Lecture 14: Multivariable Control of Robot Manipulators

- Example: Robust Motion Control

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- Adaptive Motion Control Based on Feedback Linearization

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- Passivity Based Motion Control

Robust Controller Based on Feedback Linearization

Given a trajectory $q = q_d(t)$, consider the closed loop system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u$$

$$\begin{aligned} u &= [M(q) + \triangle M]\, v + [C(q,\dot{q}) + \triangle C]\,\dot{q} + [g(q) + \triangle g] \\ v &= \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t)) + w \\ w &= \begin{cases} -\rho(e)\frac{z}{\sqrt{z^T z}}, & \text{if } z = B^T P e \neq 0 \\ 0, & \text{if } z = B^T P e = 0 \end{cases} \end{aligned}$$

where $\rho(\cdot)$ is any function that satisfies to inequality

$$\rho(e) \geq \frac{1}{1-\alpha}\left[\gamma_1 \|e\| + \gamma_2 \|e\|^2 + \gamma_3\right]$$

where constants α, γ_1-γ_2 are from the inequality

$$\|\eta(\cdot)\| \leq \alpha \|w\| + \gamma_1 \|e\| + \gamma_2 \|e\|^2 + \gamma_3, \quad \alpha < 1$$

and

$$\eta(q,\dot{q},v) = M^{-1}[\triangle M v + \triangle C\dot{q} + \triangle g], \quad e = \begin{bmatrix} (q - q_d) \\ (\dot{q} - \dot{q}_d) \end{bmatrix}$$

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$$\eta = M^{-1}\left[\triangle M v + \triangle C \dot{q} + \triangle g\right], \quad e = \begin{bmatrix} (q - q_d) \\ (\dot{q} - \dot{q}_d) \end{bmatrix}$$

Chattering Phenomenon for Robust Controller

If the control variable is scalar then the expression

$$w = \begin{cases} -\rho(e)\frac{z}{\sqrt{z^T z}}, & \text{if } z = B^T P e \neq 0 \\ 0, & \text{if } z = B^T P e = 0 \end{cases}$$

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This means that

If $z(t)$ is changing its sign

$$\Downarrow$$

the control signal has to be changed

instantaneously with non-zero increment!

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One of possible choices to avoid the discontinuity in controller is

$$w = \begin{cases} -\rho(e) \cdot \mathrm{sign}(z), & \text{if } z = B^T P e,\ |z| > \delta \\ -\frac{1}{\delta}\rho(e) \cdot z, & \text{if } z = B^T P e,\ |z| \leq \delta \end{cases}$$

Re-Design of Robust Controller

The closed loop system can be rewritten as

$$\frac{d}{dt}e = Ae + B\left[w(e) + \eta(e, w(e))\right], \quad w(\cdot) \in \mathbb{R}^1, \quad \eta(\cdot) \in \mathbb{R}^1$$

where we have assumed that

$$\|\eta(e, w(e))\| \leq \rho(e)$$

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Here

$$A = \begin{bmatrix} 0_n & I_n \\ -K_p & -K_d \end{bmatrix}, \quad B = \begin{bmatrix} 0_n \\ I_n \end{bmatrix}, \quad e = \begin{bmatrix} (q - q_d) \\ (\dot{q} - \dot{q}_d) \end{bmatrix}$$

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where we have assumed that

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Matrix A is stable, therefore $\forall\, Q > 0$, $\exists\, P = P^T > 0$ such that

$$A^T P + PA = -Q$$

Consider a Lyapunov function candidate as $V = e^T P e$, then

$$\begin{aligned} \frac{d}{dt}V &= \frac{d}{dt}e^T Pe + e^T P \frac{d}{dt}e \\ &= e^T(A^T P + PA)e + 2e^T PB\left[w + \eta(e, w)\right] \end{aligned}$$

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Consider a Lyapunov function candidate as $V = e^T P e$, then

$$\begin{aligned} \frac{d}{dt}V &= \frac{d}{dt}e^T P e + e^T P \frac{d}{dt}e \\ &= -e^T Q e + 2e^T PB\left[w + \eta(e, w)\right] \end{aligned}$$

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where we have assumed that

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If

$$w(e) = \begin{cases} -\rho(e) \cdot \mathrm{sign}(z), & \text{if } z = B^T P e, \ |z| > \delta \\ -\frac{1}{\delta}\rho(e) \cdot z, & \text{if } z = B^T P e, \ |z| \leq \delta \end{cases}$$

then for those e that satisfies $\|B^T P e\| > \delta$ we have

$$\begin{aligned} \frac{d}{dt}V &= -e^T Q e + 2e^T P B\left[w + \eta(e, w)\right] \\ &< 2z\left[-\rho(e) \cdot \mathrm{sign}(z) + \eta(e, w)\right] = -2|z|\left[\rho(e) - \frac{\eta(e, w)}{\mathrm{sign}(z)}\right] \end{aligned}$$

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then for those e that satisfies $\|B^T P e\| \leq \delta$ we have

$$\begin{aligned} \frac{d}{dt}V &= -e^T Q e + 2e^T P B\left[w + \eta(e, w)\right] \\ &= -e^T Q e + 2z\left[-\frac{1}{\delta}\rho(e) \cdot z + \eta(e, w)\right] \end{aligned}$$

