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Feedback Linearization of Nonlinear Systems: Robustness and Adaptive Control.

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Feedback Linearization of Nonlinear Systems: Robustness and Adaptive Control

A Dissertation

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To my wife …

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Dissertation Abstract

Feedback linearization provides an effective means of designing nonlinear control systems. This method permits one to have an exactly equivalent linear system by using a coordinate transformation and state feedback. Once the nonlinear system is transformed to a linear system, one can proceed with well developed control technologies for linear systems. Feedback linearization is based on a model of the real system. If there is mismatch between the model and the real plant, feedback linearization does not yield an exactly linear system. The question of robustness then arises: will a controller based on the model be stable when applied to the real plant ?

We have developed a theoretical approach to analyze robustness of feedback linearization of SISO (Single-Input Single-Output) systems. We have also considered the dimensional reduction of a high dimensional model which is not a standard singularly perturbed system. Specifically we have found sufficient conditions for boundedness and convergence of the system trajectories when feedback linearization based on a nominal mathematical model is applied to an uncertain real plant which may have parametric and structural uncertainties as well as unmodeled dynamics. The developed approach does not require the restrictive conditions which are commonly used in the previously developed methods of robustness analysis.

Furthermore, for parametric uncertainties a nonlinear adaptive control of feedback linearizable processes is proposed. The main feature of the proposed nonlinear adaptive control system is that it is relatively straightforward and simple. For this adaptive control system we have found sufficient conditions for stability of the output regulation and tracking of feedback linearizable systems using the second method of Lyapunov.

Examples of the robustness analysis and the adaptive control for unstable chemical and biochemical reactors are given.

Chapter I

Introduction

For nonlinear systems, controller design technologies are not yet well developed even though there are many excellent procedures for linear systems. In chemical engineering the conventional way of designing controllers for nonlinear systems is linearization at a local point using a truncated Taylor series expansion. This kind of linearization yields good results for some chemical processes. Once the nonlinear system is transformed into the approximated linear system, one can proceed with well-developed control technologies for linear systems.

However, the linearization using the truncated Taylor series expansion holds only around the local point where the Taylor series expansion is taken. If a system is highly nonlinear (such as an exothermic chemical reactor or a pH process) or if the operating range is very wide (for example, during start-up or shut-down), the above linearization method may give poor results.

Recently, studies in nonlinear control systems using a differential geometric approach [Hunt et al., 1983, Isidori, 1989] have provided an effective means of designing nonlinear control systems. The importance of the differential geometric approach can be compared with the Laplace transform, complex variable theory and linear algebra in relation to linear systems [Isidori, 1989]. As one of the differential geometric approaches, feedback linearization has attracted the interest of chemical process control engineers. This method is distinctly different from the conventional linearization using Taylor series. It permits one to have an exactly equivalent linear system by using a coordinate transformation and state feedback. In other words, feedback linearization makes it possible to transform a nonlinear system into a linear system without any kind of approximation. In this case also, once we have an equivalent linear system, we can proceed with well-developed design technologies for linear systems. In chemical engineering this method has been applied to many processes such as an exothermic chemical reaction in CSTR (Continuous Stirred Tank Reactor) and a batch reactor [Hoo and Kantor, 1985, Kravaris and Chung, 1987, Calvet and Arkun, 1988, Alvarez et al., 1989]. These studies have shown that for some nonlinear chemical processes feedback linearization results in much better performance than the conventional method based on the Taylor series approximation.

Conceptually, feedback linearization is based on use of a mathematical model to achieve exact cancellation of the nonlinear part of the real plant. If there exists mismatch between the mathematical model and the real plant, feedback linearization does not yield an exactly linear system. This can be seen from the following feedback linearization of a simple one-dimensional nonlinear system. Feedback linearization for higher-dimensional systems will be reviewed in the next chapter.

Consider the following one-dimensional nonlinear system

$$\frac{dx}{dt} = f(x) + g(x)\,u$$

where f(x) and g(x) are nonlinear differentiable functions.

Suppose that $g(x) \neq 0$. Then with the following state feedback

$$u = \frac{1}{g(x)}(-\,f(x) + v)$$

where v is a new input

we have the following equivalent linear system

$$\frac{dx}{dt} = v.$$

That is, the above state feedback cancels the nonlinear terms exactly.

Now, suppose that modelling of the the real plant is uncertain. For the above uncertain real plant, assume we have the following nominal mathematical model

$$\frac{dx}{dt} = \hat{f}(x) + \hat{g}(x)\, u$$

Since the linearizing state feedback is calculated from the given mathematical model, we have

$$u = \frac{1}{\hat{g}(x)}\left(-\hat{f}(x) + v\right)$$

If we apply the above state feedback based on the nominal mathematical model of the real plant we have the following perturbed system

$$\frac{dx}{dt} = v + \left(f(x) - \frac{g(x)}{\hat{g}(x)}\hat{f}(x)\right) + \left(\frac{g(x)}{\hat{g}(x)} - 1\right) v$$

Therefore we need a perfect mathematical model for the real plant in order for cancellation of the nonlinear terms to occur. However, in practice, it is difficult or impossible to model a real plant perfectly. Therefore it is very important to investigate whether feedback linearization based on an imperfect mathematical model yields acceptable performance. In control terminology, this property is called robustness. In general a control system is designed to guarantee stability under the assumption that there is no model-plant mismatch. Then robustness is analyzed, since in almost all cases a real plant has some uncertainties.

In chemical engineering, most of the studies on feedback linearization are concerned with direct applications without any systematic analysis of robustness. Sometimes these researches examine robustness through numerical simulation of each possible case of uncertainty. For example, suppose that the real plant cannot be modeled exactly because one of the parameters of the model is uncertain. Only the parametric error bound is given. In this case, these studies examine the system response by numerically solving differential equations with each possible parametric value within the given parametric error bound. This method is very tedious and also it is very difficult to make the results general. Therefore it is desirable to have a systematic general way to analyze robustness of feedback linearization.

Generally we are confronted with three types of model-plant mismatch; (1) parametric uncertainties, (2) structural uncertainties, and (3) unmodeled dynamics. Structural uncertainties denote the uncertainties which cannot be represented as parametric error but do not affect the dimensionality of the system. For example, disturbances, measurement errors or uncertainties in the form of the kinetic model of a chemical reaction system can be considered as structural uncertainties.

With only parametric and structural uncertainties we can at least model a real plant in the same dimensional space; that is, parametric and structural uncertainties do not give a model with different dimensionality than the real plant. Unmodeled dynamics may be introduced when a high-dimensional model is simplified to a reduced-dimensional model. For example, when control valve dynamics are ignored or when a multi-component chemical reaction system is simplified using the steady-state approximation, unmodeled dynamics are introduced.

In this dissertation we have developed a systematic approach to analyze robustness of feedback linearization of single-input single-output (SISO) systems for parametric and structural uncertainties as well as for unmodeled dynamics. Furthermore, for parametric uncertainties, an adaptive control system has been proposed for feedback linearizable systems. The dissertation is organized as follows.

In Chapter 2 we have briefly reviewed feedback linearization.

In Chapter 3 we have developed a theoretical approach to analyze robustness of feedback linearization for parametric and structural uncertainties. Specifically we have found sufficient conditions for boundedness and convergence of the system trajectories when the feedback linearization based on the nominal mathematical model is applied to a real plant which has parametric and structural uncertainties. The main feature of the approach developed in this dissertation is that it can be applied to a more general class of model-plant mismatch than previous studies.

As an example we have chosen a first order exothermic reaction in a CSTR. We have applied the developed approach to find sufficient conditions for boundedness and convergence of concentration and temperature for a parametric uncertainty in the reaction rate constant, for an unmeasured disturbance in feed concentration, and for a measurement error in concentration in the reactor.

In Chapter 4 we have investigated robustness of feedback linearization for parametric and structural uncertainties as well as unmodeled dynamics. We have also considered the dimensional reduction of a high-dimensional model which is not a standard singularly perturbed system [Kokotovic et al., 1986]. With uncertainties including unmodeled dynamics, the robustness analysis of the feedback linearization becomes more difficult. Very little research has been done on this subject. Our main objective in this chapter is to find sufficient conditions for boundedness and convergence of the system trajectories when the feedback linearization based on the reduced dimensional model is applied to the uncertain high-dimensional real plant.

In order to do this, we expand the theoretical approach developed in Chapter 3. In this case, parametric and structural uncertainties do not necessarily require the restrictive conditions which are commonly assumed in the previously developed methods of robustness analysis. However, we consider only the case that unmodeled dynamics has a special structure.

As an application, a multi-component exothermic chemical reaction in a CSTR has been chosen. It is assumed that there exists an uncertainty in the reaction rate constant. In this system the feedback linearization of the original high-dimensional model is very difficult, but can be easily done for the reduced-dimensional model. The coordinate transformation and linearizing state feedback based on the reduced-dimensional model have been applied to the real plant and boundedness and convergence of system trajectories have been investigated.

In Chapter 5 we have proposed an adaptive controller for feedback linearizable systems with parametric uncertainties. The main feature of the proposed adaptive control system is that it is relatively straightforward and simple. For this adaptive control system sufficient conditions for asymptotic stability have been found based on the Lyapunov stability theorem.

This adaptive control system has been applied to the output regulation and tracking of unstable CSTR and CSTBR (Continuous Stirred Tank Biological Reactor). By these examples it has been demonstrated that the proposed adaptive approach can be an efficient control method for nonlinear chemical processes with parametric uncertainties.

Chapter II

Feedback Linearization

2.1. Introduction

In the past decade one of the most significant developments in control theory for nonlinear systems is the differential geometric approach. With this approach many nonlinear control problems such as decoupling, output regulation and tracking, and shaping of the input-output response have been solved successfully. This approach is the one of the most active research fields in nonlinear system control.

In the differential geometric approach, feedback linearization has been used in designing controllers for SISO (Single-Input Single-Output) highly nonlinear systems such as unstable chemical and biochemical reactors [Hoo and Kantor, 1985, 1986, Kravaris and Chung, 1987, Alvarez-Gallegos, 1988, Calvet and Arkun, 1988, Alvarez et al., 1989]. Feedback linearization of SISO nonlinear systems is conveniently divided into two categories: input-output linearization and exact state-space linearization.

Input-output linearization in SISO systems provides a basis for more sophisticated and complicated control schemes for a nonlinear system. This approach transforms a certain class of nonlinear systems into a linear system in the input-output sense by a proper coordinate change and a linearizing state feedback. A very clear explanation of this approach can be found in Isidori's textbook (1989). In chemical engineering, Kravaris and Chung (1987), Kravaris (1988) and Daoutidis and Kravaris (1989) have applied this method to design control systems for nonlinear chemical reactors.

The exact state-space linearization for SISO systems has attracted great interest among chemical engineers. The exact state-space linearization problem involves finding a proper coordinate transformation and state feedback so that a given nonlinear system is transformed into a linear system in state space. Even though input-output linearization can be done relatively easily, exact state-space linearization may require solving simultaneous first-order partial differential equations. However this does not mean that input-output linearization is preferable to exact state-space linearization. As indicated eariler, each approach has its own specific objective, that is, linearity in the input-output sense or in the whole state space.

Hunt, Su and Mayer (1983) have found necessary and sufficient conditions for exact state-space linearization by combining the results of several versions of the global inverse function theorem [Boothby, 1975]. Zak and Maccarley (1986) have reviewed the different transformation algorithms and Hunt, Luksic and Su (1986) have studied exact state-space linearization with an output. For the linearization of discrete-time systems Lee et al. (1986, 1987, 1988) can be cited.

In chemical engineering Hoo and Kantor (1985, 1986) have applied the exact state-space linearization method to the control of nonlinear chemical and biochemical reactors. Later Alvarez-Gallegos (1988), Calvet and Arkun (1988), Alvarez, Alvarez and Gonzalez (1989), Niranjan and San (1988) have also applied this method to the control of chemical systems.

For MIMO (Multi-Input Multi-Output) systems, until now, decoupling control theory has been used in the chemical engineering field [Kravaris and Soroush, 1990, Castro et al., 1990]. The differential geometric approach to input-output decoupling problems of a linear system has been studied by Wonham and Morse (1970). For a nonlinear system Porter (1970) and Freund (1973) have studied this problem. Extension to a nonlinear system with the geometric concept of Wonham and Morse has been done by many authors [Isidori et al., 1981, Ha and Gilbert, 1986, Li and Feng,1987]. In particular Ha and Gilbert (1986) have studied the class of decoupling control law. In other words, they have investigated what types of state feedback can be applied. This concept has been used by Kravaris and Soroush (1990) in the control of a semibatch copolymerization reactor.

In this dissertation we consider robustness of feedback linearization only for SISO systems. Therefore even though decoupling theory may be very useful in the control of chemical processes it is beyond the scope of this dissertation.

In the following sections input-output linearization and exact state-space linearization are reviewed briefly, under the assumption that there is no model-plant mismatch. In this review we mainly follow the theoretical development of Isidori (1989) and Hunt, Su and Mayer (1983).

2.2. Input-Output Linearization

Consider the following n-dimensional SISO nonlinear system

$$\frac{dx}{dt} = f(x) + g(x)\,u \tag{2.2.1}$$
$$y = h(x)$$

$$\text{where } x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n, \quad y \in \mathbb{R}, \quad u \in \mathbb{R}$$

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix}, \qquad g(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_n(x) \end{bmatrix}$$

and $\mathbb{R}$ = set of real numbers

$\mathbb{R}^n$ = n-dimensional Euclidian space

Suppose that the vector functions f(x) and g(x) and the scalar function h(x) are infinitely differentiable. For the above system (2.2.1) we will define relative degree, which plays an important rôle in feedback linearization. Before doing this, let us define Lie derivatives.

The Lie derivative of h along the vector field f, denoted by $L_f h(x)$, is defined by

$$L_f h(x) = \frac{\partial h(x)}{\partial x} f(x),$$

where the partial derivative of scalar function h(x) with respect to the n-dimensional vector x, i.e., $\frac{\partial h(x)}{\partial x}$ is defined by

$$\frac{\partial h(x)}{\partial x} = \begin{bmatrix} \frac{\partial h(x)}{\partial x_1} & \frac{\partial h(x)}{\partial x_2} & \cdots & \frac{\partial h(x)}{\partial x_n} \end{bmatrix}$$

Since

$$f(x) = [f_1(x) \quad f_2(x) \quad \cdots \quad f_n(x)]^T$$

where $[\cdots]^T$ = transpose of the vector $[\cdots]$

we have

$$L_f h(x) = \frac{\partial h(x)}{\partial x} f(x) = \begin{bmatrix} \frac{\partial h(x)}{\partial x_1} & \frac{\partial h(x)}{\partial x_2} & \cdots & \frac{\partial h(x)}{\partial x_n} \end{bmatrix} \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix}$$

$$= \frac{\partial h(x)}{\partial x_1} f_1(x) + \frac{\partial h(x)}{\partial x_2} f_2(x) + \cdots + \frac{\partial h(x)}{\partial x_n} f_n(x).$$

Similarly, the Lie derivative of $L_f h(x)$ along f, denoted by $L_f^2 h(x)$, is defined by

$$L_f^2 h(x) = \left(\frac{\partial}{\partial x} (L_f h(x)) \right) f(x).$$

Of course, repeated use of this operation is possible. Thus for any nonnegative integer k

$$L_f^k h(x) = \left(\frac{\partial}{\partial x} \left(L_f^{k-1} h(x) \right) \right) f(x)$$

with $L_f^0 h(x) = h(x)$ (i.e., when $k = 0$).

In the same way, Lie derivative $L_g L_f h(x)$ is defined as follows

$$L_g L_f h(x) = \left(\frac{\partial}{\partial x} (L_f h(x)) \right) g(x).$$

Now the relative degree of the above n-dimensional nonlinear system (2.2.1) is defined in the following way.

Definition 2.1: Let U be a neighborhood of any point, x^0. System (2.2.1) is said to have relative degree r ($= \rho + 1$) at a point x^0 if

(i) $L_g L_f^{\rho-i} h(x) = 0$, for every $x \in U$ and for every $i = 1, 2, \cdots, \rho-1$

(ii) $L_g L_f^{\rho} h(x^0) \neq 0$

(if $L_g h(x^0) \neq 0$ then $r = 1$) ///

The following example shows how feedback linearization can be done when the relative degree is the same as the dimensionality of a nonlinear system.

Example 2.1: Consider the following 2-dimensional SISO nonlinear system

$$\dot{x} = f(x) + g(x)\,u \tag{2.2.2}$$
$$y = h(x)$$

where $\dot{x}$ = time derivative of x

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2, \quad u \in \mathbb{R} \text{ and } y \in \mathbb{R}$$

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}, \quad g(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix}$$

and the vector functions f(x) and g(x) and the scalar function h(x) are infinitely differentiable.

Suppose that the above system (2.2.2) has relative degree 2, the same as the dimensionality of the system. This means that by the previous definition 2.1

$$\text{(i) } L_g h(x) = \frac{\partial h(x)}{\partial x}\, g(x) = 0$$
$$\text{(ii) } L_g L_f h(x) = \left(\frac{\partial}{\partial x}(L_f h(x))\right) g(x) \neq 0$$

Taking the derivative of the output h(x) with respect to time, t

$$\frac{dh(x)}{dt} = \frac{\partial h(x)}{\partial x}\frac{dx}{dt} \quad \text{(by the chain rule)} \tag{2.2.3}$$
$$= \frac{\partial h(x)}{\partial x}\,[f(x) + g(x)\,u]$$
$$= \frac{\partial h(x)}{\partial x}\, f(x) + \frac{\partial h(x)}{\partial x}\, g(x)u$$

Since the system has relative degree 2 we know that

$$L_g h(x) = \frac{\partial h(x)}{\partial x} g(x) = 0$$

Define the new coordinate system:

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = T(x) = \begin{bmatrix} T_1(x) \\ T_2(x) \end{bmatrix} = \begin{bmatrix} h(x) \\ L_f h(x) \end{bmatrix}$$

Then equation (2.2.3) can be written

$$\frac{dz_1}{dt} = z_2$$

Similarly taking the derivative of z_2 with respect to time we have

$$\frac{dz_2}{dt} = \frac{\partial}{\partial x}\left(\frac{\partial h(x)}{\partial x} f(x)\right)\frac{dx}{dt} \tag{2.2.4}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial h(x)}{\partial x} f(x)\right)[f(x) + g(x)\, u]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial h(x)}{\partial x} f(x)\right) f(x) + \frac{\partial}{\partial x}\left(\frac{\partial h(x)}{\partial x} f(x)\right) g(x)u$$

Define $$A^*(x) = \frac{\partial}{\partial x}\left(\frac{\partial h(x)}{\partial x} f(x)\right) f(x)$$

$$D^*(x) = \frac{\partial}{\partial x}\left(\frac{\partial h(x)}{\partial x} f(x)\right) g(x)$$

Then we have

$$\frac{dz_2}{dt} = A^*(x) + D^*(x)u \tag{2.2.5}$$

And also since the relative degree of the system is 2

$$D^*(x) \neq 0.$$

Therefore the following linearizing state feedback can be defined

$$u = \alpha(x) + \beta(x)\, v \tag{2.2.6}$$

where v = new input

$$\alpha(x) = \frac{-1}{D^*(x)} A^*(x)$$

$$\beta(x) = \frac{1}{D^*(x)}$$

Applying the above state feedback (2.2.6) to equation (2.2.5) we have

$$\frac{dz_2}{dt} = v$$

That is, the linearizing state feedback (2.2.6) exactly cancels the nonlinear terms in equation (2.2.5). Therefore we have the following linear system

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

In this way we can get an equivalent linear system using the coordinate transformation and state feedback. The following diagram shows feedback linearization, i.e., the closed loop system, inside the box, is linear.

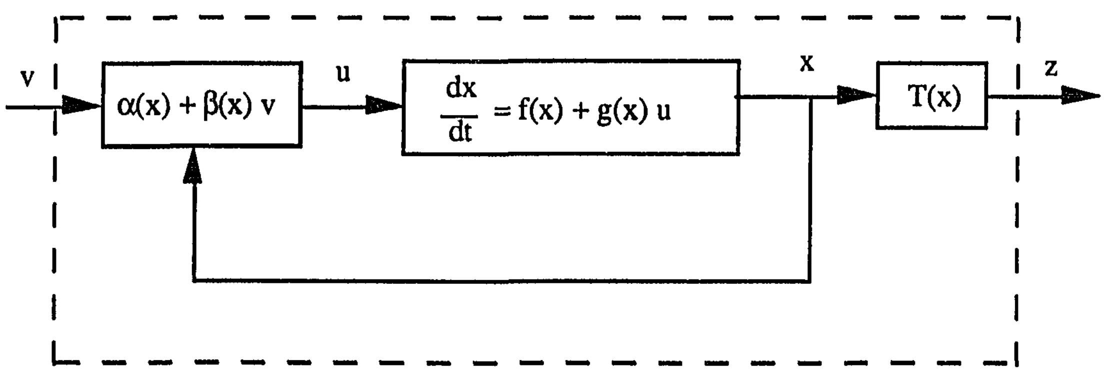

Fig. 2.1 Feedback linearization

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In this example we can see that if the relative degree is exactly equal to the dimension of the state space, feedback linearization can be done very easily. Now we consider a more general case. The following formal statement illustrates a way in which the coordinate transformation can be completed in case the relative degree r is less than or equal to the dimension of the state space.

Proposition 2.1 : [Isidori, 1989, proposition 1.4, p 149]

Suppose system (2.2.1) has relative degree r, which is less than or equal to the dimensionality of the system, n, at x^0. Let

$$T_i(x) = L_f^{i-1} h(x), \quad i = 1, 2, \cdots, r$$

If $r < n$ then it is always possible to find $(n - r)$ additional functions $T_j(x)$, $j = r+1, r+2, \ldots, n$ such that the mapping

$$T(x) = [\, T_1(x) \;\; T_2(x) \cdots T_n(x) \,]^T$$

has a Jacobian matrix which is nonsingular at x^0. Moreover it is always possible to choose $T_j(x)$, $j = r+1, \ldots, n$ in such a way that

$$L_g T_j(x) = 0, \quad j = r+1, \cdots, n \text{ and for all x around } x^0 \qquad ///$$

Remark 2.1: From the above proposition 2.1 we can see that by the inverse function theorem [see Appendix VI, or Boothby, 1970, p 42] the mapping $T : U \rightarrow \mathbb{R}^n$ is a diffeomorphism, that is, it is invertible, i.e. there exists a function $T^{-1}(x)$ such that

$$T^{-1}(T(x)) = x$$

and $T(x)$ and $T^{-1}(x)$ are both smooth mappings, i.e. have continuous partial derivatives of any order. Therefore it is a local coordinate transformation. ///

The following illustration shows how feedback linearization can be done for a more general class of SISO nonlinear system.

Suppose that the above nonlinear system (2.2.1) has relative degree $r \le n$. From proposition 2.1 the coordinate transformations are defined by

$$
\begin{aligned}
z_1 &= h(x) \\
z_2 &= L_f h(x) \\
&\vdots \\
z_r &= L_f^{r-1} h(x)
\end{aligned}
\tag{2.2.7}
$$

The derivative of z_1 with respect to time, t, is

$$
\begin{aligned}
\dot{z}_1 &= \frac{\partial h(x)}{\partial x}\frac{dx}{dt} = \frac{\partial h(x)}{\partial x}(f(x) + g(x)\,u) \\
&= L_f h(x) + L_g h(x)\,u
\end{aligned}
$$

If $r > 1$ then $L_g h(x) = 0$. Therefore

$$\dot{z}_1 = z_2$$

Similarly the derivative of z_2 with respect to time is

$$
\begin{aligned}
\dot{z}_2 &= \left(\frac{\partial}{\partial x}\left(L_f h(x)\right)\right)\frac{dx}{dt} \\
&= L_f^2 h(x) + L_g L_f h(x)\,u
\end{aligned}
$$

Again if $r > 2$ then $L_g L_f h(x) = 0$. Therefore we have

$$\dot{z}_2 = z_3$$

Continuing in this way, we get

$$\dot{z}_{r-1} = z_r$$

$$\dot{z}_r = L_f^r h(x) + L_g L_f^{r-1} h(x)\, u \qquad (2.2.8)$$

Choose the following linearizing state feedback

$$u = \alpha(x) + \beta(x)v, \qquad (2.2.9)$$

where $\alpha(x) = -D^*(x)^{-1} A^*(x)$

$\beta(x) = D^*(x)^{-1}$

v = new input

$D^*(x) = L_g L_f^{r-1} h(x)$

$A^*(x) = L_f^r h(x)$

It is noted that since the system has relative degree r, $D^*(x)$ is not zero.

It can easily be seen that the above state feedback cancels the nonlinear terms in equation (2.2.8) so that we have

$$\dot{z}_r = v$$

As far as the other new coordinates are concerned, we cannot expect any special structure for the corresponding equations, if nothing else has been specified. However if the new coordinates $z_j = T_j(x)$, $j = r+1, r+2, ..., n$ have been chosen in such a way that

$$L_g T_j(x) = 0, \quad j = r+1, r+2, ..., n$$

then we can see easily that

$$\dot{z}_j = L_f T_j(x), \quad j = r+1, r+2, ..., n \qquad (2.2.10)$$

Setting $q_j(z) = L_f T_j\left(T^{-1}(z)\right)$

where $T = \left[T_1 \;\; T_2 \;\cdots\; T_n\right]^T$

$T^{-1}(z)$ = inverse of $T(x)$, that is, $T^{-1}(T(x)) = x$ (remember that T is diffeomorphism).

equation (2.2.10) can be written

$$\dot{z}_j = q_j(z), \quad j = r+1, r+2, \ldots, n$$

Define

$$z^1 = [\, T_1, \cdots, T_r]^T$$

$$z^2 = [\, T_{r+1}, \cdots, T_n]^T$$

$$z = [\, T_1 \cdots T_r \cdots T_n \,]^T$$

Then system (2.2.1) is decomposed into an r-dimensional linear subsystem associated with an (n-r)-dimensional system, possibly nonlinear:

$$\dot{z}^1 = Az^1 + bv \tag{2.2.11}$$

$$\dot{z}^2 = Q(z)$$

$$y = Cz^1$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ & \cdot & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

$$b = [\, 0 \;\; 0 \;\; 0 \;\; \ldots\ldots 0 \;\; 1]^T$$

$$C = [\, 1 \;\; 0 \;\; 0 \;\; \ldots\ldots 0 \;\; 0]$$

$$Q(z) = [q_{r+1}(z) \;\; q_{r+2}(z) \;\; \cdots \;\; q_n(z)]^T$$

The above equation (2.2.11) shows that the output of the system, $y = h(x)$, is related to the new input, v, linearly, where the dynamics $\dot{z}^2 = Q(z)$ does not affect the output. Thus the input-output behavior is linear.

Usually the new input v has the following form:

$$v = Kz^1 \tag{2.2.12}$$

where $K = [b_1, \dots, b_r]$

With this new input the above system (2.2.11) can be written

$$\dot{z}^1 = (A+bK)z^1 \qquad (2.2.13)$$

$$\dot{z}^2 = Q(z)$$

$$y = Cz^1$$

The constants b_i, $i = 1, 2, \dots, r$ are assigned such that all eigenvalues of the matrix (A+bK) have negative real parts so that the system (2.2.13) is asymptotically stable when zero dynamics [Isidori, 1989, p 172], i.e. $\dot{z}^2 = Q(z)\,|_{z^1 = 0}$, is asymptotically stable. This asymptotic stability can be proved by using the central manifold theorem [Isodori, 1989, p 434].

With the new input v of (2.2.12) the input-output relationship of system (2.2.11) can be derived easily

$$\frac{d^r y}{dt^r} = b_1 y + b_2 \frac{dy}{dt} + \dots + b_r \frac{d^{r-1} y}{dt^{r-1}} \qquad (2.2.14)$$

Sometimes the new input v can have another additive term:

$$v = Kz^1 + k\,w \qquad (2.2.15)$$

where w = additional new input

and k = constant.

With this input (2.2.15) and the proper value of vector K and constant k the input-output relationship of system (2.2.11) can be written

$$w = \beta_0 y + \beta_1 \frac{dy}{dt} + \dots + \beta_r \frac{d^r y}{dt^r}$$

where β_i, $i = 0, 1, \cdots, r$ are constants.

This is the same as Kravaris and Chung's input/output linearization [Kravaris and Chung, 1987].

Example 2.2: [Kravaris, 1988]

Consider a CSTR in which an isothermal, liquid-phase, multicomponent chemical reaction is being carried out. The modeling equation can be represented as follows. The detailed procedure can be found in the above mentioned reference.

$$\frac{dx_1}{dt} = -(1 + Da_1)\, x_1 + (2Da_2x_{2D})\, x_2 + Da_2\, x_2^2$$

$$\frac{dx_2}{dt} = Da_1\, x_1 - (1 + 2Da_2x_{2D} + 2Da_3x_{2D})\, x_2 - (Da_2 + Da_3)\, x_2^2 + u$$

$$\frac{dx_3}{dt} = Da_3\, x_2^2 + 2Da_3x_{2D}\, x_2 - x_3$$

$$y = x_3$$

Where Da_1, Da_2, Da_3 and x_{2D} are constants.

Define

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix} = \begin{bmatrix} -(1 + Da_1)\, x_1 + (2Da_2x_{2D})\, x_2 + Da_2\, x_2^2 \\ Da_1\, x_1 - (1 + 2Da_2x_{2D} + 2Da_3x_{2D})\, x_2 - (Da_2 + Da_3)\, x_2^2 \\ Da_3\, x_2^2 + 2Da_3x_{2D}\, x_2 - x_3 \end{bmatrix}$$

$$g(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$h(x) = x_3$$

Since

$$L_g h(x) = 0$$

$$L_g L_f h(x) = \frac{\partial f_3(x)}{\partial x_2} \quad (\text{ since } L_f h(x) = f_3(x) \text{ and } g_1(x) = g_3(x) = 0\ )$$

$$= 2Da_3\, x_2 + 2Da_3 x_{2D} \neq 0$$

The above system has relative degree 2. Therefore the relative degree is less than the dimensionality of the system.

As indicated in proposition 2.1, define the coordinate transformation

$$z_1 = h(x) = x_3$$

$$z_2 = L_f h(x) = f_3(x)$$

The time derivative of z_1 is obviously

$$\frac{dz_1}{dt} = z_2$$

Similarly the time derivative of z_2 is

$$\frac{dz_2}{dt} = \frac{\partial}{\partial x}\big(L_f h(x)\big)\frac{dx}{dt}$$

$$= \frac{\partial}{\partial x}\big(L_f h(x)\big)\left[f(x) + g(x)\, u\right]$$

$$= \frac{\partial f_3}{\partial x_2} f_2(x) + \frac{\partial f_3}{\partial x_3} f_3(x) + \frac{\partial f_3}{\partial x_2} u \quad (\text{ since } \frac{\partial f_3}{\partial x_1} = 0,\ g_1(x) = g_3(x) = 0)$$

$$= \big(2Da_3\, x_2 + 2Da_3 x_{2D}\big)$$

$$\cdot \left\{ Da_1\, x_1 - (1 + 2Da_2 x_{2D} + 2Da_3 x_{2D})\, x_2 - (Da_2 + Da_3)\, x_2^2 \right\}$$

$$- \left\{ Da_3\, x_2^2 + 2Da_3 x_{2D}\, x_2 - x_3 \right\} + \big(2Da_3\, x_2 + 2Da_3 x_{2D}\big)\, u$$

Define

$$A^*(x) = \left(2Da_3\, x_2 + 2Da_3 x_{2D}\right)$$
$$\cdot \left\{ Da_1\, x_1 - (1 + 2Da_2 x_{2D} + 2Da_3 x_{2D})\, x_2 - (Da_2 + Da_3)\, x_2^2 \right\}$$
$$- \left\{ Da_3\, x_2^2 + 2Da_3 x_{2D}\, x_2 - x_3 \right\}$$

$$D^*(x) = \left(2Da_3\, x_2 + 2Da_3 x_{2D}\right)$$

Then the following linearizing state feedback

$$u = \alpha(x) + \beta(x)v$$

$$\text{where } \alpha(x) = \frac{-A^*(x)}{D^*(x)}$$

$$\beta(x) = \frac{1}{D^*(x)}$$

exactly cancels the nonlinear terms so that we have

$$\frac{dz_2}{dt} = v$$

Now, we seek another coordinate transformation $T_3(x)$ such that

$$L_g T_3(x) = \frac{\partial T_3(x)}{\partial x}\, g(x) = 0$$

It is easily seen that $T_3(x) = x_1$ satisfies this condition. And also the mapping $T(x) = \left[T_1(x) \quad T_2(x) \quad T_3(x)\right]^T$, where $T_1(x) = x_3$, $T_2(x) = f_3(x)$, has a nonsingular Jacobian matrix. Therefore it is a local coordinate transformation.

The time derivative of $z_3 = T_3(x) = x_1$ is

$$\frac{dz_3}{dt} = -\,(1+Da_1)z_3 + (2Da_2 x_{2D})\, x_2 + Da_2\, x_2^2$$

Since $(2Da_2 x_{2D})\, x_2 + Da_2\, x_2^2 = f_3(x) + x_3 = z_2 + z_1$

we have finally

$$\frac{dz_3}{dt} = z_1 + z_2 - (1+Da_1)z_3$$

Therefore finally we have the following equivalent system

$$\frac{dz_1}{dt} = z_2$$
$$\frac{dz_2}{dt} = v$$
$$\frac{dz_3}{dt} = z_1 + z_2 - (1+Da_1)z_3$$
$$y = z_1$$

In this system, fortunately the dynamics of state z_3 becomes linear. Actually this is a very special case. In general the equation for $\frac{dz_3}{dt}$ is nonlinear. In any case state z_3 does not affect the output. Therefore the above system is linear in the input-output sense.

2.3. Exact State-Space Linearization

The previous approach transforms a given n-dimensional nonlinear system into an r-dimensional linear subsystem, i.e. linear in the input-output sense. Exact state-space linearization has the objective of transforming an n-dimensional nonlinear system into a linear and controllable system in state space.

Consider again the following n-dimensional nonlinear system

$$\dot{x} = f(x) + g(x)\, u \qquad (2.3.1)$$
$$y = h(x)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y \in \mathbb{R}$ and the vector functions f(x) and g(x) and the scalar function h(x) are infinitely differentiable.

Suppose that this system has relative degree n. In this case as explained in the previous section, with the following coordinate transformation and state feedback

Coordinate transformation:

$$T_i(x) = L_f^{i-1}h(x), \quad i = 1, 2, \cdots, n \tag{2.3.2}$$

State feedback:

$$u = \alpha(x) + \beta(x)v, \tag{2.3.3}$$

where $\alpha(x) = -D^*(x)^{-1} A^*(x)$

$\beta(x) = D^*(x)^{-1}$

$v =$ new input

$D^*(x) = L_g L_f^{n-1} h(x)$

$A^*(x) = L_f^n h(x)$

we have the following equivalent linear system

$$\dot{z} = Az + bv \tag{2.3.4}$$

$$y = Cz$$

where $z = [\, T_1\ T_2 \cdots T_n \,]^T$

and the (n x n) matrix A and the vectors b and C are defined in (2.2.11)

Therefore if the relative degree is exactly equal to the dimension of the state space then we can get an equivalent linear system in state space.

As we have seen, the relative degree of a nonlinear system is dependent on the output, $y = h(x)$. In process control problems output y is generally given. With the given output we can find the relative degree of a nonlinear system and determine whether the system can be transformed into equation (2.3.4), i.e. a linear and controllable system.

However, even though the relative degree is strictly less than the dimension of the state space with a given output, as Kantor (1986) indicated, we can in some cases redefine the output so that the the relative degree is exactly equal to the dimension of the state space. In other words even though with a given output the system has relative degree less than the dimension of the state space, with a redefined output the system can be transformed into a linear and controllable system in state space. In this case we may have the following question: under what conditions we can redefine an output so that an n-dimensional system has relative degree n ? In the differential geometric approach this problem is called a State Space Exact Linearization Problem. For this problem many references [Hunt, 1983, 1986, Hoo, 1985, 1986, Kravaris, 1987, Isidori, 1989] can be cited.

Now, consider the exact state-space linearization of a CSTR, which will be used frequently in the rest of this work.

Example 2.3: Consider the following first order exothermic reaction in a CSTR. Basically this is the same problem as the previous Example 2.1. The mathematical model in dimensionless form is

$$\frac{dx_1}{d\tau} = (-x_1 + \alpha_c) - (x_1 - \alpha_c + 1)\exp\left[D - \frac{\upsilon^2}{x_2 - \alpha_T + \upsilon}\right] \tag{2.3.5}$$

$$\frac{dx_2}{d\tau} = (-x_2 + \alpha_T) - B(x_1 - \alpha_c + 1)\exp\left[D - \frac{\upsilon^2}{x_2 - \alpha_T + \upsilon}\right] - \gamma\,(x_2 - \alpha_T + \alpha_W) + \gamma\, u$$

$$y = x_1$$

The detailed procedure to obtain the above mathematical model will be shown in Chapter 3 in this dissertation.

Define

$$f(x) = [f_1(x) \quad f_2(x)]^T$$

$$f_1(x) = (-x_1 + \alpha_c) - (x_1 - \alpha_c + 1)\exp\left[D - \frac{\upsilon^2}{x_2 - \alpha_T + \upsilon}\right]$$

$$f_2(x) = (-x_2 + \alpha_T) - B(x_1 - \alpha_c + 1)\exp\left[D - \frac{\upsilon^2}{x_2 - \alpha_T + \upsilon}\right] - \gamma(x_2 - \alpha_T + \alpha_W)$$

$$g(x) = [g_1(x) \quad g_2(x)]^T$$

$$g_1(x) = 0$$

$$g_2(x) = \gamma$$

$$h(x) = x_1$$

Then the above equation (2.3.5) can be written simply

$$\dot{x} = f(x) + g(x)\,u$$

$$y = h(x)$$

By a simple calculation we have

$$\text{(i) } L_g h(x) = \frac{\partial h(x)}{\partial x} g(x) = 0$$

$$\text{(ii) } L_g L_f h(x) = \left(\frac{\partial}{\partial x}(L_f h(x))\right) g(x) = \frac{\partial f_1(x)}{\partial x_2} g_2(x) \quad (\text{since } g_1(x) = 0)$$

$$= \frac{-(x_1 - \alpha_c + 1)\,\upsilon^2}{(x_2 - \alpha_T + \upsilon)^2}\exp\left[D - \frac{\upsilon^2}{x_2 - \alpha_T + \upsilon}\right]\gamma$$

As we can see in later Chapter 3, physically x_1 is never greater than α_c. Therefore in the set $\mathbb{U}$ defined by

$$\mathbb{U} = \{(x_1, x_2) \mid \alpha_c - 1 < x_1 \le \alpha_c, \quad \alpha_T - \upsilon < x_2 < \infty\}$$

we can see that

$$L_g L_f h(x) \neq 0$$

Therefore this system has relative degree 2 in the set $\mathbb{U}$.

In the same way of the previous Example 2.1 with the following coordinate transformation and state feedback:

Coordinate transformation:

$$z_1 = y = x_1$$
$$z_2 = L_f h(x) = f_1(x)$$

State feedback:

$$u = \alpha(x) + \beta(x)\, v$$

where

$$\alpha(x) = -\left[\frac{\partial f_1(x)}{\partial x_2} g_2(x)\right]^{-1}\left[\frac{\partial f_1(x)}{\partial x_1} f_1(x) + \frac{\partial f_1(x)}{\partial x_2} f_2(x)\right]$$
$$\beta(x) = \left[\frac{\partial f_1(x)}{\partial x_2} g_2(x)\right]^{-1}$$

and

$$\frac{\partial f_1(x)}{\partial x_1} = -1 - E(x_2, D)$$
$$\frac{\partial f_1(x)}{\partial x_2} = \frac{-(x_1 - \alpha_c + 1)\, \upsilon^2 E(x_2, D)}{(x_2 - \alpha_T + \upsilon)^2}$$

$$E(x_2, D) = \exp\left[D - \frac{\upsilon^2}{x_2 - \alpha_T + \upsilon}\right]$$

we have the following linear and controllable system:

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$ ///

CHAPTER III

Robustness Analysis of Feedback Linearization for Parametric and Structural Uncertainties

3.1. Introduction

Nonlinear process control using feedback linearization has attracted a great deal of attention among control engineers. In contrast to the truncated Taylor series approximation, this method permits one to linearize locally or globally a certain class of nonlinear systems without any kind of approximation. Hoo and Kantor (1985, 1986), Kravaris and Chung (1987), Kravaris (1988), Alvarez-Gallegos (1988), Calvet and Arkun (1988), Niranjan and San (1988), Daoutidis and Kravaris (1989), and Alvarez, Alvarez and Gonzalez (1989) have utilized this method to control some highly nonlinear chemical processes. From these researches we can see that on some nonlinear systems feedback linearization yields much better performance than conventional methods, such as a linear control based on the Taylor series approximation.

