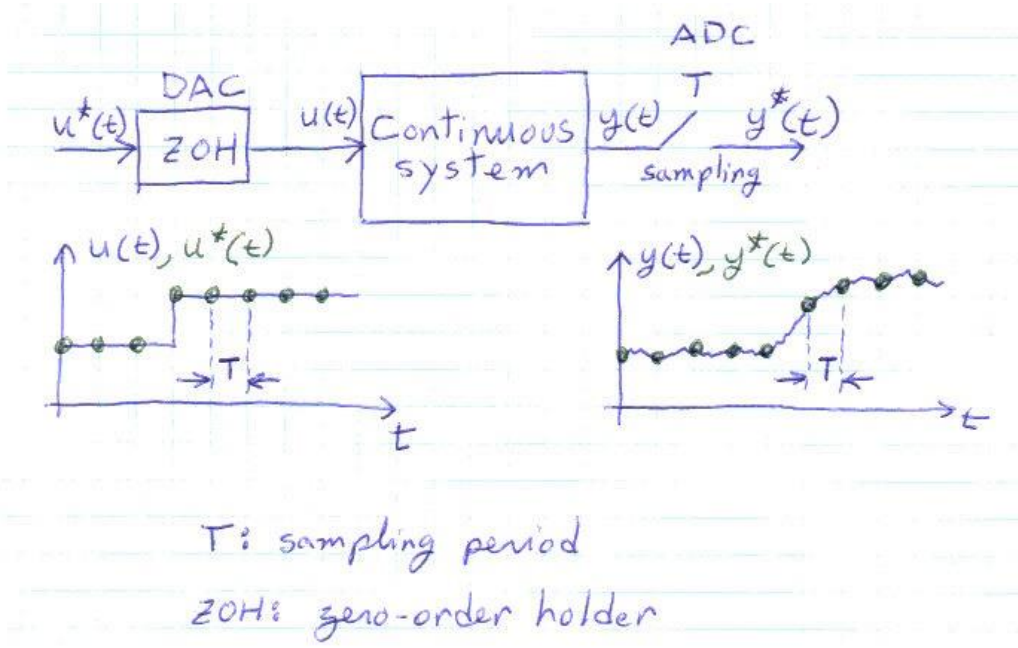
DISCRÉTISATION D'UNE FONCTION DE TRANSFERT

© A. Desbiens, 2020

Discrete systems



Continuous systems

$$Y(s) = \mathcal{L}y(t) \qquad U(s) = \mathcal{L}u(t)$$

$$\frac{Y(s)}{U(s)} = G(s)$$

Continuous transfer function (zero initial conditions)

Same as the differential equation

Discrete systems – Transfer function

Discrete systems

$$Y(z) = \mathcal{L}y^*(t) \qquad \text{with } z^{-1} = e^{-Ts} \text{ (delay of one sampling period)}$$

$$U(z) = \mathcal{L}u^*(t)$$

Discrete transfer function: $\boxed{\dfrac{Y(z)}{U(z)} = G(z)}$ (zero initial conditions)

$$\frac{Y(z)}{U(z)} = G(z) = \frac{b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

$$Y(z) = -a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + b_1 z^{-1} U(z) + b_2 z^{-2} U(z)$$

$$y(k) = -a_1 y(k-1) + a_2 y(k-2) + b_1 u(k-1) + b_2 u(k-2)$$

Static regime: $y(k) = y(k - 1) = y(k - 2) = \text{etc.}$ => $z = 1$

Static gain of the transfer function: $G(1)$

Discrete systems – Transfer function

$$\frac{Y(s)}{U(s)}=\frac{K}{1+\tau s} \quad \text{ZOH, } T\text{:} \quad \frac{Y(z)}{U(z)}=\frac{K(1-e^{-T/\tau})z^{-1}}{1-e^{-T/\tau}z^{-1}}=\frac{bz^{-1}}{1+az^{-1}}$$

$$Y(z)=-az^{-1}Y(z)+bz^{-1}U(z)$$

$$y(k)=-ay(k-1)+bu(k-1)$$

t	k	u	y
$-T$	-1	0	0
0	0	1	$-a0+b0=0$
T	1	1	$-a0+b1=b$
$2T$	2	1	$-ab+b1=b(1-a)$

Discrete systems – Transfer function

$$\frac{Y(s)}{U(s)}=\frac{Ke^{-\theta s}}{1+\tau s}\quad \text{ZOH, } T\text{:}\quad \frac{Y(z)}{U(z)}=\frac{K(1-e^{-T/\tau})z^{-1}z^{-\theta/T}}{1-e^{-T/\tau}z^{-1}}=\frac{bz^{-1-\theta/T}}{1+az^{-1}}$$

The value of θ is assumed to be a multiple of T (if not: modified z transform)

$$Y(z)=-az^{-1}Y(z)+bz^{-1-\theta/T}U(z)$$

$$y(k)=-ay(k-1)+bu(k-1-\theta/T)$$

Discretization:

- Process: ZOH, no direct transmission, $y(k) \neq \mathrm{f}^{\mathrm{n}}(u(k))$
- Controller: Tustin (bilinear), direct transmission, $y(k) = \mathrm{f}^{\mathrm{n}}(u(k))$
- Matlab:
  - Gpz=c2d(Gp,T,'zoh');
  - Gcz=c2d(Gc,T,'Tustin');

Discrete systems – Transfer function

Stability

- A continuous transfer function is stable all its poles have their real part negative
- $z = e^{Ts} = e^{T(a + bj)} = e^{aT}e^{jbT}$ => $|z| = e^{aT}$
- If $a < 0$ => $|z| < 1$
- A discrete transfer function is stable all its poles lie within the unit circle (i.e. modulus < 1)

System with integration

- Continuous transfer function: one pole at $s = 0$
- Discrete transfer function: one pole at $z = e^{T0} = 1$