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then for those e that satisfies $\|B^T P e\| \leq \delta$ we have

$$\begin{aligned} \frac{d}{dt}V &= -e^T Q e + 2e^T P B\left[w + \eta(e, w)\right] \\ &= -e^T Q e + 2\Big[-\frac{1}{\delta}\rho(e) \cdot |z|^2 + \underbrace{z \cdot \eta(e, w)}_{\leq |z| \cdot |\eta|}\Big] \end{aligned}$$

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$$\begin{aligned} \frac{d}{dt}V &= -e^T Q e + 2 e^T P B \left[w + \eta(e, w)\right] \\ &\le -e^T Q e + 2\left[-\frac{1}{\delta}\rho(e) \cdot |z|^2 + \rho(e)|z|\right] \end{aligned}$$

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then for those e that satisfies $\|B^T P e\| \le \delta$ we have

$$\begin{aligned} \frac{d}{dt}V &= -e^T Q e + 2e^T P B\left[w + \eta(e, w)\right] \\ &\le -e^T Q e + 2\rho(e) \cdot \underbrace{\left[-\frac{1}{\delta} \cdot |z|^2 + |z|\right]} \end{aligned}$$

What is the maximum of this expression?

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then for those e that satisfies $\|B^T P e\| \le \delta$ we have

$$\begin{aligned} \frac{d}{dt}V &= -e^T Q e + 2e^T P B\left[w + \eta(e, w)\right] \\ &\le -e^T Q e + 2\rho(e) \cdot \underbrace{\left[-\frac{1}{\delta} \cdot \left|\frac{\delta}{2}\right|^2 + \left|\frac{\delta}{2}\right|\right]}_{\text{the maximum is that}} \end{aligned}$$

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then for those e that satisfies $\|B^T P e\| \le \delta$ we have

$$\begin{aligned} \frac{d}{dt}V &= -e^T Q e + 2e^T P B\left[w + \eta(e, w)\right] \\ &\le -e^T Q e + 2\rho(e) \cdot \frac{\delta}{4} = -e^T Q e + \rho(e) \cdot \frac{\delta}{2} \end{aligned}$$

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then for those e that satisfies $\|B^T P e\| \le \delta$ we have

$$\frac{d}{dt}V \quad \le \quad -e^T Q e + 2\rho(e) \cdot \frac{\delta}{4} = -e^T Q e + \rho(e) \cdot \frac{\delta}{2} \le 0$$

for those e that satisfies: $e^T Q e > \frac{\delta}{2}\rho(e)$

Re-Design of Robust Controller

To conclude:

- To avoid chattering, the next modification is suggested

$$w(e) = \begin{cases} -\rho(e)\cdot \mathrm{sign}(z), & \text{if } z = B^T P e,\ |z| > \delta \\ -\dfrac{1}{\delta}\rho(e)\cdot z, & \text{if } z = B^T P e,\ |z| \le \delta \end{cases}$$

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- With such modification we cannot prove asymptotic stability of the closed loop system :(

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- But we were able to show that the value of Lyapunov function will be discreasing everywhere, except some vicinity of the origin of error dynamics.

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- The size of this set can be modified, and depends on δ.

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- With such modification we cannot prove asymptotic stability of the closed loop system :(
- But we were able to show that the value of Lyapunov function will be discreasing everywhere, except some vicinity of the origin of error dynamics.
- The size of this set can be modified, and depends on δ.
- Such property is called ultimate boundedness

Example:

Consider a target reference $q_d(t)$ and a system

$$(M + \triangle M)\ddot{q} = u, \quad M = 1, \quad \triangle M = \varepsilon$$

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The system is linear so that transforms $u \longrightarrow v \longrightarrow w$ are

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If for instance, $K_p = K_d = 1$, then the transformed system is

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$$\boxed{(\ddot{q} - \ddot{q}_d(t)) + (\dot{q} - \dot{q}_d(t)) + (q - q_d(t)) = w + \eta(\cdot)}$$

with

$$\boxed{\eta = \frac{\varepsilon}{1+\varepsilon}\left[\ddot{q}_d(t) - (q - q_d(t)) - (\dot{q} - \dot{q}_d(t)) + w\right]}$$

Example (Cont'd):

To proceed with robust design we need

- Rewrite the system

$$(\ddot{q} - \ddot{q}_d(t)) + (\dot{q} - \dot{q}_d(t)) + (q - q_d(t)) = w + \eta(\cdot)$$

into a state-space form

$$\frac{d}{dt}e = Ae + B\left[w + \eta(\cdot)\right], \quad e = \begin{bmatrix} (q - q_d) \\ (\dot{q} - \dot{q}_d) \end{bmatrix}$$