Conceptually, feedback linearization is based on the exact cancellation of the nonlinear terms by the nonlinear feedback. If there exists mismatch between the model and real system, feedback linearization does not yields an exactly linear system. Therefore we need a perfect mathematical model for the real plant in order for cancellation of the nonlinear terms to occur. However, it is difficult or impossible to model a real plant perfectly. So it is very important to investigate robustness of feedback linearization.

When there exists model-plant mismatch, feedback linearization decomposes a nonlinear system into a perturbed system composed of a linear and a perturbed nonlinear term because the coordinate transformation and linearizing state feedback are calculated based on the given nominal mathematical model. Robustness of feedback linearization depends on the characteristics of this perturbed nonlinear term.

Many attempts have been made to design a stable robust controller for uncertain feedback linearizable processes [Khorasani, 1989, Calvet and Arkun, 1989, Esfandiari and Khalil, 1989, Calvet and Arkun, 1989, Ha, 1989, Kravaris and Palanki, 1988, Spong, 1986, Abdallah and Lewis, 1987, Chen and Leitmann, 1987, Tarn et al, 1984]. However these approaches are based on the following restrictive conditions: the matching condition (see Appendix IV), global Lipschitz continuity of nonlinear uncertainties (see Appendix V), or same equilibrium point for mathematical model and real plant for all possible model-plant mismatch. In many practical situations these assumptions are hardly satisfied. For example, the invariant set of the transformed system by feedback linearization can be a limit cycle for a certain class of model-plant mismatch, that is, the linearized system is unstable around the equilibrium point of the nominal system. Generally in this case, it is very difficult or impossible to design robust state feedback which guarantees asymptotic stability. Therefore it is necessary to analyze robustness of feedback linearization without restrictive assumptions such as those mentioned above.

In this chapter we have developed a theoretical approach to analyze robustness of feedback linearization for a more general class of parametric and structural uncertainties, but without unmodeled dynamics. Robustness for uncertainties including unmodeled dynamics will be considered in the next chapter. Specifically we have found sufficient conditions for boundedness and convergence of the system trajectories when feedback linearization based on the nominal mathematical model is applied to an uncertain real plant which has parametric and structural uncertainties. The developed approach does not require the above restrictive conditions and may be useful in designing a robust state feedback for given model-plant mismatch.

As an example, we have chosen a first order irreversible exothermic reaction in a CSTR and analyzed robustness using the developed approach for parametric uncertainties, for an unmeasured disturbance and for a measurement error.

3.2. Literature Review

When there exists model-plant mismatch feedback linearization yields a perturbed system, not an exactly linear system. Usually robust stabilization or robustness analysis of a perturbed system depends on the characteristics of the perturbed nonlinearity.

When the perturbed nonlinearity satisfies the matching condition, robust state regulation is always possible if there is no limit of the input. On this class of model-plant mismatch Gutman (1979, 1985) and Corless and Leitmann (1988) have developed a method of robust stabilization using the second method of Lyapunov. This stabilization method is called Lyapunov Min-Max approach. The concept of Lyapunov Min-Max approach has been a basis for designing a robust controller for a perturbed linear system with matched uncertainties and has been applied to design a robust controller for robot manipulators by Hached (1988) and Shoureshi (1990).

With the concept of differential geometric approach Spong (1986) has considered the robust stabilization of an exact state-space linearizable process when model-plant mismatch satisfies the matching condition. In this work a composite controller which guarantees stability of the system has been designed using the small gain theorem [Desoer and Vidyasagar, 1975] and the concept of Lyapunov Min- Max approach.

Further development of the concept of Lyapunov Min-Max approach to robust output tracking for input-output linearizable processes has been made by Ha and Gilbert (1987), Ha (1989) and Behtash (1990). By the work of Ha and Gilbert (1987) the ordinary matching condition can be considerably weakened for a robust output tracking problem, that is, output tracking error is ultimately bounded in the presence of model-plant mismatch. This problem has been studied more extensively by Ha (1989).

Behtash (1990) has introduced the generalized matching condition which is a significant generalization of the ordinary matching condition. For this class of uncertainties he has employed the high-gain and the sliding mode control strategies.

In some sense, the matching condition is not only sufficient but also necessary for the robust *state* regulation [Petersen, 1985, Ha, 1989]. From the work of Chen (1990) we can see that the application of a Lyapunov Min-Max approach to a nonlinear system with unmatched nonlinearity results in performance limit.

Chen and Leitmann (1987) have considered the controller design to assure practical stability in the absence of the ordinary matching condition. Using Lyapunov stability theory they have proposed a design procedure for uncertain dynamic systems under some restrictive conditions. Unlike the case with the matching condition this procedure does not guarantee asymptotic stability of the system.

However, even for unmatched uncertainties, if the equilibrium point of a nominal model is the same as the real plant (if a real plant has a unique equilibrium point) and is invariant for every possible model-plant mismatch then the nonlinear system can be asymptotically stabilized under some conditions. Zak (1990) has studied this problem. In his work, under the assumption that the equilibrium point is invariant for all possible uncertainties, a stabilizing feedback has been proposed based on the constructive usage of Lyapunov functions and the Bellman-Gronwell lemma. However as indicated by Siljak (1989) the above assumption is hardly satisfied in many cases, and one should consider stability and changes of the equilibrium as a joint problem.

As a related problem, robustness analysis in state space is also very important to control engineers. Mathematical development for the analysis of a perturbed linear system in state space can be found in many textbooks [Hahn, 1967, Struble, 1962, Vidyasagar, 1978, Miller and Michel, 1982]. The small gain theorem and total stability theorem are invoked by many authors [Spong and Vidyasagar, 1987, Abdallah and Lewis, 1988] to analyse robustness of a nonlinear system.

When model-plant mismatch satisfies the matching condition, Abdallah and Lewis (1988) have found sufficient conditions for the uniform boundedness of the state variables using Lyapunov and total stability theorem. In practice this kind of analysis is necessary for a feedback linearizable process with unmatched uncertainties, since in this case generally it is almost impossible to design a robust *state* regulator.

Kokotovic and Marino (1986) have also studied stability of nonlinear systems. Under fairly weakened assumptions on model-plant mismatch they have shown that neglected nonlinearity can create a limit cycle around the asymptotically stable equilibrium point of a nominal system.

In the chemical engineering field Kravaris and Palanki (1988) have studied a robust output regulation problem with the matching condition for an input-output linearizable process. Similar to the work of Ha and Gilbert (1987) they have designed robust state feedback under a weaker condition than the ordinary matching condition and have applied this method to a chemical reactor.

For exact state-space linearizable processes, Calvet and Arkun (1989) have studied the design of robust stabilizing controllers for matched and unmatched parametric uncertainties. For these uncertainties they have found sufficient conditions to guarantee δ-stabilization of a system with high-gain linear state feedback based on the second method of Lyapunov. For unmatched uncertainties they have shown that with the same type of high-gain linear state feedback asymptotic stability can not be guaranteed. In this work they have considered the much more practical situation that the equilibrium point of a mathematical model is not the same as a real plant and also not invariant for possible model-plant mismatch. However, they have assumed the global Lipschitz continuity of the mismatch, which may restrict the applicability of this method. For example, as will be shown in the next section, for some parametric uncertainties in a CSTR, the global Lipschitz constant of the model-plant mismatch is so big that the result becomes very conservative and useless practically.

3.3. Theoretical Analysis

If a mathematical model has some error compared to a real plant, feedback linearization by the procedure in chapter 2 may not yield an exactly linear system. In this section we will investigate robustness of feedback linearization for a fairly general class of parametric and structural uncertainties, which do not require matching condition (see Appendix IV), the global Lipschitz continuity of uncertainties (see Appendix V), or the same equilibrium point for the mathematical model and the real plant for all possible uncertainties.

3.3.1. Feedback Linearization of an Uncertain System

Consider the n-dimensional SISO real plant

Real plant:

$$\dot{x} = f(t,x) + g(t,x)\,u \qquad (3.3.1)$$

$$y = h(x)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y \in \mathbb{R}$

$f(t, x)$, $g(t, x)$ and $h(x)$ are infinitely differentiable.

Suppose that for the real plant (3.3.1) we have the following mathematical model

Mathematical model:

$$\dot{x} = \hat{f}(x) + \hat{g}(x)\,u \qquad (3.3.2)$$

$$y = \hat{h}(x)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y \in \mathbb{R}$

$\hat{f}(x)$, $\hat{g}(x)$ and $\hat{h}(x)$ are infinitely differentiable.

In this section we assume that for the time-varying real plant we have a time invariant mathematical model: for instance, the real plant may have time-varying uncertainties such as disturbances which depend on time explicitly and the mathematical model ignores these time-varying terms.

From now on we will denote the vector fields f, g and the scalar field h for the real plant, partially unknown, and $\hat{f}$, $\hat{g}$ and $\hat{h}$ for the mathematical model.

Define the model-plant mismatch of f, g and h as follows:

$\Delta f \equiv f - \hat{f}$

$\Delta g \equiv g - \hat{g}$

$\Delta h \equiv h - \hat{h}$.

Suppose that mathematical model (3.3.2) has relative degree $\rho+1 = r$ ($\leq n$). As shown in the previous Chapter 2 for this mathematical model (3.3.2) the coordinate transformation $z_i = T_i(x)$ is defined by

$$z_i = T_i(x) = L_{\hat{f}}^{i-1}\hat{h}(x), \qquad i = 1, 2, \cdots, \rho+1 \tag{3.3.3}$$

Here feedback linearization , i.e. finding a linearizing state feedback and coordinate transformation, is based on the mathematical model instead of the real plant. So the relative degree of the mathematical model is not necessarily the same as for the real plant.

Let us apply the coordinate transformation and state feedback based on mathematical model (3.3.2) to real plant (3.3.1).

The time derivatives of each z_i are

$$\dot{z}_1 = z_2 + \frac{\partial \hat{h}(x)}{\partial x}\Delta f + \frac{\partial \hat{h}(x)}{\partial x}\Delta g\, u \tag{3.3.4}$$

$$\vdots$$

$$\dot{z}_\rho = z_{\rho+1} + \frac{\partial}{\partial x}\left[L_{\hat{f}}^{\rho-1}\hat{h}(x)\right]\Delta f + \frac{\partial}{\partial x}\left[L_{\hat{f}}^{\rho-1}\hat{h}(x)\right]\Delta g\;\; u$$

and

$$\dot{z}_{\rho+1} = \hat{A}^*(x) + \frac{\partial}{\partial x}\left[L_{\hat{f}}^{\rho}\hat{h}(x)\right]\Delta f + \left[\hat{D}^*(x) + \frac{\partial}{\partial x}\left(L_{\hat{f}}^{\rho}\hat{h}(x)\right)\Delta g\right]u\,. \tag{3.3.5}$$

where $\widehat{A}^*(x) \equiv L_{\hat{f}}^{\rho+1}\,\hat{h}(x)$

$\widehat{D}^*(x) \equiv L_{\hat{g}}\,L_{\hat{f}}^{\rho}\,\hat{h}(x)$

The state feedback u, which is also based on the mathematical model (3.3.2), is, from equation (2.2.9)

$$u = \alpha(x) + \beta(x)\, v \tag{3.3.6}$$

where $\alpha(x) = -\ \widehat{D}^*(x)^{-1}\widehat{A}^*(x)$

$\beta(x) = \widehat{D}^*(x)^{-1}$

Applying this state feedback u with $v = Kz$ into equation (3.3.5), then we obtain

$$\dot{z}_{\rho+1} = \sum_{k=1}^{\rho+1} b_k\, L_{\hat{f}}^{k-1}\,\hat{h}(x) + \frac{\partial}{\partial x}\left[L_{\hat{f}}^{\rho}\hat{h}(x)\right]\Delta f + \frac{\partial}{\partial x}\left[L_{\hat{f}}^{\rho}\hat{h}(x)\right]\Delta g\ (\alpha(x) + \beta(x)\ Kz) \tag{3.3.7}$$

where $K = [\ b_1\, b_2 \cdots b_r\ ]$,

and b_i, $i = 1, 2, \cdots, r$, are constants.

As shown in Chapter 2, if relative degree r is less than the dimensionality of the system, n, the coordinate transformation $T(x) = [T_1(x)\ \ T_2(x)\ \ \cdots\ \ T_r(x)]^T$ alone does not complete the linearization. In this case, we need additional coordinate transformations, $T_{r+1}(x), T_{r+2}(x), \cdots, T_n(x)$ as shown in Proposition 2.1. From proposition 2.1 we know that it is always possible to find coodinate transformations, $T_{r+1}(x), T_{r+2}(x), \cdots, T_n(x)$ such that

$$L_g T_j(x) = 0, \quad \text{for } j = r+1, r+2, \ldots, n.$$

The time derivatives of T_j, $j = r+1, r+2, \ldots, n$, are

$$\dot{T}_j = \frac{\partial T_j(x)}{\partial x}\frac{dx}{dt}$$
$$= \frac{\partial T_j(x)}{\partial x}(f(x) + g(x)\,u)$$
$$= \frac{\partial T_j(x)}{\partial x}\left\{\hat{f}(x) + \Delta f + \left(\hat{g}(x) + \Delta g\right)u\right\}$$
$$= \frac{\partial T_j(x)}{\partial x}\hat{f}(x) + \frac{\partial T_j(x)}{\partial x}\left(\Delta f + \Delta g\,u\right) \quad (\text{ since } L_g T_j(x) = 0\ ) \tag{3.3.8}$$

Define

$$z^1 = [\,T_1 \ \cdots \ T_r]^T$$
$$z^2 = [\,T_{r+1} \ \cdots \ T_n]^T$$
$$z = [\,T_1 \ \cdots \ T_r \ T_{r+1} \ \cdots \ T_n]^T$$

Combining equations (3.3.4), (3.3.5), (3.3.7) and (3.3.8) we can express the linearized system of the real plant as follows:

$$\dot{z}^1 = (\bar{A} + bK)\,z^1 + \varphi(t, x) \tag{3.3.9}$$
$$\dot{z}^2 = Q(x) + \mu(t, x)$$
$$y = Cz^1 + \Delta h(x)$$

$$\text{where } \ \bar{A}+bK = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \dots & 1 \\ b_1 & b_2 & b_3 & \dots & b_r \end{bmatrix}$$

$$b = [\,0 \ \ 0 \ \ 0 \ \ \dots\dots 0 \ \ 1]^T$$
$$C = [\,1 \ \ 0 \ \ 0 \ \ \dots\dots 0 \ \ 0]$$
$$\varphi(t, x) = \left[\varphi_1(t, x) \ \ \varphi_2(t, x) \ \cdots \ \varphi_r(t, x)\right]^T$$
$$\varphi_i\,(t, x) = \frac{\partial}{\partial x}\left[L_{\hat{f}}^{i-1}\hat{h}(x)\right]\Delta f + \frac{\partial}{\partial x}\left[L_{\hat{f}}^{i-1}\hat{h}(x)\right]\Delta g\ (\,\alpha(x) + \beta(x)\,Kz\,)\Big|_{z = T(x)},$$
$$i = 1, \dots, r$$

$$Q(x) = \left[q_{r+1}(x) \quad q_{r+2}(x) \cdots q_n(x)\right]^T$$
$$q_j(x) = \frac{\partial T_j(x)}{\partial x}\hat{f}(x), \qquad j = r+1, r+2, \ldots, n$$
$$\mu(t, x) = \left[\Delta q_{r+1}(t, x) \quad \Delta q_{r+2}(t, x) \cdots \Delta q_n(t, x)\right]^T$$
$$\Delta q_j(t, x) = \frac{\partial T_j(x)}{\partial x}\left(\Delta f + \Delta g\, u\right), \qquad j = r+1, r+2, \ldots, n$$

Since the transformation, T, is a diffeomorphism (see Remark 2.1 in Section 2.2, p15) we can define nonlinear functions η, Θ, ρ and ε as follows:

$$\psi(t, z) = \varphi(t, T^{-1}(z))$$
$$\Theta(z) = Q(T^{-1}(z))$$
$$\rho(t, z) = \mu(t, T^{-1}(z))$$
$$\varepsilon(z) = \Delta h(T^{-1}(z))$$

where T^{-1} is the inverse of T.

With these notations the above system (3.3.9) can be written as follows:

$$\dot{z}^1 = (\bar{A} + bK)\, z^1 + \psi(t, z) \tag{3.3.10}$$
$$\dot{z}^2 = \Theta(z) + \rho(t, z)$$
$$y = Cz^1 + \varepsilon(z)$$

Note that this representation in the z-coordinate system is only for convenience of the theoretical development. As we can see later, we do not need to find the inverse of the coordinate transformation, T, when we apply the developed approach to the robustness analysis.

As we indicated in Chapter 2, dynamics $\dot{z}^2 = Q(z)$ do not affect the output $y = z_1 = h(x)$ when the mathematical model is perfect; that is, there exists no model-plant mismatch. However from the above differential equation (3.3.10) we can see that when there exists a model-plant mismatch this dynamics $\dot{z}^2 = Q(z)$ may affect the output if the perturbed nonlinear term $\phi(t, z)$ is a function of z^2. Because of this fact input-output linearization has difficulty in analyzing the robustness in state space for a general class of model-plant mismatch.

Let us simplify the above equation (3.3.10). If dynamics $\dot{z}^2 = \Theta(z)$ are linearized by the Taylor series expansion then the real plant can be finally represented by the following simple form; that is, the perturbed linear system

$$\dot{z} = Az + \eta(t, z) \tag{3.3.11}$$
$$y = Cz^1 + \varepsilon(z)$$

Of course if the system is exact state-space linearizable; that is, the relative degree of the mathematical model is equal to the dimension of the state space, then equation (3.3.10) directly has the form of equation (3.3.11).

3.3.2. Robustness Analysis

By analyzing system (3.3.11) we want to find conditions which guarantee that the solution $z(t)$ is bounded for every time $t \geq 0$ and converges into a set around $z = 0$ with the appropriately chosen K in $v = Kz$.

First we consider the special case that $z = 0$ is a trivial solution; that is, $\eta(t, 0) = 0$ for every time $t \in [0, \infty)$. This means that the equilibrium point, $z = 0$ (whenever a system has an equilibrium point then it can be transformed to zero as its equilibrium point by shifting the axis), is invariant for all possible parametric and structural uncertainties. However the perturbed nonlinear term $\eta(t, z)$ is not necessarily globally Lipschitz continuous in z (see Appendix V) and also does not need to satisfy matching condition (see Appendix IV).

In this case we have developed the following theorem to analyze stability of the system (3.3.11). Actually very similar theorems can be found in many textbooks [Miller and Michel, 1982, Struble, 1962].

Theorem 1

Consider the following n-dimensional nonlinear system

$$\frac{dz}{dt} = Az + \eta(t, z) \text{ with } z(t) = z(0) \text{ at } t = 0 \tag{3.3.12}$$

where $z \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$

$\eta : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ is continuous

Suppose that the nonlinear system (3.3.12) satisfies the following assumptions:

Assumption 1.1 : There exists a finite nonnegative constant M such that

$$\| \eta(t, z) \| \leq M \| z \| \qquad \forall\, z \in B_r(z = 0)\,,\ t \in [0, \infty)$$

(This assumption can be possible only when $\eta(t, 0) = 0$ for every time, t.)

where $B_r(z = 0)$ = a ball of radius r centered at $z = 0$, defined by

$B_r(z = 0) \equiv \{ \| z \| \leq r \,;\, r < \infty \}$

$\| \cdot \|$ = Euclidian norm.

Assumption 1.2 : $\frac{dz}{dt} = Az$ is asymptotically stable.

$\Rightarrow$ There exist constants $\alpha \geq 1$, $k > 0$ such that

$$\| e^{At} \| \leq \alpha\, e^{-kt} \qquad \forall\, t \in [\, 0, \infty\,).$$

Then for any $\| z(0) \| < r / \alpha$, if $\alpha M / k < 1$ then $\| z(t) \| \leq r \ \ \forall\, t \in [\, 0, \infty\,)$ and moreover $\lim_{t \to \infty} z(t) = 0$. ///

Proof: Any solution of the system (3.3.12) satisfies the following integral equation

$$z(t) = e^{At} z(0) + \int_0^t e^{A(t - \tau)}\, \eta(\tau, z(\tau))\, d\tau$$

Taking the norm on both sides of the above equation then

$$\|z(t)\| \le \| e^{At} \| \, \|z(0)\| + \int_0^t \|e^{A(t-\tau)}\| \, \|\eta(\tau, z(\tau))\| \, d\tau$$

$$\le \alpha \, \|z(0)\| \, e^{-kt} + \int_0^t \alpha \, e^{-k(t-\tau)} \, M \, \| z(\tau) \| \, d\tau \qquad \forall \, z \in B_r(z=0)$$

Therefore

$$\| z(t) \| \, e^{kt} \le \alpha \, \|z(0)\| + \int_0^t \alpha \, Me^{k\tau} \, \| z(\tau) \| \, d\tau \qquad \forall \, z \in B_r(z=0)$$

We can see that if $\|z(0)\| < r / \alpha$, then $\|z(\tau)\| \le r$ for some interval $0 \le \tau \le t$, with $t > 0$ (from the given nonlinear system (3.3.12) it is obvious that z(t) is continuous because z(t) is differentiable for every $t \ge 0$).

Applying Gronwall's lemma (see Appendix II) for $0 \le \tau \le t$

$$\| z(t) \| \, e^{kt} \le \alpha \, \|z(0)\| \; \exp\left[\int_0^t \alpha \, M \, d\tau\right]$$

$$= \alpha \, \|z(0)\| \, e^{\alpha M \, t}$$

$$\therefore \qquad \| z(t) \| \; \le \alpha \, \|z(0)\| \, e^{(\alpha M - k)\, t}.$$

Now if $\alpha M - k < 0 \left(\Leftrightarrow \dfrac{\alpha M}{k} < 1 \right)$, the above inequality shows that if the condition $\| z(t) \| \le \alpha \, \| z(0) \|$ is initially satisfied it will be maintained thereafter. Clearly Gronwall's lemma applies for every $t \ge 0$, so

$$\| z(t) \| \le r \quad \text{for every} \;\; t \in [0,\infty) \;\; \text{and} \;\; \lim_{t \to \infty} \| z(t) \| \to 0. \qquad ///$$

Remark 3.1: Consider the problem of determining k and α in assumption 1.2. For a given matrix $A \in \mathbb{R}^{nxn}$, which has the eigenvalues λ_i, $i = 1, 2, \ldots\ldots, n$, there exists a transformation Q such that [Chen, 1984, p35]

$$e^{At} = Q\, e^{\Lambda t}\, Q^{-1}$$

where $Q = [\, v_1 \;\; v_2 \;\ldots\ldots\; v_n \,]$

v_i = eigenvector associated with eigenvalue λ_i, $i = 1, 2, \ldots, n$

Λ = matrix in Jordan form

Taking the norm on both sides of the above equation we have

$$\| e^{At} \| \leq \| Q \| \, \| e^{\Lambda t} \| \, \| Q^{-1} \|$$

Since $\bar{\sigma}(Q) = \| Q \|$

$\bar{\sigma}(Q^{-1}) = 1 / \underline{\sigma}(Q)$

$\| e^{\Lambda t} \| \leq \exp(\bar{\lambda}_{max}\, t)$

where $\bar{\sigma}(Q)$ = maximum singular value of Q

$\underline{\sigma}(Q)$ = minimum singular value of Q

$\bar{\lambda}_{max}$ = maximum value of $Re(\lambda_i)$, $i = 1, 2, \ldots, n$

we have

$$\| e^{At} \| \leq \bar{\sigma}(Q) / \underline{\sigma}(Q) \; \exp(\bar{\lambda}_{max}\, t).$$

Therefore we can choose

$$\alpha = \bar{\sigma}(Q) / \underline{\sigma}(Q)$$

$$k = | \bar{\lambda}_{max} |$$

///

For all possible parametric and structural uncertainties, if the equilibrium point, $z = 0$, is invariant then Theorem 1 can be applied directly to analyze robustness. However, in many practical cases, the equilibrium point of the system (3.3.11) changes with each possible model-plant mismatch. For example, as we will show in the next example of a CSTR, when an uncertainty is given as a parametric bound instead of a specific constant parametric value the equilibrium point of the CSTR changes with each different parametric value. To analyze robustness for this case, we may use Theorem 1 at each equilibrium point for each possible parametric value and combine the result to cover all possible parametric changes. However this approach usually requires too much tedious calculation.

In this case by a simple extension of Theorem 1 we can reduce the necessary calculation. Actually this extension is straightforward and conceptually almost the same as the above approach. The following Theorem 2 shows this extension. However Theorem 2 is more conservative than Theorem 1.

Theorem 2:

Consider the same nonlinear system (3.3.12). Suppose that the equilibrium point, d, that is, $Ad + \eta(t, d) = 0$ for every time $t \in [0,\infty)$, exists in the ball $B_{r_d}(z = 0)$ (that is, unique but uncertain equilibrium point, d, exists in the ball $B_{r_d}(z = 0)$).

Let $y = z - d$. Suppose that the system (3.3.12) satisfies the following assumptions:

Assumption 2.1: There exists a finite nonnegative constant M such that

$$\| Ad + \eta(t, y+d) \| \le M \| y \|$$

for every $y \in B_r(y = 0)$, $d \in B_{r_d}(z = 0)$ and $t \in [0, \infty)$

where $B_r(y = 0)$ = a ball of radius r centered at $y = 0$.

Assumption 2.2 : $\frac{dz}{dt} = Az$ is asymptotically stable.

$\Rightarrow$ There exist constants $\alpha \geq 1$, $k > 0$ such that

$$\| e^{At} \| \leq \alpha e^{-kt} \quad \forall t \in [0, \infty).$$

Then for any $\| z(0) \| < \frac{r}{\alpha} - r_d$ if $\frac{\alpha M}{k} < 1$ then $\| z(t) \| \leq r + r_d$ for every $t \in [0, \infty)$ and moreover $\lim_{t \to \infty} z(t) \in B_{r_d}(z = 0)$. ///

Proof: For any $d \in B_{r_d}(z=0)$, with $y = z - d$ the system (3.3.12) can be written

$$\frac{dy}{dt} = Ay + Ad + \eta(t, y+d)$$

If $\| Ad + \eta(t, y + d) \| \leq M \|y\| \quad \forall y \in B_r(y=0),\ t \in [0, \infty)$ and $\frac{\alpha M}{k} < 1$ then by Theorem 1, for every $\| y(0) \| < \frac{r}{\alpha}$, $\| y(t) \| \leq r \quad \forall t \in [0, \infty)$ and $\lim_{t \to \infty} \| y(t) \| = 0$.

Since $$B_{\frac{r}{\alpha} - r_d}(z=0) \subset \bigcap_{\forall d \in B_{r_d}(z=0)} B_{\frac{r}{\alpha}}(z=d)$$

$$B_{r + r_d}(z=0) \supset \bigcup_{\forall d \in B_{r_d}(z=0)} B_r(z=d)$$

where $B_r(z=d)$ = ball of radius r centered at $z = d$

$\cup$ = union

$\cap$ = intersection

we can conclude that for every $\| z(0) \| < \frac{r}{\alpha} - r_d$ if $\frac{\alpha M}{k} < 1$ then $\| z(t) \| \leq r + r_d$ for every $t \in [0, \infty)$ and $\lim_{t \to \infty} z(t) \in B_{r_d}(z = 0)$. ///

Sometimes there may exist multiple equilibrium points for a certain class of model-plant mismatch. Even though this happens, applying Theorem 2 we do not need to find all of the multiple equilibrium points. We need only the nearest equilibrium point from the point $z = 0$ which is the equilibrium point of a nominal system. The details are treated in the next application.

Now consider a more general case that there exists no constant d such that $Ad + \eta(t, d) = 0$ but $\eta(t, 0)$ is absolutely bounded for every time $t \in [0, \infty)$. For this kind of parametric or structural uncertainty the above Theorem 1 or 2 can not be used. In the CSTR problem which will be treated in the next section, this situation may happen, for example, when there exists time-varying unmeasured disturbance, measurement or parametric error. The following Theorem 3 can be used in this case.

Theorem 3:

Consider the same nonlinear system (3.3.12). Suppose that the system (3.3.12) satisfies the following assumptions:

Assumption 3.1: There exist finite nonnegative constants m and δ_η in $B_r(z = 0) \subset \mathbb{R}^n$ such that

(i) $\| \eta(t, z) - \eta(t, y) \| \leq m \| z - y \|$ for every $z, y \in B_r(z = 0)$ and for every $t \in [0, \infty)$

(ii) $\| \eta(t, 0) \| \leq \delta_\eta$ for every $t \in [0, \infty)$.

Assumption 3.2: $\dfrac{dz}{dt} = A z$ is asymptotically stable.

$\Rightarrow$ There exist constants $\alpha \geq 1$, $k > 0$ such that

$$\| e^{At} \| \leq \alpha e^{-kt} \quad \forall t \in [0, \infty).$$

Then for any r_i such that $\dfrac{\delta_\eta \alpha}{k - m\alpha} \leq r_i < r$, if $\alpha \| z(0) \| \leq r_i$ and $(mr + \delta_\eta) \dfrac{\alpha}{k} < r$ then there exists a unique solution, $z(t)$ of the system (3.3.12) and this solution is in the set $\mathbb{S}_i$ and moreover $\lim\limits_{t \to \infty} \| z(t) \| \leq r_d$, where $r_d \equiv \dfrac{\delta_\eta \alpha}{k - m\alpha}$. The set $\mathbb{S}_i$ is defined by

$$\mathbb{S}_i = \left\{ z(t) \, ; \, \| z(t) \|_c \le r_i \, , \, \frac{\delta_\eta \alpha}{k - m\alpha} \le r_i < r \right\}$$

$$\text{where} \quad \| z(t) \|_c \equiv \sup_{t \in [0, \infty)} \| z(t) \| \qquad ///$$

Proof: This proof follows a similar argument to that found in Anderson et al. (1986) and Vidyasagar (1978). We will prove this theorem in three steps. First using the Contraction mapping theorem [see Appendix III, or Vidyasagar, 1978, Theorem 25, p78] we will prove that there exists exactly one solution in $\mathbb{S}_i$. Second, we will show that the solution is unique in the closed ball $B_r(z = 0)$ for some time interval so that the solution is unique in $\mathbb{R}^n$ and finally every solution in $\mathbb{S}_i$ converges to the closed ball $B_{r_d}(z = 0)$ defined by

$$B_{r_d}(z = 0) \equiv \left\{ z \in \mathbb{R}^n \, ; \; \| z \| \le r_d \equiv \frac{\delta_\eta \alpha}{k - m\alpha} \right\}.$$

Step 1; existence of exactly one solution in $\mathbb{S}_i$.

The solution of equation (3.3.12), provided it exists, will be

$$z(t) = e^{At} z(0) + \int_0^t e^{A(t-\tau)} \eta(\tau, z(\tau)) \, d\tau \tag{3.3.13}$$

Define the mapping T by

$$Tz(t) = e^{At} z(0) + \int_0^t e^{A(t-\tau)} \eta(\tau, z(\tau)) \, d\tau \tag{3.3.14}$$

Now with this mapping T we will apply the contraction mapping theorem [see Appendix III]. By the contraction mapping theorem if the map T satisfies the following conditions

(i) T maps $\mathbb{S}_i$ into itself

(ii) There exists a constant $\rho < 1$ such that

$\| Tz - Ty \|_c \leq \rho \| z - y \|_c$ for every $z, y \in \mathbb{S}_i$

that is, the mapping T is contraction

then there exists a unique fixed point $z^*(t)$, defined by $Tz^*(t) = z^*(t)$, in $\mathbb{S}_i$. In other words there exists exactly one solution of the nonlinear system (3.3.12) in $B_{r_i}(z = 0)$, defined by $B_{r_i}(z = 0) \equiv \left\{ z \in \mathbb{R}^n ;\ \| z \| \leq r_i \right\}$, for every time $t \in [0, \infty)$.

First we will show that T maps $\mathbb{S}_i$ into itself.

Let $z(t) \in \mathbb{S}_i$. Then from equation (3.3.15)

$$\| Tz(t) \| \leq \| e^{At} \| \ \| z(0) \| + \int_0^t \| e^{A(t-\tau)} \| \ \| \eta(\tau, z(\tau)) \| \, d\tau \tag{3.3.15}$$

$$\leq \alpha e^{-kt} \| z(0) \| + \int_0^t \alpha e^{-k(t-\tau)} [\, m \| z(\tau) \| + \delta_\eta] \, d\tau$$

$$\leq \alpha e^{-kt} \| z(0) \| + (m r_i + \delta_\eta) \int_0^t \alpha e^{-k(t-\tau)} \, d\tau \qquad (\text{since } z(t) \in \mathbb{S}_i)$$

$$\leq \alpha e^{-kt} \| z(0) \| + (m r_i + \delta_\eta) \frac{\alpha}{k} (1 - e^{-kt})$$

Therefore for any r_i such that $\frac{\delta_\eta \alpha}{k - m\alpha} \leq r_i < r$ if $\alpha \| z(0) \| \leq r_i < r$ and $(mr + \delta_\eta) \frac{\alpha}{k} < r$ then it is obvious that $r > \frac{\delta_\eta \alpha}{k - m\alpha}$ since $r > 0$ and $(m\alpha - k) < 0$. Since $\frac{\delta_\eta \alpha}{k - m\alpha} \leq r_i < r$ we have $(m r_i + \delta_\eta) \frac{\alpha}{k} \leq r_i$.

Therefore $\quad \| Tz(t) \| \leq r_i e^{-kt} + r_i (1 - e^{-kt}) \leq r_i \qquad \forall\, t \in [0, \infty)$

$\Rightarrow \| Tz(t) \|_c \in \mathbb{S}_i$.

So T maps $\mathbb{S}_i$ into itself.

Next, we will show that T is a contraction, i.e. there exists a constant $\rho < 1$ such that

$$\| Tz(t) - Ty(t) \|_c \leq \rho \, \|z(t) - y(t)\|_c \quad \text{for every } z, y \in \mathbb{S}_i$$

From equation (3.3.14)

$$Tz(t) - Ty(t) = \int_0^t e^{A(t-\tau)} \left(\eta(\tau, z(\tau)) - \eta(\tau, y(\tau)) \right) d\tau \tag{3.3.16}$$

Taking the norm on both sides

$$\| Tz(t) - Ty(t) \| \leq \int_0^t \|e^{A(t-\tau)}\| \, \| \eta(\tau, z(\tau)) - \eta(\tau, y(\tau)) \| \, d\tau \tag{3.3.17}$$

$$\leq \int_0^t \alpha e^{-k(t-\tau)} \, m \, \| z(\tau) - y(\tau) \| \, d\tau$$

Taking the sup for every $t \in [0, \infty)$ on both sides

$$\sup_{t \in [0,\infty)} \| Tz(t) - Ty(t) \| \tag{3.3.18}$$

$$\leq \sup_{t \in [0,\infty)} \int_0^t \alpha e^{-k(t-\tau)} \, m \, \| z(\tau) - y(\tau) \| \, d\tau$$

$$\leq m \, \| z(t) - y(t) \|_c \sup_{t \in [0,\infty)} \frac{\alpha}{k} \left(1 - e^{-kt} \right)$$

$$\leq \frac{\alpha \, m}{k} \| z(t) - y(t) \|_c$$

So if $(m\, r + \delta_\eta) \frac{\alpha}{k} < r$, then $\frac{m\,\alpha}{k} + \frac{\delta_\eta \alpha}{k\, r} < 1$ and since $\frac{\delta_\eta \alpha}{k\, r} \geq 0$ we can see that $\frac{m\,\alpha}{k} < 1$.

Let $\rho = \frac{m\,\alpha}{k}$. Then there exists a constant $\rho < 1$ such that

$$\| Tz(t) - Ty(t) \|_c \leq \rho \, \| z(t) - y(t) \|_c \quad \text{for every } z(t), y(t) \in \mathbb{S}_i$$

So by the contraction mapping theorem, there exists exactly one solution of the nonlinear system (3.3.12) in $\mathbb{S}_i$.

With only the local contraction mapping theorem we cannot exclude the possibility of the existence of a solution outside of $\mathbb{S}_i$. So we want to show that the solution of the initial value problem (3.3.12) is unique in $B_r(z = 0)$.

Step 2; uniqueness of the solution.

Let z(t) and x(t) be any two solutions of the system (3.3.12) with the same initial value. Then we can write from (3.3.13):

$$z(t) = e^{At}z(0) + \int_0^t e^{A(t-\tau)}\eta(\tau, z(\tau))\, d\tau$$

$$x(t) = e^{At}z(0) + \int_0^t e^{A(t-\tau)}\eta(\tau, x(\tau))\, d\tau$$

From these equations we have

$$z(t) - x(t) = \int_0^t e^{A(t-\tau)}[\, \eta(\tau, z(\tau)) - \eta(\tau, x(\tau))\,]\, d\tau$$

If $\|z(0)\| \leq \frac{r_1}{\alpha} < \frac{r}{\alpha}$ then there exists a finite time t_1 such that for a interval $0 \leq \tau \leq t_1$, $\|z(\tau)\| \leq r$ and $\|x(\tau)\| \leq r$. Therefore by assumption 3.1 and 3.2 we can see that

$$\|z(t) - x(t)\| \leq \int_0^t \alpha\, e^{-k(t-\tau)} m \|z(\tau) - x(\tau)\|\, d\tau \qquad \text{for } t \leq t_1$$

This equation can be written

$$\|z(t) - x(t)\|\, e^{kt} \leq \int_0^t \alpha\, m\, e^{k\tau} \|z(\tau) - x(\tau)\|\, d\tau \qquad \text{for } t \leq t_1$$

By the Gronwall lemma (see Appendix II) we can see that

$\|z(t) - x(t)\| = 0$ whenever $\| z(t) \| \le r$ and $\| x(t) \| \le r$

(because in this case the nonnegative constant c in the Gronwall lemma in Appendix II is zero).

That is, the z(t) is unique in $B_r(z = 0)$ for some time interval.

Previously we proved that there exists exactly one solution in $\mathbb{S}_i$. Let's consider another possible solution p(t) outside of $B_{r_i}(z = 0)$ for some time. Suppose that there exists this kind of solution p(t). Then for some time interval there exists a solution in the closed set $(B_r(z = 0) - B_{r_i}(z = 0))$ (remember that this is a dense set.). This contradicts the fact that there exists exactly one solution in $\mathbb{S}_i$ and uniqueness in $B_r(z =0)$; that is, if for some time interval there exists a solution p(t) in the closed set $(B_r(z = 0) - B_{r_i}(z = 0))$ then it is not unique in $B_r(z = 0)$.

Therefore we can conclude that for any r_i such that

$$\frac{\delta_\eta \alpha}{k - m\alpha} \le r_i < r$$

if $\alpha \| z(0) \| \le r_i < r$ and $(mr + \delta_\eta) \frac{\alpha}{k} < r$ then there exists exactly one solution of system (3.3.12) and it remains in $\mathbb{S}_i$.

Step 3; convergence of the solution in $\mathbb{S}_i$

Finally let's consider the convergence of the solution of (3.3.12). Suppose z(t) and y(t) denote solutions in $\mathbb{S}_i$ with different initial conditions such that $\| z(0) \| \le \frac{r_i}{\alpha}$ and $\|y(0)\| \le \frac{r_i}{\alpha}$ but $z(0) \neq y(0)$.

From (3.3.15) we have

$$y(t) - z(t) = e^{At}(\, y(0) - z(0)\,) + \int_0^t e^{A(t-\tau)}[\, \eta(\tau, y(\tau)) - \eta(\tau, z(\tau))\,]\, d\tau$$

Since z(t) and y(t) are in $\mathbb{S}_i$

$$\| y(t) - z(t) \| \le \alpha e^{-kt} \| y(0) - z(0) \| + \int_0^t \alpha m\, e^{-k(t-\tau)} \| y(\tau) - z(\tau) \|\, d\tau$$

By the Gronwall lemma (see Appendix II)

$$\| y(t) - z(t) \| e^{kt} \le \alpha \| y(0) - z(0) \| e^{\alpha m t} \quad \text{for every } t \in [0, \infty)$$

and this means that

$$\|y(t) - z(t) \| \le \alpha \|y(0) - z(0) \| e^{(\alpha m - k)t} \quad \text{for every } t \in [0, \infty).$$

Since $(\alpha m - k) < 0$ $\| y(t) - z(t) \| \to 0$ as $t \to \infty$. This convergence holds for any r_i such that $\dfrac{\delta_\eta \alpha}{k - m\alpha} \le r_i < r$. Therefore we can see that $\lim\limits_{t \to \infty} \| z(t) \| \le r_d$ where $r_d = \dfrac{\delta_\eta \alpha}{k - m\alpha}$.

///

Remark 3.2: Theorem 1 can be also derived directly from Theorem 3 with δ_η=0. However it must be noted that Theorem 1 does not require the incremental Lipschitz continuity of $\eta(t, z)$.

