Example: Robust Motion Control

- Example: Robust Motion Control
- Adaptive Motion Control Based on Feedback Linearization

- Example: Robust Motion Control
- Adaptive Motion Control Based on Feedback Linearization
- Passivity Based Motion Control

Robust Controller Based on Feedback Linearization

Given a trajectory $q = q_d(t)$, consider the closed loop system

$$egin{array}{ll} M(q)\ddot{q}+C(q,\dot{q})\dot{q}+g(q)=oldsymbol{u} \ oldsymbol{u} &=& \left[M(q)+igtriangle M
ight]oldsymbol{v}+\left[C(q,\dot{q})+igtriangle C
ight]\dot{q}+\left[g(q)+igtriangle g
ight] \ oldsymbol{v} &=& \ddot{q}_d(t)-K_p(q-q_d(t))-K_d(\dot{q}-\dot{q}_d(t))+oldsymbol{w} \ oldsymbol{v} &=& egin{cases} -oldsymbol{
ho}(e)rac{z}{\sqrt{z^Tz}}, & ext{if } z=B^TPe \neq 0 \ 0, & ext{if } z=B^TPe=0 \end{cases} \end{array}$$

where $\rho(\cdot)$ is any function that satisfies to inequality

$$oldsymbol{
ho(e)} \geq rac{1}{1-lpha} \left[\gamma_1 \left\lVert e
ight
Vert + \gamma_2 \left\lVert e
ight
Vert^2 + \gamma_3
ight]$$

where constants α , γ_1 - γ_2 are from the inequality

$$\|\eta(\cdot)\| \le \alpha \|\mathbf{w}\| + \gamma_1 \|e\| + \gamma_2 \|e\|^2 + \gamma_3, \quad \alpha < 1$$

and

$$\eta(q,\dot{q},v) = M^{-1} \left[riangle M oldsymbol{v} + riangle C \dot{q} + riangle g
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$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \mathbf{u}$$
 $\mathbf{u} = [M(q) + \triangle M]\mathbf{v} + [C(q,\dot{q}) + \triangle C]\dot{q} + [g(q) + \triangle g]$
 $\mathbf{v} = \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t)) + \mathbf{w}$

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$$\eta = M^{-1} \left[\triangle M {\color{red} v} + \triangle C {\dot{q}} + \triangle g
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Chattering Phenomenon for Robust Controller

If the control variable is scalar then the expression

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ight.$$

becomes

$$m{w} = \left\{ egin{array}{ll} -m{
ho}(m{e}) \cdot \mathrm{sign}(m{z}), & \mathrm{if} \ m{z} = m{B}^{\mathrm{\scriptscriptstyle T}} m{P} m{e}
eq 0 \\ 0, & \mathrm{if} \ m{z} = m{B}^{\mathrm{\scriptscriptstyle T}} m{P} m{e} = 0 \end{array}
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ight.$$

This means that

If z(t) is changing its sign



the control signal has to be changed instantaneously with non-zero increment!

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One of possible choices to avoid the discontinuity in controller is

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ho}(oldsymbol{e}) \cdot oldsymbol{z}, & ext{if } oldsymbol{z} &= B^{\scriptscriptstyle T} P e, \ |oldsymbol{z}| \leq \delta \ \end{array}
ight.$$

The closed loop system can be rewritten as

$$rac{d}{dt}e = Ae + B\left[oldsymbol{w(e)} + \eta(e,oldsymbol{w(e)})
ight], \quad oldsymbol{w(\cdot)} \in \mathbb{R}^1, \quad \eta(\cdot) \in \mathbb{R}^1$$

where we have assumed that

$$\|\eta(e, w(e))\| \leq \rho(e)$$

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Here

$$A = \left[egin{array}{ccc} 0_n & I_n \ -K_p & -K_d \end{array}
ight], \quad B = \left[egin{array}{ccc} 0_n \ I_n \end{array}
ight], \quad e = \left[egin{array}{ccc} (q-q_d) \ (\dot{q}-\dot{q}_d) \end{array}
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Matrix A is stable, therefore $\forall\,Q>0,\,\exists\,P=P^{\scriptscriptstyle T}>0$ such that

$$A^{\scriptscriptstyle T}P + PA = -Q$$

Consider a Lyapunov function candidate as $V = e^{T}Pe$, then

$$\frac{d}{dt}V = \frac{d}{dt}e^{T}Pe + e^{T}P\frac{d}{dt}e$$

$$= e^{T}(A^{T}P + PA)e + 2e^{T}PB\left[\mathbf{w} + \eta(e, \mathbf{w})\right]$$

The closed loop system can be rewritten as

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$$\frac{d}{dt}V = \frac{d}{dt}e^{T}Pe + e^{T}P\frac{d}{dt}e$$

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ight. \end{aligned}$$

$$egin{array}{lll} rac{d}{dt}V &=& -e^{\scriptscriptstyle T}Qe + 2e^{\scriptscriptstyle T}PB\left[oldsymbol{w} + \eta(e,oldsymbol{w})
ight] \\ &<& 2z\left[-
ho(e)\cdot ext{sign}(z) + \eta(e,oldsymbol{w})
ight] = -2|z| \left[oldsymbol{
ho}(e) - rac{\eta(e,oldsymbol{w})}{ ext{sign}(z)}
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ight.$$

$$\frac{d}{dt}V = -e^{T}Qe + 2e^{T}PB\left[\mathbf{w} + \eta(e, \mathbf{w})\right]$$

$$= -e^{T}Qe + 2\left[-\frac{1}{\delta}\boldsymbol{\rho}(e) \cdot |z|^{2} + \underline{z} \cdot \eta(e, \mathbf{w})\right]$$

$$< |z| \cdot |\eta|$$

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$$\frac{d}{dt}V = -e^{T}Qe + 2e^{T}PB\left[\mathbf{w} + \eta(e, \mathbf{w})\right]$$

$$\leq -e^{T}Qe + 2\rho(\mathbf{e}) \cdot \frac{\delta}{4} = -e^{T}Qe + \rho(\mathbf{e}) \cdot \frac{\delta}{2}$$