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- Solve the Lyapunov equation

$$A^T P + PA = -Q, \quad \text{let's say for }, Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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- Solve the Lyapunov equation

$$A^T P + PA = -Q, \quad \text{let's say for }, Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Find a range of ε so that the function

$$\eta = \frac{\varepsilon}{1+\varepsilon}\left[\ddot{q}_d(t) - (q - q_d(t)) - (\dot{q} - \dot{q}_d(t)) + w\right]$$

  satisfies the bound

$$\|\eta(e, w)\| \le \alpha\,\|w\| + \gamma_1\,\|e\| + \gamma_2\,\|e\|^2 + \gamma_3, \quad \alpha < 1$$

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Adaptive Feedback Linearization

Given a mechanical system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u$$

and the desired trajectory $q_d = q_d(t)$, introduce the controller

$$\begin{aligned} u &= \hat{M}(q)v + \hat{C}(q,\dot{q})\dot{q} + \hat{g}(q) \\ v &= \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t)) + w \end{aligned}$$

What is the difference with the robust design:

- In robust design, the coefficients of $\hat{M}(\cdot)$, $\hat{C}(\cdot)$, $\hat{g}(\cdot)$ were fixed.
- Now, they are variables to tune and $w = 0$!

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- Now, they are variables to tune and $w = 0$!

Let us find the dynamical equations for updating values of $\hat{M}(\cdot)$, $\hat{C}(\cdot)$, $\hat{g}(\cdot)$ provided that we measure $q(t)$, $\dot{q}(t)$, $\ddot{q}(t)$.

Adaptive Feedback Linearization

Let us rewrite the system

$$M(q)\ddot{q}+C(q,\dot{q})\dot{q}+g(q) = u, \quad u = \hat{M}(q)v+\hat{C}(q,\dot{q})\dot{q}+\hat{g}(q)$$

$$v = \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t))$$

in the form

$$\hat{M}(q)\ddot{q} + \left[M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) - \hat{M}(q)\ddot{q}\right] =$$

$$= \hat{M}(q)v + \hat{C}(q,\dot{q})\dot{q} + \hat{g}(q)$$

Adaptive Feedback Linearization

Let us rewrite the system

$$M(q)\ddot{q}+C(q,\dot{q})\dot{q}+g(q) = u, \quad u = \hat{M}(q)v+\hat{C}(q,\dot{q})\dot{q}+\hat{g}(q)$$

$$v = \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t))$$

in the form

$$\hat{M}(q)\ddot{q} + \Big[M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) - \hat{M}(q)\ddot{q}\Big] =$$

$$= \hat{M}(q)v + \hat{C}(q,\dot{q})\dot{q} + \hat{g}(q)$$

$$\hat{M}(q)\ddot{q} = \hat{M}(q)v + Y(q,\dot{q},\ddot{q})\left[\theta - \hat{\theta}\right]$$

where

- θ is the vector of true parameters of the model
- $\hat{\theta}$ is the vector of estimates
- $Y(\cdot)$ is the vector function, which values we can compute measuring $q(t)$, $\dot{q}(t)$, $\ddot{q}(t)$ and $q_d(t)$ and its derivatives

Adaptive Feedback Linearization

Let us rewrite the system

$$M(q)\ddot{q}+C(q,\dot{q})\dot{q}+g(q)=u, \quad u=\hat{M}(q)v+\hat{C}(q,\dot{q})\dot{q}+\hat{g}(q)$$

$$v=\ddot{q}_d(t)-K_p(q-q_d(t))-K_d(\dot{q}-\dot{q}_d(t))$$

in the form

$$\hat{M}(q)\ddot{q}+\left[M(q)\ddot{q}+C(q,\dot{q})\dot{q}+g(q)-\hat{M}(q)\ddot{q}\right]=$$

$$= \quad \hat{M}(q)v+\hat{C}(q,\dot{q})\dot{q}+\hat{g}(q)$$

$$\ddot{q}=v+\hat{M}(q)^{-1}Y(q,\dot{q},\ddot{q})\left[\theta-\hat{\theta}\right]$$

where

- θ is the vector of true parameters of the model
- $\hat{\theta}$ is the vector of estimates
- $Y(\cdot)$ is the vector function, which values we can compute measuring $q(t)$, $\dot{q}(t)$, $\ddot{q}(t)$ and $q_d(t)$ and its derivatives

Adaptive Feedback Linearization

Let us rewrite the system

$$M(q)\ddot{q}+C(q,\dot{q})\dot{q}+g(q) = u, \quad u = \hat{M}(q)v+\hat{C}(q,\dot{q})\dot{q}+\hat{g}(q)$$

$$v = \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t))$$

in the form

$$\hat{M}(q)\ddot{q} + \Big[M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) - \hat{M}(q)\ddot{q}\Big] =$$

$$= \hat{M}(q)v + \hat{C}(q,\dot{q})\dot{q} + \hat{g}(q)$$

With the controller the equation

$$\ddot{q} = v + \hat{M}(q)^{-1}Y(q,\dot{q},\ddot{q})\left[\hat{\theta} - \theta\right]$$

becomes

$$(\ddot{q}-\ddot{q}_d)+K_d(\dot{q}-\dot{q}_d)+K_p(q-q_d) = \hat{M}(q)^{-1}Y(q,\dot{q},\ddot{q})\left[\hat{\theta} - \theta\right]$$

Adaptive Feedback Linearization

The equation

$$(\ddot{q}-\ddot{q}_d)+K_d(\dot{q}-\dot{q}_d)+K_p(q-q_d) = \hat{M}(q)^{-1}Y(q,\dot{q},\ddot{q})\left[\hat{\theta}-\theta\right]$$

has variables we can change! How to do this update?

