, \quad \alpha < 1$$

Let us look at the second term

$$\frac{d}{dt}V \quad = \quad -e^T Q e + 2\boxed{e^T PB\left[w + \eta(e, w)\right]}$$

when w has the form

$$w = -\rho(\cdot)\frac{z}{\sqrt{z^T z}}, \qquad \boxed{z = B^T P e}, \qquad \boxed{\rho \text{ is a function to choose}}$$

Robust Motion Control Based on Feedback Linearization

To continue with design of w for the system

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$$w = -\rho(\cdot)\frac{z}{\sqrt{z^T z}}, \qquad \boxed{z = B^T P e}, \qquad \boxed{\rho \text{ is a function to choose}}$$

$$z^T\left(-\rho\frac{z}{\sqrt{z^T z}} + \eta\right) \leq -\rho\|z\| + \|z\|\,\|\eta\| = \|z\|\left(-\rho + \|\eta\|\right)$$

Robust Motion Control Based on Feedback Linearization

To sum up, we search for a scalar function $\rho(\cdot)$ such that

$$\boxed{(-\rho + \|\eta\|) \leq 0} \;\Leftrightarrow\; \boxed{\|\eta\| \leq \rho}$$

and

$$\|\eta(e, w)\| \leq \alpha\,\|w\| + \gamma_1\,\|e\| + \gamma_2\,\|e\|^2 + \gamma_3, \quad \alpha < 1$$

with

$$w = -\rho(\cdot)\frac{z}{\sqrt{z^T z}}$$

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with

$$w = -\rho(\cdot) \frac{z}{\sqrt{z^T z}}$$

These two inequalities imply the next one

$$\alpha \left\| \rho(\cdot) \frac{z}{\sqrt{z^T z}} \right\| + \gamma_1 \|e\| + \gamma_2 \|e\|^2 + \gamma_3 \le \rho(\cdot)$$

Robust Motion Control Based on Feedback Linearization

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Robust Motion Control Based on Feedback Linearization

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with

$$w = -\rho(\cdot) \frac{z}{\sqrt{z^T z}}$$

These two inequalities imply the next one

$$\alpha \rho(\cdot) + \gamma_1 \|e\| + \gamma_2 \|e\|^2 + \gamma_3 \leq \rho(\cdot)$$

$$\boxed{\rho(e) \geq \frac{1}{1-\alpha} \left[\gamma_1 \|e\| + \gamma_2 \|e\|^2 + \gamma_3 \right]}$$

Final Form of the Controller

Given a trajectory $q = q_d(t)$, consider the closed loop system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u$$

$$\begin{aligned} u &= [M(q) + \triangle M]\, v + [C(q,\dot{q}) + \triangle C]\, \dot{q} + [g(q) + \triangle g] \\ v &= \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t)) \boxed{+w} \\ w &= \begin{cases} -\rho(e)\frac{z}{\sqrt{z^T z}}, & \text{if } z = B^T P e \neq 0 \\ 0, & \text{if } z = B^T P e = 0 \end{cases} \end{aligned}$$

where $\rho(\cdot)$ is any function that satisfies to inequality

$$\rho(e) \geq \frac{1}{1-\alpha}\left[\gamma_1 \|e\| + \gamma_2 \|e\|^2 + \gamma_3\right]$$

where constants α, γ_1-γ_2 are from the inequality

$$\|\eta(\cdot)\| \leq \alpha \|w\| + \gamma_1 \|e\| + \gamma_2 \|e\|^2 + \gamma_3, \quad \alpha < 1$$

and

$$\eta(q,\dot{q},v) = M^{-1}[\triangle M v + \triangle C\dot{q} + \triangle g], \quad e = \begin{bmatrix} (q - q_d) \\ (\dot{q} - \dot{q}_d) \end{bmatrix}$$