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$$\frac{d}{dt}V \leq -e^{\scriptscriptstyle T}Qe + 2\rho(e) \cdot \frac{\delta}{4} = -e^{\scriptscriptstyle T}Qe + \rho(e) \cdot \frac{\delta}{2} \leq 0$$
 for those e that satisfies: $e^{\scriptscriptstyle T}Qe > \frac{\delta}{2}\rho(e)$

To conclude:

$$m{w(e)} = \left\{ egin{array}{l} -m{
ho}(e) \cdot ext{sign}(z), & ext{if } z = B^{\scriptscriptstyle T} P e, \ |z| > \delta \ -rac{1}{\delta}m{
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To conclude:

To avoid chattering, the next modification is suggested

$$egin{aligned} oldsymbol{w}(oldsymbol{e}) &= \left\{ egin{aligned} -oldsymbol{
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 With such modification we cannot prove asymptotic stability of the closed loop system:(

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- But we were able to show that the value of Lyapunov function will be discreasing everywhere, except some vicinity of the origin of error dynamics.

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- With such modification we cannot prove asymptotic stability of the closed loop system:(
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- The size of this set can be modified, and depends on δ .

To conclude:

$$m{w}(m{e}) = \left\{egin{array}{l} -m{
ho}(m{e}) \cdot \mathrm{sign}(m{z}), & ext{if } m{z} = m{B}^{\mathrm{\scriptscriptstyle T}} m{P} m{e}, \; |m{z}| > \delta \ -rac{1}{\delta} m{
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- But we were able to show that the value of Lyapunov function will be discreasing everywhere, except some vicinity of the origin of error dynamics.
- The size of this set can be modified, and depends on δ .
- Such property is called <u>ultimate boundedness</u>

Consider a target reference $q_d(t)$ and a system

$$(M + \triangle M)\ddot{q} = \mathbf{u}, \quad M = 1, \quad \triangle M = \varepsilon$$

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The system is linear so that transforms $oldsymbol{u}
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$$u = Mv = v$$

$$\mathbf{v} = \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t)) + \mathbf{w}$$

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If for instance, $K_p=K_d=1$, then the transformed system is

$$(1+\varepsilon)\ddot{q} = \ddot{q}_d(t) - (q - q_d(t)) - (\dot{q} - \dot{q}_d(t)) + \mathbf{w}$$

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$$(\ddot{q} - \ddot{q}_d(t)) + (\dot{q} - \dot{q}_d(t)) + (q - q_d(t)) = \mathbf{w} + \eta(\cdot)$$

with

$$\eta = rac{arepsilon}{1+arepsilon} \left[\ddot{q}_d(t) - (q-q_d(t)) - (\dot{q}-\dot{q}_d(t)) + oldsymbol{w}
ight]$$

Example (Cont'd):

To proceed with robust design we need

Rewrite the system

$$(\ddot{q} - \ddot{q}_d(t)) + (\dot{q} - \dot{q}_d(t)) + (q - q_d(t)) = \mathbf{w} + \eta(\cdot)$$

into a state-space form

$$rac{d}{dt}e = Ae + B\left[m{w} + \eta(\cdot)
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ight]$$

Solve the Lyapunov equation

$$A^{ \mathrm{\scriptscriptstyle T} } P + P A = - Q, \quad ext{let's say for }, Q = \left[egin{array}{cc} 1 & 0 \ 0 & 1 \end{array}
ight]$$

Example (Cont'd):

To proceed with robust design we need

Rewrite the system

$$(\ddot{q} - \ddot{q}_d(t)) + (\dot{q} - \dot{q}_d(t)) + (q - q_d(t)) = \mathbf{w} + \eta(\cdot)$$

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Solve the Lyapunov equation

$$A^{\scriptscriptstyle T}P+PA=-Q,\quad ext{let's say for },Q=\left[egin{array}{cc} 1 & 0 \ 0 & 1 \end{array}
ight]$$

• Find a range of ε so that the function

$$\eta = rac{arepsilon}{1+arepsilon} \left[\ddot{q}_d(t) - (q-q_d(t)) - (\dot{q}-\dot{q}_d(t)) + rac{oldsymbol{w}}{2}
ight]$$

satisfies the bound

$$\|\eta(e, \mathbf{w})\| \le \alpha \|\mathbf{w}\| + \gamma_1 \|e\| + \gamma_2 \|e\|^2 + \gamma_3, \quad \alpha < 1$$

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Adaptive Feedback Linearization

Given a mechanical system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \mathbf{u}$$

and the desired trajectory $q_d = q_d(t)$, introduce the controller

$$egin{array}{lll} oldsymbol{u} &=& \hat{M}(q)oldsymbol{v} + \hat{C}(q,\dot{q})\dot{q} + \hat{g}(q) \ oldsymbol{v} &=& \ddot{q}_d(t) - K_p(q-q_d(t)) - K_d(\dot{q}-\dot{q}_d(t)) + oldsymbol{w} \end{array}$$

What is the difference with the robust design:

- In robust design, the coefficients of $\hat{M}(\cdot)$, $\hat{C}(\cdot)$, $\hat{g}(\cdot)$ were fixed.
- Now, they are variables to tune and w = 0!