Adaptive Feedback Linearization

The equation

$$(\ddot{q}-\ddot{q}_d)+K_d(\dot{q}-\dot{q}_d)+K_p(q-q_d) = \boxed{\hat{M}(q)^{-1}Y(q,\dot{q},\ddot{q})}\left[\hat{\theta}-\theta\right]$$

has variables we can change! How to do this update?

Let us rewrite the system into the state space form

$$\frac{d}{dt}e = \underbrace{\begin{bmatrix} 0 & I \\ -K_p & -K_d \end{bmatrix}}_{=A} e + \underbrace{\begin{bmatrix} 0 \\ I \end{bmatrix}}_{=B} \Phi\left[\hat{\theta}-\theta\right], \quad e = \begin{bmatrix} (q-q_d(t)) \\ (\dot{q}-\dot{q}_d(t)) \end{bmatrix}$$

Adaptive Feedback Linearization

The equation

$$(\ddot{q}-\ddot{q}_d)+K_d(\dot{q}-\dot{q}_d)+K_p(q-q_d) = \hat{M}(q)^{-1}Y(q,\dot{q},\ddot{q})\left[\hat{\theta}-\theta\right]$$

has variables we can change! How to do this update?

Let us rewrite the system into the state space form

$$\frac{d}{dt}e = Ae + B\Phi\left[\hat{\theta}-\theta\right]$$

Solve the Lyapunov equation

$$A^T P + PA = -Q < 0, \quad P > 0$$

and consider the Lyapunov function candidate

$$V = e^T P e + \frac{1}{2}\left[\hat{\theta}-\theta\right]^T \Gamma \left[\hat{\theta}-\theta\right], \quad \Gamma > 0$$

Adaptive Feedback Linearization

The equation

$$(\ddot{q}-\ddot{q}_d)+K_d(\dot{q}-\dot{q}_d)+K_p(q-q_d) = \boxed{\hat{M}(q)^{-1}Y(q,\dot{q},\ddot{q})} \left[\hat{\theta}-\theta\right]$$

has variables we can change! How to do this update?

Let us rewrite the system into the state space form

$$\frac{d}{dt}e = Ae + B\Phi\left[\hat{\theta}-\theta\right]$$

Then

$$\begin{aligned}\frac{d}{dt}V &= \frac{d}{dt}\{e^T Pe\} + \frac{d}{dt}\left\{\frac{1}{2}\left[\hat{\theta}-\theta\right]^T \Gamma \left[\hat{\theta}-\theta\right]\right\} \\ &= \left[Ae + B\Phi\left[\hat{\theta}-\theta\right]\right]^T Pe + e^T P\left[Ae + B\Phi\left[\hat{\theta}-\theta\right]\right] + \\ &\quad + \frac{1}{2}\left[\frac{d}{dt}\hat{\theta} - \mathbf{0}\right]^T \Gamma\left[\hat{\theta}-\theta\right] + \frac{1}{2}\left[\hat{\theta}-\theta\right]^T \Gamma\left[\frac{d}{dt}\hat{\theta} - \mathbf{0}\right]\end{aligned}$$

Adaptive Feedback Linearization

The equation

$$(\ddot{q}-\ddot{q}_d)+K_d(\dot{q}-\dot{q}_d)+K_p(q-q_d) = \boxed{\hat{M}(q)^{-1}Y(q,\dot{q},\ddot{q})}\left[\hat{\theta}-\theta\right]$$

has variables we can change! How to do this update?

Let us rewrite the system into the state space form

$$\frac{d}{dt}e = Ae + B\Phi\left[\hat{\theta}-\theta\right]$$

Then

$$\begin{aligned}
\frac{d}{dt}V &= \left[Ae + B\Phi\left[\hat{\theta}-\theta\right]\right]^T Pe + e^T P\left[Ae + B\Phi\left[\hat{\theta}-\theta\right]\right] + \\
&\quad + \frac{1}{2}\left[\frac{d}{dt}\hat{\theta} - 0\right]^T \Gamma\left[\hat{\theta}-\theta\right] + \frac{1}{2}\left[\hat{\theta}-\theta\right]^T \Gamma\left[\frac{d}{dt}\hat{\theta} - 0\right] \\
&= e^T\left[A^T P + PA\right]e + \left[\hat{\theta}-\theta\right]^T\left\{\Phi^T B^T Pe + \Gamma\frac{d}{dt}\hat{\theta}\right\}
\end{aligned}$$

Adaptive Feedback Linearization

The equation

$$(\ddot{q}-\ddot{q}_d)+K_d(\dot{q}-\dot{q}_d)+K_p(q-q_d) = \boxed{\hat{M}(q)^{-1}Y(q,\dot{q},\ddot{q})}\left[\theta-\hat{\theta}\right]$$

has variables we can change! How to do this update?