///

Remark 3.3: To find the Lipschitz constant m in a closed ball, $B_r(z = 0)$, the following theorem is very useful.

Theorem 3.1 [Rudin, 1984]

Suppose f maps a convex open set $\mathbb{E} \subset \mathbb{R}^n$ into $\mathbb{R}^m$, f is differentiable in $\mathbb{E}$, and there is a real number m such that

$$\left\| \frac{\partial f(x)}{\partial x} \right\| \le m \qquad \text{for every } x \in \mathbb{E},$$

then

$\| f(b) - f(a) \| \leq m \| b - a \|$ for every $a, b \in \mathbb{E}$

where $\frac{\partial f(x)}{\partial x}$ is the Jacobian of f(x). ///

From this theorem it follows that if $\left\| \frac{\partial f(t, x)}{\partial x} \right\| \leq m$ for every $x \in \mathbb{E}$ and every $t \in [0,\infty)$ then $\| f(t, b) - f(t, a) \| \leq m \| b - a \|$ for every $a, b \in \mathbb{E}$ and every $t \in [0, \infty)$.

///

Remark 3.4: Suppose we can find a constant $r > 0$ for a given k, α and δ_η satisfying assumption 3.1 in Theorem 3. Let's consider subset $B_{r'}(z = 0) \subset B_r(z = 0)$. Then we can always find Lipschitz constant $m' \leq m$ in $B_{r'}(z = 0)$. Therefore we can find minimum and maximum r denoted by r_{min} and r_{max} respectively satisfying the conditions of Theorem 3.

Let's denote Lipschitz constant m_{min} in closed ball $B_{r_{min}}(z = 0)$. Then by Theorem 3 for any initial condition $\alpha \| z(0) \| < r_{min}$, $\lim_{t \to \infty} \| z(t) \| \leq r_{dmin}$, where $r_{dmin} = \frac{\delta_\eta \alpha}{k - m_{min}\,\alpha}$. Since every solution in $B_{r_{max}}(z = 0)$ converges we can conclude that for any initial condition $\alpha \| z(0) \| < r_{max}$, $\lim_{t \to \infty} \| z(t) \| \leq r_{dmin}$. ///

This Theorem 3 has some benefit in the sense that it requires very weak conditions on parametric or structural uncertainties. However sometimes it may suffer from conservativeness; that is, the boundedness and convergence of solution trajectories are guaranteed only for small magnitude of perturbed nonlinear term $\eta(t, x)$ or a small region of an initial condition even though for a larger magnitude of $\eta(t, x)$ or a bigger region of initial condition the real system has a bounded solution.

Compared with Theorem 3, Theorem 2 requires the calculation of the region where each possible equilibrium point for a given model-plant mismatch exists. So Theorem 3 can be applied to more general cases of parametric and structural uncertainties. Practically this Theorem 3 can be used for the less restrictive cases such as a time varying parametric uncertainty, an unmeasured disturbance and a measurement error to find sufficient conditions for the ultimate boundedness; that is, for given any $\bar{d} > d$ and any $r \in [0, r_0)$, there is a $\bar{t}(\bar{d}, r) \in [0, \infty)$ such that for every solution $x(.) : [t_0, \infty) \rightarrow \mathbb{R}^n$, $x(t_0) = x_0$, $\|x_0\| \leq r \Rightarrow \|x(t)\| \leq \bar{d}$ for every $t \geq t_0 + \bar{t}(\bar{d}, r)$. For this kind of model-plant mismatch the above Theorem 1 or 2 can not be used since in this case the equilibrium point cannot be found generally. However if we can use Theorem 2 we can decrease the conservativeness in the sense that we can check the boundedness of solution trajectories for a larger magnitude of $\eta(t, z)$ or a bigger range of initial condition; that is, we can find a less conservative domain of attraction. Of course Theorem 2 gives an almost exact bound of the solution when time goes to infinity.

Next we consider the practical application of linearization and its robustness analysis using the above theorems.

3.4. Application: First Order Exothermic Reaction in a CSTR

3.4.1. Introduction

In this section we apply the feedback linearization method to a first order exothermic reaction in a CSTR and analyze robustness for a parametric uncertainty in the reaction rate constant, for an unmeasured disturbance in feed concentration and for measurement error in concentration in the reactor, using the theorems developed in the previous section.

3.4.2. Mathematical Model of a First Order Exothermic Reaction in a CSTR

A first order exothermic reaction in a CSTR is mathematically modeled by

$$\frac{dC}{dt} = \frac{q}{V}(C_0 - C) - k_0 C \exp\left[-\frac{E}{RT}\right] \tag{3.4.1}$$

$$\frac{dT}{dt} = \frac{q}{V}(T_0 - T) - \frac{\Delta H}{\rho C_p} k_0 C \exp\left[-\frac{E}{RT}\right] - \frac{UA_R}{\rho C_p V}(T - T_c)$$

where reactant concentration, C, is the controlled variable and coolant temperature, T_c, is chosen as the manipulated variable. Values of parameters and operating conditions are shown in Table 3.1 [Foster and Stevens, 1967]. With these operating conditions the system (3.4.1) has two stable and one unstable steady states:

T = 374.11 K,	C = 7.019×10^{-3} gmole/cc,	stable
T = 397.30 K,	C = 3.970×10^{-3} gmole/cc,	unstable
T = 415.65 K,	C = 1.558×10^{-3} gmole/cc,	stable

It is assumed that the unstable point is the desired operating point denoted by T^d and C^d.

Table 3.1 System parameters and operating conditions for a first order exothermic reaction in a CSTR

System parameters	
$k_0 = 3.0 \times 10^{11}$ / sec	$E = 2.5088 \times 10^4$ cal / gmole
$R = 1.987$ cal / gmole. K	$\Delta H = -11101.32$ cal / gmole
$\rho C_p = 1$ cal / cc . K	$V = 3.048 \times 10^6$ cc
$U = 0.014$ cal / sec cm^2 K	$A_R = 4.645 \times 10^5$ cm^2
Operating conditions	
$T_0 = 383.3$ K	$T_c = 330.0$ K
$C_0 = 8.016 \times 10^{-3}$ gmole / cc	$q = 1.4158 \times 10^4$ cc / sec
$T^d = 397.30$ K	$C^d = 3.97 \times 10^{-3}$ gmole / cc

With the dimensionless variables defined in Table 3.2 the system (3.4.1) can be written as follows:

$$\frac{dx_1}{d\tau} = (-x_1 + \alpha_c) - (x_1 - \alpha_c + 1)\exp\left[D - \frac{\upsilon^2}{x_2 - \alpha_T + \upsilon}\right] + d_c \qquad (3.4.2)$$

$$\frac{dx_2}{d\tau} = (-x_2 + \alpha_T) - B(x_1 - \alpha_c + 1)\exp\left[D - \frac{\upsilon^2}{x_2 - \alpha_T + \upsilon}\right] - \gamma(x_2 - \alpha_T + \alpha_W) + \gamma u + d_T$$

$$y = x_1$$

Table 3.2 Dimensionless variables for a first order exothermic reaction in a CSTR

$x_1 = \dfrac{C - C^d}{\overline{C}_0}$	$x_2 = \dfrac{T - T^d}{\overline{T}_0}\upsilon$
$u = \dfrac{T_c - \overline{T}_c}{\overline{T}_0}\upsilon$	$\tau = t\dfrac{q}{V}$
$\upsilon = \dfrac{E}{R\overline{T}_0} = 32.9404$	$B = \dfrac{\Delta H\,\overline{C}_0}{\rho\, C_p \overline{T}_0}\upsilon = -\,7.6476$
$D = \ln\left(\dfrac{k_0 V}{q}\right) = 31.799$	$\gamma = \dfrac{U\,A_R}{q\,\rho\,C_p} = 0.4593$
$\alpha_T = \dfrac{\overline{T}_0 - T^d}{\overline{T}_0}\upsilon = -\,1.2031$	$\alpha_c = \dfrac{\overline{C}_0 - C^d}{\overline{C}_0} = 0.5047$
$\alpha_W = \dfrac{\overline{T}_0 - \overline{T}_c}{\overline{T}_0}\upsilon = 4.5805$	
$d_T = \dfrac{T_0 - \overline{T}_0}{\overline{T}_0}\upsilon$	$d_c = \dfrac{C_0 - \overline{C}_0}{\overline{C}_0}$

In the dimensionless variables of Table 3.2 $\overline{T}_0$, $\overline{T}_c$ and $\overline{C}_0$ are the nominal values of T_0, T_c and C_0, and d_c and d_T mean the inlet disturbances of concentration and temperature respectively.

Let $f_1(x) = (-x_1 + \alpha_c) - (x_1 - \alpha_c + 1)\exp\left[D - \dfrac{\upsilon^2}{x_2 - \alpha_T + \upsilon}\right] + d_c$

$$f_2(x) = (-x_2 + \alpha_T) - B(x_1 - \alpha_c + 1)\exp\left[D - \frac{\upsilon^2}{x_2 - \alpha_T + \upsilon}\right] - \gamma\,(x_2 - \alpha_T + \alpha_W) + d_T$$

$$g_1(x) = 0$$

$$g_2(x) = \gamma$$

$$h(x) = x_1$$

Then the system (3.4.2) can be written compactly

$$\dot{x} = f(x) + g(x)\,u \qquad (3.4.3)$$
$$y = h(x)$$
$$\text{where } f(x) = [f_1(x) \;\; f_2(x)]^T$$
$$g(x) = [g_1(x) \;\; g_2(x)]^T$$

This nonlinear system (3.4.3) has been linearized as shown in the previous section under the assumption that there is no model-plant mismatch.

3.4.3. Feedback Linearization of the CSTR without Model-Plant Mismatch

In this section we assume that there is no model-plant mismatch, that is, the system (3.4.3) represents the real plant with the parametric value in Table 3.1 and no disturbance; i.e., $d_c = d_T = 0$.

Define the set $\mathbb{U}$:

$$\mathbb{U} = \{(x_1, x_2) \mid \alpha_c - 1 < x_1 \le \alpha_c, \;\; \alpha_T - \upsilon < x_2 < \infty\}$$

(corresponding to $\mathbb{U} = \{(C, T) \mid 0.0 < C \le C_0, \text{ gmole/cc}, \;\; 0.0 < T < \infty, \text{ K}\}$)

By simple calculation we can see that the system (3.4.3) has relative degree $r = 2$ for all x which belong to the set $\mathbb{U}$. If the state variables, x, are outside of the set $\mathbb{U}$ then feedback linearization with static feedback cannot be applied.

Now let us initially assume that the system response never goes outside the set $\mathbb{U}$. This assumption is quite reasonable since the situation C = 0.0 gmole/cc may not happen in the actual operation of CSTR and the actual lower bound of absolute temperature, T, never can be absolute 0 K. Actually this assumption can be checked automatically after the robustness analysis. When there is no model-plant mismatch, feedback linearization yields a linear and controllable system. Therefore, by an elementary stability analysis we can check the upper and lower bound of x. When there is a model-plant mismatch with the given initial condition we can find the solution boundedness using the theoretical approach developed in the previous section.

From equation (2.2.9) in the previous chapter the linearizing state feedback u is

$$u = \alpha(x) + \beta(x)\, v \tag{3.4.4}$$

where

$$\alpha(x) = -\left[\frac{\partial f_1(x)}{\partial x_2} g_2(x)\right]^{-1}\left[\frac{\partial f_1(x)}{\partial x_1} f_1(x) + \frac{\partial f_1(x)}{\partial x_2} f_2(x)\right]$$

$$\beta(x) = \left[\frac{\partial f_1(x)}{\partial x_2} g_2(x)\right]^{-1}$$

and

$$\frac{\partial f_1(x)}{\partial x_1} = -1 - E(x_2, D)$$

$$\frac{\partial f_1(x)}{\partial x_2} = \frac{-(x_1 - \alpha_c + 1)\, \upsilon^2 E(x_2, D)}{(x_2 - \alpha_T + \upsilon)^2}$$

$$E(x_2, D) = \exp\left[D - \frac{\upsilon^2}{x_2 - \alpha_T + \upsilon}\right]$$

With the following coordinate transformation

$$z_1 = y = x_1 \tag{3.4.5}$$

$$z_2 = L_f h(x) = f_1(x)$$

the system (3.4.3) can be represented by

$$\dot{z} = (A+bK)\, z \tag{3.4.6}$$

$$y = Cz$$

when $v = Kz$, where $K = [b_1 \quad b_2]$

where

$$A+bK = \begin{bmatrix} 0 & 1 \\ b_1 & b_2 \end{bmatrix}$$

$$C = [1 \;\; 0]$$

If every eigenvalue of (A+bK) has negative real part then the above linearized system (3.4.6) is asymptotically stable. However this asymptotic stability is based on the fact that we have a perfect model, i.e. no model-plant mismatch. If a model-plant mismatch exists then this asymptotic stability may not hold. In the next section we will investigate robustness of this system.

3.4.4. Robustness Analysis

3.4.4.1. Parametric Uncertainty in the Reaction Rate Constant

In this section we will investigate the effect of parametric uncertainties. Let us assume that we have modeled the CSTR within $\pm$ 2 % error in reaction rate constant k_0, that is, the real value of k_0 is in the closed set of $[2.94 \times 10^{11},\ 3.06 \times 10^{11}]$, sec^{-1}, corresponding bound of dimensionless variable D is [31.7788, 31.8188], and the nominal value of k_0 is 3.0×10^{11} /sec. Let f(x) correspond to the real plant with $\pm$ 2% error in k_0 and $\hat{f}(x)$ to the model with nominal k_0. In this case $d_c = d_T = 0$. $\hat{D}$ denotes the nominal value of D.

For parametric uncertainties we can consider two possible cases, that is, (1) an unknown parameter has any fixed value in the given bounds, i.e. it is time invariant, or (2) an unknown parameter is time varying in the given bounds. For case (1), the previously developed Theorem 1 or 2 can be used since we can find the equilibrium point for each possible value of the uncertain parameter. Of course we can also use Theorem 3. In this case it is not necessary to calculate the equilibrium point for each possible parametric value. In this example we assume that k_0 has any fixed value in the given closed set.

Since row vector { $dh(x), dL_{\hat{f}}h(x)$ }, where $d\lambda(x) \equiv \frac{\partial\lambda(x)}{\partial x}$, has rank 2, relative degree $r = 2$ in $\mathbb{U}$. Therefore $z = T(x)$, defined by

$$z_1 = T_1(x) = x_1 \qquad (3.4.7)$$
$$z_2 = T_2(x) = \hat{f}_1(x)$$

is a diffeomorphism.

Define $\eta(z) = \varphi(T^{-1}(z))$, where model-plant mismatch, φ defined in (3.3.9), is not a function of time explicitly. Then feedback linearization yields by the procedure of the previous section (3.3)

$$\dot{z} = (A+bK)\,z + \eta(z) \qquad (3.4.8)$$

where

$$\varphi_1(x) = \Delta f_1(x) = (x_1 - \alpha_c + 1)\left[E(x_2, \hat{D}) - E(x_2, D)\right]$$
$$\varphi_2(x) = \frac{\partial\hat{f}_1(x)}{\partial x_1}\Delta f_1(x) + \frac{\partial\hat{f}_1(x)}{\partial x_2}\Delta f_2(x)$$
$$\frac{\partial\hat{f}_1(x)}{\partial x_1} = -1 - E(x_2, \hat{D})$$
$$\frac{\partial\hat{f}_1(x)}{\partial x_2} = \frac{-(x_1 - \alpha_c + 1)\,\upsilon^2 E(x_2, \hat{D})}{(x_2 - \alpha_T + \upsilon)^2}$$
$$\Delta f_2(x) = -B\,(x_1 - \alpha_c + 1)\left[E(x_2, \hat{D}) - E(x_2, D)\right]$$
$$E(x_2, D) = \exp\left[D - \frac{\upsilon^2}{x_2 - \alpha_T + \upsilon}\right]$$
$$E(x_2, \hat{D}) = \exp\left[\hat{D} - \frac{\upsilon^2}{x_2 - \alpha_T + \upsilon}\right]$$

Now let us apply the previous Theorem 2. To do this we find the bound of the possible equilibrium point, d (in the z-coordinate system) for each D in the neighborhood of $z = 0$, which is the equilibrium point of the nominal system, such that

$$(A+ bK)d + \eta(d) = 0 \qquad (3.4.9)$$

Since $z = T(x)$ and $\eta(T(x)) = \varphi(x)$ equation (3.4.9) is the same as

$$(A + bK)T(c) + \varphi(c) = 0 \tag{3.4.10}$$

where $c = [c_1, c_2]^T$ is the corresponding equilibrium point in the x-coordinate system.

By analyzing equation (3.4.10) we can see that in a small region around the point $x = 0$ there exists an equilibrium point, c, for each possible parameter D. There may exist other equilibrium points in a bigger region around $x = 0$. However this case cannot be a problem by the following argument.

When there exist more than one equilibrium points, we can transform one of the equilibrium points which is the nearest to the point $x = 0$ to zero by shifting the axis. Let us denote any other equilibrium point by d'. With the transformed equilibrium point $x = 0$, if all assumptions and conditions of Theorem 1 are satisfied, then the domain of attraction for the equilibrium point $x = 0$ will never overlay the domain of attraction for any other equilibrium point, d'. If this is not true then for some initial points such that $\| z(0) \| < \frac{r}{\alpha} - r_d$ there may exist a solution such that

$$\lim_{t \to \infty} \| z(t) \| = d' \neq 0$$

and this is contradicted by Theorem 1. Therefore for multiple equilibrium points, if we take the nearest equilibrium point to zero, where zero is the equilibrium point of the nominal system, then we can apply the above Theorem 2.

From equation (3.4.10), we can find easily the equilibrium point c and the corresponding d in the z-coordinate system. For the ± 2 % k_0 error, i.e. for $31.7788 \leq D \leq 31.8188$, with $b_1 = -2.1$, $b_2 = -2.0$ Fig.3.1 shows the equilibrium points c in the neighborhood of zero for each value of D (by the program: OUTX1DD).

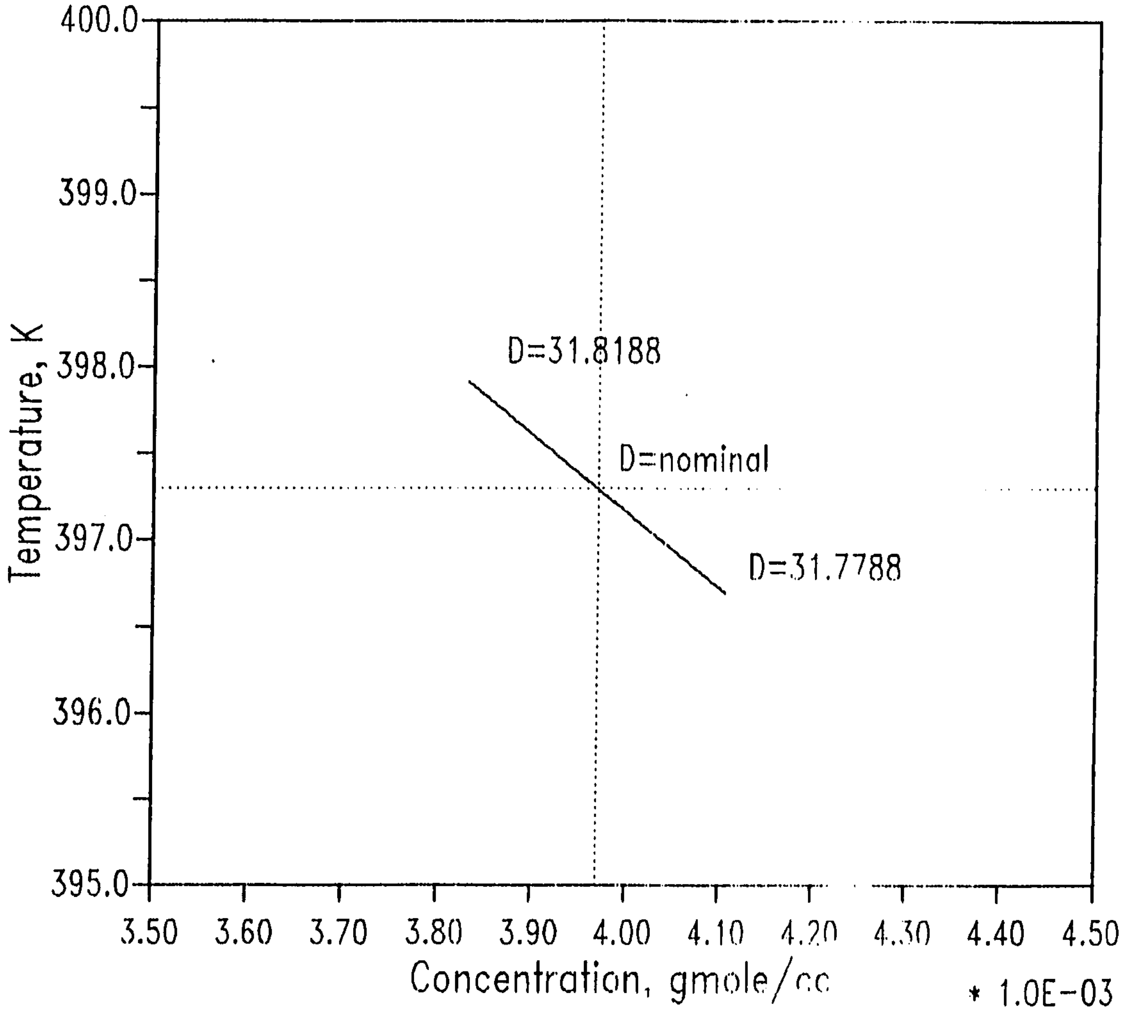

Fig. 3.1 Equilibrium point for each parametric value of D where $b_1 = -2.1$ and $b_2 = -2.0$

Here, we can apply Theorem 1 at every equilibrium point shown in Fig.3.1. In this case we must combine the result for each equilibrium point to get the maximum possible upper bound of the solution for the given parametric uncertainty. This approach requires too much calculation and is very tedious work. Therefore, in this work, we apply Theorem 2.

In order to apply Theorem 2 we need to know the radius of the ball $B_{r_d}(z = 0)$ where all possible equilibrium points exist. In this example we can see that from Fig.3.1 that when D=31.8188 or 31.7788 the equilibrium point, c or d in equation (3.4.9) and (3.4.10), is placed at the furthest distance from z = 0.

When D = 31.8188,

$$c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -0.17361 \times 10^{-1} \\ 0.52671 \times 10^{-1} \end{bmatrix}, \qquad d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} -0.17361 \times 10^{-1} \\ 0.10240 \times 10^{-1} \end{bmatrix}$$

$$\text{which corresponds to} \quad \begin{bmatrix} C \\ T \end{bmatrix} = \begin{bmatrix} 3.831 \times 10^{-3}\text{ , gmole/cc} \\ 397.91\text{ , K} \end{bmatrix}.$$

From this we can find the radius r_d of the ball $B_{r_d}(z=0)$ such that for every D in [31.7788, 31.8188] equilibrium point d is in $B_{r_d}(z=0)$: $r_d = 0.021$.

Now, we find the Lipschitz constant M in Theorem 2 such that

$$\| Ad + \eta(y+d) \| \leq M \|y\| \qquad \text{for every } y \in B_r(y=0) \qquad (3.4.11)$$
$$\text{for every } d \in B_{r_d}(z=0)$$

where $y = [y_1, y_2]^T$

$$y_1 = z_1 - d_1 = T_1(x) - d_1 = x_1 - d_1$$
$$y_2 = z_2 - d_2 = T_2(x) - d_2 = \hat{f}_1(x) - d_2$$

where the y-axis is obtained by shifting the z - axis so that the equilibrium point is zero. The above equation (3.4.11) can be written with the function $\varphi(x)$; that is, in the x-coordinate system

$$\| Ad + \eta(y+d) \| = \| Ad + \eta(z) \| = \| Ad + \varphi(x) \| \qquad (3.4.12)$$

and the Euclidian norm of y also can be represented by

$$\| y \| = \sqrt{(z_1-d_1)^2 + (z_2 - d_2)^2}$$
$$= \sqrt{(x_1-d_1)^2 + (\hat{f}_1(x) - d_2)^2}$$

This relation makes the calculation of M of equation (3.4.11) easier since it is not necessary to convert $\varphi(x)$ into $\eta(z)$. In other words, we do not need to find the inverse of the coordinate transformation. With the previously calculated sufficiently many values of d, we can find the constant M by a simple computation.

When we choose $b_1 = -2.1$, $b_2 = -2.0$ by remark 3.1 $\alpha = 2.566$ and $k = 1.0$ (program name: EIGEN). With the ± 2 % k_0 error we can find by a simple search method (program name: SLOPWH) that for every $z \in B_{r+r_d}(z=0)$, where $r+r_d \leq 0.412$, $M = 0.3897 < 1.0 / 2.566 = k / \alpha$. This means that $M\alpha / k < 1$. Therefore we can conclude that by Theorem 2 for any $r+r_d \leq 0.412$, if $\| z(0) \| < r/\alpha - r_d$ then the solution of the nonlinear differential equation $\| z(t) \| < r + r_d$ for every $t \in [0, \infty)$ and $\lim_{t \to \infty} \| z(t) \| \in B_{r_d}(z = 0)$ where $r_d = 0.021$.

For example if $\| z(0) \| < 0.131$ then $\| z(t) \| \leq 0.412$ for every $t \in [0, \infty)$ and also converges into the ball $B_{r_d}(z = 0)$, where $r_d = 0.021$. Actually in this case solution z(t) converges to a point on the line in Fig.3.1.

Fig.3.2 shows the solution trajectories are bounded and converge to the ball $B_{r_d}(z=0)$ for the above case, where line 1 is the solution trajectory when the initial condition of concentration, $C = 2.944 \times 10^{-3}$ gmole / cc and temperature, T = 404.51 K (corresponding to $x_1(0) = -0.128$, $x_2(0) = 0.620$) and D = 31.8188 and line 2 is when the initial concentration, $C = 5.004 \times 10^{-3}$ gmole / cc and temperature, T = 391.60 K (corresponding to $x_1(0) = 0.129, x_2(0) = -0.49$) and $D = 31.7788$ (program name: OUTX1RO for trajectories, CHCONT2 for contours).

In this way we can analyse the robustness for a given parametric uncertainty using Theorem 2.

Now let us consider the same problem using Theorem 3. Actually in this case k_0 need not be time invariant. Moreover we do not need any information about the equilibrium point.

Define the set $B_D = \{D \,;\, 31.7788 \leq D \leq 31.8188\}$. First let us find the nonnegative constants m and δ_η such that

$$\| \eta(z) - \eta(w) \| \leq m \| z - w \| \qquad \text{for every } z, w \in B_r(z=0) \text{ and } D \in B_D$$

$$\| \eta(0) \| \leq \delta_\eta \tag{3.4.13}$$

By the previous remark 3.3 the constant m can be chosen such that

$$\left\| \frac{\partial \eta(z)}{\partial z} \right\| \leq m \qquad \text{for every } z \in B_r(z=0) \text{ and } D \in B_D$$

By the inverse function theorem [see Appendix VI, or Boothby, 1975] for $z = T(x)$, since T is a diffeomorphism

$$\frac{\partial T^{-1}(z)}{\partial z} = \left[\frac{\partial T(x)}{\partial x} \right]^{-1}$$

With this Jacobian of T Jacobian $\eta(z)$, $\frac{\partial \eta(z)}{\partial z}$ can be written by the chain rule

$$\left. \frac{\partial \eta(z)}{\partial z} \right|_{z = T(x)} = \left[\frac{\partial \varphi(x)}{\partial x} \right] \left[\frac{\partial T(x)}{\partial x} \right]^{-1} \tag{3.4.14}$$

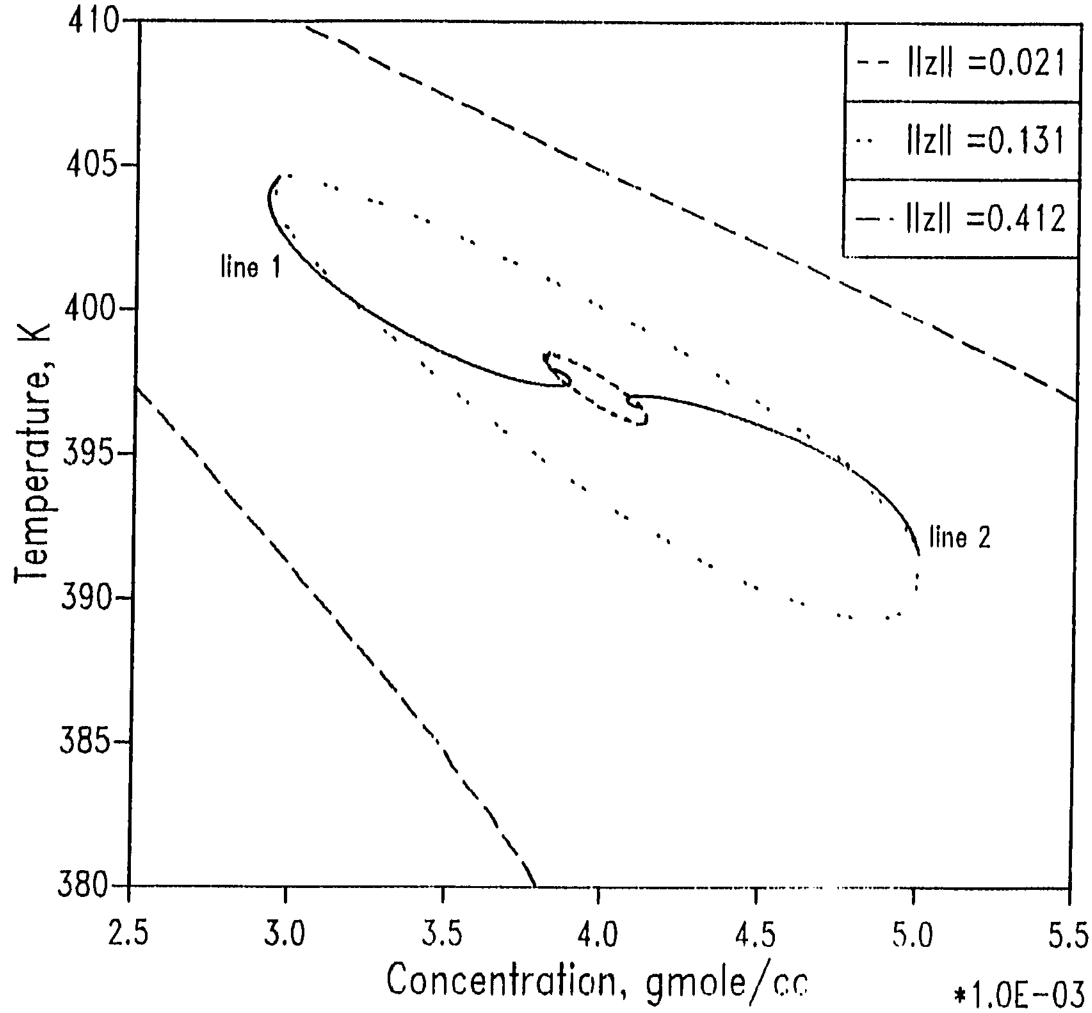

Fig. 3.2 Estimated bounded and converging region of the solution trajectories for parametric uncertainty in k_0 using Theorem 2, where $b_1 = -2.1$ and $b_2 = -2.0$ and

line 1: initial condition, $C = 2.944 \times 10^{-3}$ gmole/cc, $T = 404.51$ K and $D = 31.8188$

line 2: initial condition, $C = 5.004 \times 10^{-3}$ gmole/cc, $T = 391.60$ K and $D = 31.7788$

From this relationship we can get

$$\frac{\partial \eta(z)}{\partial z} = \begin{bmatrix} \dfrac{\partial \eta_1(z)}{\partial z_1} & \dfrac{\partial \eta_1(z)}{\partial z_2} \\[2ex] \dfrac{\partial \eta_2(z)}{\partial z_1} & \dfrac{\partial \eta_2(z)}{\partial z_2} \end{bmatrix}$$

where

$$\frac{\partial \eta_1(z)}{\partial z_1} = \frac{E(x_2, D) - E(x_2, \hat{D})}{E(x_2, \hat{D})}$$

$$\frac{\partial \eta_1(z)}{\partial z_2} = \frac{E(x_2, D) - E(x_2, \hat{D})}{E(x_2, \hat{D})}$$

$$\frac{\partial \eta_2(z)}{\partial z_1} = \left[E(x_2, D) - E(x_2, \hat{D})\right] \left\{ - \frac{\left[1 + E(x_2, \hat{D})\right]^2}{E(x_2, \hat{D})} + \frac{2\,B\,(x_1 - \alpha_c + 1)\left[1 + E(x_2, \hat{D})\right]}{(x_2 - \alpha_T + \upsilon)} - \frac{2\,B\,\upsilon^2 (x_1 - \alpha_c + 1)}{(x_2 - \alpha_T + \upsilon)^2} \right\}$$

$$\frac{\partial \eta_2(z)}{\partial z_2} = \left[E(x_2, D) - E(x_2, \hat{D})\right] \cdot \left\{ -1 - \frac{\left[1 + E(x_2, \hat{D})\right]}{E(x_2, \hat{D})} + \frac{2\,B\,(x_1 - \alpha_c + 1)}{(x_2 - \alpha_T + \upsilon)} - \frac{2\,B\,\upsilon^2 (x_1 - \alpha_c + 1)}{(x_2 - \alpha_T + \upsilon)^2} \right\}$$

for detail calculation see Appendix I.

Finally the Lipschitz constant m is chosen that

$$m = \max_{\substack{z \in B_r(z=0) \\ D \in B_D}} \left\| \frac{\partial \eta(z)}{\partial z} \right\| \tag{3.4.15}$$

The other constant δ_η in equation (3.4.13) can be calculated from the following simple equation

$$\| \eta(0) \| = \| \varphi(0) \| = \sqrt{\varphi_1(0)^2 + \varphi_2(0)^2} \tag{3.4.16}$$

where

$$\varphi_1(0) = (1 - \alpha_c)\left\{\exp\left[\hat{D} - \frac{\upsilon^2}{(\upsilon - \alpha_T)}\right] - \exp\left[D - \frac{\upsilon^2}{(\upsilon - \alpha_T)}\right]\right\}$$

$$\varphi_2(0) = \left\{-1 - \exp\left[\hat{D} - \frac{\upsilon^2}{(\upsilon - \alpha_T)}\right]\right\}\varphi_1(0) + \frac{(1 - \alpha_c)\,\upsilon^2}{(\upsilon - \alpha_T)^2}$$

$$\cdot\exp\left[\hat{D} - \frac{\upsilon^2}{(\upsilon - \alpha_T)}\right] B(1 - \alpha_c)\left\{\exp\left[D - \frac{\upsilon^2}{(\upsilon - \alpha_T)}\right] - \exp\left[\hat{D} - \frac{\upsilon^2}{(\upsilon - \alpha_T)}\right]\right\}$$

For the same parametric bound of D, i.e. $31.7788 \le D \le 31.8188$, when we choose $b_1 = -2.1$ and $b_2 = -2.0$, from the above equation $\delta_\eta = 0.0188$. And we can use equation (3.4.15) to find the Lipschitz constant, m; that is, the maximum singular value of the matrix $\frac{\partial\eta(z)}{\partial z}$ is calculated for a given r such that $z \in B_r(z = 0)$ and for a given parametric bound (program name for m and δ_η : LIPSCH1).

Table 3.3 shows the Lipschitz constant, m, $\delta_\eta\alpha / (k - m\alpha)$ and the permissible maximum m, defined by m_c such that $(mr + \delta_\eta)\,\alpha / k < r$, for each r with the given parametric error.

Table 3.3: Lipschitz constant for each radius r for the parametric uncertainty in k_0 when $b_1 = -2.1$, $b_2 = -2.0$, where $\delta_\eta = 0.0188$

r	m	$\frac{\delta_\eta\alpha}{k - m\alpha}$	m_c
0.292	0.305	0.222	0.325
0.222	0.139	0.075	0.305
0.075	0.111	0.067	0.138
0.067	0.11	0.067	0.11

First, we find the maximum r such that $(mr + \delta_\eta)\, \alpha / k < r$. By some trials we can see that when $r > 0.292$ the Lipschitz constant m is greater than m_c so we cannot use Theorem 3. Let's define $rd(m) \equiv \delta_\eta \alpha / (k - m\alpha)$. From Table 3.3 we can see that when $r = 0.292$, $rd(m) = 0.222$. By Theorem 3 we can say that for any r_i such that $0.222 \le r_i < 0.292$ if $\alpha \|z(0)\| \le r_i$ then the solution z(t) belongs to the ball $B_{r_i}(z = 0)$ for every $t \in [0, \infty)$.

Next, let us find the minimum r such that $(mr + \delta_\eta)\, \alpha/k < r$. Practically this minimum r can be found by successively applying Theorem 3. We know that when $r = 0.292$, $rd(m) = 0.222$. Now, let's apply Theorem 3 with $r = 0.222$. When $r = 0.222$, $rd(m) = 0.075$. In this case the Lipschitz constant m depends on r and moreover it decreases with the smaller r. Continue this step until Theorem 3 can be applied. Choose again $r = 0.075$ then $rd(m) = 0.067$ and when $r = 0.067$, m is the same as m_c (see Table 3.3), where rd(m) is still 0.067. When we choose r less than 0.067, m is greater than m_c and we cannot use Theorem 3. So we can see that minimum r is 0.067.

In this way, we can analyze robustness of feedback linearization using Theorem 3. In this case we can say that for every r_i such that $0.067 \le r_i < 0.292$ if $\| z(0) \| \le \dfrac{r_i}{2.566}$ then the solution of the system $\| z(t) \| \le r_i$ for every $t \in [0, \infty)$, and, moreover, $\lim_{t \to \infty} \| z(t) \| \le r_d$, where $r_d = 0.067$.

For example, if $\| z(0) \| < 0.114$ then the solution of the system $\| z(t) \| < 0.292$ for every $t \in [0, \infty)$ and converges into the ball $B_{r_d}(z = 0)$ where $r_d = 0.067$. Fig.3.3 shows this result, where line 1 is the solution when the initial condition of concentration $C = 4.867 \times 10^{-3}$ gmole / cc and temperature T = 392.30 K (corresponding to $x_1(0) = 0.112$, $x_2(0) = -0.43$) and D = 31.8188. Line 2 is when the initial condition of concentration $C = 3.136 \times 10^{-3}$ gmole / cc and temperature T = 403.58 K (corresponding to $x_1(0) = -0.104$, $x_2(0) = 0.54$) and $D = 31.7788$ (program: OUTX1RO for trajectories, CHCONT2 for contours). In this figure we can see that obviously the solution $\| z(t) \| < 0.292$ and converges to the ball $B_{r_d}(z = 0)$, $r_d = 0.067$.

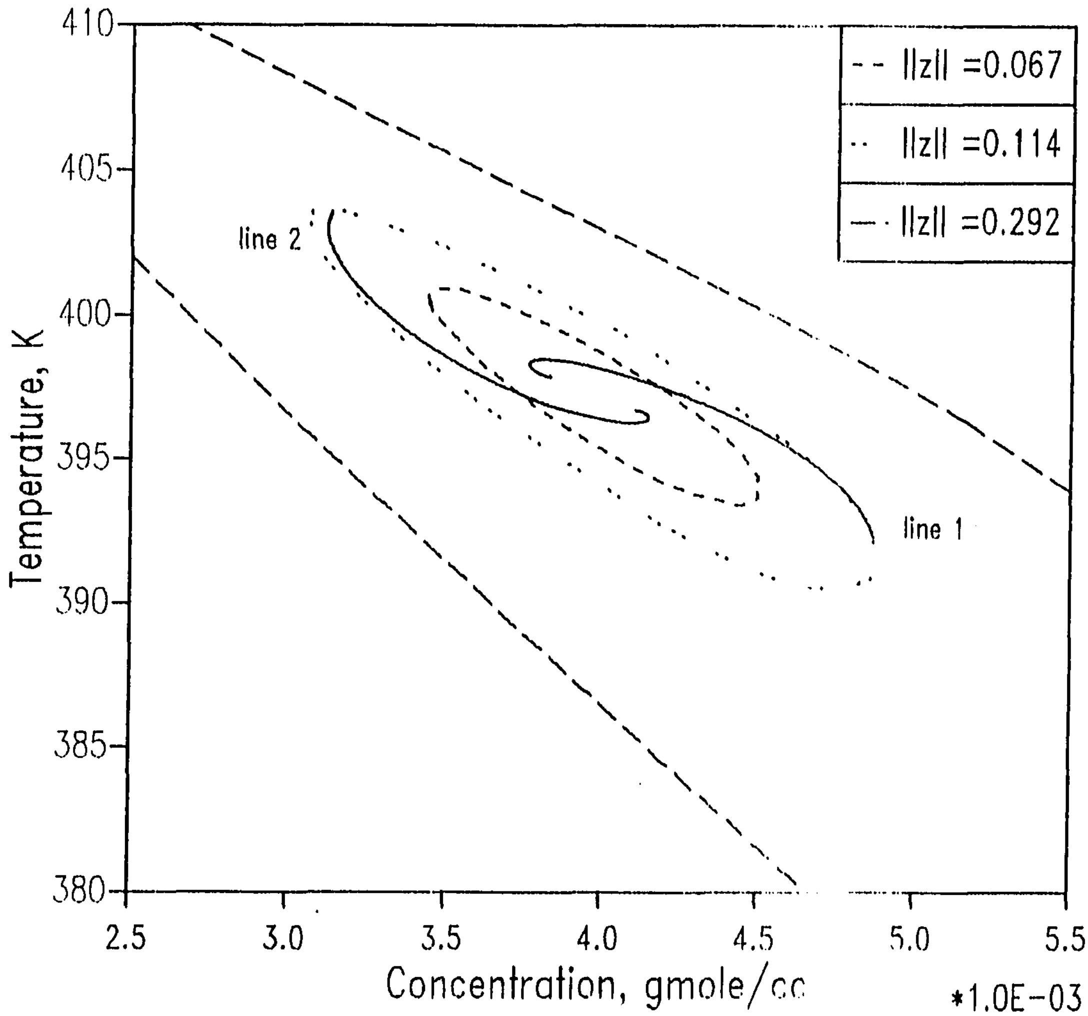

Fig. 3.3 : Estimated bounded and converging region of the solution trajectories for the parametric uncertainty in k_0 using Theorem 3, where

$b_1 = -2.1$ and $b_2 = -2.0$ and

line 1: initial condition, $C = 4.867 \times 10^{-3}$ gmole/cc, $T = 392.30$ K and $D = 31.8188$

line 2: initial condition, $C = 3.136 \times 10^{-3}$ gmole/cc, $T = 403.58$ K and $D = 31.7788$

3.4.4.2. Unmeasured Disturbance in Feed Concentration

Let us consider the effect of unmeasured disturbance d_c which may be a function of time. In this section, other parameters are assumed to have nominal values. An unmeasured disturbance can be considered as a structural uncertainty and other structural uncertainties can be treated similarly.