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$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \mathbf{u}$$

and the desired trajectory $q_d = q_d(t)$, introduce the controller

$$egin{array}{lll} oldsymbol{u} &=& \hat{M}(q)oldsymbol{v} + \hat{C}(q,\dot{q})\dot{q} + \hat{g}(q) \\ oldsymbol{v} &=& \ddot{q}_d(t) - K_p(q-q_d(t)) - K_d(\dot{q}-\dot{q}_d(t)) + oldsymbol{w} \end{array}$$

What is the difference with the robust design:

- In robust design, the coefficients of $\hat{M}(\cdot)$, $\hat{C}(\cdot)$, $\hat{g}(\cdot)$ were fixed.
- Now, they are variables to tune and w = 0!

Let us find the dynamical equations for updating values of $\hat{M}(\cdot)$, $\hat{C}(\cdot)$, $\hat{g}(\cdot)$ provided that we measure q(t), $\dot{q}(t)$, $\ddot{q}(t)$.

Adaptive Feedback Linearization

Let us rewrite the system

$$M(q)\ddot{q}+C(q,\dot{q})\dot{q}+g(q)=\mathbf{u},\quad \mathbf{u}=\hat{M}(q)\mathbf{v}+\hat{C}(q,\dot{q})\dot{q}+\hat{g}(q)$$

$$\mathbf{v} = \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t))$$

in the form

$$egin{align} \hat{M}(q)\ddot{q} + \left[M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) - \hat{M}(q)\ddot{q}
ight] = \ &= \hat{M}(q)\pmb{v} + \hat{C}(q,\dot{q})\dot{q} + \hat{g}(q) \end{split}$$

Let us rewrite the system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \mathbf{u}, \quad \mathbf{u} = \hat{M}(q)\mathbf{v} + \hat{C}(q,\dot{q})\dot{q} + \hat{g}(q)$$

$$\mathbf{v} = \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t))$$

in the form

$$egin{align} \hat{M}(q)\ddot{q} + \left[M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) - \hat{M}(q)\ddot{q}
ight] = \ &= \hat{M}(q)\pmb{v} + \hat{C}(q,\dot{q})\dot{q} + \hat{g}(q) \end{split}$$

$$\hat{M}(q)\ddot{q} = \hat{M}(q) {\color{red} oldsymbol{v}} + Y(q, \dot{q}, \ddot{q}) \left[heta - \hat{ heta}
ight]$$

where

- θ is the vector of true parameters of the model
- $\hat{\theta}$ is the vector of estimates
- $Y(\cdot)$ is the vector function, which values we can compute measuring q(t), $\dot{q}(t)$, $\ddot{q}(t)$ and $q_d(t)$ and its derivatives

Let us rewrite the system

$$M(q)\ddot{q}+C(q,\dot{q})\dot{q}+g(q)=\mathbf{u},\quad \mathbf{u}=\hat{M}(q)\mathbf{v}+\hat{C}(q,\dot{q})\dot{q}+\hat{g}(q)$$

$$\mathbf{v} = \ddot{q}_d(t) - K_p(q - q_d(t)) - K_d(\dot{q} - \dot{q}_d(t))$$

in the form

$$egin{align} \hat{M}(q)\ddot{q} + \left[M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) - \hat{M}(q)\ddot{q}
ight] = \ &= \hat{M}(q)\pmb{v} + \hat{C}(q,\dot{q})\dot{q} + \hat{g}(q) \end{split}$$

$$\ddot{q} = {\color{red} v} + \hat{M}(q)^{-1} Y(q,\dot{q},\ddot{q}) \left[heta - \hat{ heta}
ight]$$

where

- θ is the vector of true parameters of the model
- $\hat{\theta}$ is the vector of estimates
- $Y(\cdot)$ is the vector function, which values we can compute measuring q(t), $\dot{q}(t)$, $\ddot{q}(t)$ and $q_d(t)$ and its derivatives

Let us rewrite the system

$$M(q)\ddot{q}+C(q,\dot{q})\dot{q}+g(q)=\mathbf{u},\quad \mathbf{u}=\hat{M}(q)\mathbf{v}+\hat{C}(q,\dot{q})\dot{q}+\hat{g}(q)$$

$$oldsymbol{v} = \ddot{q}_d(t) - K_p(q-q_d(t)) - K_d(\dot{q}-\dot{q}_d(t))$$

$$m{v}=\ddot{q}_d(t)-K_p(q-q_d(t))-K_d(\dot{q}-\dot{q}_d(t))$$
 in the form $\hat{M}(q)\ddot{q}+\left[M(q)\ddot{q}+C(q,\dot{q})\dot{q}+g(q)-\hat{M}(q)\ddot{q}
ight]= \ = \hat{M}(q)m{v}+\hat{C}(q,\dot{q})\dot{q}+\hat{g}(q)$

With the controlled the equation

$$\ddot{q} = \hat{m{v}} + \hat{M}(q)^{-1} Y(q,\dot{q},\ddot{q}) \left[\hat{ heta} - heta
ight]$$

becomes

$$(\ddot{q}-\ddot{q}_d)+K_d(\dot{q}-\dot{q}_d)+K_p(q-q_d)=\hat{M}(q)^{-1}Y(q,\dot{q},\ddot{q})\left[\hat{ heta}- heta
ight]$$

The equation

$$(\ddot{q} - \ddot{q}_d) + K_d(\dot{q} - \dot{q}_d) + K_p(q - q_d) = \hat{M}(q)^{-1}Y(q, \dot{q}, \ddot{q}) \left[\hat{\theta} - \theta\right]$$

has variables we can change! How to do this update?