Let us rewrite the system into the state space form

$$\frac{d}{dt}e = Ae + B\Phi\left[\theta-\hat{\theta}\right]$$

Then

$$\begin{aligned}\frac{d}{dt}V &= \left[Ae + B\Phi\left[\hat{\theta}-\theta\right]\right]^T Pe + e^T P\left[Ae + B\Phi\left[\hat{\theta}-\theta\right]\right] + \\ &\quad + \frac{1}{2}\left[\frac{d}{dt}\hat{\theta} - 0\right]^T \Gamma\left[\hat{\theta}-\theta\right] + \frac{1}{2}\left[\hat{\theta}-\theta\right]^T \Gamma\left[\frac{d}{dt}\hat{\theta} - 0\right] \\ &= \underbrace{e^T\left[A^T P + PA\right]e}_{=-e^T Qe} + \left[\hat{\theta}-\theta\right]^T \underbrace{\boxed{\left\{\Phi^T B^T Pe + \Gamma\frac{d}{dt}\hat{\theta}\right\}}}_{=0} \leq 0\end{aligned}$$

Adaptive Feedback Linearization

With the proposed update law for $\hat{\theta}$ the closed loop system is

$$\frac{d}{dt}e = Ae + B\Phi\left[\theta - \hat{\theta}\right], \quad \frac{d}{dt}\hat{\theta} = -\Gamma^{-1}\Phi^{T}B^{T}Pe$$

Adaptive Feedback Linearization

With the proposed update law for $\hat{\theta}$ the closed loop system is

$$\frac{d}{dt}e = Ae + B\Phi\left[\theta - \hat{\theta}\right], \quad \frac{d}{dt}\hat{\theta} = -\Gamma^{-1}\Phi^{T}B^{T}Pe$$

The inequality

$$\frac{d}{dt}V = \frac{d}{dt}\left[e^{T}Pe + \frac{1}{2}\left[\hat{\theta} - \theta\right]^{T}\Gamma\left[\hat{\theta} - \theta\right]\right] = -e^{T}Qe$$

implies that (thanks to Barbalat lemma)

$e(t) \longrightarrow 0$ as $t \longrightarrow +\infty$ and $(\hat{\theta}(t) - \theta)$ is bounded

Adaptive Feedback Linearization

With the proposed update law for $\hat{\theta}$ the closed loop system is

$$\frac{d}{dt}e = Ae + B\Phi\left[\theta - \hat{\theta}\right], \quad \frac{d}{dt}\hat{\theta} = -\Gamma^{-1}\Phi^T B^T P e$$

The inequality

$$\frac{d}{dt}V = \frac{d}{dt}\left[e^T P e + \frac{1}{2}\left[\hat{\theta} - \theta\right]^T \Gamma \left[\hat{\theta} - \theta\right]\right] = -e^T Q e$$

implies that (thanks to Barbalat lemma)

$e(t) \to 0$ as $t \to +\infty$ and $(\hat{\theta}(t) - \theta)$ is bounded

It is necessary to remember that

- we have to measure $\ddot{q}$ for computing the regressor;
- the matrix $\hat{M}(q)$ at each time moment should be invertible.

Lecture 14: Multivariable Control of Robot Manipulators

- Example: Robust Motion Control
- Adaptive Motion Control Based on Feedback Linearization
- Passivity Based Motion Control

Passivity Based Motion Control

Given a trajectory $q = q_d(t)$, consider the system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u$$

with the controller

$$u = M(q)a + C(q,\dot{q})b + g(q) - Kr, \quad K = \text{diag}\{k_1, \ldots, k_n\}$$

where variables a, b and r are to be chosen

Passivity Based Motion Control

Given a trajectory $q = q_d(t)$, consider the system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u$$

with the controller

$$u = M(q)a + C(q,\dot{q})b + g(q) - Kr, \quad K = \text{diag}\,\{k_1, \ldots, k_n\}$$

where variables a, b and r are to be chosen

If we substitute the controller, we obtain

$$M(q)\left[\ddot{q} - a\right] + C(q,\dot{q})\left[\dot{q} - b\right] + Kr = 0$$

Passivity Based Motion Control

Given a trajectory $q = q_d(t)$, consider the system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u$$

with the controller

$$u = M(q)a + C(q,\dot{q})b + g(q) - Kr, \quad K = \text{diag}\,\{k_1, \ldots, k_n\}$$

where variables a, b and r are to be chosen

If we substitute the controller, we obtain

$$M(q)\left[\ddot{q} - a\right] + C(q,\dot{q})\left[\dot{q} - b\right] + Kr = 0$$

Let us impose some relations between signals

$$\frac{d}{dt}r = \ddot{q} - a, \qquad r = \dot{q} - b$$

Passivity Based Motion Control

Given a trajectory $q = q_d(t)$, consider the system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u$$

with the controller

$$u = M(q)a + C(q,\dot{q})b + g(q) - Kr, \quad K = \text{diag}\,\{k_1, \dots, k_n\}$$

where variables a, b and r are to be chosen

If we substitute the controller, we obtain

$$M(q)\left[\ddot{q} - a\right] + C(q,\dot{q})\left[\dot{q} - b\right] + Kr = 0$$