In this example we assume that feed concentration C_0 has the following deviation:

$7.949 \times 10^{-3} \leq C_0 \leq 8.083 \times 10^{-3}$ [gmole/cc]

(corresponding value of d_c in Table 3.2 ; $|d_c| \leq 0.83 \times 10^{-2}$)

In this case, the perturbed nonlinear term $\varphi(t,x) = [\,\varphi_1(t,x)\ \ \varphi_2(t,x)]^T$ can be written by the procedure of the previous section (3.3)

$$\varphi_1(t,x) = \frac{\partial h(x)}{\partial x}\Delta f = d_c \tag{3.4.17}$$

$$\varphi_2(t,x) = \frac{\partial}{\partial x}[L_f^{\hat{}}h(x)]\,\Delta f = [-1 - E(x_2, D)]\,d_c$$

$$\text{where } E(x_2, D) = \exp\left(D - \frac{\upsilon^2}{x_2 - \alpha_T + \upsilon}\right)$$

$$\Delta f = f - \hat{f} = \begin{bmatrix} d_c \\ 0 \end{bmatrix}$$

Define the set B_{dc} such that

$$B_{dc} = \left\{d_c\,;\, |d_c| \leq 0.83 \times 10^{-2}\right\}$$

and find the Lipschitz constant m and δ_η for every $z \in B_r(z=0)$ and $d_c \in B_{dc}$. By the method used in the previous section we can calculate easily

$$\frac{\partial \eta(t,z)}{\partial z} = \begin{bmatrix} \dfrac{\partial \eta_1(t,z)}{\partial z_1} & \dfrac{\partial \eta_1(t,z)}{\partial z_2} \\[2ex] \dfrac{\partial \eta_2(t,z)}{\partial z_1} & \dfrac{\partial \eta_2(t,z)}{\partial z_2} \end{bmatrix} \tag{3.4.18}$$

where $\frac{\partial\eta_1(t,z)}{\partial z_1} = 0, \quad \frac{\partial\eta_1(t,z)}{\partial z_2} = 0$

$$\frac{\partial\eta_2(t,z)}{\partial z_1} = \left(\frac{1 + E(x_2, D)}{x_1 - \alpha_c + 1}\right) d_c$$

$$\frac{\partial\eta_2(t,z)}{\partial z_2} = \frac{d_c}{x_1 - \alpha_c + 1}$$

With this Jacobian of $\eta(t, z)$ we can find the Lipschitz constant m and δ_η (program name: LIPSCH4). For the given bound of d_c we can find easily that $\delta_\eta \leq 0.0188$. Table 3.4 shows the Lipschitz constant m, $\delta_\eta\alpha / (k - m\alpha)$ and m_c defined in the previous section when we choose $b_1 = -2.1$ and $b_2 = -2.0$.

Table 3.4 Lipschitz constant for each radius r for unmeasured disturbance in feed concentration when $b_1 = -2.1$, $b_2 = -2.0$, where $\delta_\eta = 0.0188$

r	m	$\frac{\delta_\eta\alpha}{k - m\alpha}$	m_c
0.33	0.2744	0.163	0.3328
0.163	0.079	0.061	0.2745
0.061	0.048	0.055	0.0819
0.055	0.044	0.054	0.0483
0.054	0.042	0.054	0.042

By the same argument as in the previous section when we choose $b_1 = -2.1$, $b_2 = -2.0$ we can say that for every r_i such that $0.054 \leq r_i < 0.33$ if $\| z(0) \| \leq r_i / 2.566$ then the solution of the system $\| z(t) \| \leq r_i$ for every $t \in [0, \infty)$ and $\lim_{t\to\infty} \| z(t) \| \leq r_d$, where $r_d = 0.054$.

As an example we assume that disturbance d_c has the following time varying form:

$$d_c = 0.83 \times 10^{-2} \sin t$$

If we choose $b_1 = -2.1$ and $b_2 = -2.0$, for example, and $\| z(0) \| < 0.129$ then by Theorem 3 the solution $\| z(t) \| < 0.33$ for every $t \in [0, \infty)$ and converges into the ball $B_{r_d}(z = 0)$, where $r_d = 0.054$. Fig.3.4 shows this, where line 1 is the solution trajectory when the initial condition of concentration $C = 3.008 \times 10^{-3}$ gmole / cc and temperature $T = 404.51$ K (corresponding to $x_1(0) = -0.12$, $x_2(0) = 0.62$) and line 2 is when the initial concentration $C = 4.956 \times 10^{-3}$ gmole / cc and temperature $T = 391.48$ K (corresponding to $x_1(0) = 0.128$, $x_2(0) = -0.5$) (program name: OUTX1DI for trajectories, CHCONT2 for contours).

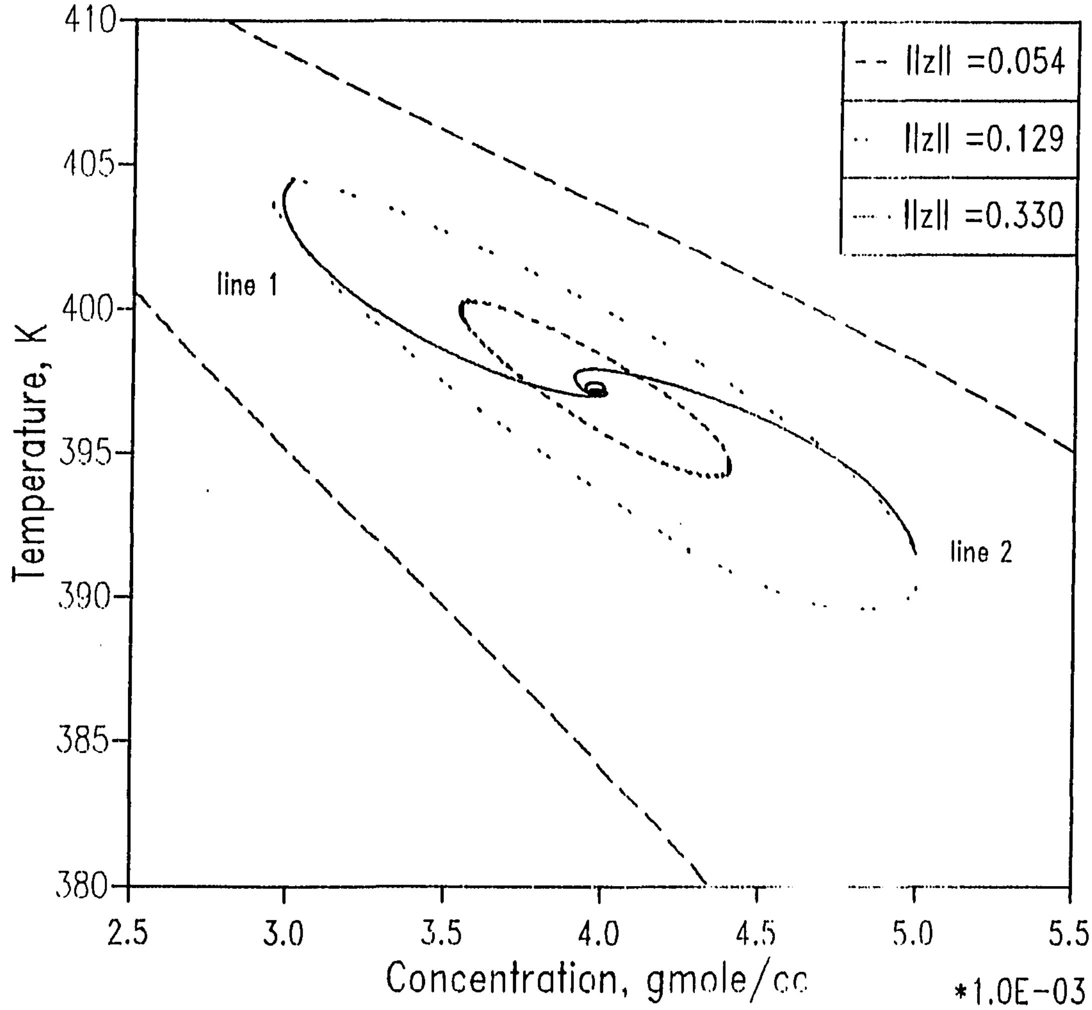

Fig. 3.4 Estimated bounded and converging region of the solution trajectories for the unmeasured disturbance in feed concentration,

where $b_1 = -2.1$ and $b_2 = -\ 2.0$ and

line 1: initial condition, $C = 3.008 \times 10^{-3}$ gmole/cc, $T = 404.51$ K

line 2: initial condition, $C = 4.956 \times 10^{-3}$ gmole/cc, $T = 391.48$ K

3.4.4.3. Measurement Error in Concentration in the Reactor

In this section we consider robustness for measurement error. We assume that the state variable x_1 has ± 1.0 % measurement error which is independent of x_1, that is, the measured variable, $\hat{x}_1 = x_1 + \Delta x_1$, where x_1 is a true value and Δx_1 is the measurement error, independent of the value of x. This measurement error can be a random variable or time varying function. For simplicity in this section we assume that there is no other model-plant mismatch except the measurement error.

Consider the coordinate transformation:

$$z_i = L_f^{i-1}h(x),\ i = 1, \cdots, \rho+1 \tag{3.4.19}$$

where z_i is based on the true state variable

Since state feedback u is based on the measured state variable, the system (3.4.3) is decomposed when v = Kz as follows:

$$\dot{z}_1 = z_2 \tag{3.4.20}$$

$$\dot{z}_2 = b_1 z_1 + b_2 z_2 + [L_g L_f h(x)]\, \Delta u$$

where $\Delta u = u - u_0$

$$u = \left[\frac{\partial f_1(\hat{x})}{\partial x_2} g_2(\hat{x})\right]^{-1} \left\{ b_1 \hat{x}_1 + b_2 f_1(\hat{x}) - A^*(\hat{x}) \right\}$$

$$u_0 = \left[\frac{\partial f_1(x)}{\partial x_2} g_2(x)\right]^{-1} \left\{ b_1 x_1 + b_2 f_1(x) - A^*(x) \right\}$$

$$A^*(\hat{x}) = \frac{\partial f_1(x)}{\partial x_1} f_1(\hat{x}) + \frac{\partial f_1(x)}{\partial x_2} f_2(\hat{x})$$

$$A^*(x) = \frac{\partial f_1(x)}{\partial x_1} f_1(x) + \frac{\partial f_1(x)}{\partial x_2} f_2(x)$$

that is, u_0 is the linearizing state feedback when there is no measurement error. Therefore in this case the perturbed nonlinear term caused by measurement error is

$$\varphi(t, x) = [\,0 \quad \varphi_2(t, x)]^T \ \text{ where } \ \varphi_2(t, x) = [L_g L_f h(x)]\, \Delta u. \tag{3.4.21}$$

Define

$B_{\Delta x} = \{ \Delta x_1;\ |\Delta x_1| < 0.01 \}$

With this $\varphi(t, x)$ and the transformation $T(x)$ in (3.4.19) we can find the Jacobian $\eta(t, z) = \varphi(t, T^{-1}(z))$ by equation (3.4.14) and from this we can find the Lipschitz constant m such that

$$m = \max_{z \in B_r} \left\| \frac{\partial \eta(t, z)}{\partial z} \right\| \quad \text{for every } \Delta x_1 \in B_{\Delta x}$$

and also from $\varphi(t, x)\,|_{x=0}$ we can find δ_η (program name : LIPSCHM).

Table 3.5 shows Lipschitz constant m, $\delta_\eta\alpha / (k - m\alpha)$ and m_c defined in the previous section when we choose $b_1 = -2.1$ and $b_2 = -2.0$, where $\delta_\eta \leq 0.0167$.

Table 3.5 Lipschitz constant for each radius r for measurement error in concentration in the reactor when $b_1 = -2.1$, $b_2 = -2.0$, where $\delta_\eta = 0.0167$

r	m	$\dfrac{\delta_\eta\alpha}{k - m\alpha}$	m_c
0.25	0.3171	0.230	0.3229
0.23	0.3063	0.200	0.3171
0.20	0.2835	0.157	0.3062
0.157	0.2428	0.114	0.2833
0.114	0.2129	0.094	0.2432
0.094	0.1913	0.084	0.2120
0.084	0.1820	0.080	0.1909
0.080	0.181	0.08	0.181

Therefore when $b_1 = -2.1$ and $b_2 = -2.0$ we can say that for every r_i such that $0.08 \le r_i < 0.25$ if $\| z(0) \| \le r_i / 2.566$ then the solution of the system $\| z(t) \| \le r_i$ for every $t \in [0, \infty)$ and $\lim_{t \to \infty} \| z(t) \| \le r_d$, where $r_d = 0.08$.

Now let's increase the feedback gain. If we choose $b_1 = -4.0$ and $b_2 = -2.7$, where $\alpha = 3.062$, $k = 1.35$, then by Theorem 3 for every r_i such that $0.042 \le r_i < 0.25$ if $\| z(0)\| \le r_i / 3.062$ then the solution of the system $\| z(t) \| \le r_i$ for every $t \in [0, \infty)$ and converges into $B_{r_d}(z = 0)$, where $r_d = 0.042$ (see Table 3.6).

Table 3.6: Lipschitz constant for each radius r for measurement error in concentration in the reactor when b_1 =-4.0, b_2 =-2.7, where $\delta_\eta = 0.0121$

r	m	$\frac{\delta_\eta \alpha}{k - m\alpha}$	m_c
0.25	0.3278	0.107	0.3925
0.107	0.1989	0.05	0.3278
0.05	0.1570	0.043	0.1989
0.043	0.1535	0.042	0.1595
0.042	0.1501	0.042	0.1528
0.041	$>m_c$		0.1458

where $>m_c$ means that m is greater than m_c at the corresponding r.

From this analysis we can see that if we increase feedback gain, r_d decreases from 0.080 to 0.042. This means that in this case the effect of measurement error can be decreased by increasing the feedback gain (actually from (3.4.20) we can see that this case of the measurement error satisfies matching condition.).

For the simulation we assume that measurement error, Δx_1 has the following form:

$\Delta x_1 = 0.01 \sin t$

If we choose $b_1 = -2.1$ and $b_2 = -2.0$ and $\| z(0) \| < 0.097$ then by Theorem 3 the solution $\| z(t) \| < 0.25$ for every $t \in [0, \infty)$ and converges into $B_{r_d}(z = 0)$, where $r_d = 0.08$. Fig.3.5 shows this, where line 1 is the solution trajectory when the initial condition of concentration, $C = 3.273 \times 10^{-3}$, gmole / cc and temperature, $T = 402.68$, K (corresponding to $x_1(0) = -0.087$, $x_2(0) = 0.462$) and line 2 is when the initial concentration, $C = 4.748 \times 10^{-3}$, gmole / cc and temperature, $T = 392.83$, K (corresponding to $x_1(0) = 0.097$, $x_2(0) = -0.384$) (program: OUTX1ME for trajectories, CHCONT2 for contours).

When $b_1 = -4.0$ and $b_2 = -2.7$, for example, if $\| z(0) \| < 0.082$ then $\|z(t) \| < 0.25$ for every $t \in [0, \infty)$ and converges into $B_{r_d}(z = 0)$ where $r_d = 0.042$. Fig.3.6 shows this, where line 1 is the solution trajectory when the initial condition of concentration, $C = 3.393 \times 10^{-3}$, gmole / cc and temperature, $T = 401.84$, K (corresponding to $x_1(0) = -0.072$, $x_2(0) = 0.39$) and line 2 is when the initial concentration, $C = 4.619 \times 10^{-3}$, gmole / cc and temperature, $T = 393.69$, K (corresponding to $x_1(0) = 0.081$, $x_2(0) = -0.31$).

Until now we have analyzed robustness of the feedback linearization of CSTR for parametric uncertainties, an unmeasured disturbance and a measurement error. For all these parametric or structural uncertainties the perturbed nonlinear term $\varphi(t, x)$ is not cancelled at $x = 0$, i.e. $\delta_\eta \neq 0$. And also $\varphi(t, x)$ does not necessarily satisfy matching condition. Moreover in this application $\| \varphi(t, x) \|$ increases as x increases. Therefore if we use the global Lipschitz condition for this case it causes severe conservativeness and, sometimes, the results become useless in practice.

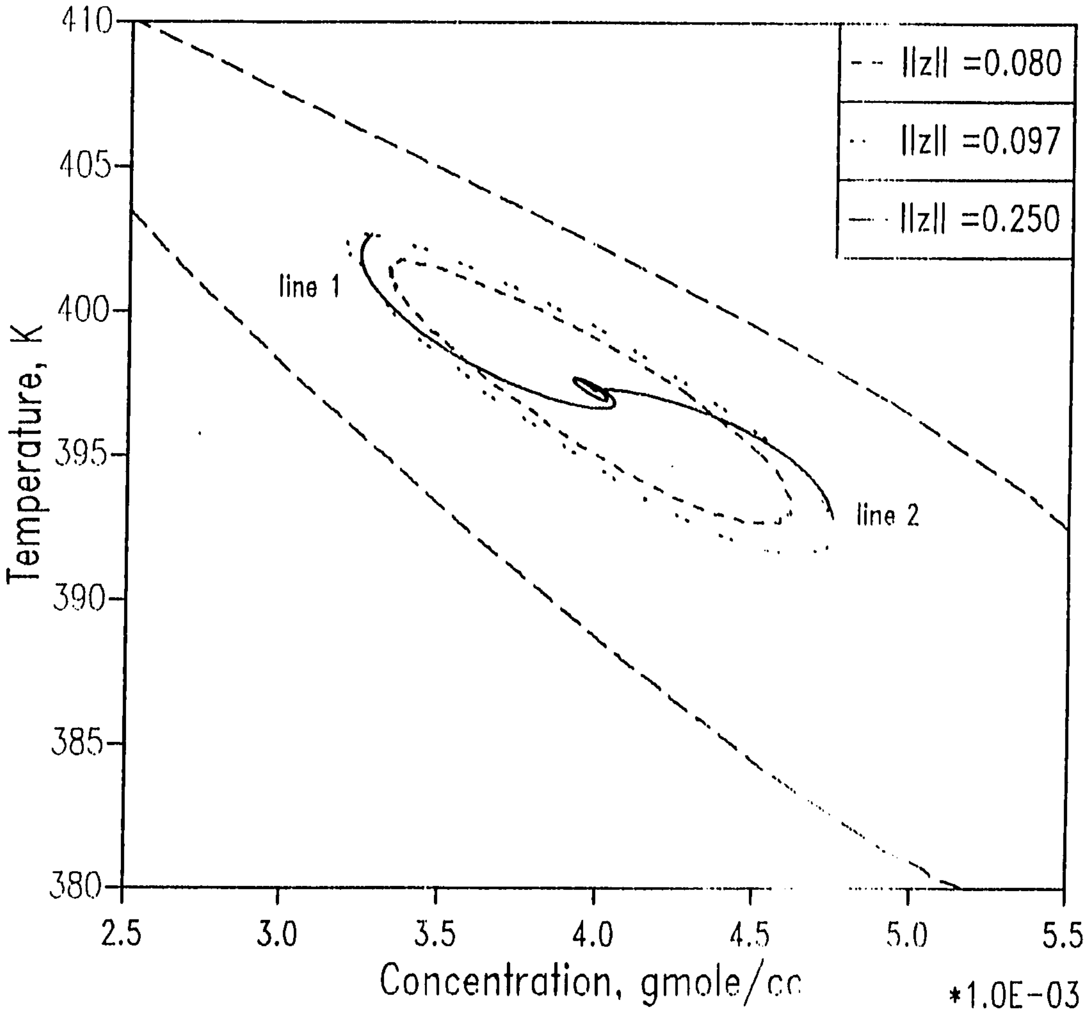

Fig. 3.5 Estimated bounded and converging region of the solution trajectories for measurement error in concentration in the reactor,

where $b_1 = -2.1$ and $b_2 = -2.0$ and

line 1: initial condition, $C = 3.273 \times 10^{-3}$ gmole/cc, $T = 402.68$ K

line 2: initial condition, $C = 4.748 \times 10^{-3}$ gmole/cc, $T = 392.83$ K

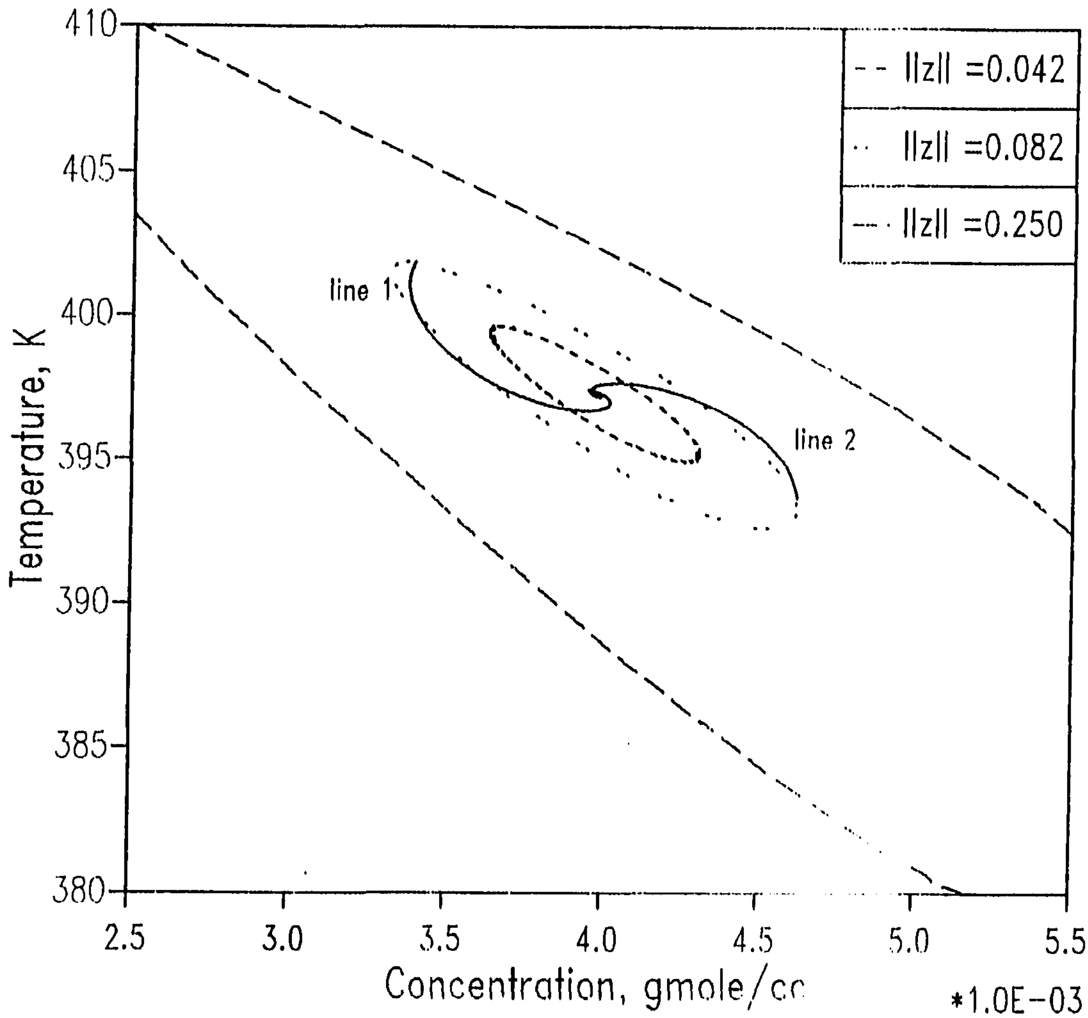

Fig. 3.6 Estimated bounded and converging region of the solution trajectories for measurement error in concentration in the reactor,

where $b_1 = -4.0$ and $b_2 = -2.7$ and

line 1: initial condition, $C = 3.393 \times 10^{-3}$ gmole/cc, $T = 401.84$ K

line 2: initial condition, $C = 4.619 \times 10^{-3}$ gmole/cc, $T = 393.69$ K

3.5. Conclusion

In this chapter we have developed a theoretical approach to analyze robustness of feedback linearization for parametric and structural uncertainties. Using this approach we can find sufficient conditions for boundedness and convergence of the system trajectories when feedback linearization based on the nominal mathematical model is applied to an uncertain real plant. The main feature of the developed approach is that it does not require the following restrictive conditions; the matching condition, global Lipschitz continuity and the same equilibrium point for the mathematical model and the real plant for all possible uncertainties.

As an example we have considered feedback linearization of a first order exothermic reaction in a CSTR and analyzed robustness for a parametric uncertainty in the reaction rate constant, for an unmeasured disturbance in feed concentration and for a measurement error in concentration in the reactor. Using the theoretical approach developed in this chapter we can estimate the upper bound and converging region of the system trajectories for a given parametric or structural uncertainty without severe conservativeness. This theoretical approach may also be utilized to design a robust state feedback.

CHAPTER IV

Robustness Analysis of Feedback Linearization for Parametric and Structural Uncertainties with Unmodeled Dynamics

4.1. Introduction

In the previous chapter robustness of feedback linearization has been investigated for parametric and structural uncertainties. In this chapter we will investigate robustness of feedback linearization for parametric and structural uncertainties as well as unmodeled dynamics.

Unmodeled dynamics may be introduced when a high dimensional model is simplified to a reduced dimensional model. This dimensional reduction sometimes tremendously simplifies the design of a control system. However, to get satisfactory performance of the control system designed based on the reduced dimensional system, certain properties of the reduced dimensional system must agree with the corresponding values in the higher dimensional real plant. In practice, dimensional reduction is not a simple or a trivial problem.

As indicated in earlier chapters, linearization by coordinate change and state feedback can yield good performance for some nonlinear systems. However, sometimes, it is very difficult or impossible to apply this technique to high-order systems. In many practical cases, it may be beneficial to apply feedback linearization to a reduced dimensional system.

With the dimensional reduction, however, robustness of feedback linearization becomes a more difficult problem. In this chapter, we have developed a theoretical approach to analyze robustness of feedback linearization for parametric and structural uncertainties with unmodeled dynamics. In our analysis, we have restricted ourselves to a special form of unmodeled dynamics; that is, a system that is linear in the state variables of the unmodeled dynamics. However, if we can find the boundedness of the state variables of unmodeled dynamics under the given bound of the state variables of a reduced dimensional model, then other types of unmodeled dynamics may also be analyzed with the same theoretical approach.

The approach developed in this chapter does not necessarily require the restrictive conditions frequently used in previous work such as matching condition, global Lipschitz continuity and an invariant equilibrium point for all possible uncertainties.

As an application, an unstable multicomponent exothermic chemical reaction in a CSTR has been chosen. It is assumed that there exists a parametric uncertainty in this reaction system. The full order mathematical model of this reaction system, assumed to have the same dimensional state as the real plant, cannot be easily linearized in the state space. For this reaction system, dimensional reduction has been done by assuming the concentration of the intermediate component, which is very reactive, is kept constant at the equilibrium state of the full-order mathematical model. For this system, we have found sufficient conditions for boundedness and convergence of system trajectories when the linearizing state feedback and coordinate transformation based on the reduced dimensional model are applied to the real plant.

4.2. Literature Review

Iwai, Fisher and Seborg (1985) recently have proposed a dimensional reduction technique for a linear system, extending Marshall's method (1966). They have applied it to the pilot plant evaporator at the University of Alberta [Wilson, 1974, Wilson et al., 1972, 73, 74] and have shown good agreement of the state variables in the reduced dimensional system with the corresponding state variables in the higher order system.

For nonlinear systems, dimensional reduction has been studied mainly on singularly perturbed systems by many authors [Kokotovic, et al., 1976, Saberi and Kharil, 1984 a, 1984 b, Khorasani and Pai, 1985, Kokotovic, et al., 1986]. The main concept of dimensional reduction in singularly perturbed systems is the invariant manifold, where system behavior is described geometrically by the rapid approach of the fast dynamics to the slow manifold. However, it is very difficult or impossible to obtain an exact closed-form of invariant manifold from the manifold condition. Therefore, an approximation to the manifold, generally, seems to be inevitable [Korasani, 1989]. In chemical engineering, the steady-state approximation of a complex reaction mechanism such as radical kinetics may be explained by a singular perturbation approach [Froment and Bischoff, 1979].

This kind of dimensional reduction may be very helpful in designing a control system. Even though some high-order dynamic systems may not be feedback linearizable in the state space, simplified reduced dimensional systems can be.

Khorasani and Kokotovic (1985), Kokotovic (1985) and Spong et al (1987) have studied feedback linearization based on a reduced dimensional model when the high-order nonlinear system can be represented by a singularly perturbed system. These studies have focused on the effect of unmodeled dynamics and have not considered other types of uncertainties such as parametric or structural uncertainties.

Robust stabilization along with feedback linearization for uncertainties with unmodeled dynamics has been initiated by Taylor et al (1989) and Khorasani (1989). They have suggested a robust state feedback for singularly perturbed nonlinear systems. Taylor et al (1989) have considered parametric uncertainties with unmodeled dynamics. In this work it is assumed that parametric uncertainties satisfy the matching condition and can be represented linearly in the unknown parameters. Taylor has developed an adaptive control scheme and analyzed stability using the Lyapunov stability theorem. Khorasani (1989) has proposed a composite control scheme to make an uncertain system robust when the linearization is based on a reduced dimensional model. He has found sufficient conditions for the ultimate boundedness of system trajectories using the various concepts of Lyapunov Min-Max approach, invariant manifold and quadratic-type Lyapunov function for a singularly perturbed system.

In this chapter we propose a systematic approach for robustness analysis of feedback linearization based on a reduced dimensional model for parametric and structural uncertainties as well as unmodeled dynamics under fairly weakened assumptions. We consider only the simple case that unmodeled dynamics can be represented by a perturbed system which is linear in the state variables of the unmodeled dynamics. In fact, this is an initial result for a broader class of uncertainties and may provide a basis for extension to more general cases.

4.3. Theoretical Analysis

4.3.1. Introduction

The system considered in this work is described by the state equation:

Real plant :

$$\dot{x} = f(x) + g(x)\,u + p(x)\,w \tag{4.3.1}$$

$$\dot{w} = d(x)\,w + q(x) \tag{4.3.2}$$

$$y = h(x)$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, $u, y \in \mathbb{R}$

In this system the functions f, g, p, d, q, h are infinitely differentiable with respect to x.

Suppose that for this real plant we have the following mathematical model:

Mathematical model:

$$\dot{x} = \hat{f}(x) + \hat{g}(x)\,u + \hat{p}(x)\,w \tag{4.3.3}$$

$$\dot{w} = d(x)\,w + \hat{q}(x) \tag{4.3.4}$$

$$y = h(x)$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, $u, y \in \mathbb{R}$

and the functions $\hat{f}$, $\hat{g}$, $\hat{p}$ and $\hat{q}$ are also infinitely differentiable.

That is, we have model-plant mismatch in the vector fields f, g, p and q. From now on we will call the dynamics (4.3.4) (or (4.3.2)) the *higher-order dynamics*. The above mathematical model, which has model-plant mismatch and (n+m)-dimensional state space, will be called the *full model.*

In the full or reduced dimensional model, which will be introduced later, without loss of generality it is assumed that $\hat{f}(0) = 0$. It is noted that when $\hat{q}(0) = 0$ the equilibrium point of the full model is $x = w = u = 0$. However, when $\hat{q}(0) \neq 0$,

$x = w = u = 0$ is not an equilibrium point of the full model. Of course, in any case the real plant may have a different equilibrium point.

4.3.2. Dimensional Reduction

Consider the higher order dynamics (4.3.4) in the mathematical model. If a solution w(t) exists, then it can be written

$$w(t) = \Phi_w(t,0)\, w(0) + \int_0^t \Phi_w(t,\tau)\, \hat{q}(x(\tau))\, d\tau$$

where x(t) is considered as a parameter and $\Phi_w(t,\tau)$ is the transition matrix of the dynamics $\dot{w} = d(x)\, w$; or equivalently, the unique solution of

$$\frac{\partial \Phi_w(t,\tau)}{\partial t} = d(x(t))\, \Phi_w(t,\tau), \qquad \Phi_w(\tau,\tau) = I$$

Suppose that $\Phi_w(t,\tau)$ satisfies the following assumption.

Assumption 4.1: If x(t) is in the ball, $B_r(z = 0)$ for every time $t \geq 0$, then there exist constants $\alpha_w \geq 1$ and $k_w > 0$ such that

$$\| \Phi_w(t,\tau) \| \leq \alpha_w\, e^{-k_w(t-\tau)} \quad \text{for every } t \geq \tau \geq 0$$

where the set $B_r(z = 0)$ is the ball centered at $z = 0$ with radius r, and the state transformation z will be defined later. ///

In practice, it is very difficult to check assumption 4.1 for a general case since the variable x is a function of time. However for some simple cases we can check assumption 4.1 easily.

First, let us consider the case that d(x) is a constant matrix. In this case if the real part of every eigenvalue of the matrix, d, is negative we can find easily the constants α_w and k_w. This case has been considered in previous Chapter 3 (see Remark 3.1, p42).

Second, when d(x) is a diagonal matrix we can also easily find the constants α_w

and k_w for the given bound of x. The diagonal matrix d(x) is represented by;

$$d(x) = \begin{bmatrix} d_1(x) & 0 & \cdots & 0 \\ 0 & d_2(x) & \cdots & 0 \\ & \vdots & & \\ 0 & \cdots & & d_m(x) \end{bmatrix}$$

Suppose that there exists a positive constant such that

$\operatorname{Re} \lambda\{ d(x) \} \leq -\sigma_s < 0$ for every fixed $x \in B_r(z = 0)$

where $\operatorname{Re} \lambda\{ \cdot \}$ = real part of the eigenvalue of $\{ \cdot \}$

that is, $d_i(x) \leq -\sigma_s < 0$, $i = 1, 2, \ldots, m$ for every fixed $x \in B_r(z = 0)$.

In this case since d(x) satisfies the commutative property [Chen, 1984, p138] and the transition matrix of the higher-order dynamics can be written

$$\Phi_w(t,\tau) = \exp\left[\int_\tau^t d(x(s))\, ds \right]$$

and we can see that if x(t) is in the ball, $B_r(z = 0)$ for every time $t \geq 0$ we can find easily the constants α_w and k_w. In fact in this case we can see that $\alpha_w = 1$ (since d(x) is diagonal) and $k_w = \sigma_s$.

Now, suppose that the constant k_w is very large. In this case we can see intuitively that if x(t) is in the ball, $B_r(z = 0)$ for every time $t \geq 0$ and the norm of $\hat{q}(x)$ is relatively small for every $x \in B_r(z = 0)$, then the solution w(t) of the full model very rapidly approaches to the equilibrium point (of the full model). Remember that when $\hat{q}(0) = 0$ the equilibrium point of the full model is also $x = w = u = 0$. However, when $\hat{q}(0) \neq 0$, $x = w = u = 0$ is not an equilibrium point of the full model. In this case under the assumption 4.1 the zero state, i.e. $x = w = 0$, can only be approximated as an equilibrium point.

Therefore, if x(t) is in the ball, $B_r(z = 0)$ for every time $t \geq 0$ the higher order dynamics can be totally ignored and the *reduced dimensional model* has the following form:

$$\dot{x} = \hat{f}(x) + \hat{g}(x)\,u \qquad (4.3.5)$$
$$y = h(x)$$

Note that since $\hat{f}(0) = 0$, $x = u = 0$ is the equilibrium point of the reduced dimensional model (4.3.5).

Actually the above argument for reducing the dimension is not quantitative and somewhat ambiguous. Mathematically, if the constant k_W goes to infinity with the finite norm of $\hat{q}(x)$ then it can be easily seen that w(t) is kept zero at almost every time, $t > 0$. Practically this may not happen. Therefore it may be very valuable to investigate whether feedback linearization based on the above dimensional reduction with parametric and structural uncertainties yields an acceptable result when k_W is still finite but very large.

Our objective in this chapter is to analyze robustness of feedback linearization with unmodeled dynamics as well as with parametric and structural uncertainties. In other words, we want to find sufficient conditions for the boundedness (to the set $B_{r'}(z = 0)$) and convergence of the solution trajectories of the real plant, x(t), when we apply the linearizing state feedback based on the reduced dimensional model obtained by the above arguments.

4.3.3. Feedback Linearization Based on the Reduced Dimensional Model

The reduced dimensional model has important benefits in designing a control system. For example, the system (4.3.5) may be easily linearizable by a coordinate change and state feedback even though feedback linearization of the full model may be impossible or, if possible, very complicated.

Let's assume that

Assumption 4.2: The n-dimensional reduced dimensional model (4.3.5) has relative degree n. ///

Under assumption 4.2 we already know from chapter 3 that the linearizing state feedback and coordinate change for the system (4.3.5) is

$$z_i = T_i(x) = L_{\hat{f}}^{i-1} h(x), \quad i = 1, 2, \ldots, \rho+1 \ (= n) \tag{4.3.6}$$

$$u = \alpha(x) + \beta(x)\, v \tag{4.3.7}$$

$$\text{where } \alpha(x) = - \hat{D}^*(x)^{-1} \hat{A}^*(x)$$

$$\beta(x) = \hat{D}^*(x)^{-1}$$

$$\hat{D}^* = L_{\hat{g}} L_{\hat{f}}^{\rho} h(x)$$

$$\hat{A}^* = L_{\hat{f}}^{\rho+1} h(x)$$

Applying the coordinate transformation (4.3.6) and state feedback (4.3.7) to the real plant (4.3.1) we have

$$\dot{z}_1 = \frac{\partial h(x)}{\partial x} \{ f(x) + g(x)\, u + p(x)\, w \}$$

Define $\Delta f(x) \equiv f(x) - \hat{f}(x)$

$\Delta g(x) \equiv g(x) - \hat{g}(x)$

Then

$$\begin{aligned} \dot{z}_1 &= \frac{\partial h(x)}{\partial x} \{ \hat{f}(x) + \Delta f(x) + (\hat{g}(x) + \Delta g(x)) u + p(x)\, w \} \\ &= L_{\hat{f}} h(x) + L_{\hat{g}} h(x)\, u + \frac{\partial h(x)}{\partial x} \{ \Delta f(x) + \Delta g(x)\, u + p(x)\, w \} \\ &= z_2 + \frac{\partial h(x)}{\partial x} \{ \Delta f(x) + \Delta g(x)\, u \} + \frac{\partial h(x)}{\partial x} p(x)\, w \end{aligned}$$

Similarly

$$\dot{z}_2 = z_3 + \frac{\partial}{\partial x}(L_f^{\wedge} h(x))\{\Delta f(x) + \Delta g(x)\, u\} + \frac{\partial}{\partial x}(L_f^{\wedge} h(x))\, p(x)\, w$$

$$\vdots$$

$$\dot{z}_{\rho+1} = L_f^{\rho+1} h(x) + \frac{\partial}{\partial x}\left(L_f^{\rho} h(x)\right)\{\Delta f(x) + \Delta g(x)\, u\} + \frac{\partial}{\partial x}\left(L_f^{\rho} h(x)\right) p(x)\, w + L_g^{\wedge} L_f^{\rho} h(x)\, u$$

Applying the linearizing state feedback (4.3.7) with $v = Kz$, where $K = [b_1 \; \ldots \; b_n]$

$$\dot{z}_{\rho+1} = \sum_{i=1}^{\rho+1} b_i z_i + \frac{\partial}{\partial x}\left(L_f^{\rho} h(x)\right)\{\Delta f(x) + \Delta g(x)\, u\} + \frac{\partial}{\partial x}\left(L_f^{\rho} h(x)\right) p(x)\, w$$

Therefore finally we have

$$\dot{z} = (A+bK)\, z + \varphi(x) + \psi(x)\, w \tag{4.3.8}$$

$$\dot{w} = d(x)\, w + q(x)$$

$$y = C\, z$$

where

$$A+bK = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1 \\ b_1 & b_2 & b_3 & \ldots & b_n \end{bmatrix}$$

$$C = [\,1 \;\; 0 \;\; 0 \; \ldots\ldots \; 0 \;\; 0\,]$$

$$\varphi(x) = \begin{bmatrix} \frac{\partial h(x)}{\partial x}\{\Delta f(x) + \Delta g(x)\, u\} \\ \frac{\partial}{\partial x}(L_f^{\wedge} h(x))\{\Delta f(x) + \Delta g(x)\, u\} \\ \vdots \\ \frac{\partial}{\partial x}\left(L_f^{\rho} h(x)\right)\{\Delta f(x) + \Delta g(x)\, u\} \end{bmatrix}$$

$$\psi(x) = \begin{bmatrix} \frac{\partial h(x)}{\partial x} p(x) \\ \frac{\partial}{\partial x}(L_f h(x))\, p(x) \\ \vdots \\ \frac{\partial}{\partial x}\left(L_f^{\rho} h(x)\right) p(x) \end{bmatrix}$$

Define

$$\xi(z) = \varphi(T^{-1}(z))$$
$$\mu(z) = \psi(T^{-1}(z))$$
$$\sigma(z) = q(T^{-1}(z))$$
$$\kappa(z) = d(T^{-1}(z))$$

As mentioned in the previous chapter, the above representation of the terms φ, ψ, q and d in the z-coordinate system is only for convenience of the theoretical development.