The equation

$$(\ddot{q}-\ddot{q}_d)+K_d(\dot{q}-\dot{q}_d)+K_p(q-q_d)=\left|\hat{M}(q)^{-1}Y(q,\dot{q},\ddot{q})
ight|\left|\hat{ heta}- heta
ight|$$

has variables we can change! How to do this apdate?

Let us rewrite the system into the state space form

$$\frac{d}{dt}e = \begin{bmatrix} 0 & I \\ -K_p & -K_d \end{bmatrix} e + \begin{bmatrix} 0 \\ I \end{bmatrix} \Phi \begin{bmatrix} \hat{\theta} - \theta \end{bmatrix}, \quad e = \begin{bmatrix} (q - q_d(t)) \\ (\dot{q} - \dot{q}_d(t)) \end{bmatrix}$$

The equation

$$(\ddot{q}-\ddot{q}_d)+K_d(\dot{q}-\dot{q}_d)+K_p(q-q_d)=\left|\hat{M}(q)^{-1}Y(q,\dot{q},\ddot{q})
ight|\left[\hat{ heta}- heta
ight]$$

has variables we can change! How to do this update?

Let us rewrite the system into the state space form

$$rac{d}{dt}e = Ae + B\Phi \left[\hat{ heta} - heta
ight]$$

Solve the Lyapunov equation

$$A^{\scriptscriptstyle T}P + PA = -Q < 0, \quad P > 0$$

and consider the Lyapunov function candidate

$$V = e^{\scriptscriptstyle T} P e + rac{1}{2} \left[\hat{ heta} - heta
ight]^{\scriptscriptstyle T} \Gamma \left[\hat{ heta} - heta
ight], \quad \Gamma > 0$$

The equation

$$(\ddot{q} - \ddot{q}_d) + K_d(\dot{q} - \dot{q}_d) + K_p(q - q_d) = \left| \hat{M}(q)^{-1} Y(q, \dot{q}, \ddot{q}) \right| \left| \hat{\theta} - \theta \right|$$

has variables we can change! How to do this update?

Let us rewrite the system into the state space form

$$rac{d}{dt}e = Ae + B\Phi \left[\hat{ heta} - heta
ight]$$

$$\begin{split} \frac{d}{dt}V &= \frac{d}{dt}\left\{e^{\scriptscriptstyle T}Pe\right\} + \frac{d}{dt}\left\{\frac{1}{2}\left[\hat{\theta} - \theta\right]^{\scriptscriptstyle T}\Gamma\left[\hat{\theta} - \theta\right]\right\} \\ &= \left[Ae + B\Phi\left[\hat{\theta} - \theta\right]\right]^{\scriptscriptstyle T}Pe + e^{\scriptscriptstyle T}P\left[Ae + B\Phi\left[\hat{\theta} - \theta\right]\right] + \\ &+ \frac{1}{2}\left[\frac{d}{dt}\hat{\theta} - \mathbf{0}\right]^{\scriptscriptstyle T}\Gamma\left[\hat{\theta} - \theta\right] + \frac{1}{2}\left[\hat{\theta} - \theta\right]^{\scriptscriptstyle T}\Gamma\left[\frac{d}{dt}\hat{\theta} - \mathbf{0}\right] \end{split}$$

The equation

$$(\ddot{q} - \ddot{q}_d) + K_d(\dot{q} - \dot{q}_d) + K_p(q - q_d) = \left| \hat{M}(q)^{-1} Y(q, \dot{q}, \ddot{q}) \right| \left| \hat{\theta} - \theta \right|$$

has variables we can change! How to do this update?

Let us rewrite the system into the state space form

$$rac{d}{dt}e = Ae + B\Phi \left[\hat{ heta} - heta
ight]$$

$$\begin{split} \frac{d}{dt}V &= \left[Ae + B\Phi\left[\hat{\theta} - \theta\right]\right]^{\mathsf{T}}Pe + e^{\mathsf{T}}P\left[Ae + B\Phi\left[\hat{\theta} - \theta\right]\right] + \\ &+ \frac{1}{2}\left[\frac{d}{dt}\hat{\theta} - \mathbf{0}\right]^{\mathsf{T}}\Gamma\left[\hat{\theta} - \theta\right] + \frac{1}{2}\left[\hat{\theta} - \theta\right]^{\mathsf{T}}\Gamma\left[\frac{d}{dt}\hat{\theta} - \mathbf{0}\right] \\ &= e^{\mathsf{T}}\left[A^{\mathsf{T}}P + PA\right]e + \left[\hat{\theta} - \theta\right]^{\mathsf{T}}\left\{\Phi^{\mathsf{T}}B^{\mathsf{T}}Pe + \Gamma\frac{d}{dt}\hat{\theta}\right\} \end{split}$$

The equation

$$(\ddot{q}-\ddot{q}_d)+K_d(\dot{q}-\dot{q}_d)+K_p(q-q_d)=\left[\hat{M}(q)^{-1}Y(q,\dot{q},\ddot{q})
ight]\left[heta-\hat{ heta}
ight]$$

has variables we can change! How to do this update?