Let us impose some relations between signals and the reference

$$\frac{d}{dt}r = \ddot{q} - a, \qquad r = \dot{q} - b, \qquad r = (\dot{q} - \dot{q}_d(t)) + \Lambda\,(q - q_d(t))$$

Passivity Based Motion Control

Given a trajectory $q = q_d(t)$, consider the system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u$$

with the controller

$$u = M(q)a + C(q,\dot{q})b + g(q) - Kr, \quad K = \text{diag}\,\{k_1, \ldots, k_n\}$$

where variables a, b and r are to be chosen

If we substitute the controller, we obtain

$$M(q)\,[\ddot{q} - a] + C(q,\dot{q})\,[\dot{q} - b] + Kr = 0$$

Let us impose some relations between signals and the reference

$$\frac{d}{dt}r = \ddot{q} - a, \qquad r = \dot{q} - b, \qquad r = (\dot{q} - \dot{q}_d(t)) + \Lambda\,(q - q_d(t))$$

Then the closed loop dynamics are

$$M(q)\frac{d}{dt}r + C(q,\dot{q})r + Kr = 0$$

Passivity Based Motion Control

To analyze the closed loop system

$$M(q)\frac{d}{dt}\boldsymbol{r} + C(q,\dot q)\boldsymbol{r} + K\boldsymbol{r} = 0$$

consider the Lyapunov function candidate

$$V = \frac{1}{2}\boldsymbol{r}^T M(q)\boldsymbol{r} + (q - q_d(t))^T K\Lambda\,(q - q_d(t))$$

Passivity Based Motion Control

To analyze the closed loop system

$$M(q)\frac{d}{dt}r + C(q,\dot q)r + Kr = 0$$

consider the Lyapunov function candidate

$$V = \frac{1}{2}r^T M(q) r + (q - q_d(t))^T K\Lambda\,(q - q_d(t))$$

Then

$$\begin{aligned}\frac{d}{dt}V \;&=\; \frac{d}{dt}\left[\frac{1}{2}r^T M(q) r\right] + \frac{d}{dt}\left[(q - q_d(t))^T K\Lambda\,(q - q_d(t))\right]\\ &=\; r^T M(q)\frac{d}{dt}[r] + \frac{1}{2}r^T\frac{d}{dt}\left[M(q)\right] r + \\ &\qquad +2\,(q - q_d(t))^T K\Lambda\frac{d}{dt}\,(q - q_d(t))\end{aligned}$$

Passivity Based Motion Control

To analyze the closed loop system

$$M(q)\frac{d}{dt}r + C(q,\dot q)r + Kr = 0$$

consider the Lyapunov function candidate

$$V = \frac{1}{2}r^T M(q) r + (q - q_d(t))^T K\Lambda\,(q - q_d(t))$$

Then

$$\begin{aligned}
\frac{d}{dt}V &= \frac{d}{dt}\left[\frac{1}{2}r^T M(q) r\right] + \frac{d}{dt}\left[(q - q_d(t))^T K\Lambda\,(q - q_d(t))\right] \\
&= r^T M(q)\frac{d}{dt}[r] + \frac{1}{2}r^T\frac{d}{dt}\left[M(q)\right] r + \\
&\qquad +2\,(q - q_d(t))^T K\Lambda\frac{d}{dt}\,(q - q_d(t)) \\
&= r^T \boxed{\left[-C(q,\dot q)r - Kr\right]} + \frac{1}{2}r^T\frac{d}{dt}\left[M(q)\right] r + \\
&\qquad +2\,(q - q_d(t))^T K\Lambda\frac{d}{dt}\,(q - q_d(t))
\end{aligned}$$

Passivity Based Motion Control

To analyze the closed loop system

$$M(q)\frac{d}{dt}r + C(q,\dot q)r + Kr = 0$$

consider the Lyapunov function candidate

$$V = \frac{1}{2}r^T M(q) r + (q - q_d(t))^T K\Lambda\,(q - q_d(t))$$

Then

$$\begin{aligned}
\frac{d}{dt}V &= \frac{d}{dt}\left[\frac{1}{2}r^T M(q) r\right] + \frac{d}{dt}\left[(q - q_d(t))^T K\Lambda\,(q - q_d(t))\right] \\
&= r^T M(q)\frac{d}{dt}[r] + \frac{1}{2}r^T\frac{d}{dt}\left[M(q)\right]r + \\
&\qquad\qquad +2\,(q - q_d(t))^T K\Lambda\frac{d}{dt}(q - q_d(t)) \\
&= -r^T K r + r^T\left[\frac{1}{2}\frac{d}{dt}M(q) - C(q,\dot q)\right]r + \\
&\qquad\qquad +2\,(q - q_d(t))^T K\Lambda\frac{d}{dt}(q - q_d(t))
\end{aligned}$$