With the above defined variables the system (4.3.8) can be written in the z - coordinate system:

$$\begin{aligned} \dot{z} &= (A+bK)\, z + \xi(z) + \mu(z)\, w \qquad \text{at } t = 0,\ z(t) = z(0),\ w(t) = w(0) \\ \dot{w} &= \kappa(z)\, w + \sigma(z) \\ y &= C\, z \end{aligned} \tag{4.3.9}$$

When d(x) is not a function of x the above system (4.3.9) may be analyzed using the previous Theorem 3 since in this case it can be represented by

$$\begin{bmatrix} \dot{z} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A+bK & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} + \begin{bmatrix} \xi(z) \\ \sigma(z) \end{bmatrix} + \begin{bmatrix} \mu(z) \\ 0 \end{bmatrix} w$$

However, in this case, severe conservativeness may occur when the eigenvalues of A+bK and d are very different.

In this section we consider the higher-order dynamics (4.3.2) separately in order to avoid this problem. Using the following Theorem 4 we can find sufficient conditions for the boundedness and convergence of the solution z(t) of the system (4.3.9).

Theorem 4 :

Consider the above system (4.3.9). Suppose that assumptions 4.1, 4.2 and following assumptions are satisfied.

Assumption 4.3: There exist finite nonnegative constants m_w and δ_w such that

$$||\sigma(z)|| \le m_w ||z|| + \delta_w \quad \text{for every } z \in B_r(z=0)$$

where $B_r(z=0)$ = ball with radius of r around z = 0.

Assumption 4.4: There exist constants $\alpha \ge 1$ and $k > 0$ such that

$$||\Phi(t,\tau)|| \le \alpha\, e^{-k(t-\tau)} \quad \text{for every } t \ge \tau \ge 0$$

where $\Phi(t,\tau)$ is the state transition matrix of the dynamics

$$\dot{z} = (A+bK)\, z$$

Assumption 4.5: There exist finite nonnegative constants m_1, δ_1 and m_2, δ_2 such that

$$||\xi(z) - \xi(y)|| \le m_1 ||z - y|| \quad \text{for every } z, y \in B_r(z=0)$$

$$||\xi(0)|| \le \delta_1$$

$$||\mu(z) - \mu(y)|| \le m_2 ||z - y|| \quad \text{for every } z, y \in B_r(z=0)$$

$$||\mu(0)|| \le \delta_2$$

That is, $\xi(z)$ and $\mu(z)$ are incrementally Lipschitz continuous in $B_r(z=0)$.

Define c^* such that

$$c^* = \begin{cases} 0 & \text{when } \| w(0) \| \le \dfrac{(m_w r + \delta_w)}{k_w} \\ \dfrac{(m_2 r + \delta_2)\, \alpha\, \alpha_w \|w(0)\|}{k_w} - \dfrac{(m_2 r + \delta_2)\, \alpha\, \alpha_w (m_w r + \delta_w)}{k_w^2} & \\ & \text{when } \| w(0) \| > \dfrac{(m_w r + \delta_w)}{k_w} \end{cases}$$

and suppose that $c^* < r$.

Then if $\alpha \| z(0) \| < r - c^*$ and $\dfrac{\alpha}{k} (m_1 r + \delta_1) + \dfrac{(m_2 r + \delta_2)\, \alpha\, \alpha_w (m_w r + \delta_w)}{k\, k_w} < r$ then

$|| z(t) || < r$ for every $t \in [0, \infty)$. And moreover $\lim\limits_{t \to \infty} \| z(t) \| \le r_d$, where

$$r_d = \begin{cases} \dfrac{-c_2 - \sqrt{c_2^2 - 4c_1c_3}}{2\,c_1} & \text{when } c_1 > 0 \\ \dfrac{c_3}{|c_2|} & \text{when } c_1 = 0 \end{cases}$$

where c_1, c_2 and c_3 are defined by

$$c_1 = \alpha\, \alpha_w m_2 m_w$$

$$c_2 = k_w \alpha\, m_1 + \alpha\, \alpha_w m_2 \delta_w + \alpha\, \alpha_w \delta_2 m_w - k\, k_w$$

$$c_3 = k_w \alpha\, \delta_1 + \alpha\, \alpha_w \delta_2 \delta_w$$

///

Proof: The proof of Theorem 4 is similar to the previous Theorem 3. First, we will show that there exists a solution z(t) in the set S, which is defined by

$$S \equiv \{ z(t)\, ;\ \| z(t) \|_c < r \}$$

Consider the following dynamics

$$\dot{z} = (A+bK)\, z + \xi(z) + \mu(z)\, w \qquad (4.3.10)$$

If a solution z(t) exists, then

$$z(t) = \Phi(t,0)\, z(0) + \int_0^t \Phi(t,\tau) \Big(\xi(z(\tau) + \mu(z(\tau))\, w\Big)\, d\tau$$

Define a mapping T by;

$$Tz(t) = \Phi(t,0)\, z(0) + \int_0^t \Phi(t,\tau) \Big(\xi(z(\tau) + \mu(z(\tau))\, w\Big)\, d\tau \tag{4.3.11}$$

Assume that $z(t) \in S$. Then by the above assumptions 4.4 and 4.5 $\| Tz(t) \|$ is

$$\begin{aligned} \| Tz(t) \| \le \alpha e^{-kt} \| z(0) \| \\ + \int_0^t \alpha e^{-k(t-\tau)} \Big(\big[m_1 \| z(\tau) \| + \delta_1\big] + \big[m_2 \| z(\tau) \| + \delta_2\big] \; \| w(\tau) \| \Big)\, d\tau \end{aligned}$$

for every $z(t) \in S$

(4.3.12)

And also if $\| z(t) \| \le r$ for every $t \ge 0$, then

$$\begin{aligned} \| w(t) \| &\le \alpha_w e^{-k_w t} \| w(0) \| + \int_0^t \alpha_w e^{-k_w(t-\tau)} \big[m_w \|z(\tau)\| + \delta_w\big]\, d\tau \\ &= \alpha_w e^{-k_w t} \| w(0) \| + \frac{\alpha_w (m_w r + \delta_w)}{k_w} \Big(1 - e^{-k_w t}\Big) \end{aligned} \tag{4.3.13}$$

Substituting (4.3.13) into (4.3.12) then

$$\begin{aligned} \|Tz(t)\| \le \alpha \, \|z(0)\|\, e^{-kt} &+ \frac{\alpha}{k} (m_1 r + \delta_1) \Big(1 - e^{-kt}\Big) \\ &+ \frac{(m_2 r + \delta_2)\, \alpha\, \alpha_w \, \|w(0)\|}{k - k_w} \Big(e^{-k_w t} - e^{-kt}\Big) \\ &+ \frac{(m_2 r + \delta_2)\, \alpha\, \alpha_w (m_w r + \delta_w)}{k\; k_w} \Big(1 - e^{-kt}\Big) \\ &- \frac{(m_2 r + \delta_2)\, \alpha\, \alpha_w (m_w r + \delta_w)}{k_w (k - \; k_w)} \Big(e^{-k_w t} - e^{-kt}\Big) \end{aligned} \tag{4.3.14}$$

Case 1: When $\| w(0) \| \le \dfrac{(m_w r + \delta_w)}{k_w}$

$$\|Tz(t)\| \le \alpha \|z(0)\| e^{-kt} + \frac{\alpha}{k}(m_1 r + \delta_1)\left(1 - e^{-kt}\right)$$
$$+ \frac{(m_2 r + \delta_2)\,\alpha\,\alpha_w (m_w r + \delta_w)}{k\ k_w}\left(1 - e^{-kt}\right)$$

Therefore if $\alpha \| z(0) \| < r$ and $\dfrac{\alpha}{k}(m_1 r + \delta_1) + \dfrac{(m_2 r + \delta_2)\,\alpha\,\alpha_w\,(m_w r + \delta_w)}{k\,k_w} < r$ then $\|Tz(t)\| < r$ for every $t \in [0, \infty)$.

Case 2: When $\| w(0) \| > \dfrac{(m_w r + \delta_w)}{k_w}$

$$\|Tz(t)\| \le (r - c^*)\, e^{-kt} + r\left(1 - e^{-kt}\right)$$
$$+ (m_2 r + \delta_2)\,\alpha\,\alpha_w \left\{ \|w(0)\| - \frac{(m_w r + \delta_w)}{k_w} \right\} \frac{\left(e^{-k_w t} - e^{-kt}\right)}{(k - k_w)} \tag{4.3.15}$$

It is easy to see that $\dfrac{\partial}{\partial t}\left(e^{-k_w t} - e^{-kt}\right) = 0$ at $t = t^* = \dfrac{\ln (k_w/k)}{k_w - k}$, that is, at $t = t^*$ right-hand side of (4.3.15) has maximum value.

If the given conditions are all satisfied then from equation (4.3.15)

$$\| Tz(t) \| \le r - c^* e^{-kt^*} + c^*\, e^{-kt^*}$$
$$\Rightarrow \| Tz(t) \| < r \quad \text{for every } t \ge 0$$

Therefore T maps S into itself.

Next, we will show that the mapping T is a contraction. From equation (4.3.11) for any $z(t), y(t) \in S$

$$\| Tz(t) - Ty(t) \| \le \int_0^t \alpha\, e^{-k(t-\tau)}\, m_1 \| z(\tau) - y(\tau) \|\, d\tau$$
$$+ \int_0^t \alpha\, e^{-k(t-\tau)}\, m_2 \| z(\tau) - y(\tau) \|$$
$$\cdot \left\{ \alpha_w\, e^{-k_w\tau} \| w(0) \| + \frac{\alpha_w(m_w r + \delta_w)}{k_w} \left(1 - e^{-k_w\tau}\right) \right\} d\tau$$

Taking a sup on both sides then

$$\sup_{t \in [0,\infty)} \| Tz(t) - Ty(t) \|$$
$$\le \| z(\tau) - y(\tau) \|_c \sup_{t \in [0,\infty)} \int_0^t \alpha\, e^{-k(t-\tau)}$$
$$\cdot \left(m_1 + m_2 \left\{ \alpha_w e^{-k_w\tau} \|w(0)\| + \frac{\alpha_w(m_w r + \delta_w)}{k_w} \left[1 - e^{-k_w\tau}\right] \right\} \right) d\tau$$

Define

$$\eta \equiv \sup_{t \in [0,\infty)} \int_0^t \alpha\, e^{-k(t-\tau)}$$
$$\cdot \left(m_1 + m_2 \left\{ \alpha_w e^{-k_w\tau} \|w(0)\| + \frac{\alpha_w(m_w r + \delta_w)}{k_w} \left[1 - e^{-k_w\tau}\right] \right\} \right) d\tau$$
$$= \sup_{t \in [0,\infty)} \left[\frac{m_1\alpha}{k} (1 - e^{-kt}) + \frac{\alpha\, \alpha_w m_2 \|w(0)\|}{k - k_w} (e^{-k_w t} - e^{-kt}) \right.$$
$$\left. + m_2 \alpha\, \alpha_w (m_w r + \delta_w) \left\{ \frac{1}{k\, k_w} (1 - e^{-kt}) - \frac{1}{k_w (k - k_w)} (e^{-k_w t} - e^{-kt}) \right\} \right]$$

Case 1: When $\| w(0) \| \le \dfrac{(m_w r + \delta_w)}{k_w}$

$$\eta \le \sup_{t \in [0,\infty)} \left[\frac{m_1\alpha}{k} (1 - e^{-kt}) + \frac{\alpha\, \alpha_w m_2 (m_w r + \delta_w)}{k\, k_w} (1 - e^{-kt}) \right]$$
$$\le \frac{m_1\alpha}{k} + \frac{\alpha\, \alpha_w m_2 (m_w r + \delta_w)}{k\, k_w}$$
$$< 1$$

Case 2: When $\| w(0) \| > \dfrac{(m_w r + \delta_w)}{k_w}$

From equation (4.3.14) since if $\| z(t) \| < r$ then $\| Tz(t) \| < r$ for every $t \geq 0$

$$\frac{\alpha}{k}(m_1 r + \delta_1)\left(1 - e^{-kt}\right) + \frac{(m_2 r + \delta_2)\,\alpha\,\alpha_w \,\|w(0)\|}{k - k_w}\left(e^{-k_w t} - e^{-kt}\right)$$
$$+ \frac{(m_2 r + \delta_2)\,\alpha\,\alpha_w (m_w r + \delta_w)}{k\ k_w}\left(1 - e^{-kt}\right)$$
$$- \frac{(m_2 r + \delta_2)\,\alpha\,\alpha_w (m_w r + \delta_w)}{k_w (k - k_w)}\left(e^{-k_w t} - e^{-kt}\right) < r \quad \text{for every } t \geq 0$$

$$\Rightarrow \frac{\alpha\, m_1 r}{k}\left(1 - e^{-kt}\right) + \frac{m_2 r\,\alpha\,\alpha_w\,(m_w r + \delta_w)}{k\,k_w}\left(1 - e^{-kt}\right)$$
$$+ \frac{\alpha\,\alpha_w m_2 r}{(k - k_w)}\left(\|w(0)\| - \frac{(m_w r + \delta_w)}{k_w}\right)\left(e^{-k_w t} - e^{-kt}\right)$$
$$+ \frac{\alpha\,\delta_1}{k}\left(1 - e^{-kt}\right) + \frac{\delta_2\,\alpha\,\alpha_w\,(m_w r + \delta_w)}{k\,k_w}\left(1 - e^{-kt}\right)$$
$$+ \frac{\alpha\,\alpha_w \delta_2}{(k - k_w)}\left(\|w(0)\| - \frac{(m_w r + \delta_w)}{k_w}\right)\left(e^{-k_w t} - e^{-kt}\right) < r \quad \text{for every } t \geq 0$$

Since it is always true that

$$\frac{\alpha\,\delta_1}{k}\left(1 - e^{-kt}\right) + \frac{\delta_2\,\alpha\,\alpha_w\,(m_w r + \delta_w)}{k\,k_w}\left(1 - e^{-kt}\right)$$
$$+ \frac{\alpha\,\alpha_w \delta_2}{(k - k_w)}\left(\|w(0)\| - \frac{(m_w r + \delta_w)}{k_w}\right)\left(e^{-k_w t} - e^{-kt}\right) \geq 0$$

$$\frac{\alpha\, m_1 r}{k}\left(1 - e^{-kt}\right) + \frac{m_2 r\,\alpha\,\alpha_w\,(m_w r + \delta_w)}{k\,k_w}\left(1 - e^{-kt}\right)$$
$$+ \frac{\alpha\,\alpha_w m_2 r}{(k - k_w)}\left(\|w(0)\| - \frac{(m_w r + \delta_w)}{k_w}\right)\left(e^{-k_w t} - e^{-kt}\right) < r$$

From this relation we can see that $\eta < 1$ for every $t \geq 0$.

Therefore the mapping T is a contraction. So we can say that by the contraction mapping theorem (see Appendix III) there exists a solution z(t) in the set S. By almost the same argument as for Theorem 3 it is easy to see that the solution is unique for a given initial condition. Finally convergence of the solution will be proved.

Let z(t) and y(t) be any two solutions of the system (4.3.10). Then from equation (4.3.11)

$$\| z(t) - y(t) \| \leq \alpha e^{-kt} \| z(0) - y(0) \| + \int_0^t \alpha m_1 e^{-k(t-\tau)} \|z(\tau) - y(\tau)\| d\tau$$
$$+ \int_0^t \alpha m_2 e^{-k(t-\tau)} \|z(\tau) - y(\tau)\|$$
$$\cdot \left\{ \alpha_w e^{-k_w\tau} \|w(0)\| + \frac{\alpha_w(m_w r + \delta_w)}{k_w}\left(1 - e^{-k_w\tau}\right)\right\} d\tau$$

Define

$$r(t) = \| z(t) - y(t) \| e^{kt}$$
$$c = \alpha \| z(0) - y(0) \|$$
$$K(\tau) = \alpha m_1 + \alpha m_2 \cdot \left\{ \alpha_w e^{-k_w\tau} \|w(0)\| + \frac{\alpha_w(m_w r + \delta_w)}{k_w}\left(1 - e^{-k_w\tau}\right)\right\}$$

Then the above equation can be written

$$r(t) \leq c + \int_0^t r(\tau) K(\tau) d\tau$$

By Bellman - Gronwell lemma [see Appendix II, Vidyasagar, 1978, p292]

$$r(t) \leq c \exp\left[\int_0^t K(\tau) d\tau\right]$$

$$\Rightarrow \| z(t) - y(t) \| \leq c \exp\left[\int_0^t K(\tau) d\tau - kt\right]$$

Let $a_1 = (\alpha m_1 - k) + \dfrac{\alpha \alpha_w m_2(m_w r + \delta_w)}{k_w}$

$$a_2 = \frac{\alpha \alpha_w m_2 \| w(0) \|}{k_w} - \frac{\alpha \alpha_w m_2(m_w r + \delta_w)}{k_w^2}$$

Then we can see that

$$\| z(t) - y(t) \| \leq c \exp\left[a_1 t + a_2(1 - e^{-k_w t})\right]$$

It is easy to see that $a_1 < 0$ by the following argument:

$$\frac{\alpha}{k}(m_1 r + \delta_1) + \frac{(m_2 r + \delta_2)\,\alpha\,\alpha_w\,(m_w r + \delta_w)}{k\,k_w} < r \quad \text{by hypothesis}$$

$$\Leftrightarrow \frac{\alpha m_1}{k} r + \frac{m_2 r\,\alpha\,\alpha_w\,(m_w r + \delta_w)}{k\,k_w} + \frac{\alpha\delta_1}{k} + \frac{\delta_2\alpha\,\alpha_w\,(m_w r + \delta_w)}{k\,k_w} < r.$$

Since $\frac{\alpha\delta_1}{k} + \frac{\delta_2\alpha\,\alpha_w\,(m_w r + \delta_w)}{k\,k_w} \geq 0$

$$\frac{\alpha\, m_1}{k} + \frac{\alpha\,\alpha_w m_2(m_w r + \delta_w)}{k_w k} < 1$$

$$\Rightarrow (\alpha\, m_1 - k) + \frac{\alpha\,\alpha_w m_2(m_w r + \delta_w)}{k_w} < 0$$

$$\therefore\ a_1 < 0.$$

And also it is easy to see that a_2 is bounded. Therefore $\| z(t) - y(t) \| \to 0$ as $t \to \infty$.

Next, let's find the value of r_d such that $\lim_{t \to \infty} \| z(t) \| \leq r_d$. Consider the following equation

$$\frac{\alpha}{k}(m_1 r + \delta_1) + \frac{(m_2 r + \delta_2)\,\alpha\,\alpha_w\,(m_w r + \delta_w)}{k\,k_w} < r \tag{4.3.16}$$

It can be seen that r_d is the minimum value of r which satisfies the above equation because of the uniqueness of the solution of the given nonlinear system.

Define

$$c_1 = \alpha\,\alpha_w m_2 m_w$$

$$c_2 = k_w \alpha\, m_1 + \alpha\,\alpha_w m_2 \delta_w + \alpha\,\alpha_w \delta_2 m_w - k\,k_w$$

$$c_3 = k_w \alpha\,\delta_1 + \alpha\,\alpha_w \delta_2 \delta_w$$

where it is obvious $c_1 \geq 0$ and $c_3 \geq 0$.

Then the above condition (4.3.16) can be written

$$c_1 r^2 + c_2 r + c_3 < 0.$$

Suppose that there exists a nonnegative constant r satisfying the above equation (in this case it is obvious that $c_2 \leq 0$). Since $c_1 \geq 0$ and $c_3 \geq 0$ the other constant r which satisfies the above condition is also nonnegative.

When $c_1 > 0$ if $\dfrac{-c_2 - \sqrt{c_2^2 - 4c_1c_3}}{2c_1} < r < \dfrac{-c_2 + \sqrt{c_2^2 - 4c_1c_3}}{2c_1}$ and

when $c_1 = 0$ if $r > \dfrac{c_3}{|c_2|}$

the above condition (4.3.16) is satisfied and from this obviously we can find r_d. ///

Remark 4.1: If $c_3 = 0$ then it is obvious that $r_d = 0$, that is, $\lim_{t \to \infty} \| z(t) \| = 0$.

4.4. Application: Multicomponent Exothermic Chemical Reaction in CSTR

4.4.1. Introduction

The theoretical approach developed in the previous section is now applied to an unstable multicomponent exothermic chemical reaction in a CSTR. The chemical reaction system considered in this section is

$$A \underset{k_2}{\overset{k_1}{\rightleftharpoons}} B \xrightarrow{k_3} C$$

with the rates of reaction given by

$$r_1 = k_1 C_A - k_2 C_B$$

$$r_2 = k_3 C_B$$

where C_A, C_B and C_C are the concentrations of each component A, B and C and

$$k_i = k_{i0} \exp\left[\frac{-E_i}{RT}\right], \quad i = 1, 2, 3$$

In this reaction it is assumed that the reaction coefficient, k_{10} is uncertain and the heat of reaction from B to C is negligible.

For this reaction system we want to control the conversion of component A by adjusting temperature in the jacket, T_c. Fig. 4.1 shows the chemical reactor control system considered in this section. In this control system feedback linearization with the measured temperature and concentration of component A gives the set point of temperature in the jacket. Practically this temperature in the jacket is controlled by hot and cold heat transfer media.

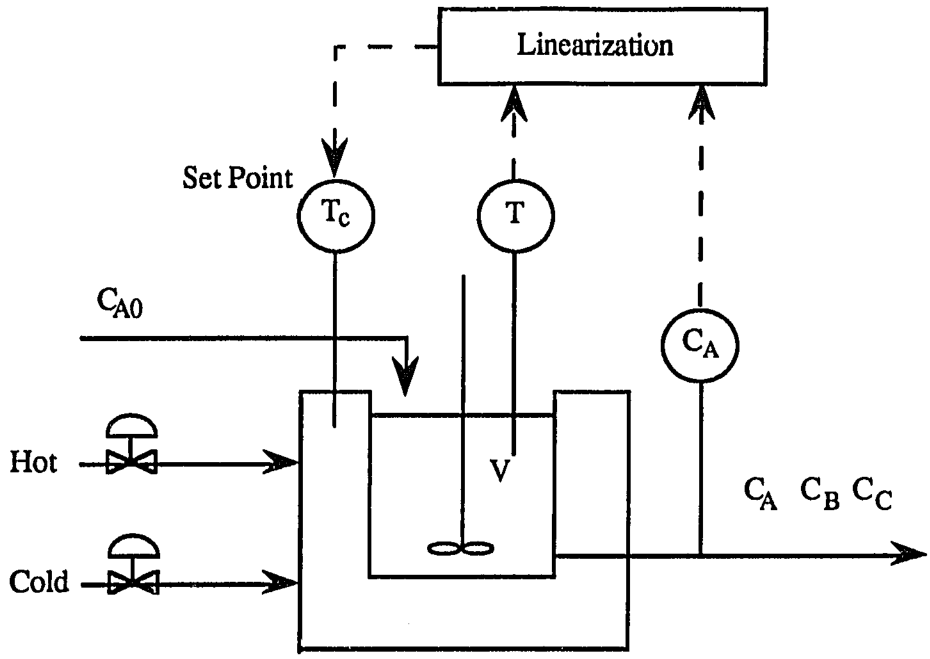

Fig. 4.1 Control system of a multicomponent exothermic chemical reaction in CSTR

4.4.2. Mathematical Model

The above multicomponent chemical reaction system can be represented by

$$\frac{dC_A}{dt} = \frac{q}{V}(C_{A0} - C_A) - (k_1C_A - k_2C_B) \tag{4.4.1}$$

$$\frac{dC_B}{dt} = \frac{q}{V}(-C_B) - (k_1C_A - k_2C_B - k_3C_B)$$

$$\frac{dT}{dt} = \frac{q}{V}(T_0 - T) - \frac{1}{\rho C_p}(\Delta H_{AB}k_1C_A - \Delta H_{AB}k_2C_B - \Delta H_{BC}k_3C_B) - \frac{UA_R}{\rho C_p V}(T - T_c)$$

The nominal values of the physical parameters in the above system are shown in Table 4.1.

Table 4.1 System parameters and operating conditions of the multicomponent chemical reaction in a CSTR

System parameters (nominal values)

$k_{10} = 3.0 \times 10^{11}$ / sec	$E_1 = 2.5088 \times 10^4$ cal / gmole
$k_{20} = 5.0 \times 10^{15}$ / sec	$E_2 = 3.6189 \times 10^4$ cal / gmole
$k_{30} = 6.0 \times 10^{13}$ / sec	$E_3 = 2.5088 \times 10^4$ cal / gmole
$R = 1.987$ cal / gmole. K	$\Delta H_{AB} = -11101.32$ cal / gmole
$\rho C_p = 1$ cal / cc . K	$\Delta H_{BC} = 0.0$ cal / gmole
$V = 3.048 \times 10^6$ cc	$U = 0.014$ cal / sec cm^2 K
$A_R = 4.645 \times 10^5$ cm^2	

Operating conditions

$T_0 = 383.3$ K	$T_c^d = 330.0$ K
$C_{A0} = 8.016 \times 10^{-3}$ gmole / cc	$q = 1.4158 \times 10^4$ cc / sec

With the given system parameters and the operating condition let us find the equilibrium points, C_A^d, C_B^d and T^d, which satisfy the following equations (remember that $\Delta H_{BC} = 0$):

$$0 = \frac{q}{V}(C_{A0} - C_A^d) - (k_1^d C_A^d - k_2^d C_B^d) \tag{4.4.2}$$

$$0 = \frac{q}{V}(- C_B^d) - (k_1^d C_A^d - k_2^d C_B^d - k_3^d C_B^d) \tag{4.4.3}$$

$$0 = \frac{q}{V}(T_0 - T^d) - \frac{1}{\rho C_p}(\Delta H_{AB} k_1^d C_A^d - \Delta H_{AB} k_2^d C_B^d) - \frac{U A_R}{\rho C_p V}(T^d - T_c^d) \tag{4.4.4}$$

$$\text{where } k_i^d = k_{i0} \exp\left[\frac{-E_i}{RT^d}\right], \; i = 1, 2, 3$$

From equations (4.4.2) and (4.4.3)

$$C_A^d = \frac{\frac{q}{V}C_{A0}}{(\frac{q}{V}+k_1^d)} + \frac{k_2^d}{(\frac{q}{V}+k_1^d)^2} \cdot \frac{\frac{q}{V}k_1^d C_{A0}}{\left\{\frac{q}{V} - \frac{k_1^d k_2^d}{(\frac{q}{V}+k_1^d)} + k_2^d + k_3^d\right\}} \tag{4.4.5}$$

$$C_B^d = \frac{\frac{q}{V}k_1^d C_{A0}}{(\frac{q}{V}+k_1^d)\left\{\frac{q}{V} - \frac{k_1^d k_2^d}{(\frac{q}{V}+k_1^d)} + k_2^d + k_3^d\right\}} \tag{4.4.6}$$

With the above C_A^d and C_B^d, equation (4.4.4) can be written

$$\left(\frac{q}{V} + \frac{UA_R}{\rho C_p V}\right)T^d - \left(\frac{q}{V}T_0 + \frac{UA_R}{\rho C_p V}T_c^d\right) = \frac{-\Delta H_{AB}}{\rho C_p}(k_1^d C_A^d - k_2^d C_B^d) \tag{4.4.7}$$

Fig. 4.2 is a plot of both sides of equation (4.4.7) against temperature and as is well known, the points of intersection of the two curves must correspond to the equilibrium points. From Fig. 4.2 we can see that there exist two stable and one unstable equilibrium points. In this example, the unstable equilibrium point is assumed to be the desired operating point, i.e.

$$C_A^d = 0.3974 \times 10^{-2} \text{ gmole / cc}$$
$$C_B^d = 0.1977 \times 10^{-4} \text{ gmole / cc}$$
$$T^d = 397.27 \text{ K}$$

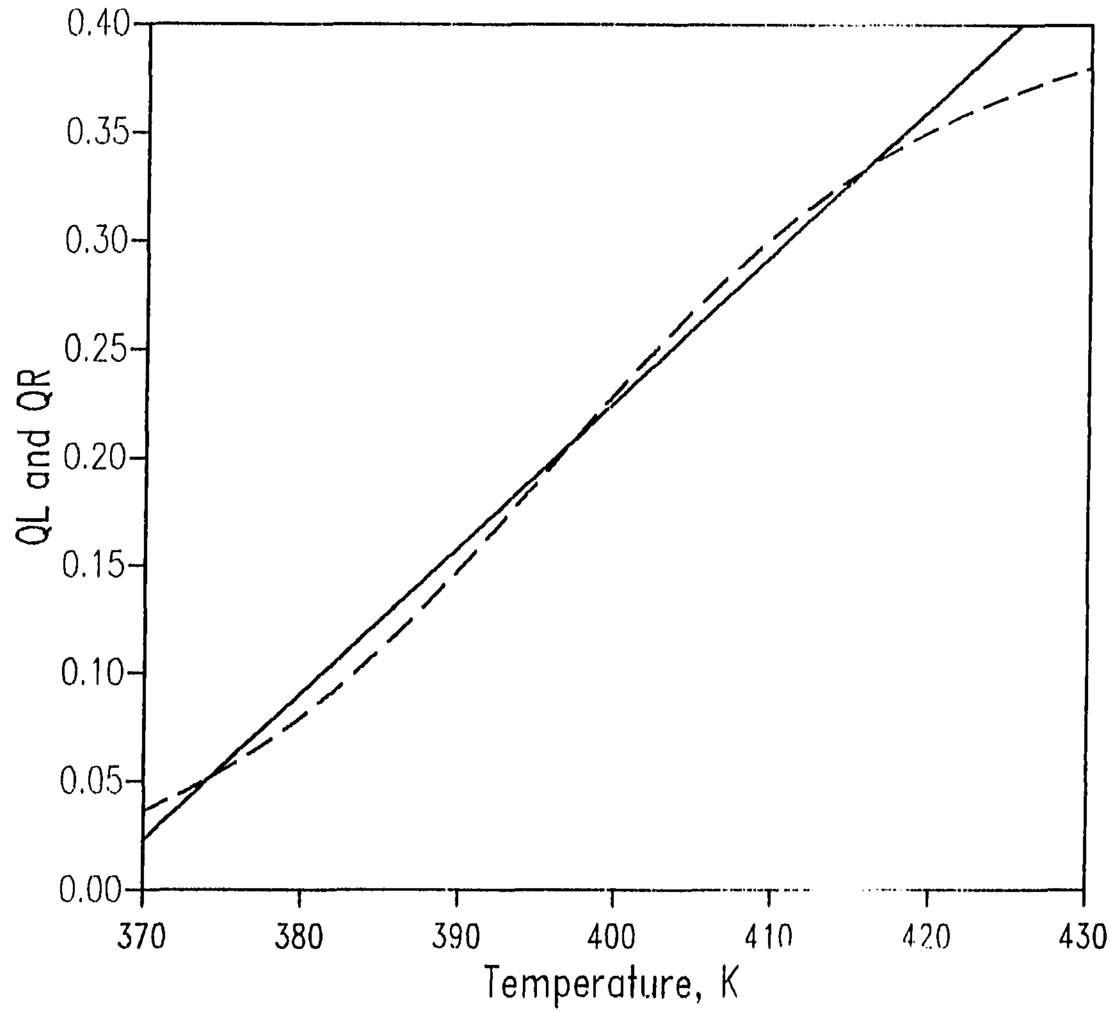

Fig. 4.2 Three steady state solutions of energy equation

where $$Q_L = \left(\frac{q}{V} + \frac{UA_R}{\rho C_p V}\right) T^d - \left(\frac{q}{V} T_0 + \frac{UA_R}{\rho C_p V} T_c^d\right)$$

$$Q_R = \frac{-\Delta H_{AB}}{\rho C_p} (k_1^d C_A^d - k_2^d C_B^d)$$

With the dimensionless variables defined in Table 4 2, we get the dimensionless mathematical model as follows:

$$\frac{dx_1}{d\tau} = (\alpha_A - x_1) - (x_1 - \alpha_A + 1) \exp\left[D_1 - \frac{v_1^2}{(x_2 - \alpha_T + v_1)}\right] + (w - \alpha_B + 1) \exp\left[D_2 - \frac{v_1 v_2}{(x_2 - \alpha_T + v_1)}\right] \tag{4.4.8}$$

$$\frac{dx_2}{d\tau} = (\alpha_T - x_2) - B(x_1 - \alpha_A + 1) \exp\left[D_1 - \frac{v_1^2}{(x_2 - \alpha_T + v_1)}\right] + B(w - \alpha_B + 1) \exp\left[D_2 - \frac{v_1 v_2}{(x_2 - \alpha_T + v_1)}\right] - \gamma (x_2 - \alpha_T + \alpha_w) + \gamma u \tag{4.4.9}$$

$$\frac{dw}{d\tau} = - \left\{1 + \exp\left[D_2 - \frac{v_1 v_2}{(x_2 - \alpha_T + v_1)}\right] + \exp\left[D_3 - \frac{v_1 v_3}{(x_2 - \alpha_T + v_1)}\right]\right\} w + (x_1 - \alpha_A + 1) \exp\left[D_1 - \frac{v_1^2}{(x_2 - \alpha_T + v_1)}\right] - (1 - \alpha_B)\left\{1 + \exp\left[D_2 - \frac{v_1 v_2}{(x_2 - \alpha_T + v_1)}\right] + \exp\left[D_3 - \frac{v_1 v_3}{(x_2 - \alpha_T + v_1)}\right]\right\} \tag{4.4.10}$$

Now suppose that in the above system reaction coefficient k_{10} is uncertain and lies within the following bounds

$$2.97 \times 10^{11} \leq k_{10} \leq 3.03 \times 10^{11} \quad (1 / \text{sec}) \quad (\pm 1.0\ \%\ \text{error})$$

$$(\ \text{corresponding to} \quad 31.7890 \leq D_1 \leq 31.8090\ )$$

From now on we will denote the nominal value of D_1 to be $\hat{D}_1$.

Table 4.2 Dimensionless variables of the multicomponent exothermic reaction in a CSTR

$x_1 = \dfrac{C_A - C_A^d}{C_{A0}}$	$x_2 = \dfrac{T - T^d}{T_0} \nu_1$
$w = \dfrac{C_B - C_B^d}{C_{A0}}$	$u = \dfrac{T_c - T_c^d}{T_0} \nu_1$
$\tau = t \dfrac{q}{V}$	
$\nu_1 = \dfrac{E_1}{RT_0} = 32.9404$	$D_1 = \ln\left(\dfrac{k_{10}V}{q}\right) = 31.7990$
$\nu_2 = \dfrac{E_2}{RT_0} = 47.5160$	$D_2 = \ln\left(\dfrac{k_{20}V}{q}\right) = 41.5202$
$\nu_3 = \dfrac{E_3}{RT_0} = 32.9404$	$D_3 = \ln\left(\dfrac{k_{30}V}{q}\right) = 37.0973$
$\alpha_A = \dfrac{C_{A0} - C_A^d}{C_{A0}} = 0.5042$	$\alpha_B = \dfrac{C_{A0} - C_B^d}{C_{A0}} = 0.9975$
$\alpha_T = \dfrac{T_0 - T^d}{T_0} \nu_1 = - 1.2006$	$\alpha_w = \dfrac{T_0 - T_c^d}{T_0} \nu_1 = 4.5805$
$B = \dfrac{\Delta H_{AB} C_{A0} \nu_1}{\rho C_p T_0} = - 7.6476$	$\gamma = \dfrac{U A_R}{q \rho C_p} = 0.4593$

Let us denote

$$f(x) = [f_1(x) \quad f_2(x)]^T$$

$$f_1(x) = (\alpha_A - x_1) - (x_1 - \alpha_A + 1) E_{D1}(x) + (w - \alpha_B + 1) E_{D2}(x)$$

$$f_2(x) = (\alpha_T - x_2) - B(x_1 - \alpha_A + 1) E_{D1}(x) + B(w - \alpha_B + 1) E_{D2}(x)$$
$$- \gamma (x_2 - \alpha_T + \alpha_w)$$

$$\hat{f}(x) = [\hat{f}_1(x) \quad \hat{f}_2(x)]^T$$

$$\hat{f}_1(x) = (\alpha_A - x_1) - (x_1 - \alpha_A + 1)\, \hat{E}_{D1}(x) + (w - \alpha_B + 1)\, E_{D2}(x)$$

$$\hat{f}_2(x) = (\alpha_T - x_2) - B(x_1 - \alpha_A + 1)\, \hat{E}_{D1}(x) + B(w - \alpha_B + 1)\, E_{D2}(x)$$
$$- \gamma\,(x_2 - \alpha_T + \alpha_w)$$

$$g(x) = [g_1(x) \quad g_2(x)]^T$$

$$\hat{g}(x) = [\hat{g}_1(x) \quad \hat{g}_2(x)]^T$$

$$g_1(x) = \hat{g}_1(x) = 0$$

$$g_2(x) = \hat{g}_2(x) = \gamma$$

$$p(x) = [p_1(x) \quad p_2(x)]^T$$

$$\hat{p}(x) = [\hat{p}_1(x) \quad \hat{p}_2(x)]^T$$

$$p_1(x) = \hat{p}_1(x) = E_{D2}(x)$$

$$p_2(x) = \hat{p}_2(x) = B\, E_{D2}(x)$$

$$d(x) = -\{1 + E_{D2}(x) + E_{D3}(x)\}$$

$$q(x) = (x_1 - \alpha_A + 1)E_{D1}(x) + (\alpha_B - 1)\{1 + E_{D2}(x) + E_{D3}(x)\}$$

$$\hat{q}(x) = (x_1 - \alpha_A + 1)\hat{E}_{D1}(x) + (\alpha_B - 1)\{1 + E_{D2}(x) + E_{D3}(x)\}$$

$$h(x) = x_1$$

where $E_{D1}(x) = \exp\left[D_1 - \dfrac{v_1^2}{(x_2 - \alpha_T + v_1)}\right]$

$$\hat{E}_{D1}(x) = \exp\left[\hat{D}_1 - \frac{v_1^2}{(x_2 - \alpha_T + v_1)}\right]$$

$$E_{D2}(x) = \exp\left[D_2 - \frac{v_1\, v_2}{(x_2 - \alpha_T + v_1)}\right]$$

$$E_{D3}(x) = \exp\left[D_3 - \frac{v_1\, v_3}{(x_2 - \alpha_T + v_1)}\right]$$

Then finally the real plant and the mathematical model can be represented by

Real plant :

$$\dot{x} = f(x) + g(x)\,u + p(x)\,w \qquad (4.4.11)$$

$$\dot{w} = d(x)\,w + q(x)$$

$$y = h(x)$$

Mathematical model:

$$\dot{x} = \hat{f}(x) + \hat{g}(x)\,u + \hat{p}(x)\,w \qquad (4.4.12)$$

$$\dot{w} = d(x)\,w + \hat{q}(x)$$

$$y = h(x)$$

In this system we can see that $\hat{f}(0) = \hat{q}(0) = 0$, i.e. $x = w = u = 0$ is the equilibrium point of the mathematical model. However $f(0) \neq 0$ and $q(0) \neq 0$, i.e. the real plant has a different equilibrium point.