Let us rewrite the system into the state space form

$$\frac{d}{dt}e = Ae + B\Phi \left[\theta - \hat{\theta}\right]$$

$$\frac{d}{dt}V = \left[Ae + B\Phi \left[\hat{\theta} - \theta\right]\right]^{T} Pe + e^{T}P \left[Ae + B\Phi \left[\hat{\theta} - \theta\right]\right] + \frac{1}{2} \left[\frac{d}{dt}\hat{\theta} - \mathbf{0}\right]^{T} \Gamma \left[\hat{\theta} - \theta\right] + \frac{1}{2} \left[\hat{\theta} - \theta\right]^{T} \Gamma \left[\frac{d}{dt}\hat{\theta} - \mathbf{0}\right]$$

$$= \underbrace{e^{T} \left[A^{T}P + PA\right] e}_{= -e^{T}Qe} + \left[\hat{\theta} - \theta\right]^{T} \underbrace{\left\{\Phi^{T}B^{T}Pe + \Gamma\frac{d}{dt}\hat{\theta}\right\}}_{= 0} \leq 0$$

With the proposed update law for $\hat{\theta}$ the closed loop system is

$$rac{d}{dt}e = Ae + B\Phi \left[heta - \hat{ heta}
ight], \quad rac{d}{dt}\hat{ heta} = -\Gamma^{-1}\Phi^{\scriptscriptstyle T}B^{\scriptscriptstyle T}Pe$$

With the proposed update law for $\hat{\theta}$ the closed loop system is

$$rac{d}{dt}e = Ae + B\Phi \left[heta - \hat{ heta}
ight], \quad rac{d}{dt}\hat{ heta} = -\Gamma^{-1}\Phi^{\scriptscriptstyle T}B^{\scriptscriptstyle T}Pe$$

The inequality

$$rac{d}{dt}V = rac{d}{dt}\left[e^{\scriptscriptstyle T}Pe + rac{1}{2}\left[\hat{ heta} - heta
ight]^{\scriptscriptstyle T}\Gamma\left[\hat{ heta} - heta
ight]
ight] = -e^{\scriptscriptstyle T}Qe$$

implies that (thanks to Barbalat lemma)

$$e(t) o 0$$
 as $t o +\infty$ and $(\hat{ heta}(t) - heta)$ is bounded

With the proposed update law for $\hat{\theta}$ the closed loop system is

$$rac{d}{dt}e = Ae + B\Phi \left[heta - \hat{ heta}
ight], \quad rac{d}{dt}\hat{ heta} = -\Gamma^{-1}\Phi^{\scriptscriptstyle T}B^{\scriptscriptstyle T}Pe$$

The inequality

$$rac{d}{dt}V = rac{d}{dt}\left[e^{\scriptscriptstyle T}Pe + rac{1}{2}\left[\hat{ heta} - heta
ight]^{\scriptscriptstyle T}\Gamma\left[\hat{ heta} - heta
ight]
ight] = -e^{\scriptscriptstyle T}Qe$$

implies that (thanks to Barbalat lemma)

$$e(t) o 0$$
 as $t o +\infty$ and $(\hat{ heta}(t) - heta)$ is bounded

It is necessary to remember that

- we have to measure \ddot{q} for computing the regressor;
- the matrix $\hat{M}(q)$ at each time moment should be invertible.

Lecture 14: Multivariable Control of Robot Manipulators

- Example: Robust Motion Control
- Adaptive Motion Control Based on Feedback Linearization
- Passivity Based Motion Control

Given a trajectory $q = q_d(t)$, consider the system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \mathbf{u}$$

with the controller

$$\mathbf{u} = M(q)\mathbf{a} + C(q, \dot{q})\mathbf{b} + g(q) - K\mathbf{r}, \quad K = \text{diag}\{k_1, \dots, k_n\}$$

where variables a, b and r are to be chosen

Given a trajectory $q = q_d(t)$, consider the system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \mathbf{u}$$

with the controller

$$\mathbf{u} = M(q)\mathbf{a} + C(q, \dot{q})\mathbf{b} + g(q) - K\mathbf{r}, \quad K = \text{diag}\{k_1, \dots, k_n\}$$

where variables a, b and r are to be chosen

If we substitute the controller, we obtain

$$M(q) \left[\ddot{q} - \mathbf{a} \right] + C(q, \dot{q}) \left[\dot{q} - \mathbf{b} \right] + K\mathbf{r} = 0$$

Given a trajectory $q = q_d(t)$, consider the system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \mathbf{u}$$

with the controller

$$\mathbf{u} = M(q)\mathbf{a} + C(q, \dot{q})\mathbf{b} + g(q) - K\mathbf{r}, \quad K = \text{diag}\{k_1, \dots, k_n\}$$

where variables a, b and r are to be chosen

If we substitute the controller, we obtain

$$M(q) \left[\ddot{q} - \mathbf{a} \right] + C(q, \dot{q}) \left[\dot{q} - \mathbf{b} \right] + K\mathbf{r} = 0$$

Let us impose some relations between signals

$$\frac{d}{dt}\mathbf{r} = \ddot{q} - \mathbf{a}, \qquad \mathbf{r} = \dot{q} - \mathbf{b}$$

Given a trajectory $q = q_d(t)$, consider the system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \mathbf{u}$$

with the controller

$$\mathbf{u} = M(q)\mathbf{a} + C(q, \dot{q})\mathbf{b} + g(q) - K\mathbf{r}, \quad K = \text{diag}\{k_1, \dots, k_n\}$$

where variables a, b and r are to be chosen

If we substitute the controller, we obtain

$$M(q) \left[\ddot{q} - \mathbf{a} \right] + C(q, \dot{q}) \left[\dot{q} - \mathbf{b} \right] + K\mathbf{r} = 0$$

Let us impose some relations between signals and the reference

$$rac{d}{dt}\mathbf{r} = \ddot{q} - \mathbf{a}, \qquad \mathbf{r} = \dot{q} - \mathbf{b}, \qquad \mathbf{r} = (\dot{q} - \dot{q}_d(t)) + \Lambda \left(q - q_d(t)\right)$$