Passivity Based Motion Control

To analyze the closed loop system

$$M(q)\frac{d}{dt}r + C(q,\dot{q})r + Kr = 0$$

consider the Lyapunov function candidate

$$V = \frac{1}{2}r^T M(q) r + (q - q_d(t))^T K\Lambda (q - q_d(t))$$

Then

$$\begin{aligned}
\frac{d}{dt}V &= \frac{d}{dt}\left[\frac{1}{2}r^T M(q) r\right] + \frac{d}{dt}\left[(q - q_d(t))^T K\Lambda (q - q_d(t))\right] \\
&= r^T M(q)\frac{d}{dt}[r] + \frac{1}{2}r^T \frac{d}{dt}[M(q)]\, r + \\
&\qquad\qquad +2\,(q - q_d(t))^T K\Lambda \frac{d}{dt}(q - q_d(t)) \\
&= -r^T K r + \underbrace{r^T\left[\frac{1}{2}\frac{d}{dt}M(q) - C(q,\dot{q})\right] r}_{=0} + \\
&\qquad\qquad +2\,(q - q_d(t))^T K\Lambda \frac{d}{dt}(q - q_d(t))
\end{aligned}$$

Passivity Based Motion Control

To analyze the closed loop system

$$M(q)\frac{d}{dt}r + C(q,\dot q)r + Kr = 0$$

consider the Lyapunov function candidate

$$V = \frac{1}{2}r^T M(q) r + (q - q_d(t))^T K\Lambda\,(q - q_d(t))$$

Then

$$\begin{aligned}
\frac{d}{dt}V &= \frac{d}{dt}\left[\frac{1}{2}r^T M(q) r\right] + \frac{d}{dt}\left[(q - q_d(t))^T K\Lambda\,(q - q_d(t))\right] \\
&= r^T M(q)\frac{d}{dt}[r] + \frac{1}{2}r^T\frac{d}{dt}\left[M(q)\right]r + \\
&\qquad +2\,(q - q_d(t))^T K\Lambda\frac{d}{dt}(q - q_d(t)) \\
&= -r^T K r + 2\,(q - q_d(t))^T K\Lambda\frac{d}{dt}(q - q_d(t))
\end{aligned}$$

where

$$r = (\dot q - \dot q_d(t)) + \Lambda\,(q - q_d(t))$$

Passivity Based Motion Control

To analyze the closed loop system

$$M(q)\frac{d}{dt}r + C(q,\dot{q})r + Kr = 0$$

consider the Lyapunov function candidate

$$V = \frac{1}{2}r^T M(q) r + (q - q_d(t))^T K\Lambda\,(q - q_d(t))$$

Then

$$\begin{aligned}
\frac{d}{dt}V &= \frac{d}{dt}\left[\frac{1}{2}r^T M(q) r\right] + \frac{d}{dt}\left[(q - q_d(t))^T K\Lambda\,(q - q_d(t))\right] \\
&= r^T M(q)\frac{d}{dt}[r] + \frac{1}{2}r^T \frac{d}{dt}\left[M(q)\right] r + \\
&\qquad\qquad +2\,(q - q_d(t))^T K\Lambda \frac{d}{dt}\,(q - q_d(t)) \\
&= -\left(\frac{d}{dt}\tilde{q} + \Lambda\tilde{q}\right)^T K\left(\frac{d}{dt}\tilde{q} + \Lambda\tilde{q}\right) + 2\tilde{q}^T K\Lambda\frac{d}{dt}\tilde{q}
\end{aligned}$$

where

$$r = \frac{d}{dt}\tilde{q} + \Lambda\tilde{q}$$

Passivity Based Motion Control

To analyze the closed loop system

$$M(q)\frac{d}{dt}r + C(q,\dot q)r + Kr = 0$$

consider the Lyapunov function candidate

$$V = \frac{1}{2}r^T M(q) r + (q - q_d(t))^T K\Lambda\,(q - q_d(t))$$

Then

$$\begin{aligned}
\frac{d}{dt}V &= \frac{d}{dt}\left[\frac{1}{2}r^T M(q) r\right] + \frac{d}{dt}\left[(q - q_d(t))^T K\Lambda\,(q - q_d(t))\right] \\
&= r^T M(q)\frac{d}{dt}[r] + \frac{1}{2}r^T\frac{d}{dt}\left[M(q)\right] r + \\
&\qquad\qquad +2\,(q - q_d(t))^T K\Lambda\frac{d}{dt}\,(q - q_d(t)) \\
&= -\left(\frac{d}{dt}\tilde q + \Lambda\tilde q\right)^T K\left(\frac{d}{dt}\tilde q + \Lambda\tilde q\right) + 2\tilde q^T K\Lambda\frac{d}{dt}\tilde q \\
&= -\left[\frac{d}{dt}\tilde q\right]^T K\frac{d}{dt}\tilde q - \tilde q^T\Lambda^T K\Lambda\tilde q < 0
\end{aligned}$$

Adaptive Passivity Based Motion Control

Given a trajectory $q = q_d(t)$, consider the system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u$$

with the controller

$$u = \hat{M}(q)a + \hat{C}(q,\dot{q})b + \hat{g}(q) - Kr, \quad K = \text{diag}\{k_1, \ldots, k_n\}$$

with the variables a, b, r defined by

$$\frac{d}{dt}r = \ddot{q} - a, \qquad r = \dot{q} - b, \qquad r = (\dot{q} - \dot{q}_d(t)) + \Lambda\,(q - q_d(t))$$