4.4.3. Reduced Dimensional Model

Consider the system (4.4.12). If a solution w(t) exists, then it can be written

$$w(t) = \Phi_w(t,0)\,w(0) + \int_0^t \Phi_w(t,\tau)\,\hat{q}(x(\tau))\,d\tau \qquad (4.4.13)$$

$$\text{where } \Phi_w(t,\tau) = \exp\left[\int_\tau^t d(x(s))\;ds\right]$$

$$= \exp\left[\int_\tau^t -\{1 + E_{D2}(x(s)) + E_{D3}(x(s))\}\,ds\right]$$

In this case we can see that for any bounded set in x_2, defined by S_X there exists a positive constant σ_s such that

$$- \{1 + E_{D2}(x) + E_{D3}(x)\} \leq -\sigma_s < 0, \quad \forall\, x \in S_x \tag{4.4.14}$$

because for any finite value of x_2, $1 + E_{D2}(x) + E_{D3}(x)$ is positive. If this constant σ_s is very large compared with the norm of $\widehat{q}(x)$ for every x in a given set, then, as explained in the previous section, we can ignore the higher-order dynamics since the solution w(t) goes very rapidly to the equilibrium point, which can be approximated to be zero. Actually in this case, as will be shown later, around the operating point, T^d, the constant σ_s has a large value compared with the norm of $\widehat{q}(x)$.

Therefore in this case we have the following reduced dimensional model

$$\begin{aligned} \dot{x} &= \hat{f}(x) + \hat{g}(x)\, u \\ y &= h(x) \end{aligned} \tag{4.4.15}$$

In fact, this dimensional reduction has already been used frequently in chemical engineering, where it is known as the steady-state approximation [Froment and Bischoff, 1979, p29]. The steady-state approximation assumes that the concentration of each intermediate, such as B in the present case, remains constant during the course of the reaction. This is a good approximation when the intermediates are very reactive and therefore are present at very small concentrations. Therefore in this system if we apply the steady-state approximation to equation (4.4.1)

$$\begin{aligned} &\frac{dC_B}{dt} \cong 0 = \frac{q}{V}(-\overline{C}_B) - (\overline{k}_1\overline{C}_A - \overline{k}_2\overline{C}_B - \overline{k}_3\overline{C}_B) \\ &\Rightarrow \overline{C}_B = \frac{\overline{k}_1\overline{C}_A}{\frac{q}{V} + \overline{k}_2 + \overline{k}_3}. \end{aligned} \tag{4.4.16}$$

where superscript $-$ indicates a concentration and temperature obtained using the steady - state approximation.

Now if $\overline{k}_2 + \overline{k}_3 >> \overline{k}_1$ then $\overline{C}_B \cong 0$, that is, we can ignore the higher-order dynamics.

4.4.4. Feedback Linearization Based on the Reduced Dimensional Model

First let us find the relative degree of the reduced dimensional system (4.4.15)

when $r = 1$ $\quad L_{\hat{g}}h(x) = 0$

when $r = 2$ $\quad L_{\hat{g}}L_{\hat{f}}h(x) = \dfrac{\partial \hat{f}_1(x)}{\partial x_2}\,\hat{g}_2(x) \quad (\text{since } \hat{g}_1(x) = 0)$

$$= \frac{-(x_1 - \alpha_A + 1)\, v_1^2\, \hat{E}_{D1}(x) + (1 - \alpha_B) v_1\, v_2\, E_{D2}(x)}{(x_2 - \alpha_T + v_1)^2}$$

Therefore if

(i) $(x_2 - \alpha_T + v_1) < \infty$, that is, x_2 is finite

(ii) $-(x_1 - \alpha_A + 1)\, v_1\, \hat{E}_{D1}(x) + (1 - \alpha_B)\, v_2\, E_{D2}(x) \neq 0$ (4.4.17)

then the reduced dimensional model has relative degree 2.

Define the set U

$$U = \{x_1, x_2\, ;\ x_1 > \frac{(1 - \alpha_B) v_2 E_{D2}(x)}{v_1\, \hat{E}_{D1}(x)} + \alpha_A - 1,\ x_2 < \infty\ \}$$

then we can see that in the set U the reduced dimensional model has relative degree 2.

Fig. 4.3 shows the condition (4.4.17) for $-10.0 \leq x_2 \leq 10.0$ (corresponding to 280.91 K $\leq T \leq$ 513.63 K). In this figure we can see that if $x_1 \geq -0.4946$ (corresponding to $C_A \geq 9.2864 \times 10^{-6}$ gmole/cc) then the reduced dimensional model is linearizable with relative degree 2 when $-10.0 \leq x_2 \leq 10.0$. Practically the range of x_2, [-10.0, 10.0] is outside of the control range and as the range of x_2 becomes smaller around $x_2 = 0$ a larger range of x_1 satisfies the condition (4.4.17).

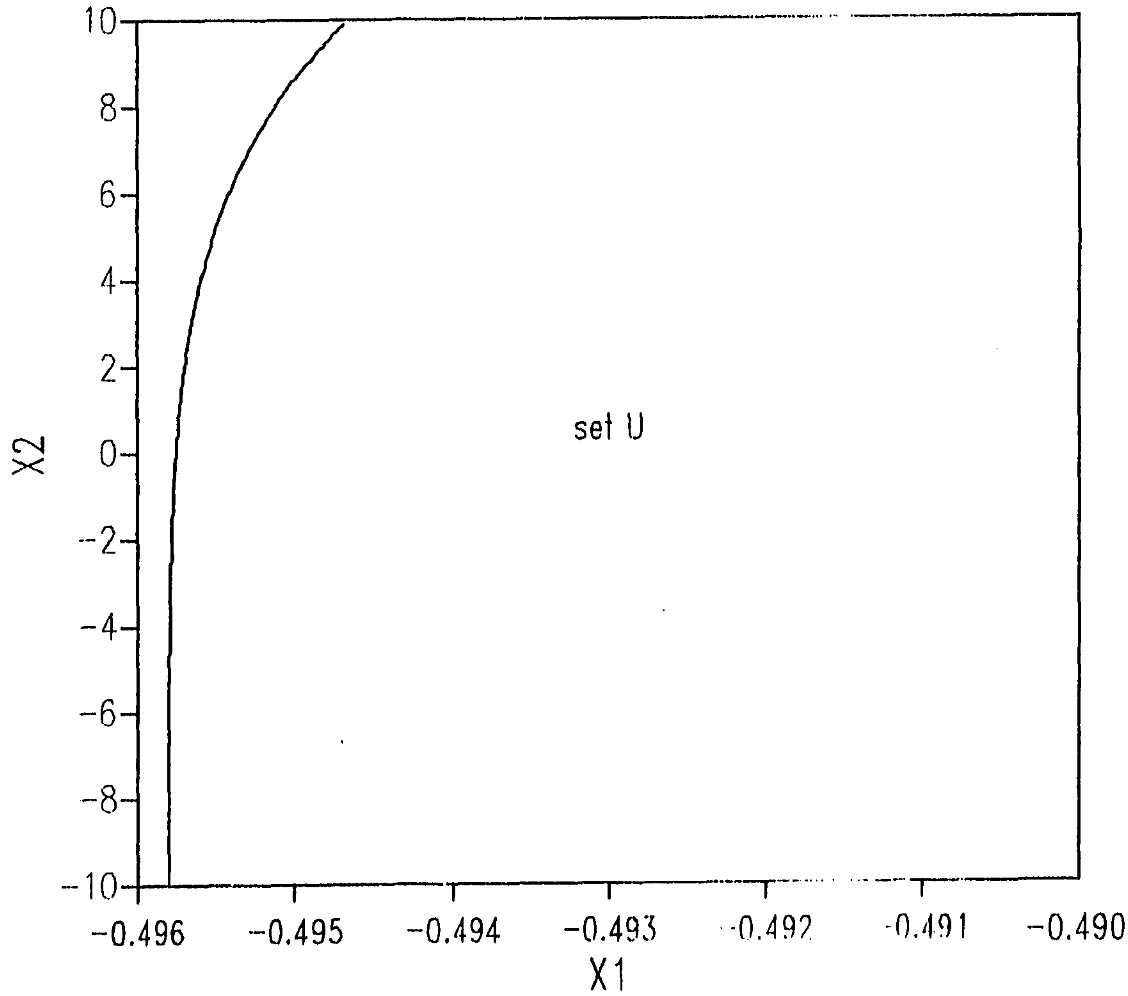

Fig. 4.3 The set U where the reduced dimensional model has relative degree 2

As in Chapter 3 the following coordinate transformation and state feedback linearize the reduced dimensional model

Coordinate transformation:

$$z_1 = T_1(x) = x_1 \qquad (4.4.18)$$

$$z_2 = T_2(x) = \hat{f}_1(x)$$

State feedback:

$$u = \alpha(x) + \beta(x)\, v \qquad (4.4.19)$$

$$\text{where } \alpha(x) = -\,\hat{D}^*(x)^{-1}\hat{A}^*(x)$$

$$\beta(x) = \hat{D}^*(x)^{-1}$$

$$\hat{D}^*(x) = L_{\hat{g}}L_{\hat{f}}h(x) = \frac{\partial \hat{f}_1(x)}{\partial x_2}\hat{g}_2(x)$$

$$\hat{A}^*(x) = L_{\hat{f}}^2 h(x) = \frac{\partial \hat{f}_1(x)}{\partial x_1}\hat{f}_1(x) + \frac{\partial \hat{f}_1(x)}{\partial x_2}\hat{f}_2(x)$$

$$v = Kz = [b_1 \quad b_2]\, z$$

By this method, we can find the linearizing coordinate transformation and state feedback based on the reduced dimensional model. Obviously, this linearizing state feedback transforms the reduced dimensional system into a linear system. For this transformed linear system we can easily apply linear control theory such as a linear optimal regulator. However, this linearizing state feedback does not transform the given real plant into a linear system. Applying the above transformation and linearizing state feedback (4.4.6) to the real plant (4.4.4) we have the following equation:

$$\dot{z} = (A + bK)z + \varphi(x) + \psi(x)\, w \qquad (4.4.20)$$

$$\dot{w} = d(x)w + q(x)$$

$$y = Cz$$

where $A + bK = \begin{bmatrix} 0 & 1 \\ b_1 & b_2 \end{bmatrix}$

$C = [1 \quad 0]$

$$\varphi(x) = \begin{bmatrix} \Delta f_1(x) \\ \dfrac{\partial \hat{f}_1(x)}{\partial x_1} \Delta f_1(x) + \dfrac{\partial \hat{f}_1(x)}{\partial x_2} \Delta f_2(x) \end{bmatrix}$$

$$\psi(x) = \begin{bmatrix} p_1(x) \\ \dfrac{\partial \hat{f}_1(x)}{\partial x_1} p_1(x) + \dfrac{\partial \hat{f}_1(x)}{\partial x_2} p_2(x) \end{bmatrix}$$

$$\begin{aligned} \Delta f_1(x) &= f_1(x) - \hat{f}_1(x) \\ &= (x_1 - \alpha_A + 1)\{\hat{E}_{D1}(x) - E_{D1}(x)\} \\ \Delta f_2(x) &= f_2(x) - \hat{f}_2(x) \\ &= B\ (x_1 - \alpha_A + 1)\{\hat{E}_{D1}(x) - E_{D1}(x)\} \end{aligned}$$

4.4.5. Robustness Analysis

Now we will analyze the system (4.4.20) using Theorem 4 in the previous section. With the given nominal values and the parametric uncertainties we can find the necessary constants as follows:

1. α_w and k_w

From (4.4.12) the state transition matrix of the higher-order dynamics is

$$\Phi_w(t,\tau) = \exp\left[\int_\tau^t - \{1 + E_{D2}(x(s)) + E_{D3}(x(s))\}\, ds\right] \qquad (4.4.21)$$

Now if there exists a constant σ_s such that

$$-\{1 + E_{D2}(x) + E_{D3}(x)\} \le -\sigma_s \qquad \forall\, x \in B_r(z = 0)$$

where $B_r(z = 0) = \{x;\ \|T(x)\| \le r\}$

then we can find easily the constants α_w and k_w such that

$$\|\Phi_w(t,\tau)\| \le \alpha_w e^{-k_w(t-\tau)}$$

Since in this case unmodeled dynamics is a single equation we can see that

$\alpha_w = 1$ and $\sigma_s = k_w$ (calculated using the program: LIP1V2).

2. m_w and δ_w such that

$$\|\sigma(z)\| \le m_w \|z\| + \delta_w \quad \forall\, z \in B_r(z=0)$$

Let us define the set S_D such that

$$S_D = \{D_1;\ 31.7890 \le D_1 \le 31.8090\}$$

The constant δ_w can be calculated by

$$\delta_w = \max_{D_1 \in S_D} |q(0)|$$

where $q(0) = |(-\alpha_A + 1) E_{D1}(0) + (\alpha_B - 1)\{1 + E_{D2}(0) + E_{D3}(0)\}|$

And also the constant m_w can be calculated by

$$m_w = \max_{\substack{x \in B_r(z=0) \\ D_1 \in S_D}} \frac{|q(x)| - \delta_w}{\|z\|}$$

The constants m_w and δ_w are calculated for a given set by the simple search method (program name: LIP2V2).

3. m_1 and δ_1

By the previous chapter we know that

$$\delta_1 = \|\varphi(0)\| = \sqrt{\varphi_1^2(0) + \varphi_2^2(0)}$$

$$m_1 = \max_{\substack{z \in B_r(z=0) \\ D_1 \in S_D}} \left\| \frac{\partial\varphi(x)}{\partial x} \left(\frac{\partial T(x)}{\partial x}\right)^{-1} \right\|$$

These constants, δ_1 and m_1, are calculated by as in Chapter 3 for a given set (by the program: LIP3V2).

4. m_2 and δ_2

Similarly to the δ_1 and the incremental Lipschitz constant m_1

$$\delta_2 = \| \psi(0) \| = \sqrt{\psi_1^2(0) + \psi_2^2(0)}$$

$$m_2 = \max_{\substack{z \in B_r(z=0) \\ D_1 \in S_D}} \left\| \frac{\partial \psi(x)}{\partial x} \left(\frac{\partial T(x)}{\partial x} \right)^{-1} \right\|$$

These constants, δ_2 and m_2, are calculated for a given set by the program "LIP4V2".

Finally when we choose $b_1 = -2.1$ and $b_2 = -2.0$, $\alpha = 2.566$ and $k = 1.0$.

By the above method, for a given parametric uncertainty with unmodeled dynamics, we can find the necessary constants as shown in Table 4.3, where LHS is defined by

$$LHS = \frac{\alpha}{k}(m_1 r + \delta_1) + \frac{(m_2 r + \delta_2)\, \alpha\, \alpha_w (m_w r + \delta_w)}{k\, k_w}$$

Table 4.3 Constants for each radius r when $b_1 = -2.1$, $b_2 = -2.0$

r	k_w	m_w	δ_w	m_1	δ_1	m_2	δ_2	LHS	r_d
0.3	23.0	4.30	0.012	0.190	0.0095	2.16	0.025	0.280	0.055
0.055	155.0	1.40	0.012	0.055	0.0095	0.29	0.025	0.032	0.028
0.028	180.0	1.17	0.012	0.053	0.0095	0.25	0.025	0.028	0.028

First, as shown in Table 4.3, when we choose $r = 0.3$ LHS is less than 0.3 so that we can apply Theorem 4. In this case $r_d = 0.055$. As we explained in Chapter 3, we apply Theorem 4 again at $r = 0.055$. In this case also, LHS is less that 0.055 and $r_d = 0.028$. When $r = 0.028$, LHS is also 0.028 and if we choose r smaller than 0.028, LHS is greater than the value of r. Thus Theorem 4 cannot be applied with a smaller value of r than 0.028. Therefore we can see that for any r_i such that $0.028 \le r_i \le 0.3$ if $\| z(0) \| < \frac{r_i}{\alpha}$ then $\| z(t) \| < r_i$ for every $t \ge 0$ and moreover $\lim_{t \to \infty} \| z(t) \| \le r_d = 0.028$.

In this case we choose $w(0) = 2.5 \times 10^{-3}$. With this value of w(0) the constant c^* defined in Theorem 4 is less than the radius r for all three cases in Table 4.3.

Fig.4.4 shows that the norm of system trajectories, $\| z(t) \| < 0.3$ and $\lim_{t \to \infty} \| z(t) \| \le r_d = 0.028$ when $\| z(0) \| \le 0.117$ where the initial condition and the true parametric value are as follows:

For line 1 : $x_1(0) = -0.116$, $x_2(0) = 0.546$, $w(0) = 2.5 \times 10^{-3}$, $D_1 = 31.789$
(corresponding to $C_A = 3.0441 \times 10^{-3}$ gmole/cc, $T = 403.62$ K and $C_B = 3.981 \times 10^{-5}$ gmole/cc)

For line 2: $x_1(0) = 0.116$, $x_2(0) = -0.451$, $w(0) = 2.5 \times 10^{-3}$, $D_1 = 31.809$
(corresponding to $C_A = 4.9039 \times 10^{-3}$ gmole/cc, $T = 392.02$ K and $C_B = 3.981 \times 10^{-5}$ gmole/cc)

Fig.4.5 shows the state variable of the unmodeled dynamics, w, for the same condition as the above line 1 in Fig. 4.4. From this figure we can see that w becomes approximately zero as time goes to infinity. From these results (program for trajectories: UNMODL, program for contour ($r = 0.3, 0.028$): CONTV2) we can see that with the given parametric uncertainty and the unmodeled dynamics, the linearization based on the reduced dimensional model results in a bounded solution and convergence to the small region around x = 0 and w = 0.

As a special case, if there are no parametric or structural uncertainties, then it is obvious that $m_l = \delta_l = \delta_w = 0$ and this results in $c_3 = 0$. Therefore in this case we can see that $r_d = 0$, that is, system trajectories x(t) and w(t) converge to zero. Fig. 4.6 shows the response when reaction coefficient k_{10} has the nominal value, i.e. only unmodeled dynamics without parametric uncertainties. This figure shows that the state variables converge to zero.

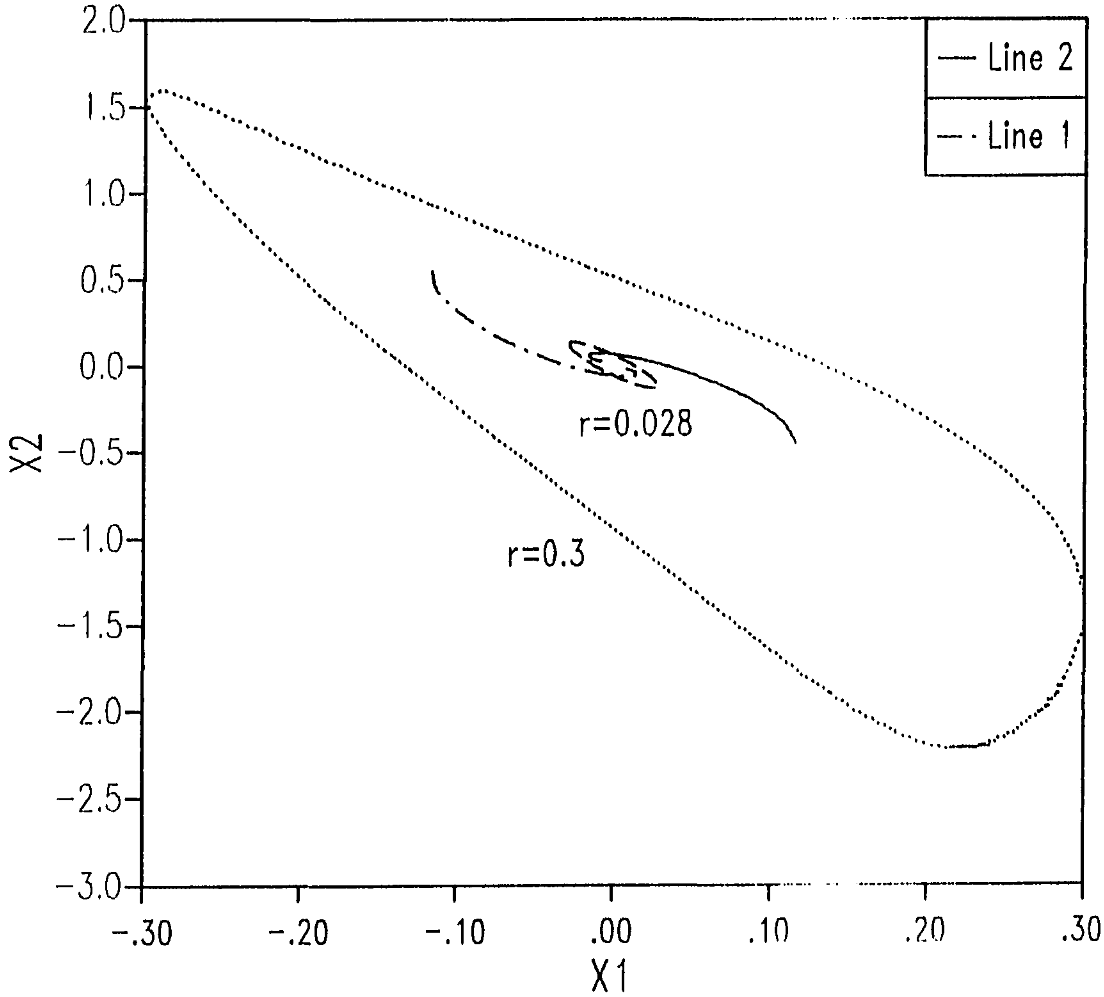

Fig. 4.4 Estimated bounded and converging region of system trajectories with unmodeled dynamics, where $b_1 = -2.1$, $b_2 = -2.0$ and

line 1 : $x_1(0) = -0.116$, $x_2(0) = 0.546$, $w(0) = 2.5 \times 10^{-3}$, $D_1 = 31.789$

line 2 : $x_1(0) = 0.116$, $x_2(0) = -0.451$, $w(0) = 2.5 \times 10^{-3}$, $D_1 = 31.809$

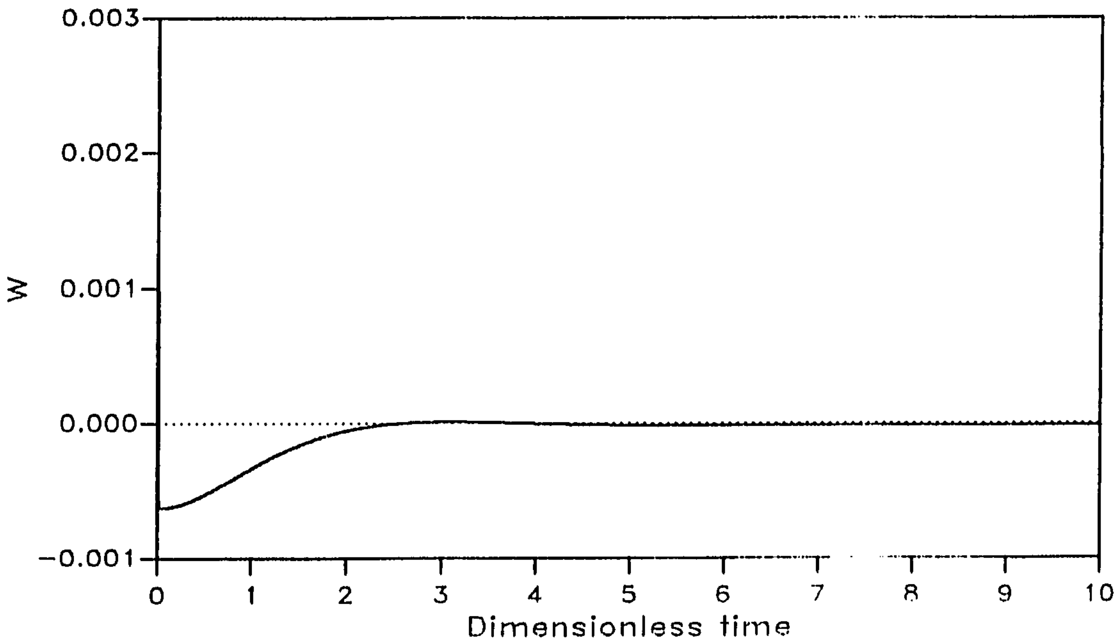

Fig. 4.5 System response of the state variable of the unmodeled dynamics, w, where $b_1 = -2.1$, $b_2 = -2.0$ and

$x_1(0) = -0.116$, $x_2(0) = 0.546$, $w(0) = 2.5 \times 10^{-3}$, $D_1 = 31.789$

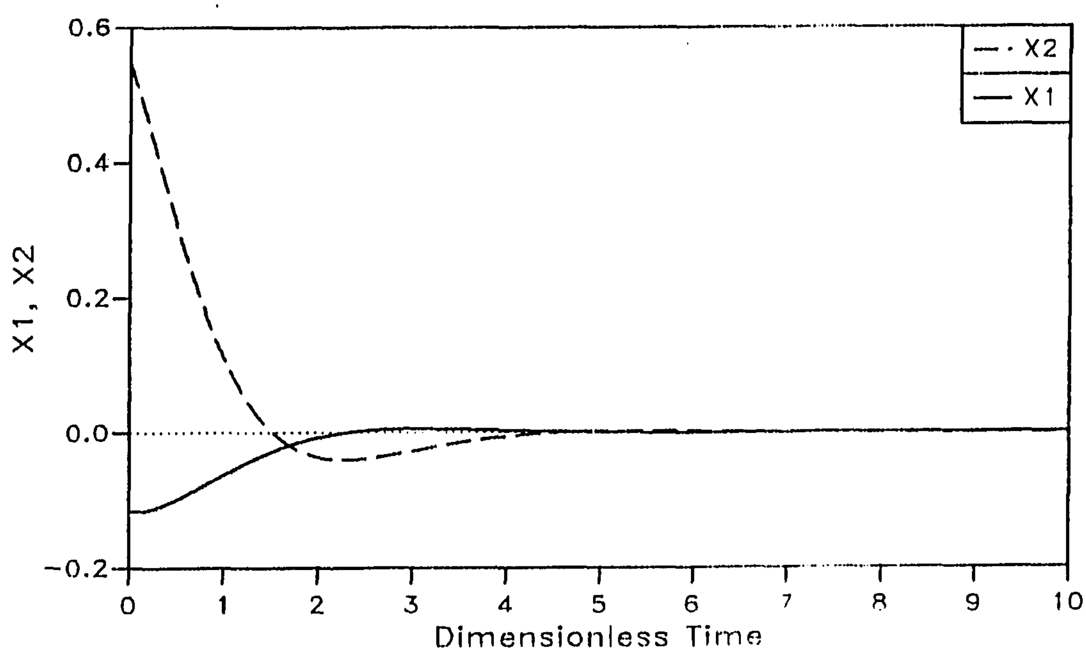

Fig. 4.6 System response when the reaction coefficient k_{10} has the nominal value where b_1 = -2.1, b_2 = -2.0 and

$x_1(0) = -0.116$, $x_2(0) = 0.546$, $w(0) = 2.5 \times 10^{-3}$

4.5. Conclusion

In this chapter we have developed a theoretical approach extending Theorem 3, developed in Chapter 3, to find sufficient conditions for boundedness and convergence of system trajectories when the coordinate change and linearizing state feedback based on a reduced dimensional model are applied to a real plant which has parametric and structural uncertainties as well as unmodeled dynamics. The parametric and structural uncertainties do not require the restrictive conditions which are commonly used in previous work, such as matching condition, the global Lipschitz continuity or the same equilibrium point for the mathematical model and the real plant for all possible uncertainties. However, the proposed approach requires that the unmodeled dynamics can be represented by a perturbed system which is linear in the state variables of the unmodeled dynamics. This approach has been applied to an unstable multicomponent chemical reaction in CSTR. This initial result of robustness analysis for uncertainties with unmodeled dynamics may be useful in developing a robust state feedback based on the reduced dimensional model for a more general class of unmodeled dynamics.

CHAPTER V

Adaptive Control of Feedback Linearizable Systems

5.1. Introduction

In the previous chapters we have investigated robustness of feedback linearization for parametric and structural uncertainties as well as unmodeled dynamics. In this chapter we have considered a parameter adaptive control for feedback linearizable systems.

The differential geometric approach makes it possible to design a nonlinear feedback control for a certain class of nonlinear systems without a conventional linearization based on the Taylor series expansion. In fact, many chemical processes are inherently nonlinear so that the direct use of a nonlinear chemical model has considerable benefit in the design of a feedback system without loss of the information about the process.

Unfortuantely, feedback linearization strongly relies on the exact cancellation of nonlinear terms. Therefore, if there exists model-plant mismatch, the cancellation is no longer exact. In the previous chapters we have seen that feedback linearization results in a perturbed nonlinear system when there exists model-plant mismatch. For parametric uncertainties, parameter adaptive control can be a method to make feedback linearizable systems robust.

Recently, outside of the chemical engineering field, the use of an adaptive parameter law with the differential geometric approach has been studied by many authors

[Nam and Arapostathis, 1988, Akhrif and Blankenship, 1988, Sastry and Isidori, 1989, Taylor et al, 1989]. However, these studies are based on restrictive conditions on parametric uncertainties: matching condition or requirement of the same equilibrium point for all possible parametric uncertainties. These restrictive conditions sometimes limit the applicability of the above methods to chemical processes.

In this chapter we have considered a parameter adaptive control for feedback linearizable systems, where parametric uncertainties can be represented linearly in the unknown parameters. The main feature in our adaptive control approach is that the linearizing coordinate transformation depends on estimated parameters so that it is updated during the parameter adaptation. The parametric adaptive law has been derived from the second method of Lyapunov. The proposed adaptive control algorithm is very straightforward and simple in the sense that it does not use concepts such as augmented error and also does not increase the number of estimated parameters. For this adaptive control scheme we find sufficient conditions for stability of output regulation and tracking.

The developed adaptive control scheme has been applied to chemical and biochemical reactors. In a further application, it is shown that the adaptive approach to output tracking can also be used to make the linearizing state feedback smooth for an uncertain nonlinear system.

We have also shown by numerical simulation that the developed adaptive control scheme may be applied to a case for which parametric uncertainties cannot be represented linearly in the unknown parameters. In this case, parametric uncertainties have been approximated so that they can be represented linearly in the unknown parameters. By the simulation results we see that for this particular system the developed adaptive control is robust to structural uncertainties.

5.2. Literature Review

The study of adaptive control by the differential geometric approach has been started very recently so that relatively few results have been published.

Nam and Arapostathis (1988) have proposed a model-reference adaptive control scheme when parametric uncertainties can be represented linearly in the unknown parameters and have found sufficient conditions for the convergence of the proposed adaptive control system. They have considered a class of well structured nonlinear systems, i.e. block triangular form, called pure-feedback systems [Su and Hunt, 1986]. The uncertainty of the vector fields, in their work, is assumed to have the following structure: (1) the uncertainties do not change the structure of the pure-feedback system, (2) nonlinear uncertainties are globally Lipschitz continuous, and (3) the equilibrium point of the system is invariant for all possible parametric uncertainties. Under these assumptions they have used the concept of augmented error [Monopoli, 1974] and proposed a parameter adaptive law based on the pseudogradient algorithm. In this approach, the linearizing coordinate transformation is a function of estimated parameters so that it is updated continuously by the parametric adaptive law. Menon and Garg (1988) have also developed a model-reference adaptive control scheme for a robot manipulator which is exact state-space linearizable.

Akhrif and Blankenship (1988) have considered the adaptive control of feedback linearizable systems. In this approach, only a nominal model is required to be exactly state-space linearizable. They have proposed a composite linear feedback of a function of the estimated parameters, which are adjusted by a parametric adaptive law constructed by the second method of Lyapunov. In this approach, the linearizing coordinate transformation and state feedback are constructed based on the nominal parameter values instead of the estimated parameters so that they are not updated during the adaptation. They have shown, for a particular nonlinear system, that this adaptive approach increases the robustness compared with a nonadaptive linear feedback. However, this approach is also based on restrictive conditions such as the above condition (3).

Taylor et al. (1989) have considered the adaptive control of nonlinear systems with unmodeled dynamics. These workers have proposed an adaptive control scheme based on the second method of Lyapunov and have shown that this adaptive control is robust to unmodeled dynamics. However, in their approach, the most restrictive assumption is that the real plant must be linearizable with a state coordinate transformation which is independent of the unknown parameters. Even though the assumptions on the system are very restrictive, this work is an initial attempt to unify the three ingredient methodologies: geometric, asymptotic and adaptive.

All of this work assumes that the system is exactly state-space linearizable. Instead of this, Sastry and Isidori (1989) have suggested an adaptive control scheme for an exponentially minimum-phase nonlinear system, which is globally input-output linearizable, using the concept of augmented error and the gradient-type algorithm. However, this method yields an increase of the number of estimated parameters. For example, if there are q unknown parameters, then the adaptive control system must update q! parameters.

5.3. Adaptive Regulation of a Feedback Linearizable Process

In this section we consider the adaptive regulation of a feedback linearizable process. We consider only parametric uncertainties, no other structural uncertainties or unmodeled dynamics. Nonlinear systems considered in this chapter are not necessarily *globally* feedback linearizable and parametric uncertainties need not satisfy the matching condition.

Consider the following nonlinear real plant

$$\dot{x} = f(x, \theta^*) + g(x, \theta^*)\, u \tag{5.3.1}$$

$$y = h(x)$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $u \in \mathbb{R}$ and $\theta^* \in \mathbb{R}^q$

and θ^* = true parameter

In this real plant, θ^* is not known exactly. However, in this work, it is assumed that bounds on θ^* are known. For the above real plant, suppose that we have the following mathematical model

$$\dot{x} = f(x, \hat{\theta}) + g(x, \hat{\theta})\, u \tag{5.3.2}$$

$$y = h(x)$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $u \in \mathbb{R}$ and $\hat{\theta} \in \mathbb{R}^q$

and $\hat{\theta}$ = estimated parameter

Note that θ^* is an unknown constant and $\hat{\theta}$ will be adjusted through an adaptive law so that it is a function of time.

We denote by $\hat{\theta}(0)$ the estimated parameter at time $t = 0$. Thus $\hat{\theta}(0)$ is a nominal value of the unknown parameter, θ^*. Without loss of generality we can assume that $x = 0$ is the equilibrium point of the mathematical model (5.3.2) with $\hat{\theta} = \hat{\theta}(0)$; that is, $f(0, \hat{\theta}(0)) = 0$. However, it is not assumed that $f(0, \theta) = 0$ for every possible value of θ. In other words, the point $x = 0$ may not be the equilibrium point of the uncertain real plant (5.3.1). This situation is much more realistic for many chemical engineering processes. This will be investigated in detail in the later examples.

We assume that the mathematical model (5.3.2) satisfies the following assumptions.

Assumption 5.1: Let a set U be a neighborhood of $x = 0$ and a set S_θ be a neighborhood of $\theta = \hat{\theta}(0)$. For every $x \in U$, $\hat{\theta} \in S_\theta$, the mathematical model (5.3.2) is feedback linearizable with relative degree, $r = n$; that is, the mathematical model (5.3.2) satisfies the following conditions:

$$L_{\hat{g}} L_{\hat{f}}^{\rho} h(x) \neq 0 \quad \text{and}$$
$$L_{\hat{g}} L_{\hat{f}}^{\rho - i} h(x) = 0, \quad i = 1, 2, \dots, \rho$$

and $n = \rho+1$

where Lie derivatives are taken only with respect to x. ///

Model-plant mismatch, Δf and Δg, for the system (5.3.1) and (5.3.2) are defined by

$$\Delta f(x, \theta^*, \hat{\theta}) \equiv f(x, \theta^*) - f(x, \hat{\theta}) \tag{5.3.3}$$
$$\Delta g(x, \theta^*, \hat{\theta}) \equiv g(x, \theta^*) - g(x, \hat{\theta})$$

As mentioned previously, Δf does not necessarily satisfy the matching condition.

Assumption 5.2: Model-plant mismatch Δf and Δg can be represented in the following way:

$$\Delta f(x, \theta^*, \hat{\theta}) = \begin{bmatrix} \Delta f_1(x, \theta^*, \hat{\theta}) \\ \Delta f_2(x, \theta^*, \hat{\theta}) \\ \vdots \\ \Delta f_n(x, \theta^*, \hat{\theta}) \end{bmatrix} = ({}^1\theta_1^* - {}^1\hat{\theta}_1) \begin{bmatrix} {}^1f_1(x) \\ {}^1f_2(x) \\ \vdots \\ {}^1f_n(x) \end{bmatrix} + ({}^1\theta_2^* - {}^1\hat{\theta}_2) \begin{bmatrix} {}^2f_1(x) \\ {}^2f_2(x) \\ \vdots \\ {}^2f_n(x) \end{bmatrix}$$

$$+ \cdots + ({}^1\theta_K^* - {}^1\hat{\theta}_K) \begin{bmatrix} {}^Kf_1(x) \\ {}^Kf_2(x) \\ \vdots \\ {}^Kf_n(x) \end{bmatrix} \tag{5.3.4}$$

$$\Delta g(x, \theta^*, \hat{\theta}) = \begin{bmatrix} \Delta g_1(x, \theta^*, \hat{\theta}) \\ \Delta g_2(x, \theta^*, \hat{\theta}) \\ \vdots \\ \Delta g_n(x, \theta^*, \hat{\theta}) \end{bmatrix} = ({}^2\theta_1^* - {}^2\hat{\theta}_1) \begin{bmatrix} {}^1g_1(x) \\ {}^1g_2(x) \\ \vdots \\ {}^1g_n(x) \end{bmatrix} + ({}^2\theta_2^* - {}^2\hat{\theta}_2) \begin{bmatrix} {}^2g_1(x) \\ {}^2g_2(x) \\ \vdots \\ {}^2g_n(x) \end{bmatrix}$$

$$+ \cdots + ({}^2\theta_J^* - {}^2\hat{\theta}_J) \begin{bmatrix} {}^Jg_1(x) \\ {}^Jg_2(x) \\ \vdots \\ {}^Jg_n(x) \end{bmatrix}$$

where ${}^1\theta_k^*$, ${}^2\theta_j^*$, $k = 1, 2, \cdots, K$, $j = 1, 2, \cdots, J$, are the uncertain true parameters in Δf and Δg respectively and ${}^1\hat{\theta}_k$, ${}^2\hat{\theta}_j$ are the corresponding estimated parameters. And ${}^kf_i(x)$, ${}^jg_i(x)$, $k = 1, 2, \cdots, K$, $j = 1, 2, \cdots, J$, and $i = 1, 2, \cdots, n$, are known functions.

Denote

$${}^kf(x) \equiv \begin{bmatrix} {}^kf_1(x) \\ {}^kf_2(x) \\ \vdots \\ {}^kf_n(x) \end{bmatrix}, \qquad {}^jg(x) \equiv \begin{bmatrix} {}^jg_1(x) \\ {}^jg_2(x) \\ \vdots \\ {}^jg_n(x) \end{bmatrix}$$

Then the above condition (5.3.4) can be written simply

$$\Delta f(x, \theta^*, \hat{\theta}) = \sum_{k=1}^{K} ({}^1\theta_k^* - {}^1\hat{\theta}_k)\, {}^kf(x) \tag{5.3.5}$$

$$\Delta g(x, \theta^*, \hat{\theta}) = \sum_{j=1}^{J} ({}^2\theta_j^* - {}^2\hat{\theta}_j)\, {}^jg(x)$$

///

From now on, we will refer to this condition as linearity in the unknown parameters. This linearity in the unknown parameters of model-plant mismatch may be satisfied, for example, when the vector functions of the system (5.3.1) and (5.3.2) have the following form:

$$f(x, \theta^*) = {}^0f(x) + \sum_{k=1}^{K} {}^1\theta_k^* \, {}^kf(x) \tag{5.3.6}$$

$$f(x, \hat{\theta}) = {}^0f(x) + \sum_{k=1}^{K} {}^1\hat{\theta}_k \, {}^kf(x)$$

$$g(x, \theta^*) = {}^0g(x) + \sum_{j=1}^{J} {}^2\theta_j^* \, {}^jg(x)$$

$$g(x, \hat{\theta}) = {}^0g(x) + \sum_{j=1}^{J} {}^2\hat{\theta}_j \, {}^jg(x)$$

where ${}^0f(x)$, ${}^0g(x)$, ${}^kf(x)$ and ${}^jg(x)$ are known vector functions.