Given a trajectory $q = q_d(t)$, consider the system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \mathbf{u}$$

with the controller

$$\mathbf{u} = M(q)\mathbf{a} + C(q, \dot{q})\mathbf{b} + g(q) - K\mathbf{r}, \quad K = \text{diag}\{k_1, \dots, k_n\}$$

where variables a, b and r are to be chosen

If we substitute the controller, we obtain

$$M(q) \left[\ddot{q} - \mathbf{a} \right] + C(q, \dot{q}) \left[\dot{q} - \mathbf{b} \right] + K\mathbf{r} = 0$$

Let us impose some relations between signals and the reference

$$\frac{d}{dt}\mathbf{r} = \ddot{q} - \mathbf{a}, \qquad \mathbf{r} = \dot{q} - \mathbf{b}, \qquad \mathbf{r} = (\dot{q} - \dot{q}_d(t)) + \Lambda (q - q_d(t))$$

Then the closed loop dynamics are

$$M(q) \frac{d}{dt} \mathbf{r} + C(q, \dot{q}) \mathbf{r} + K \mathbf{r} = 0$$

To analyze the closed loop system

$$M(q)\frac{d}{dt}\mathbf{r} + C(q,\dot{q})\mathbf{r} + K\mathbf{r} = 0$$

consider the Lyapunov function candidate

$$V = \frac{1}{2} \mathbf{r}^{\mathrm{\scriptscriptstyle T}} M(q) \mathbf{r} + (q - q_d(t))^{\mathrm{\scriptscriptstyle T}} K \Lambda (q - q_d(t))$$

To analyze the closed loop system

$$M(q)\frac{d}{dt}\mathbf{r} + C(q,\dot{q})\mathbf{r} + K\mathbf{r} = 0$$

consider the Lyapunov function candidate

$$V = rac{1}{2} oldsymbol{r}^{\scriptscriptstyle T} M(q) oldsymbol{r} + (q - q_d(t))^{\scriptscriptstyle T} K \Lambda \left(q - q_d(t)
ight)$$

$$\frac{d}{dt}V = \frac{d}{dt} \left[\frac{1}{2} \boldsymbol{r}^{T} M(q) \boldsymbol{r} \right] + \frac{d}{dt} \left[(q - q_{d}(t))^{T} K \Lambda \left(q - q_{d}(t) \right) \right]
= \boldsymbol{r}^{T} M(q) \frac{d}{dt} [\boldsymbol{r}] + \frac{1}{2} \boldsymbol{r}^{T} \frac{d}{dt} \left[M(q) \right] \boldsymbol{r} +
+ 2 \left(q - q_{d}(t) \right)^{T} K \Lambda \frac{d}{dt} \left(q - q_{d}(t) \right)$$

To analyze the closed loop system

$$M(q)\frac{d}{dt}\mathbf{r} + C(q,\dot{q})\mathbf{r} + K\mathbf{r} = 0$$

consider the Lyapunov function candidate

$$V = rac{1}{2} oldsymbol{r}^{\scriptscriptstyle T} M(q) oldsymbol{r} + (q - q_d(t))^{\scriptscriptstyle T} K \Lambda \left(q - q_d(t)
ight)$$

$$\frac{d}{dt}V = \frac{d}{dt} \left[\frac{1}{2} \boldsymbol{r}^{T} M(q) \boldsymbol{r} \right] + \frac{d}{dt} \left[(q - q_{d}(t))^{T} K \Lambda \left(q - q_{d}(t) \right) \right]
= \boldsymbol{r}^{T} M(q) \frac{d}{dt} [\boldsymbol{r}] + \frac{1}{2} \boldsymbol{r}^{T} \frac{d}{dt} \left[M(q) \right] \boldsymbol{r} +
+ 2 \left(q - q_{d}(t) \right)^{T} K \Lambda \frac{d}{dt} \left(q - q_{d}(t) \right)
= \boldsymbol{r}^{T} \left[-C(q, \dot{\boldsymbol{q}}) \boldsymbol{r} - K \boldsymbol{r} \right] + \frac{1}{2} \boldsymbol{r}^{T} \frac{d}{dt} \left[M(q) \right] \boldsymbol{r} +
+ 2 \left(q - q_{d}(t) \right)^{T} K \Lambda \frac{d}{dt} \left(q - q_{d}(t) \right)$$

To analyze the closed loop system

$$M(q)\frac{d}{dt}\mathbf{r} + C(q,\dot{q})\mathbf{r} + K\mathbf{r} = 0$$

consider the Lyapunov function candidate

$$V = rac{1}{2} oldsymbol{r}^{\scriptscriptstyle T} M(q) oldsymbol{r} + (q - q_d(t))^{\scriptscriptstyle T} K \Lambda \left(q - q_d(t)
ight)$$

$$\frac{d}{dt}V = \frac{d}{dt} \left[\frac{1}{2} \boldsymbol{r}^{T} M(q) \boldsymbol{r} \right] + \frac{d}{dt} \left[(q - q_{d}(t))^{T} K \Lambda \left(q - q_{d}(t) \right) \right]
= \boldsymbol{r}^{T} M(q) \frac{d}{dt} [\boldsymbol{r}] + \frac{1}{2} \boldsymbol{r}^{T} \frac{d}{dt} \left[M(q) \right] \boldsymbol{r} +
+ 2 \left(q - q_{d}(t) \right)^{T} K \Lambda \frac{d}{dt} \left(q - q_{d}(t) \right)
= -\boldsymbol{r}^{T} K \boldsymbol{r} + \left[\boldsymbol{r}^{T} \left[\frac{1}{2} \frac{d}{dt} M(q) - C(q, \dot{q}) \right] \boldsymbol{r} \right] +
+ 2 \left(q - q_{d}(t) \right)^{T} K \Lambda \frac{d}{dt} \left(q - q_{d}(t) \right)$$