Adaptive Passivity Based Motion Control

Given a trajectory $q = q_d(t)$, consider the system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u$$

with the controller

$$u = \hat{M}(q)a + \hat{C}(q,\dot{q})b + \hat{g}(q) - Kr, \quad K = \text{diag}\,\{k_1, \ldots, k_n\}$$

with the variables a, b, r defined by

$$\frac{d}{dt}r = \ddot{q} - a, \qquad r = \dot{q} - b, \qquad r = (\dot{q} - \dot{q}_d(t)) + \Lambda\,(q - q_d(t))$$

The dynamics can be rewritten as

$$M(q)\left[\ddot{q} - a\right] + C(q,\dot{q})\left[\dot{q} - b\right] + Kr =$$

$$= \left[\hat{M}(q) - M(q)\right] a + \left[\hat{C}(q,\dot{q}) - C(q,\dot{q})\right] b + \left[\hat{g}(p) - g(p)\right]$$

$$= Y(q,\dot{q},a,b)\left[\hat{\theta} - \theta\right]$$

Adaptive Passivity Based Motion Control

Given a trajectory $q = q_d(t)$, consider the system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u$$

with the controller

$$u = \hat{M}(q)a + \hat{C}(q,\dot{q})b + \hat{g}(q) - Kr, \quad K = \text{diag}\{k_1, \ldots, k_n\}$$

with the variables a, b, r defined by

$$\frac{d}{dt}r = \ddot{q} - a, \qquad r = \dot{q} - b, \qquad r = (\dot{q} - \dot{q}_d(t)) + \Lambda\,(q - q_d(t))$$

The dynamics can be rewritten as

$$M(q)\frac{d}{dt}r + C(q,\dot{q})r + Kr =$$

$$= \left[\hat{M}(q) - M(q)\right]a + \left[\hat{C}(q,\dot{q}) - C(q,\dot{q})\right]b + \left[\hat{g}(p) - g(p)\right]$$

$$= Y(q,\dot{q},a,b)\left[\hat{\theta} - \theta\right]$$

Adaptive Passivity Based Motion Control

To find the update law for $\hat{\theta}$-variable for the system

$$M(q)\frac{d}{dt}r + C(q,\dot{q})r + Kr = Y(q,\dot{q},a,b)\left[\hat{\theta} - \theta\right]$$

we will use the Lyapunov function candidate

$$V = \frac{1}{2}r^T M(q) r + (q - q_d(t))^T K\Lambda\,(q - q_d(t)) + \frac{1}{2}\left[\hat{\theta} - \theta\right]^T \Gamma\left[\hat{\theta} - \theta\right]$$

Adaptive Passivity Based Motion Control

To find the update law for $\hat{\theta}$-variable for the system

$$M(q)\frac{d}{dt}r + C(q,\dot{q})r + Kr = Y(q,\dot{q},a,b)\left[\hat{\theta} - \theta\right]$$

we will use the Lyapunov function candidate

$$V = \frac{1}{2}r^T M(q) r + (q - q_d(t))^T K\Lambda\,(q - q_d(t)) + \frac{1}{2}\left[\hat{\theta} - \theta\right]^T \Gamma \left[\hat{\theta} - \theta\right]$$

Its time-derivative along a solution of the system is

$$\begin{aligned}\frac{d}{dt}V &= -\dot{\tilde{q}}^T K \dot{\tilde{q}} - \tilde{q}^T \Lambda^T K \Lambda \tilde{q} + \\ &\qquad + \left[\hat{\theta} - \theta\right]^T \left\{\Gamma \frac{d}{dt}\hat{\theta} + Y(q,\dot{q},a,b)^T r\right\}\end{aligned}$$

Adaptive Passivity Based Motion Control

To find the update law for $\hat{\theta}$-variable for the system

$$M(q)\frac{d}{dt}r + C(q,\dot{q})r + Kr = Y(q,\dot{q},a,b)\left[\hat{\theta} - \theta\right]$$

we will use the Lyapunov function candidate

$$V = \frac{1}{2}r^T M(q) r + (q - q_d(t))^T K\Lambda\,(q - q_d(t)) + \frac{1}{2}\left[\hat{\theta} - \theta\right]^T \Gamma \left[\hat{\theta} - \theta\right]$$

Its time-derivative along a solution of the system is

$$\begin{aligned}\frac{d}{dt}V &= -\dot{\tilde{q}}^T K \dot{\tilde{q}} - \tilde{q}^T \Lambda^T K \Lambda \tilde{q} + \\ &\quad + \left[\hat{\theta} - \theta\right]^T \underbrace{\left\{\Gamma \frac{d}{dt}\hat{\theta} + Y(q,\dot{q},a,b)^T r\right\}}_{=0} \le 0\end{aligned}$$

$$\frac{d}{dt}\hat{\theta} = -\Gamma^{-1} Y(q,\dot{q},a,b)^T r$$