With condition (5.3.5) we now apply the feedback linearization method to the real plant (5.3.1). If the nominal value of θ^*, $\hat{\theta}(0)$, is the same as the true parameter; that is, if we have a perfect mathematical model for the real plant, and also if, in the neighborhood, U, of $x = 0$, the system is feedback linearizable with relative degree $r = n$, then we already know that the coordinate transformation $z_i = T_i(x)$, $i = 1, 2, ..., n$, where $z_i = L_f^{i-1}h(x)$, and state feedback $u = \alpha(x) + \beta(x)\, v$ linearize the system exactly and the equilibrium point, $z = 0$, is asymptotically stable with the properly chosen K such that $v = Kz$.

In the adaptive control system, where θ^* is uncertain, we must change the coordinate transformation and state feedback which now depends on the value of $\hat{\theta}(t)$. Remember that $\hat{\theta}(t)$ will be adjusted by an adaptive parameter update law and thus is a function of time. As we can see later, this approach simplifies the adaptive control scheme in the sense of the number of adjusted parameters. In other words, this approach does not increase the number of adaptive parameters. Also, this approach can treat the situation in which the system is not globally feedback linearizable.

Let us denote C^∞ (infinitely differentiable)-vector fields, f, g, $\hat{f}$ and $\hat{g}$ as follows:

$$f \equiv f(x, \theta^*), \quad g \equiv g(x, \theta^*)$$
$$\hat{f} \equiv f(x, \hat{\theta}), \quad \hat{g} \equiv g(x, \hat{\theta})$$

Consider the following coordinate transformation, which depends on the value of the estimated parameters:

$$z_i = T_i(x, \hat{\theta}) = L_{\hat{f}}^{i-1} h(x), \quad i = 1, 2, \cdots, \rho+1 = n \tag{5.3.7}$$

Then the derivative of z_1 is

$$\dot{z}_1 = \frac{dh(x)}{dt} = \frac{\partial h(x)}{\partial x}\frac{dx}{dt} \tag{5.3.8}$$

Note that h(x) is not a function of $\hat{\theta}(t)$. Therefore,

$$\begin{aligned}\dot{z}_1 &= \frac{\partial h(x)}{\partial x}\left\{f(x, \theta^*) + g(x, \theta^*)u\right\}\\ &= \frac{\partial h(x)}{\partial x}\left\{\left(f(x, \hat{\theta}) + \Delta f(x, \theta^*, \hat{\theta})\right) + \left(g(x, \hat{\theta}) + \Delta g(x, \theta^*, \hat{\theta})\right)u\right\}\end{aligned} \tag{5.3.9}$$

By assumption 5.1 and 5.2, the above equation (5.3.9) can be written for every $x \in U$ and $\hat{\theta} \in S_\theta$

$$\dot{z}_1 = z_2 + \sum_{k=1}^{K} ({}^1\theta_k^* - {}^1\hat{\theta}_k) L_{kf} h(x) + \left\{\sum_{j=1}^{J} ({}^2\theta_j^* - {}^2\hat{\theta}_j) L_{jg} h(x)\right\} u \tag{5.3.10}$$

In a similar manner, we get

$$\begin{aligned}\dot{z}_2 = z_3 &+ \sum_{k=1}^{K} ({}^1\theta_k^* - {}^1\hat{\theta}_k) L_{kf} L_{\hat{f}} h(x)\\ &+ \left\{\sum_{j=1}^{J} ({}^2\theta_j^* - {}^2\hat{\theta}_j) L_{jg} L_{\hat{f}} h(x)\right\} u + \frac{\partial}{\partial \hat{\theta}}(L_{\hat{f}} h(x)) \frac{d\hat{\theta}}{dt}\end{aligned} \tag{5.3.11}$$

$$\vdots$$

$$\dot{z}_{\rho+1} = L_{\hat{f}}^{\rho+1} h(x) + \sum_{k=1}^{K} ({}^1\theta_k^* - {}^1\hat{\theta}_k)\, L_{kf} L_{\hat{f}}^{\rho} h(x) + \left\{ L_{\hat{g}} L_{\hat{f}}^{\rho} h(x) + \sum_{j=1}^{J} ({}^2\theta_j^* - {}^2\hat{\theta}_j)\, L_{jg} L_{\hat{f}}^{\rho} h(x) \right\} u + \frac{\partial}{\partial \hat{\theta}} \left(L_{\hat{f}}^{\rho} h(x) \right) \frac{d\hat{\theta}}{dt} \qquad (5.3.12)$$

Since z_i, $i = 2, 3, \ldots, \rho+1$ are functions of $\hat{\theta}(t)$, there are time derivative terms in equations (5.3.11) and (5.3.12).

If we apply the following linearizing state feedback, u, with $v = Kz$ to equation (5.3.12)

$$u = \alpha(x, \hat{\theta}) + \beta(x, \hat{\theta})\, v \qquad (5.3.13)$$

where $\alpha(x, \hat{\theta}) = - \hat{D}^*(x, \hat{\theta})^{-1} \hat{A}^*(x, \hat{\theta})$

$\beta(x, \hat{\theta}) = \hat{D}^*(x, \hat{\theta})^{-1}$

$\hat{D}^*(x, \hat{\theta}) = L_{\hat{g}} L_{\hat{f}}^{\rho} h(x)$

$\hat{A}^*(x, \hat{\theta}) = L_{\hat{f}}^{\rho+1} h(x)$

$K = [b_1 \ b_2 \ \ldots \ b_{\rho+1}]$,

then finally we have

$$\dot{z}_{\rho+1} = \sum_{k=1}^{\rho+1} b_k L_{\hat{f}}^{k-1} h(x) + \sum_{k=1}^{K} ({}^1\theta_k^* - {}^1\hat{\theta}_k)\, L_{kf} L_{\hat{f}}^{\rho} h(x) + \left\{ \sum_{j=1}^{J} ({}^2\theta_j^* - {}^2\hat{\theta}_j)\, L_{jg} L_{\hat{f}}^{\rho} h(x) \right\} u + \frac{\partial}{\partial \hat{\theta}} \left(L_{\hat{f}}^{\rho} h(x) \right) \frac{d\hat{\theta}}{dt} \qquad (5.3.14)$$

Combining the above equations (5.3.10), (5.3.11) and (5.3.14), we can write compactly

$$\dot{z} = (A+bK)\, z + \Phi(x,\hat{\theta})\, (\theta^* - \hat{\theta}) + \Pi(x,\hat{\theta}) \frac{d\hat{\theta}}{dt} \qquad (5.3.15)$$

where

$z = \left[z_1 \ z_2 \ \cdots \ z_{\rho+1} \right]^T$

$$A+bK = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ b_1 & b_2 & b_3 & \cdots & b_{\rho+1} \end{bmatrix}$$

$$\Phi(x, \hat{\theta}) = \begin{bmatrix} L_{1f}h & \cdots & L_{Kf}h & L_{1g}h\,u & \cdots & L_{Jg}h\,u \\ L_{1f}L_{\hat{f}}h & \cdots & L_{Kf}L_{\hat{f}}h & L_{1g}L_{\hat{f}}h\,u & \cdots & L_{Jg}L_{\hat{f}}h\,u \\ & \vdots & & & \vdots & \\ L_{1f}L_{\hat{f}}^{\rho}h & \cdots & L_{Kf}L_{\hat{f}}^{\rho}h & L_{1g}L_{\hat{f}}^{\rho}h\,u & \cdots & L_{Jg}L_{\hat{f}}^{\rho}h\,u \end{bmatrix}$$

$$\Pi(x, \hat{\theta}) = \begin{bmatrix} \mathbf{o} \\ \frac{\partial}{\partial\hat{\theta}}(L_{\hat{f}}h) \\ \vdots \\ \frac{\partial}{\partial\hat{\theta}}\left(L_{\hat{f}}^{\rho}h\right) \end{bmatrix}$$

where $\Pi(x, \hat{\theta})$ is a matrix since

$$\frac{\partial}{\partial\hat{\theta}}(L_{\hat{f}}h) = \left[\frac{\partial}{\partial^1\hat{\theta}_1}(L_{\hat{f}}h) \cdots \frac{\partial}{\partial^2\hat{\theta}_J}(L_{\hat{f}}h)\right]$$

$$\mathbf{o} = [\,0\,0\,0 \cdots 0]$$

$$(\theta^* - \hat{\theta}) = \left[\left({}^1\theta_1^* - {}^1\hat{\theta}_1\right), \ldots, \left({}^1\theta_I^* - {}^1\hat{\theta}_I\right), \left({}^2\theta_1^* - {}^2\hat{\theta}_1\right), \ldots, \left({}^2\theta_J^* - {}^2\hat{\theta}_J\right)\right]^T$$

By assumption 5.1, the coordinate transformation $z = T(x, \hat{\theta})$ has a nonsingular Jacobian with respect to x at every point $x \in U$ for any fixed $\hat{\theta} \in S_\theta$, i.e. $T(x, \hat{\theta})$ is a diffeomorphism for any fixed $\hat{\theta} \in S_\theta$. Therefore, we can write the above equation (5.3.15) in the z-coordinate system. Actually, this representation in the z-coordinate system is only for the convenience of the theoretical development.

Define

$$\Theta(z, \hat{\theta}) = \Phi\,(x, \hat{\theta}) \tag{5.3.16}$$

$$\Gamma(z, \hat{\theta}) = \Pi(x, \hat{\theta})$$

Then the above equation (5.3.15) can be written:

$$\dot{z} = (A+bK)\, z + \Theta(z, \hat{\theta})\, (\theta^* - \hat{\theta}) + \Gamma(z, \hat{\theta}) \frac{d\hat{\theta}}{dt} \tag{5.3.17}$$

That is, the real plant (5.3.1) can be written as (5.3.17) by the coordinate transformation (5.3.7) and the linearizing state feedback (5.3.13).

Now, choose the following simple adaptive parameter estimation law:

$$\frac{d\hat{\theta}}{dt} = Q^{-1}\Theta^T(z, \hat{\theta})Pz \tag{5.3.18}$$

where Q and P are positive definite matrices and P is chosen such that

$$(A+bK)^T P + P(A+bK) = -I$$

When all eigenvalues of the matrix A+bK have negative real parts, the above Lyapunov matrix equation has a unique solution of positive definite P [Vidyasagar, 1978, Theorem 55, p175].

With this adaptive law, we now investigate the stability of the nonlinear adaptive system (5.3.17) and (5.3.18). To do this, consider the following Lyapunov candidate function for the adaptive control system:

$$V = z^T P z + (\theta^* - \hat{\theta})^T Q (\theta^* - \hat{\theta}) \tag{5.3.19}$$

The derivative of V along the trajectories of the adaptive system (5.3.17) and (5.3.18) is

$$\begin{aligned}\dot{V} = z^T\left[(A+bK)^T P + P(A+bK)\right] z + (\theta^* - \hat{\theta})^T \Theta^T(z, \hat{\theta})Pz + z^T P \Theta(z, \hat{\theta})\, (\theta^* - \hat{\theta}) \\ + \left(\frac{d\hat{\theta}}{dt}\right)^T \Gamma^T(z, \hat{\theta})\, Pz + z^T P \Gamma(z, \hat{\theta})\frac{d\hat{\theta}}{dt} - \left(\frac{d\hat{\theta}}{dt}\right)^T Q(\theta^* - \hat{\theta}) - (\theta^* - \hat{\theta})^T Q \frac{d\hat{\theta}}{dt}\end{aligned} \tag{5.3.20}$$

Applying the adaptive law (5.3.18) to equation (5.3.20) then

$$\dot{V} = z^T\left[-I + P\left(\Theta(z,\hat{\theta})Q^{-1}\Gamma^T(z,\hat{\theta}) + \Gamma(z,\hat{\theta})Q^{-1}\Theta^T(z,\hat{\theta})\right)P\right] z \tag{5.3.21}$$

where the adaptation law (5.3.18) cancels the unknown parameter term $(\theta^* - \hat{\theta})$ from equation (5.3.20).

Define

$$B(z,\hat{\theta}) = \Theta(z,\hat{\theta})Q^{-1}\Gamma^T(z,\hat{\theta})$$

then $B(z,\hat{\theta}) + B^T(z,\hat{\theta})$ is symmetric and equation (5.3.21) can be written

$$-\dot{V} = z^T\left[I - P\left(B(z,\hat{\theta}) + B^T(z,\hat{\theta})\right)P\right] z \tag{5.3.22}$$

Define a set Ω_c, where $\phi \equiv \theta^* - \hat{\theta}$, such that

$$\Omega_c \equiv \left\{z, \phi;\ z^TPz + \phi^TQ\phi \le c\right\}$$
$$= \left\{x, \hat{\theta};\ T(x,\hat{\theta})^T PT(x,\hat{\theta}) + (\theta^* - \hat{\theta})^T Q(\theta^* - \hat{\theta}) \le c\right\}$$

where c is a positive constant such that $\Omega_c \subset U \times S_\theta$.

Then we can analyze the above adaptive control system using LaSalle's theorem.

LaSalle's Theorem [LaSalle, 1968, Vidyasagar, 1978, p 157, Lemma 81]

Consider the autonomous system

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n \tag{5.3.23}$$

We assume that f is a continuous function. Let V: $\mathbb{R}^n \to \mathbb{R}$ be continuously differentiable, and suppose that for some $c > 0$, the set

$$\Omega \equiv \left\{x \in \mathbb{R}^n;\ V(x) \le c\right\}$$

is bounded. Suppose that V is bounded below over the set Ω and that $\dot{V}(x) \le 0\ \forall\ x \in \Omega$. Let S denote the subset of Ω defined by

$$S = \{x \in \Omega \,;\, \dot{V}(x) = 0\}$$

and let M be the largest invariant set of (5.3.23) contained in the set S. Then whenever the initial condition x(0) is in the set Ω, the solution x(t) of (5.3.23) approaches M as $t \to \infty$. ///

Proposition 1: Consider the above nonlinear adaptive control system (5.3.17) and (5.3.18). If the matrix $\left[I - P\left(B(z, \hat{\theta}) + B^T(z, \hat{\theta})\right) P\right]$ is positive definite for every $z, \phi \in \Omega_c$ and the initial condition of z and $\hat{\theta}$ is in the set Ω_c, then the solution z(t), $\hat{\theta}(t)$ are in the set Ω_c $\forall\, t \in [\,0, \infty\,)$ and moreover $\lim_{t \to \infty} z(t) = 0$. ///

Proof : This proposition can be proved using LaSalle's theorem. If the initial condition of z and $\hat{\theta}$ is in the set Ω_c, then by equation (5.3.22) and the condition that the matrix $\left[I - P\left(B(z, \hat{\theta}) + B^T(z, \hat{\theta})\right) P\right]$ is positive definite for every $z, \phi \in \Omega_c$

$$\dot{V}(z,\phi) \leq 0 \qquad \forall\, z,\phi \in \Omega_c.$$

That is, V is a nonincreasing function. Therefore every solution of (5.3.17) and (5.3.18) is in the set Ω_c for every $t \in [\,0, \infty\,)$.

Let us define

$$S = \{z, \hat{\theta} \in \Omega_c \,;\, \dot{V}(z,\hat{\theta}) = 0\}$$

Since

$$\begin{aligned} &\dot{V}(z,\hat{\theta}) = 0 \\ &\Leftrightarrow z^T\left[I - P\left(B(z, \hat{\theta}) + B^T(z, \hat{\theta})\right) P\right] z = 0 \\ &\Rightarrow z = 0 \ \ (\text{since } \left[I - P\left(B(z, \hat{\theta}) + B^T(z, \hat{\theta})\right) P\right] \text{ is positive definite for every} \\ &\qquad\qquad z, \phi \in \Omega_c), \end{aligned}$$

the set S can be written

$$S = \left\{ z, \hat{\theta} \in \Omega_c \, ; \, z = 0 \right\}$$

Actually, the set S is composed of an infinite number of points $(z, \hat{\theta}) \in \Omega_c$ with $z = 0$. This means that $h(x) = L_{\hat{f}}h(x) = \cdots = L_{\hat{f}}^{\rho}h(x) = 0$ and in turn $\Theta(z = 0, \hat{\theta}) = 0$. Therefore, $z = 0$ is an invariant set of the system (5.3.17) and (5.3.18). Therefore, by LaSalle's theorem, $\lim_{t \to \infty} z(t) = 0$. ///

Remark 5.1: In this proposition 1 it cannot be guaranteed that

$$\hat{\theta}(t) \to \theta^* \quad \text{as } t \to \infty. \qquad ///$$

Remark 5.2: We can always find a positive definite matrix Q in equation (5.3.18) such that the matrix $\left[I - P\left(B(z, \hat{\theta}) + B^T(z, \hat{\theta})\right)P\right]$ is positive definite in a given set of x and $\hat{\theta}$ if a system is globally feedback linearizable. Actually, Q^{-1} is a gain matrix of the adaptive control. If Q^{-1} is very small, then the adaptive control is sluggish, and if it is very large, then the control response becomes excessively oscillatory. And obviously, if $Q^{-1} = 0$, then it is simply nonadaptive linearizing control. ///

It is noted that the above proposition 1 is only a sufficient condition for stability of the proposed adaptive control system. Practically, we can adjust Q for the best performance with the given performance objectives.

Until now, we have consider the adaptive regulation problem. Next, we will consider the adaptive output tracking problem.

5.4. Adaptive Output Tracking of a Feedback Linearizable Process

In this section, we consider an adaptive output tracking problem; that is, output, y, follows the given desired trajectory, $y_d(t)$, which is a C^∞ (infinitely differentiable) - function.

We consider the same system (5.3.1) and (5.3.2) with parametric uncertainties. Under assumption 5.1 in section 5.3, the mathematical model (5.3.2) is linearizable with relative degree n for every $x, \hat{\theta} \in U \times S_\theta$, defined in the previous section 5.3.

For output tracking, we apply the following state feedback u instead of the feedback (5.3.13) [Isidori, 1989]:

$$u = \alpha(x, \hat{\theta}) + y_d^{(\rho+1)}(t) + \beta(x, \hat{\theta})\,(v - v_t) \tag{5.4.1}$$

where $y_d^{(i)}(t) = i$ - th derivative of $y_d(t)$ with respect to t

$v_t = K\, dy$

$dy = [\, y_d(t) \;\; y_d^{(1)}(t) \;\; \cdots \;\; y_d^{(\rho)}(t)\,]^T$

$K = [\, b_1 \;\; b_2 \;\; \cdots \;\; b_{\rho+1}]^T$

$\alpha(x, \hat{\theta})$, $\beta(x, \hat{\theta})$ and v are defined in (5.3.13)

It is well known that the above state feedback (5.4.1) with the transformation (5.3.7) linearizes the mathematical model (5.3.2) into the following form [Isidori, 1989]:

$$\dot{e} = (A+bK)\, e \tag{5.4.2}$$

where $e = [\, e_1 \;\; e_2 \;\; \cdots \;\; e_{\rho+1}]^T$

$e_i = L_f^{i-1}h(x) - y_d^{(i-1)}(t) \qquad i = 1, 2, \ldots \;, \rho+1$

That is, output, $y = h(x)$, follows $y_d(t)$ asymptotically with properly chosen K.

Suppose that the system (5.3.1) and (5.3.2) satisfies assumption 5.2 in section

5.3. Applying the coordinate transformation (5.3.7) and the state feedback (5.4.1) to the real plant (5.3.1) we have

$$
\begin{aligned}
e_1 &= h(x) - y_d(t) \\
\dot{e}_1 &= \frac{\partial h(x)}{\partial x}\frac{dx}{dt} - y_d^{(1)}(t) \\
&= \frac{\partial h(x)}{\partial x}\left\{\left(f(x,\hat{\theta}) + \Delta f(x,\theta^*,\hat{\theta})\right) + \left(g(x,\hat{\theta}) + \Delta g(x,\theta^*,\hat{\theta})\right)u\right\} - y_d^{(1)}(t) \\
&= L_{\hat{f}}h(x) - y_d^{(1)}(t) + \sum_k^K \left({}^1\theta_k^* - {}^1\hat{\theta}_k\right) L_{kf}h(x) + (L_{\Delta g}h(x))\,u \\
&= e_2 + \sum_k^K \left({}^1\theta_k^* - {}^1\hat{\theta}_k\right) L_{kf}h(x) + (L_{\Delta g}h(x))\,u
\end{aligned} \tag{5.4.3}
$$

In a similar manner, we get

$$
\dot{e}_2 = e_3 + \sum_k^K \left({}^1\theta_k^* - {}^1\hat{\theta}_k\right) L_{kf}L_{\hat{f}}h(x) + (L_{\Delta g}L_{\hat{f}}h(x))\,u + \frac{\partial}{\partial\hat{\theta}}(L_{\hat{f}}h(x))\frac{d\hat{\theta}}{dt} \tag{5.4.4}
$$

$$\vdots$$

$$
\begin{aligned}
\dot{e}_{\rho+1} &= L_f^{\rho+1}h(x) - y_d^{(\rho+1)}(t) + \sum_k^K \left({}^1\theta_k^* - {}^1\hat{\theta}_k\right) L_{kf}L_f^{\rho}h(x) + \left(L_{\hat{g}}L_f^{\rho}h(x)\right)u \\
&\quad + \left(L_{\Delta g}L_f^{\rho}h(x)\right)u + \frac{\partial}{\partial\hat{\theta}}\left(L_f^{\rho}h(x)\right)\frac{d\hat{\theta}}{dt}
\end{aligned} \tag{5.4.5}
$$

If we apply the state feedback (5.4.1) to (5.4.5), then

$$
\dot{e}_{\rho+1} = \sum_{k=1}^{\rho+1} b_k e_k + \sum_{k=1}^{K} \left({}^1\theta_k^* - {}^1\hat{\theta}_k\right) L_{kf}L_f^{\rho}h(x) + \left(L_{\Delta g}L_f^{\rho}h(x)\right)u + \frac{\partial}{\partial\hat{\theta}}\left(L_f^{\rho}h(x)\right)\frac{d\hat{\theta}}{dt} \tag{5.4.6}
$$

Combining the above equations (5.4.3), (5.4.4) and (5.4.6) we obtain

$$
\dot{e} = (A + bK)\,e + \Phi(x,\hat{\theta}, t)\,(\theta^* - \hat{\theta}) + \Pi(x,\hat{\theta})\frac{d\hat{\theta}}{dt} \tag{5.4.7}
$$

where matrices Φ and Π are defined in (5.3.15).

Note that $\Phi(x,\hat{\theta}, t)$ depends on time, t, explicitly, i.e., it is a time-varying function.

Now, consider the following adaptive parameter estimation law:

$$\frac{d\hat{\theta}}{dt} = Q^{-1}\Phi^T(x,\hat{\theta},t)\,P\,e \tag{5.4.8}$$

To analyze the stability of the adaptive control system (5.4.7) and (5.4.8), let us consider the following Lyapunov candidate function V:

$$V = e^TPe + (\theta^* - \hat{\theta})^TQ(\theta^* - \hat{\theta}) \tag{5.4.9}$$

The derivative of V along the trajectories of the adaptive system is

$$\dot{V} = e^T\left[(A+bK)^TP + P(A+bK)\right]e + (\theta^* - \hat{\theta})^T\Phi^T(x,\hat{\theta},t)Pe + e^TP\Phi(x,\hat{\theta},t)(\theta^* - \hat{\theta}) + \left(\frac{d\hat{\theta}}{dt}\right)^T\Pi^T(x,\hat{\theta})Pe + e^TP\Pi(x,\hat{\theta})\frac{d\hat{\theta}}{dt} - \left(\frac{d\hat{\theta}}{dt}\right)^TQ(\theta^* - \hat{\theta}) - (\theta^* - \hat{\theta})^TQ\frac{d\hat{\theta}}{dt} \tag{5.4.10}$$

Applying the adaptive law (5.4.8) to equation (5.4.10) we obtain

$$\dot{V} = e^T\left[-I + P\left(\Phi(x,\hat{\theta},t)Q^{-1}\Pi^T(x,\hat{\theta}) + \Pi(x,\hat{\theta})Q^{-1}\Phi^T(x,\hat{\theta},t)\right)P\right]e \tag{5.4.11}$$

Define

$$B(x,\hat{\theta},t) = \Phi(x,\hat{\theta},t)Q^{-1}\Pi^T(x,\hat{\theta})$$

then we have

$$-\dot{V} = e^T\left[I - P\left(B(x,\hat{\theta},t) + B^T(x,\hat{\theta},t)\right)P\right]e \tag{5.4.12}$$

Define a set Ω_c, where $\phi = \theta^* - \hat{\theta}$, such that

$$\Omega_c \equiv \left\{e, \phi;\ e^TPe + \phi^TQ\phi \le c\right\}$$

where c is a positive constant such that $\Omega_c \subset U \times S_\theta$ for every $t \ge 0$. That is, in the set Ω_c, the given nonlinear system is exactly state-space linearizable. Generally, this condition, i.e., $\Omega_c \subset U \times S_\theta$ for every $t \ge 0$, can be checked following the given desired trajectory.

Now, we can apply the following proposition 2.

Proposition 2: If the matrix $\left[I - P\left(B(x, \hat{\theta}, t) + B^T(x, \hat{\theta}, t)\right)P\right]$ is positive definite for every $e, \phi \in \Omega_c$ and $t \geq 0$ and the initial condition of e and $\hat{\theta}$ is in the set Ω_c, then the solution $e(t)$, $\hat{\theta}(t)$ are in the set Ω_c $\forall\, t \in [\,0, \infty)$ and moreover $\lim_{t \to \infty} e(t) = 0$. This means that $y(t) \to y_d(t)$ as $t \to \infty$. ///

Proof: We can prove the above proposition 2 by standard argument of Lyapunov theory [Vidyasagar, 1978, Gavel and Siljak, 1989]. Since for every $e, \hat{\theta} \in \Omega_c$ and $t \geq 0$ $\dot{V} \leq 0$, V is a nonincreasing function. Therefore, if $V(0) \leq c$, then $V(t) \leq c$ for every $t \geq 0$. It is also obvious that $\dot{V}$ is uniformly continuous on Ω_c.

Since V(t) is nonincreasing and bounded below (obviously V is nonnegative),

$$\lim_{t \to \infty} V(t) = \inf V(t) \equiv V_f \geq 0$$

Since $\displaystyle\lim_{t \to \infty} \int_0^t \dot{V}(\tau)\, d\tau = V(\infty) - V(0) < \infty$,

$$\lim_{t \to \infty} \dot{V}(t) = 0 \Rightarrow \lim_{t \to \infty} e(t) = 0$$ ///

Note that the vector e is a function of x, $\hat{\theta}$ and t, and has a nonsingular Jacobian matrix with respect to x in Ω_c for every fixed $\hat{\theta}$ and t by assumption 5.1 in the section 5.3. So, the coordinate transformation e is a local diffeomorphism.

In order to apply proposition 2, we must check the condition that the matrix $\left[I - P\left(B(x, \hat{\theta}, t) + B^T(x, \hat{\theta}, t)\right)P\right]$ is positive definite for every $e, \phi \in \Omega_c$ and $t \geq 0$. Since $B(x, \hat{\theta}, t)$ is a function of t explicitly, it is very difficult to check the above condition generally. However, if $\Delta g = 0$, then $B(x, \hat{\theta}, t)$ depends only on x and $\hat{\theta}$. In this case, we can check the above condition easily. For practical purposes the desired output trajectory is a periodic function or a function which converges to a point. Therefore we can find the set Ω_x easily such that

$$x, \hat{\theta} \in \Omega_x \Rightarrow e, \hat{\theta} \in \Omega_c \;\; \forall\, t \geq 0 \tag{5.4.13}$$

At a fixed time $t = t_0$, we can find the set Ω_c. Along the desired trajectory $dy = [\, y_d(t) \;\; y_d^{(1)}(t) \cdots y_d^{(\rho+1)}(t)\,]^T$ if we combine these sets Ω_c, then we can find the set Ω_x, and then we can check easily, in the x-coordinate system, the condition that the matrix $\left[I - P\left(B(x, \hat{\theta}, t) + B^T(x, \hat{\theta}, t)\right) P\right]$ is positive definite for every $e, \phi \in \Omega_c$ and $t \geq 0$. Fig.5.1 shows this schematically.

When there exists only a mismatch $\Delta g \in$ span $\{g\}$, we may apply the Lyapunov Min - Max approach [Corless and Leitmann, 1988, Gutman, 1985, 1979]. If there is also Δf caused by parametric uncertainties, then we may apply the composite control, i.e. adaptive for Δf and Lyapunov Min - Max approach for Δg. However, in this dissertation, this is not pursued.

Sometimes, feedback linearization requires a very high magnitude of a state feedback, u, which may go outside of the allowable bound of manipulated variables. The output tracking can be utilized to decrease the highest magnitude of the linearizing state feedback, u, if an initial condition can be estimated reasonably. In other words, it can make a state feedback, u, smooth so that it can be applied easily in a real application. This has been demonstrated in the next application.

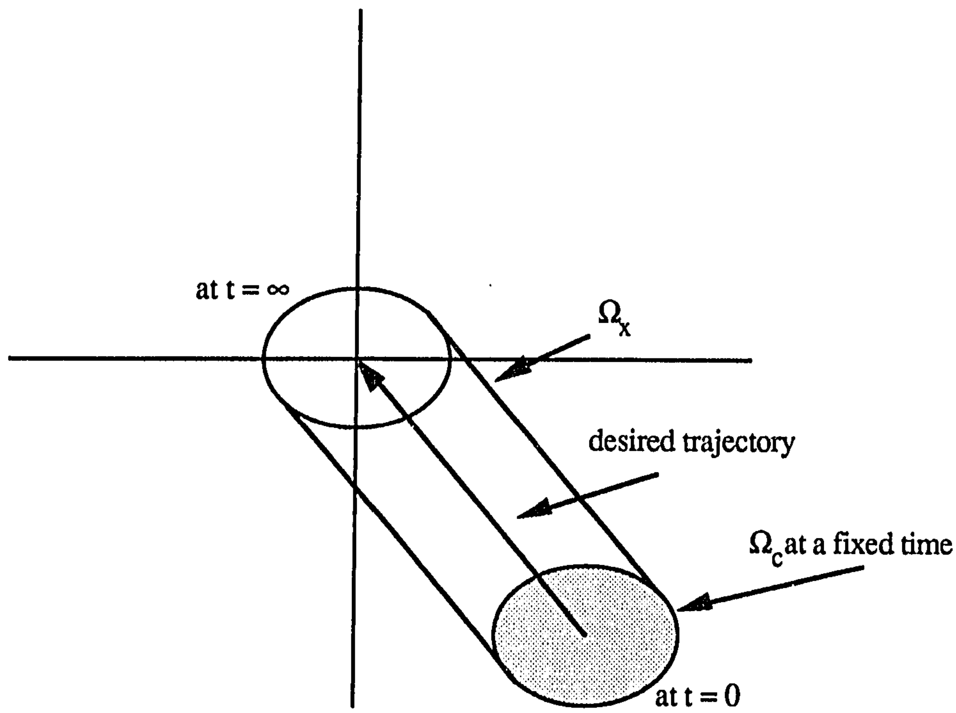

Fig. 5.1 The set Ω_c and Ω_x for output tracking.

5.5. Applications of Adaptive Regulation

5.5.1. First Order Exothermic Reaction in a CSTR

Mathematical model

Consider a first order exothermic reaction in a CSTR, as in Chapter 3

$$\frac{dx_1}{d\tau} = (-x_1 + \alpha_c) - (x_1-\alpha_c+1)\exp\left[D - \frac{\upsilon^2}{x_2-\alpha_T+\upsilon}\right] \tag{5.5.1}$$

$$\frac{dx_2}{d\tau} = (-x_2 + \alpha_T) - B(x_1-\alpha_c+1)\exp\left[D - \frac{\upsilon^2}{x_2-\alpha_T+\upsilon}\right] - \gamma(x_2-\alpha_T+\alpha_W) + \gamma u$$

$$y = x_1$$

In this system, it is assumed that reaction rate constant k_0 of the first order exothermic reaction has an error; that is, the dimensionless parameter D, defined in section 3.4, is uncertain. It is obvious that model-plant mismatch caused by this parametric error does not satisfy the matching condition (see Appendix IV).

Denote D^* the true parameter, $\widehat{D}(t)$ an estimated parameter, and $\widehat{D}(0)$ a nominal value of D^*. It is assumed that we know only the parametric bounds of D^*.

Define $\theta^* = e^{D^*}$, $\widehat{\theta}(t) = e^{\widehat{D}(t)}$ and $\widehat{\theta}(0) = e^{\widehat{D}(0)}$. With this notation the real plant and the mathematical model of the system (5.5.1) can be written

Real plant:

$$\dot{x} = f(x, \theta^*) + g(x)\,u \tag{5.5.2}$$

$$y = h(x)$$

Mathematical model:

$$\dot{x} = f(x, \widehat{\theta}) + g(x)\,u \tag{5.5.3}$$

$$y = h(x)$$

where

$$f(x, \theta^*) = \left[f_1(x, \theta^*), f_2(x, \theta^*)\right]^T$$

$$f_1(x, \theta^*) = (-x_1 + \alpha_c) - (x_1\text{-}\alpha_c+1) \exp\left(- \frac{\upsilon^2}{x_2\text{-}\alpha_T+ \upsilon}\right)\theta^*$$

$$f_2(x, \theta^*) = (-x_2 + \alpha_T) - B(x_1\text{-}\alpha_c+1) \exp\left(- \frac{\upsilon^2}{x_2\text{-}\alpha_T+ \upsilon}\right)\theta^* - \gamma\,(x_2\text{-}\alpha_T+\alpha_W)$$

$$f(x, \hat{\theta}) = \left[f_1(x, \hat{\theta}), f_2(x, \hat{\theta})\right]^T$$

$$f_1(x, \hat{\theta}) = (-x_1 + \alpha_c) - (x_1\text{-}\alpha_c+1) \exp\left(- \frac{\upsilon^2}{x_2\text{-}\alpha_T+ \upsilon}\right)\hat{\theta}$$

$$f_2(x, \hat{\theta}) = (-x_2 + \alpha_T) - B(x_1\text{-}\alpha_c+1) \exp\left(- \frac{\upsilon^2}{x_2\text{-}\alpha_T+ \upsilon}\right)\hat{\theta} - \gamma\,(x_2\text{-}\alpha_T+\alpha_W)$$

$$g(x,\theta^*) = [g_1(x,\theta^*)\;\; g_2(x,\theta^*)]^T$$

$$g(x, \hat{\theta}) = \left[g_1(x, \hat{\theta}), g_2(x, \hat{\theta})\right]^T$$

$$g_1(x,\theta^*) = g_1(x,\hat{\theta}) = 0$$

$$g_2(x,\theta^*) = g_2(x,\hat{\theta}) = \gamma$$

$$h(x) = x_1$$

and system parameters are given in the previous section 3.4.

The nominal value of θ^*, $\hat{\theta}(0)$, is $\exp(31.799)$ and $f(0,\hat{\theta}(0)) = 0$; that is, $x = 0$ is the equilibrium point of the systems (5.5.2) and (5.5.3) when $\theta^* = \hat{\theta}(t) = \hat{\theta}(0)$. However, this equilibrium point changes according to the value of θ^*; that is $f(0, \theta^*) \neq 0$ when $\theta^* \neq \hat{\theta}(0)$. Therefore, $f(0, \theta^*) \neq 0$ for some $\theta^* \in S_\theta$, where S_θ is a neighborhood of $\theta = \hat{\theta}(0)$; that is, the real plant may have a different equilibrium point.

From equation (5.5.2) and (5.5.3), model-plant mismatch is by definition (5.3.3) in the previous section 5.3

$$\Delta f(x, \theta^*, \hat{\theta}) = \left[\Delta f_1(x, \theta^*, \hat{\theta}), \Delta f_2(x, \theta^*, \hat{\theta})\right]^T \tag{5.5.4}$$

$$\Delta g = 0$$

where

$$\Delta f_1(x, \theta^*, \hat{\theta}) = -(x_1-\alpha_c+1)\exp\left(-\frac{\upsilon^2}{x_2-\alpha_T+\upsilon}\right)(\theta^* - \hat{\theta})$$

$$\Delta f_2(x, \theta^*, \hat{\theta}) = -B\,(x_1-\alpha_c+1)\exp\left(-\frac{\upsilon^2}{x_2-\alpha_T+\upsilon}\right)(\theta^* - \hat{\theta})$$

and this model-plant mismatch can be represented in the following form:

$$\Delta f(x, \theta^*, \hat{\theta}) = (\theta^* - \hat{\theta})\begin{bmatrix} {}^1f_1(x) \\ {}^1f_2(x) \end{bmatrix} \tag{5.5.5}$$

where $${}^1f_1(x) = -(x_1-\alpha_c+1)\exp\left(-\frac{\upsilon^2}{x_2-\alpha_T+\upsilon}\right)$$

$${}^1f_2(x) = -B(x_1-\alpha_c+1)\exp\left(-\frac{\upsilon^2}{x_2-\alpha_T+\upsilon}\right).$$

Therefore model-plant mismatch satisfies linearity in the unknown parameters, where ${}^1f_1(x)$ and ${}^1f_2(x)$ are known functions, but does not satisfy the matching condition. From now on, we will denote $\hat{f}$ and $\hat{g}$ the vector fields for the vector functions $f(x,\hat{\theta})$ and $g(x,\hat{\theta})$ which are infinitely differentiable.

Linearization

Now, let us apply the feedback linearization method to the real plant (5.5.2). To do this it is necessary to check for what values of x and $\hat{\theta}$ the mathematical model (5.5.3) can be linearized with relative degree 2. It is easy to see that if the system (5.5.3) satisfies the following conditions:

$$L_{\hat{g}}h(x) = 0 \text{ and } L_{\hat{g}}L_{\hat{f}}h(x) \neq 0$$

then the system is linearizable with relative degree 2.

Since

$$L_{\hat{g}}h(x) = g_1(x,\hat{\theta}) = 0 \quad \forall\, x \in \mathbb{R}^n, \hat{\theta} \in \mathbb{R}$$

$$L_{\hat{g}}L_{\hat{f}}h(x) = \frac{-\upsilon^2(x_1 - \alpha_c + 1)\,\hat{\theta}}{\left(x_2-\alpha_T+\upsilon\right)^2}\exp\left(-\frac{\upsilon^2}{x_2-\alpha_T+\upsilon}\right)\gamma,$$

we know directly if all of the following conditions are satisfied, then the mathematical model (5.5.3) is linearizable with relative degree 2

(1) $x_1 > \alpha_c - 1$

(2) $\alpha_T - \upsilon < x_2 < \infty$

(3) $\hat{\theta} \neq 0.$

As already indicated in the previous Chapter 3, the condition $x_1 = \alpha_c - 1$ means that concentration in the reactor is absolutely zero, and $x_2 = \alpha_T - \upsilon$ means that reactor temperature is 0 K (absolute zero, see the definition of the dimensionless variables in the previous section (3.4)). Also, since $\theta = \exp\,[k_0\ V/q]$, if k_0 has a finite value, then the above condition (3) is satisfied.

Define the set U and S_θ such that

$$U = \left\{x_1, x_2 \in \mathbb{R};\ x_1 > \alpha_c - 1,\ \alpha_T - \upsilon < x_2 < \infty\right\}$$

$$S_\theta = \left\{\hat{\theta} \in \mathbb{R};\ \hat{\theta} > 0\right\}$$

Then for every $x, \hat{\theta} \in U x S_\theta$, the mathematical model (5.5.3) is linearizable with relative degree 2.

Applying the coordinate change (5.3.7) and the state feedback (5.3.13) to the real plant (5.5.2) we have the following equation:

$$\dot{z}_1 = z_2 + (\theta^* - \hat{\theta})L1_f h(x) \tag{5.5.6}$$

$$\dot{z}_2 = b_1 z_1 + b_2 z_2 + (\theta^* - \hat{\theta})L1_f L_{\hat{f}} h(x) + \frac{\partial}{\partial\hat{\theta}}(L_{\hat{f}} h(x))\frac{d\hat{\theta}}{dt}$$

$$y = z_1$$

where

$$z_1 = h(x) = x_1$$

$$z_2 = L_{\hat{f}} h(x) = f_1(x,\hat{\theta})$$

$$L1_f h(x) = {}^1f_1(x)$$

$$L1_f L_{\hat{f}} h(x) = \frac{\partial f_1(x,\hat{\theta})}{\partial x_1}\ {}^1f_1(x) + \frac{\partial f_1(x,\hat{\theta})}{\partial x_2}\ {}^1f_2(x)$$

$$\frac{\partial f_1(x,\hat{\theta})}{\partial x_1} = -1 - \hat{\theta}\exp\left(-\frac{\upsilon^2}{x_2 - \alpha_T + \upsilon}\right)$$

$$\frac{\partial f_1(x,\hat{\theta})}{\partial x_2} = \frac{-\upsilon^2(x_1 - \alpha_c + 1)\hat{\theta}}{(x_2 - \alpha_T + \upsilon)^2}\exp\left(-\frac{\upsilon^2}{x_2 - \alpha_T + \upsilon}\right)$$

$$\frac{\partial}{\partial\hat{\theta}}(L_{\hat{f}} h(x)) = \frac{\partial f_1(x,\hat{\theta})}{\partial\hat{\theta}} = -(x_1 - \alpha_c + 1)\exp\left(-\frac{\upsilon^2}{x_2 - \alpha_T + \upsilon}\right)$$

The above equation (5.5.6) can be written compactly

$$\dot{z} = (A+bK)z + \Phi(x,\hat{\theta})(\theta^* - \hat{\theta}) + \Pi(x,\hat{\theta})\frac{d\hat{\theta}}{dt} \tag{5.5.7}$$

where $A+bK = \begin{bmatrix} 0 & 1 \\ b_1 & b_2 \end{bmatrix}$

$$\Phi(x,\hat{\theta}) = \begin{bmatrix} L1_f h(x) \\ L1_f L_{\hat{f}} h(x) \end{bmatrix}$$

$$\Pi(x,\hat{\theta}) = \begin{bmatrix} 0 \\ \frac{\partial}{\partial\hat{\theta}}(L_{\hat{f}} h(x)) \end{bmatrix}$$

Adaptive control

Now, we apply the following adaptive law which is the same as (5.3.18)

$$\frac{d\hat{\theta}}{dt} = Q^{-1}\Phi^T(x, \hat{\theta})P\, z \tag{5.5.8}$$

and analyze the stability of the system (5.5.7) and (5.5.8) using proposition 1 in section 5.3.