To analyze the closed loop system

$$M(q)\frac{d}{dt}\mathbf{r} + C(q,\dot{q})\mathbf{r} + K\mathbf{r} = 0$$

consider the Lyapunov function candidate

$$V = rac{1}{2} oldsymbol{r}^{\scriptscriptstyle T} M(q) oldsymbol{r} + (q - q_d(t))^{\scriptscriptstyle T} K \Lambda \left(q - q_d(t)
ight)$$

$$\frac{d}{dt}V = \frac{d}{dt} \left[\frac{1}{2} \boldsymbol{r}^{T} M(q) \boldsymbol{r} \right] + \frac{d}{dt} \left[(q - q_{d}(t))^{T} K \Lambda \left(q - q_{d}(t) \right) \right]
= \boldsymbol{r}^{T} M(q) \frac{d}{dt} [\boldsymbol{r}] + \frac{1}{2} \boldsymbol{r}^{T} \frac{d}{dt} \left[M(q) \right] \boldsymbol{r} +
+ 2 \left(q - q_{d}(t) \right)^{T} K \Lambda \frac{d}{dt} \left(q - q_{d}(t) \right)
= -\boldsymbol{r}^{T} K \boldsymbol{r} + \underbrace{\boldsymbol{r}^{T} \left[\frac{1}{2} \frac{d}{dt} M(q) - C(q, \dot{q}) \right] \boldsymbol{r}}_{=0} +
= 0
+ 2 \left(q - q_{d}(t) \right)^{T} K \Lambda \frac{d}{dt} \left(q - q_{d}(t) \right)$$

To analyze the closed loop system

$$M(q)\frac{d}{dt}\mathbf{r} + C(q,\dot{q})\mathbf{r} + K\mathbf{r} = 0$$

consider the Lyapunov function candidate

$$V = rac{1}{2} oldsymbol{r}^{\scriptscriptstyle T} M(q) oldsymbol{r} + (q - q_d(t))^{\scriptscriptstyle T} K \Lambda \left(q - q_d(t)
ight)$$

Then

$$\frac{d}{dt}V = \frac{d}{dt} \left[\frac{1}{2} \boldsymbol{r}^{T} M(q) \boldsymbol{r} \right] + \frac{d}{dt} \left[(q - q_{d}(t))^{T} K \Lambda \left(q - q_{d}(t) \right) \right]
= \boldsymbol{r}^{T} M(q) \frac{d}{dt} [\boldsymbol{r}] + \frac{1}{2} \boldsymbol{r}^{T} \frac{d}{dt} \left[M(q) \right] \boldsymbol{r} +
+ 2 \left(q - q_{d}(t) \right)^{T} K \Lambda \frac{d}{dt} \left(q - q_{d}(t) \right)
= -\boldsymbol{r}^{T} K \boldsymbol{r} + 2 \left(q - q_{d}(t) \right)^{T} K \Lambda \frac{d}{dt} \left(q - q_{d}(t) \right)$$

where

$$\mathbf{r} = (\dot{q} - \dot{q}_d(t)) + \Lambda (q - q_d(t))$$

To analyze the closed loop system

$$M(q)\frac{d}{dt}\mathbf{r} + C(q,\dot{q})\mathbf{r} + K\mathbf{r} = 0$$

consider the Lyapunov function candidate

$$V = rac{1}{2} oldsymbol{r}^{\scriptscriptstyle T} M(q) oldsymbol{r} + (q - q_{oldsymbol{d}}(t))^{\scriptscriptstyle T} \, K \Lambda \left(q - q_{oldsymbol{d}}(t)
ight)$$

Then

$$\frac{d}{dt}V = \frac{d}{dt} \left[\frac{1}{2} \boldsymbol{r}^{T} M(q) \boldsymbol{r} \right] + \frac{d}{dt} \left[(q - q_{d}(t))^{T} K \Lambda \left(q - q_{d}(t) \right) \right]
= \boldsymbol{r}^{T} M(q) \frac{d}{dt} [\boldsymbol{r}] + \frac{1}{2} \boldsymbol{r}^{T} \frac{d}{dt} \left[M(q) \right] \boldsymbol{r} +
+ 2 \left(q - q_{d}(t) \right)^{T} K \Lambda \frac{d}{dt} \left(q - q_{d}(t) \right)
= - \left(\frac{d}{dt} \tilde{q} + \Lambda \tilde{q} \right)^{T} K \left(\frac{d}{dt} \tilde{q} + \Lambda \tilde{q} \right) + 2 \tilde{q}^{T} K \Lambda \frac{d}{dt} \tilde{q}$$

where

$$m{r}=rac{d}{dt} ilde{q}+\Lambda ilde{q}$$

To analyze the closed loop system

$$M(q)\frac{d}{dt}\mathbf{r} + C(q,\dot{q})\mathbf{r} + K\mathbf{r} = 0$$

consider the Lyapunov function candidate

$$V = rac{1}{2} oldsymbol{r}^{\scriptscriptstyle T} M(q) oldsymbol{r} + (q - q_{oldsymbol{d}}(t))^{\scriptscriptstyle T} \, K \Lambda \left(q - q_{oldsymbol{d}}(t)
ight)$$

$$\frac{d}{dt}V = \frac{d}{dt} \left[\frac{1}{2} \boldsymbol{r}^{T} M(q) \boldsymbol{r} \right] + \frac{d}{dt} \left[(q - q_{d}(t))^{T} K \Lambda \left(q - q_{d}(t) \right) \right]
= \boldsymbol{r}^{T} M(q) \frac{d}{dt} [\boldsymbol{r}] + \frac{1}{2} \boldsymbol{r}^{T} \frac{d}{dt} \left[M(q) \right] \boldsymbol{r} +
+ 2 \left(q - q_{d}(t) \right)^{T} K \Lambda \frac{d}{dt} \left(q - q_{d}(t) \right)
= - \left(\frac{d}{dt} \tilde{q} + \Lambda \tilde{q} \right)^{T} K \left(\frac{d}{dt} \tilde{q} + \Lambda \tilde{q} \right) + 2 \tilde{q}^{T} K \Lambda \frac{d}{dt} \tilde{q}
= - \left[\frac{d}{dt} \tilde{q} \right]^{T} K \frac{d}{dt} \tilde{q} - \tilde{q}^{T} \Lambda^{T} K \Lambda \tilde{q} < 0$$

Given a trajectory $q = q_d(t)$, consider the system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \mathbf{u}$$

with the controller

$$\mathbf{u} = \hat{M}(q)\mathbf{a} + \hat{C}(q, \dot{q})\mathbf{b} + \hat{g}(q) - K\mathbf{r}, \quad K = \text{diag}\{k_1, \dots, k_n\}$$

with the variables a, b, r defined by

$$rac{d}{dt}oldsymbol{r}=\ddot{q}-oldsymbol{a}, \qquad oldsymbol{r}=\dot{q}-oldsymbol{b}, \qquad oldsymbol{r}=(\dot{q}-\dot{q}_d(t))+\Lambda\left(q-q_d(t)
ight)$$

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The dynamics can be rewritten as

$$\begin{split} M(q) \left[\ddot{q} - \mathbf{a} \right] + C(q, \dot{q}) \left[\dot{q} - \mathbf{b} \right] + K\mathbf{r} &= \\ &= \left[\hat{M}(q) - M(q) \right] \mathbf{a} + \left[\hat{C}(q, \dot{q}) - C(q, \dot{q}) \right] \mathbf{b} + \left[\hat{g}(p) - g(p) \right] \\ &= \left[Y(q, \dot{q}, \mathbf{a}, \mathbf{b}) \left[\hat{\theta} - \theta \right] \end{split}$$

Given a trajectory $q = q_d(t)$, consider the system

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The dynamics can be rewritten as

$$egin{aligned} M(q)rac{d}{dt}m{r}+C(q,\dot{q})m{r}+Km{r} = \ &= \left[\hat{M}(q)-M(q)
ight]m{a}+\left[\hat{C}(q,\dot{q})-C(q,\dot{q})
ight]m{b}+\left[\hat{g}(p)-g(p)
ight] \ &= \left.Y(q,\dot{q},m{a},m{b})\left[\hat{ heta}- heta
ight] \end{aligned}$$

To find the update law for $\hat{\theta}$ -variable for the system

$$M(q)rac{d}{dt}oldsymbol{r}+C(q,\dot{q})oldsymbol{r}+Koldsymbol{r}=Y(q,\dot{q},oldsymbol{a},oldsymbol{b})\left[\hat{ heta}- heta
ight]$$

we will use the Lyapunov function candidate

$$V = rac{1}{2} oldsymbol{r}^{\scriptscriptstyle T} M(q) oldsymbol{r} + (q - q_d(t))^{\scriptscriptstyle T} K \Lambda \left(q - q_d(t)
ight) + rac{1}{2} \left[\hat{ heta} - heta
ight]^{\scriptscriptstyle T} \Gamma \left[\hat{ heta} - heta
ight]$$

To find the update law for $\hat{\theta}$ -variable for the system

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ight) + rac{1}{2} \left[\hat{ heta} - heta
ight]^{\scriptscriptstyle T} \Gamma \left[\hat{ heta} - heta
ight]$$

Its time-derivative along a solution of the system is

$$egin{array}{ll} rac{d}{dt}V &=& -\dot{ ilde{q}}^{\scriptscriptstyle T}K\dot{ ilde{q}}- ilde{q}^{\scriptscriptstyle T}\Lambda^{\scriptscriptstyle T}K\Lambda ilde{q}+\ &+\left[\hat{ heta}- heta
ight]^{\scriptscriptstyle T}\left\{\Gammarac{d}{dt}\hat{ heta}+Y(q,\dot{q},oldsymbol{a},oldsymbol{b})^{\scriptscriptstyle T}oldsymbol{r}
ight\} \end{array}$$

To find the update law for $\hat{\theta}$ -variable for the system

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ight) + rac{1}{2} \left[\hat{ heta} - heta
ight]^{\scriptscriptstyle T} \Gamma \left[\hat{ heta} - heta
ight]$$

Its time-derivative along a solution of the system is

$$\frac{d}{dt}V = -\dot{\tilde{q}}^{T}K\dot{\tilde{q}} - \tilde{q}^{T}\Lambda^{T}K\Lambda\tilde{q} + \left[\hat{\theta} - \theta\right]^{T}\underbrace{\left\{\Gamma\frac{d}{dt}\hat{\theta} + Y(q,\dot{q},\boldsymbol{a},\boldsymbol{b})^{T}\boldsymbol{r}\right\}}_{=0} \leq 0$$

$$rac{d}{dt}\hat{ heta} = -\Gamma^{-1}Y(q,\dot{q},oldsymbol{a},oldsymbol{b})^{\scriptscriptstyle T}oldsymbol{r}$$