Suppose that the initial value of the state variables $x_1(0) = 0.2$, $x_2(0) = -1.0$, and the nominal value of θ^*, $\hat{\theta}(0) = \exp(31.799)$. Assume that we know the parametric value of θ^* has the following bound:

$$\exp(31.739) \le \theta^* \le \exp(31.859)$$

(corresponding to $2.8253 \times 10^{11} \le k_0 \le 3.1855 \times 10^{11}$, sec^{-1}).

With the above initial values of x and $\hat{\theta}$, we can calculate the initial value of z:

$$\begin{aligned} z_1(0) &= x_1(0) = 0.2 \\ z_2(0) &= f_1(x(0), \hat{\theta}(0)) = 0.033 \end{aligned} \tag{5.5.9}$$

With the given matrix (A+bK) in (5.5.7), we can find easily a positive definite matrix P such that

$$\begin{aligned} &(A+bK)^T P + P(A+bK) = -I \\ &\text{where } P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}, \quad p_{12} = p_{21} \\ &p_{11} = \frac{b_1^2 - b_1 + b_2^2}{2\, b_1 b_2} \\ &p_{12} = -\frac{1}{2\, b_1} \\ &p_{22} = \frac{1 - b_1}{2\, b_1 b_2} \end{aligned} \tag{5.5.10}$$

Suppose that we choose $b_1 = -2.1$, $b_2 = -2.0$ then, by the above equation (5.5.10),

$$P = \begin{bmatrix} 1.25 & 0.238 \\ 0.238 & 0.369 \end{bmatrix} \tag{5.5.11}$$

Consider the Lyapunov function V defined in equation (5.3.19). In this system, V can be written

$$V = p_{11}z_1^2 + 2\,p_{12}z_1z_2 + p_{22}z_2^2 + Q\,(\theta^* - \hat{\theta})^2 \tag{5.5.12}$$

Now, suppose that we choose $Q = 0.7 \times 10^{-27}$. Then the initial value of V, V(0), is

$$\begin{aligned} V(0) &= p_{11}z_1(0)^2 + 2\,p_{12}z_1(0)z_2(0) + p_{22}z_2(0)^2 + Q\,(\,\theta^* - \hat{\theta}(0)\,)^2 \\ &\leq 0.0535 + (0.7 \times 10^{-27})\left(\theta^* - \exp(31.799)\right)^2 \Big|_{\theta^* = \exp(31.859)} \\ &= 0.0647 \end{aligned}$$

So, V(0) cannot be any higher than 0.0647.

Consider the set Ω_c such that

$$\Omega_c \equiv \left\{z, \hat{\theta};\ V \leq c,\ c = 0.0647\right\}$$

By Proposition 1, if the matrix $[I - P(\,B(z, \hat{\theta}) + B^T(z, \hat{\theta})\,)P]$ is positive definite for every $z, \hat{\theta} \in \Omega_c$, then the trajectories z(t) and $\hat{\theta}(t)$ of the adaptive control system (5.5.7) and (5.5.8) are in the set Ω_c for every $t \geq 0$ and moreover $\lim_{t \to \infty} z = 0$, which means that $\lim_{t \to \infty} y = 0$.

Before checking positive definiteness of the above matrix $[I - P(\,B(z, \hat{\theta}) + B^T(z, \hat{\theta})\,)P]$, let us make sure that the set Ω_c is a proper subset of $U \times S_\theta$; that is, the set Ω_c does not include the point $x_1 = \alpha_c - 1$ and $\hat{\theta} \leq 0$ (obviously x_2 never can be $\alpha_T - \upsilon$; i.e., temperature in the reactor can never be absolute zero and also since Ω_c is finite, x_2 has a finite value). To do this, it is sufficient to find the upper and lower bounds of x_1

and $\hat{\theta}$ in the set Ω_c.

It is obvious that the possible maximum and minimum values of $\hat{\theta}$ in the set Ω_c occur when $z_1 = z_2 = 0$ since P is a positive definite matrix. Therefore, from the equation: $Q(\theta^* - \hat{\theta})^2 \leq c; c = 0.0647$, we can find the possible maximum range of $\hat{\theta}$:

$$\exp(31.57) \leq \hat{\theta} \leq \exp(31.99).$$

The maximum possible range of z can also be found obviously from the following equation:

$$p_{11}z_1^2 + 2p_{12}z_1z_2 + p_{22}z_2^2 \leq c$$

Define a set S_z such that

$$S_z \equiv \left\{z;\ p_{11}z_1^2 + 2p_{12}z_1z_2 + p_{22}z_2^2 \leq c\right\} \tag{5.5.13}$$

Now, we want to check whether the set S_z contains the point $z_1 = x_1 = \alpha_c - 1 = -0.4593$ when $c = 0.0647$. Fig.5.2 shows the set S_z for each value of c (prepared using Macintosh program: MATLAB). Note that, in this case, we have interest in only the state variable x_1. Therefore, we draw the set S_z in the z-coordinate system because $z_1 = x_1$. From Fig.5.2 we can see that $\Omega_c \subset UxS_\theta$ whenever $c \leq 0.269$. That is, the system is linearizable with relative degree 2 for every z, $\hat{\theta} \in \Omega_c$, when $c = 0.0647$.

Now, let us check that the matrix $[I - P(B(z, \hat{\theta}) + B^T(z, \hat{\theta}))P]$ is positive definite for every z, $\hat{\theta} \in \Omega_c$, where $c = 0.0647$. Since $B(z, \hat{\theta}) = \Theta(z, \hat{\theta})\, Q^{-1}\, \Gamma^T(z, \hat{\theta})$, we can write $B(z, \hat{\theta})$ in the x-coordinate system, which makes the necessary calculation easy

$$\begin{aligned} B(x, \hat{\theta}) &= \Phi(x, \hat{\theta})\, Q^{-1}\, \Pi^T(x, \hat{\theta}) \\ &= \begin{bmatrix} L_{1f}h(x) \\ L_{1f}L_{\hat{f}}h(x) \end{bmatrix} Q^{-1} \begin{bmatrix} 0 & \dfrac{\partial}{\partial\hat{\theta}}(L_{\hat{f}}h(x)) \end{bmatrix} \end{aligned}$$

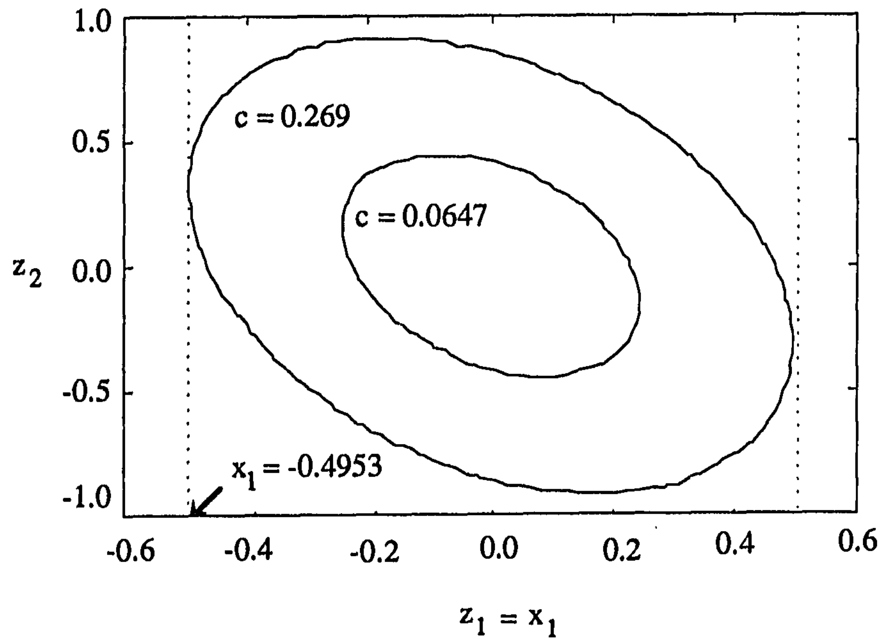

Fig. 5.2 The set S_Z for each given constant c

And

$$B + B^T = \begin{bmatrix} 0 & b_{12} \\ b_{12} & b_{22} \end{bmatrix}$$

$$\text{where}\quad b_{12} = Q^{-1}\left(L_{1f}h(x)\right)\frac{\partial}{\partial\hat{\theta}}\left(L_{\hat{f}}h(x)\right)$$

$$b_{22} = 2\,Q^{-1}\left(L_{1f}L_{\hat{f}}h(x)\right)\frac{\partial}{\partial\hat{\theta}}\left(L_{\hat{f}}h(x)\right)$$

Therefore we have

$$\left[I - P(B + B^T)P\right] = \begin{bmatrix} 1 - c_{11} & -c_{12} \\ -c_{12} & 1 - c_{22} \end{bmatrix}$$

$$\text{where}\quad c_{11} = p_{12}b_{12}p_{11} + (p_{11}b_{12} + p_{12}b_{22})p_{12}$$

$$c_{12} = p_{12}b_{12}p_{12} + (p_{11}b_{12} + p_{12}b_{22})p_{22}$$

$$c_{22} = p_{22}b_{12}p_{12} + (p_{12}b_{12} + p_{22}b_{22})p_{22}$$

Positive definiteness of the above matrix can be easily checked since if the leading principal minors, $(1-c_{11})$ and $(1-c_{11})(1-c_{22}) - c_{12}^2$, are positive for every $x, \hat{\theta} \in \Omega_c$, then the matrix $\left[I - P(B + B^T)P\right]$ is positive definite in Ω_c. In this case, the leading principal minors are so complicated that it is very difficult to check the positiveness analytically. In this work, we check the positiveness of the leading principal minors using a numerical search method (program: IPBP1) with the given parametric bounds and initial condition; that is, for sufficiently large number of $x, \hat{\theta} \in \Omega_c$ the positiveness of the above leading principal minors is checked. By this method, we conclude that the matrix $\left[I - P(B + B^T)P\right]$ is positive definite in Ω_c when $c \leq 0.0647$.

Simulation

Suppose that real plant has the true value $\theta^* = \exp(31.859)$, which is unknown, but belongs to the previously given bound. With the previously given initial value of x and $\hat{\theta}$ and $b_1 = -2.1$, $b_2 = -2.0$, we simulate the adaptive control system (program: OUTX1AD).

Fig.5.3 shows the response in the z-coordinate system with $Q = 0.7 \times 10^{-27}$. This figure shows that the system response is bounded and converges to $z = 0$. Fig. 5.4 shows the estimated parameter and in this case the estimated parameter converges to the true value, $\theta^* = \exp(31.859)$. Fig. 5.5 shows the response in the x- coordinate system with the same condition as Fig. 5.3. We can see that the output x_1 converges to zero. Fig. 5.6 compares the output response, $y = x_1$, with the nonadaptive control, i.e. nonadaptive feedback linearization considered in the previous chapter, (program: OUTX1RO, OUTX1AD). In this figure, we can see that the adaptive control results in much better performance; that is, the adaptive control shows good regulation in output. However, the nonadaptive control results in offset (remember that the equilibrium point of the system changes with the parametric value of D).

As we indicated earlier, the value of Q is closely related to the performance of the adaptive control system. Fig. 5.7 shows the adaptive control response when $Q = 0.7 \times 10^{-28}$. As expected, this large gain ($= Q^{-1}$) results in an oscillatory response of x even though it is bounded in the set Ω_c and converges to zero, i.e. $y = 0$. Fig. 5.8 shows the control response when $Q = 0.7 \times 10^{-26}$. This low gain results in a sluggish response.

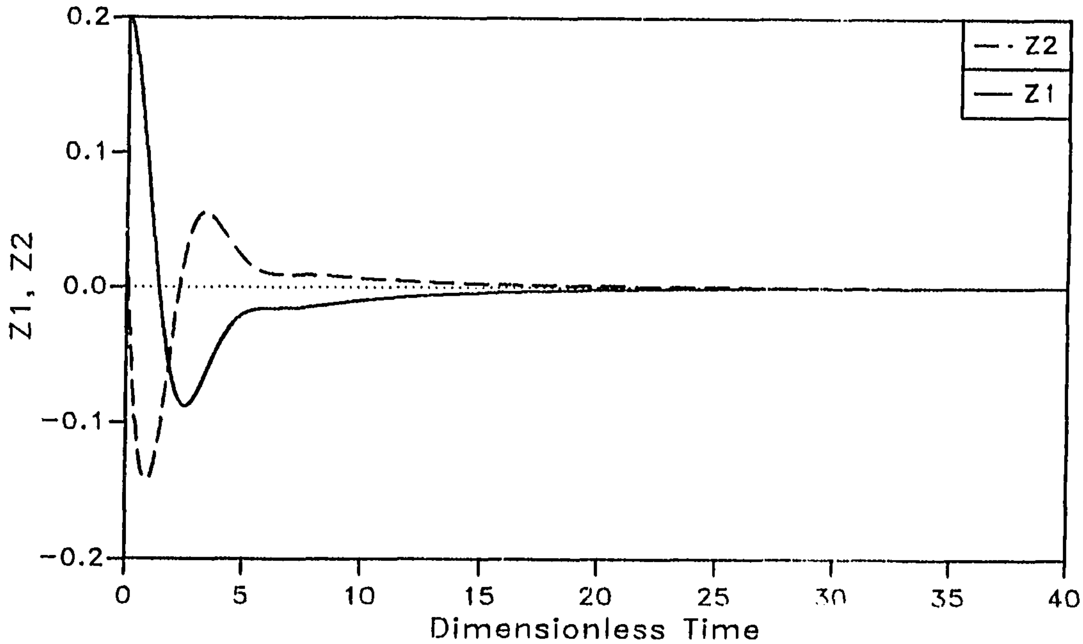

Fig. 5.3 Adaptive control response in the z-coordinate system for uncertainty in the reaction rate constant of CSTR, where

$x_1(0) = 0.2$, $x_2(0) = -1.0$, $Q = 0.7 \times 10^{-27}$ and $\theta^* = \exp(31.859)$

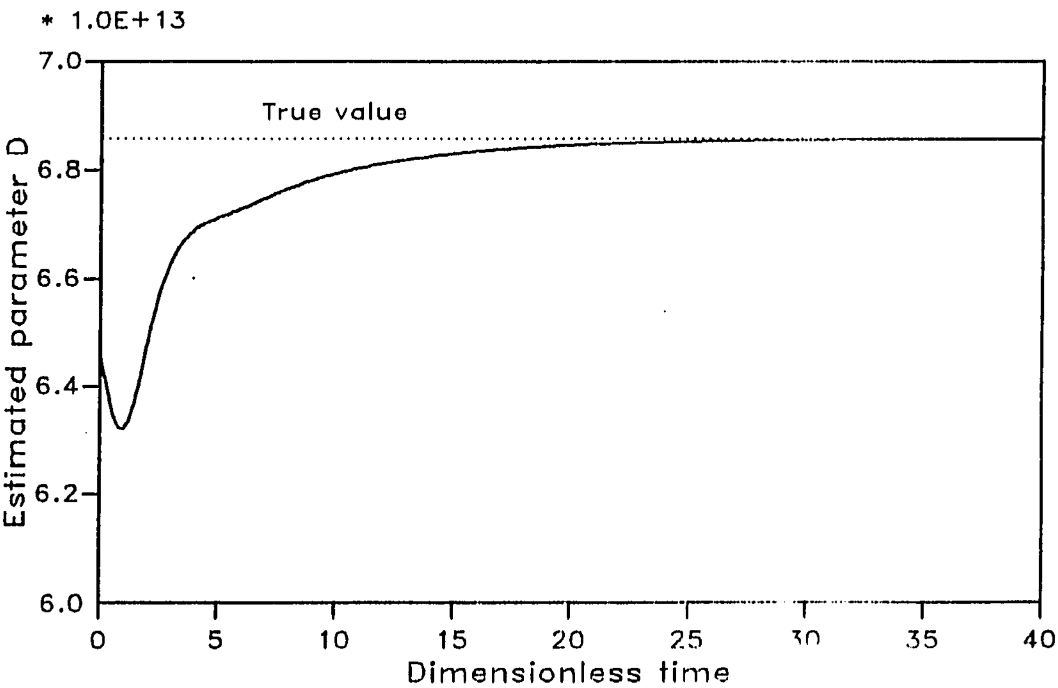

Fig. 5.4 Estimated parameter, $\hat{D}$, where

$x_1(0) = 0.2$, $x_2(0) = -1.0$, $Q = 0.7 \times 10^{-27}$ and $\theta^* = \exp(31.859)$

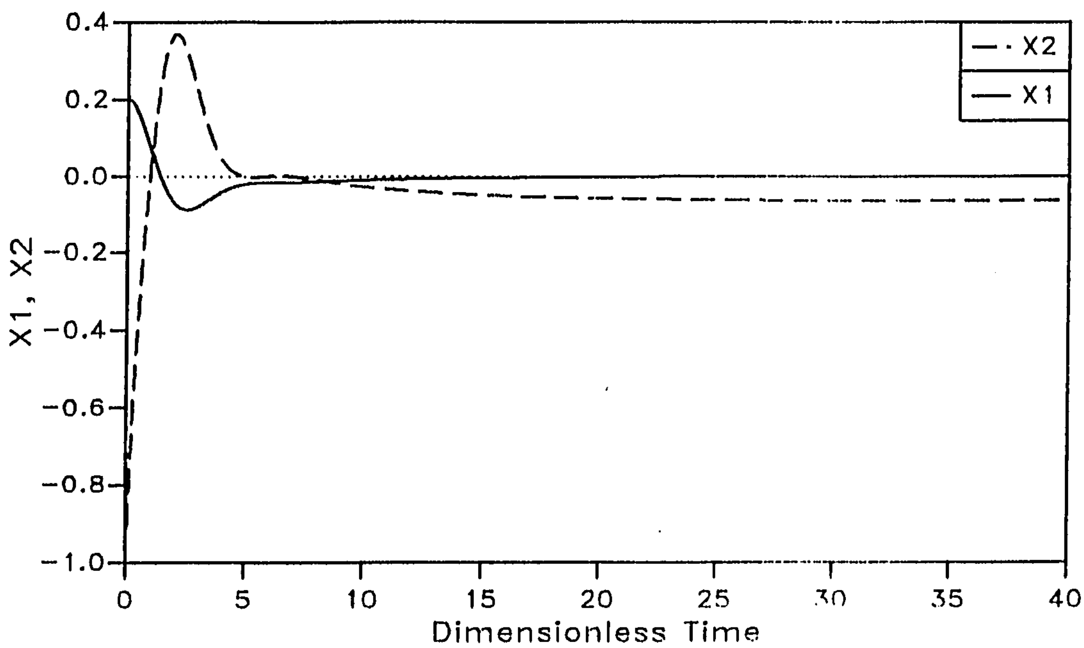

Fig. 5.5 Adaptive control response for uncertainty in the reaction rate constant of CSTR in the x-coordinate system, where

$x_1(0) = 0.2$, $x_2(0) = -1.0$, $Q = 0.7 \times 10^{-27}$ and $\theta^* = \exp(31.859)$

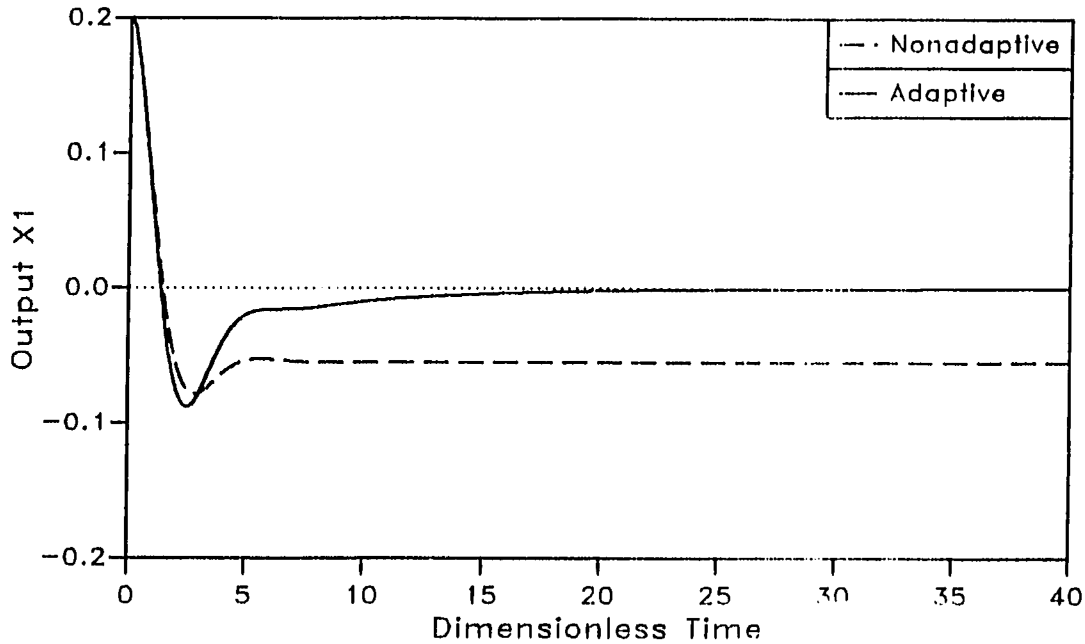

Fig. 5.6 Comparison between the adaptive and the nonadaptive control for uncertainty in the reaction rate constant of CSTR, where $x_1(0) = 0.2$, $x_2(0) = -1.0$, $Q = 0.7 \times 10^{-27}$ and $\theta^* = \exp(31.859)$

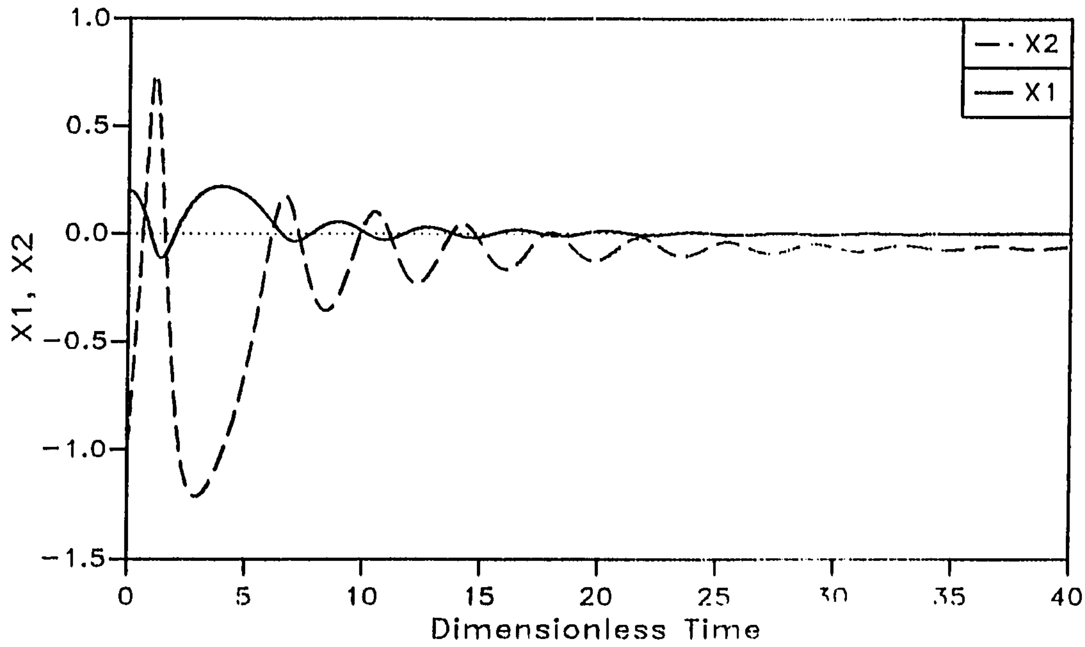

Fig. 5.7 Adaptive control response for uncertainty in the reaction rate constant of CSTR with a large gain, Q^{-1}, where

$x_1(0) = 0.2$, $x_2(0) = -1.0$, $Q = 0.7 \times 10^{-28}$ and $\theta^* = \exp(31.859)$

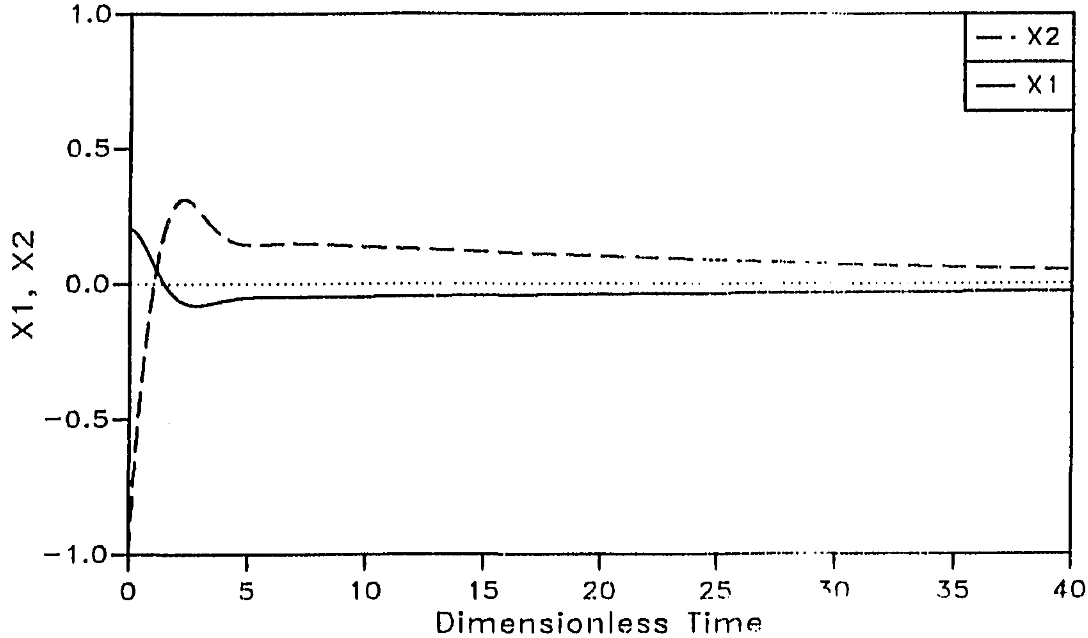

Fig. 5.8 Adaptive control response for uncertainty in the reaction rate constant of CSTR with the small gain, Q^{-1}, where

$x_1(0) = 0.2$, $x_2(0) = -1.0$, $Q = 0.7 \times 10^{-26}$ and $\theta^* = \exp(31.859)$

5.5.2. Biological Reaction in a CSTBR

In this section we consider adaptive regulation of an unstable biological reactor. Hoo and Kantor (1986) have applied feedback linearization to the growth of the methanol utilizing microorganism Methylomonas in a CSTBR (Continuous Stirred Tank Biological Reactor) studied by Dibiasio, et al (1981). Fig. 5.9 shows the diagram of system.

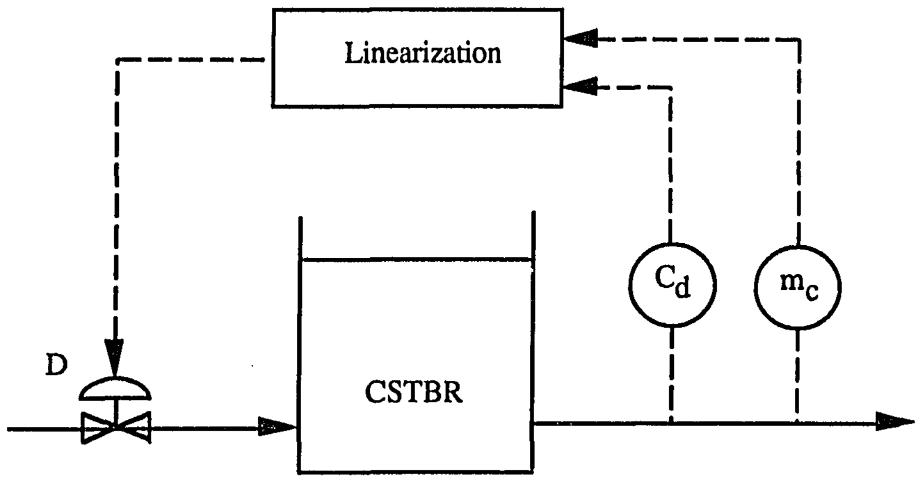

Fig. 5.9 Control system of a CSTBR

Mathematical model

The equations of the CSTBR are given by [Hoo and Kantor, 1986]

$$\frac{dC_d}{dt} = \mu(m_c)\, C_d - D\, C_d \tag{5.5.14}$$
$$\frac{dm_c}{dt} = -\sigma(m_c)\, C_d + D\,(m_{cf} - m_c)$$

where C_d = cell density

m_c = methanol concentration

D = dilution rate [hr^{-1}]

m_{cf} = concentration of substrate in feed where the specific growth rate $\mu(m_c)$ and the substrate consumption rate $\sigma(m_c)$ are given by

$$\mu(m_c) = \frac{a_1 m_c (a_2 + a_3 m_c)}{K(m_c)} \qquad (5.5.15)$$

$$\sigma(m_c) = \frac{m_c (a_4 + a_5 m_c + a_6 m_c^2)}{K(m_c)}$$

$$K(m_c) = 0.000849 + m_c + 0.406 m_c^2$$

The system parameters a_i, $i = 1, ..., 6$ are

$$a_1 = 0.504, \quad a_2 = 1.0, \quad a_3 = - 0.204$$

$$a_4 = 1.32, \quad a_5 = 3.86, \quad a_6 = - 0.661$$

With the given parameters, the above system (5.5.14) has multiple steady states, and when the concentration of substrate in feed, $m_{cf} = 1.8$, the system has an unstable steady state, $C_d = 0.24$ and $m_c = 0.3888$ (program: BIOSET), which is chosen as the set point in this problem. Let's denote the set point to be C_{de}, m_{ce} and the normal value of D to be D_e, which has the value of 0.4, i.e., $C_{de} = 0.24$, $m_{ce} = 0.3888$ and $D_e = 0.4$.

Define new state and input variables as follows:

$$x_1 = C_d - C_{de}$$

$$x_2 = m_c - m_{ce}$$

$$u = D - D_e$$

Then the above system (5.5.14) can be represented by

$$\dot{x}_1 = f_1(x) + g_1(x)\, u \qquad (5.5.16)$$

$$\dot{x}_2 = f_2(x) + g_2(x)\, u$$

where $f_1(x) = \mu(x_2)(x_1 + C_{de}) - D_e (x_1 + C_{de})$

$$f_2(x) = - \sigma(x_2)(x_1 + C_{de}) + D_e (m_{cf} - x_2 - m_{ce})$$

$$g_1(x) = - (x_1 + C_{de})$$

$$g_2(x) = m_{cf} - x_2 - m_{ce}$$

$$\mu(x_2) = \frac{1}{K(x_2)} \{a_1(x_2 + m_{ce}) [a_2 + a_3 (x_2 + m_{ce})]\}$$

$$\sigma(x_2) = \frac{1}{K(x_2)} \left\{(x_2 + m_{ce}) \left[a_4 + a_5 (x_2 + m_{ce}) + a_6 (x_2 + m_{ce})^2\right]\right\}$$

$$K(x_2) = 0.000849 + (x_2 + m_{ce}) + 0.406 (x_2 + m_{ce})^2$$

Hoo and Kantor (1986) showed that the above system (5.5.16) is exactly state-space linearizable. This means that there exists a real-valued function h(x) such that the following system

$$\dot{x} = f(x) + g(x)\, u$$
$$y = h(x)$$

is linearizable with relative degree 2 [Isidori, 1989]. In this system, the real valued function h(x) must satisfy the following condition:

$$L_g h(x) = 0 \quad \text{and} \quad L_g L_f h(x) \neq 0 \tag{5.5.17}$$

If we choose h(x) to be

$$h(x) = \frac{(m_{cf} - x_2 - m_{ce})}{(x_1 + C_{de})} - \frac{(m_{cf} - m_{ce})}{C_{de}} \tag{5.5.18}$$

then the above condition (5.5.16) is satisfied locally. Later, we will define the set where the system is linearizable with relative degree 2. Note that the meaning of h(x), as indicated by Hoo and Kantor (1986), is the inverse apparent yield of the CSTBR.

In this example, we consider two possible cases; (1) the parameter a_2 is uncertain, and (2) parameters a_3 and a_6 are uncertain. First, we will consider case (1).

Case 1: Single Parametric Error: a_2 is Uncertain

Let' denote the true value of a_2 to be θ^*, which is unknown, the estimated value to be $\hat{\theta}$ and the nominal value of θ^* to be $\hat{\theta}(0) = \bar{\theta}$. With this notation the real plant and the mathematical model can be written as follows:

Real plant:

$$\dot{x} = f(x,\theta^*) + g(x,\theta^*)\,u \qquad (5.5.19)$$

$$y = h(x)$$

Mathematical model:

$$\dot{x} = f(x,\hat{\theta}) + g(x,\hat{\theta})\,u \qquad (5.5.20)$$

$$y = h(x)$$

where

$$f(x,\theta^*) = [f_1(x,\theta^*) \quad f_2(x,\theta^*)]^T$$

$$f_1(x,\theta^*) = \mu(x_2,\theta^*)\,(x_1+C_{de}) - D_e(x_1+C_{de})$$

$$f_2(x,\theta^*) = -\sigma(x_2)\,(x_1+C_{de}) + D_e(m_{cf} - x_2 - m_{ce})$$

$$f(x,\hat{\theta}) = [f_1(x,\hat{\theta}) \quad f_2(x,\hat{\theta})]^T$$

$$f_1(x,\hat{\theta}) = \mu(x_2,\hat{\theta})\,(x_1+C_{de}) - D_e(x_1+C_{de})$$

$$f_2(x,\hat{\theta}) = -\sigma(x_2)\,(x_1+C_{de}) + D_e(m_{cf} - x_2 - m_{ce})$$

$$\mu(x_2,\theta^*) = \frac{1}{K(x_2)}\left\{a_1(x_2+m_{ce})\left[\theta^* + a_3(x_2+m_{ce})\right]\right\}$$

$$\mu(x_2,\hat{\theta}) = \frac{1}{K(x_2)}\left\{a_1(x_2+m_{ce})\left[\hat{\theta} + a_3(x_2+m_{ce})\right]\right\}$$

$$g(x,\theta^*) = [g_1(x,\theta^*) \quad g_2(x,\theta^*)]^T$$

$$g_1(x,\theta^*) = -(x_1 + C_{de})$$

$g_2(x,\theta^*) = m_{cf} - x_2 - m_{ce}$

$g(x,\hat{\theta}) = [g_1(x,\hat{\theta}) \quad g_2(x,\hat{\theta})]^T$

$g_1(x,\hat{\theta}) = g_1(x,\theta^*)$

$g_2(x,\hat{\theta}) = g_2(x,\theta^*)$

From the above equations the model-plant mismatch defined in (5.3.3) can be represented by

$$\Delta f(x, \theta^*, \hat{\theta}) = (\theta^* - \hat{\theta}) \begin{bmatrix} {}^1f_1(x) \\ {}^1f_2(x) \end{bmatrix} \tag{5.5.21}$$

$\Delta g = 0$

where ${}^1f_1(x) = \dfrac{1}{K(x_2)} a_1 (x_2 + m_{ce}) (x_1 + C_{de})$

${}^1f_2(x) = 0$

Therefore, the model-plant mismatch satisfies linearity in the unknown parameter. Denote the vector fields f, g for the vector functions $f(x,\theta^*)$, $g(x,\theta^*)$, and $\hat{f}$, $\hat{g}$ for $f(x,\hat{\theta})$, $g(x,\hat{\theta})$. We can see that the above model-plant mismatch (5.5.21) does not satisfy matching condition; that is,

$$\Delta f \notin \text{span}\{g\} \tag{5.5.22}$$

Linearization

First, let us check the linearizability of the mathematical model (5.5.20). If the following conditions are satisfied

$$L_{\hat{g}}h(x) = 0, \quad L_{\hat{g}}L_{\hat{f}}h(x) \neq 0 \tag{5.5.23}$$

then the system (5.5.20) is linearizable with relative degree 2. The first condition can be checked easily and it is always zero. For the second condition, we have

$$L_{\hat{g}}L_{\hat{f}}h(x) = \frac{\partial(L_{\hat{f}}h)}{\partial x_1} g_1(x,\hat{\theta}) + \frac{\partial(L_{\hat{f}}h)}{\partial x_2} g_2(x,\hat{\theta}) \tag{5.5.24}$$

where

$$L_{\hat{f}}h(x) = \frac{-(m_{cf} - x_2 - m_{ce})}{(x_1 + C_{de})} \mu(x_2,\hat{\theta}) + \sigma(x_2)$$

$$\frac{\partial}{\partial x_1}\left(L_{\hat{f}}h(x)\right) = \frac{(m_{cf} - x_2 - m_{ce})\, \mu(x_2,\hat{\theta})}{(x_1 + C_{de})^2}$$

$$\frac{\partial}{\partial x_2}\left(L_{\hat{f}}h(x)\right) = \frac{1}{(x_1 + C_{dc})}\left\{\mu(x_2,\hat{\theta}) - (m_{cf} - x_2 - m_{ce})\, \mu'(x_2,\hat{\theta})\right\} + \sigma'(x_2)$$

$$\mu'(x_2,\hat{\theta}) = \frac{\partial \mu(x_2,\hat{\theta})}{\partial x_2}$$

$$= \frac{a_1(\hat{\theta} + a_3[x_2 + m_{cc}]) + a_1\,(x_2 + m_{cc})\, a_3}{K(x_2)}$$

$$- \frac{a_1(x_2 + m_{cc})\,(\hat{\theta} + a_3[x_2 + m_{ce}])\, K'(x_2)}{[K(x_2)]^2}$$

$$\sigma'(x_2) = \frac{\partial \sigma(x_2)}{\partial x_2}$$

$$= \frac{1}{K(x_2)}\left\{a_4 + a_5\,(x_2 + m_{ce}) + a_6\,(x_2 + m_{ce})^2\right.$$

$$\left. + (x_2 + m_{ce})\left[a_5 + 2\,a_6(x_2 + m_{ce})\right]\right\}$$

$$- \frac{K'(x_2)}{[K(x_2)]^2}\left\{(x_2 + m_{ce})\left[a_4 + a_5\,(x_2 + m_{ce}) + a_6(x_2 + m_{ce})^2\right]\right\}$$

$$K'(x_2) = 1 + 0.812\,(x_2 + m_{ce})$$

From the above equation (5.5.24) we can see that if one of the following conditions is satisfied, then the linearization is not solvable

(1) $m_{cf} - x_2 - m_{ce} = 0$ (5.5.25)

(2) $\sigma'(x_2)(x_1 + C_{de}) = (m_{cf} - x_2 - m_{ce})\, \mu'(x_2,\hat{\theta})$

Practically, it is not necessary to consider condition (1) because it corresponds to the situation of absolutely no growth of the cells. So, we need to consider only condition (2) [Hoo and Kantor, 1986]. Later, this condition will be checked.

From the previous section we can see that, in the region where the above conditions (1) and (2) are not satisfied, the linearizing coordinate transformation and state feedback for the model (5.5.20) are as follows:

$$z = T(x,\hat{\theta}) = \begin{bmatrix} h(x) \\ L_{\hat{f}}h(x) \end{bmatrix} \tag{5.5.26}$$

$$u = -\left(L_{\hat{g}}L_{\hat{f}}h(x)\right)^{-1}\left\{L_{\hat{f}}^{2}h(x) + b_1 z_1 + b_2 z_2\right\} \tag{5.5.27}$$

With the above transformation and linearizing feedback, the real plant (5.5.19) can be represented by

$$\dot{z} = (A+bK)z + \Phi(x,\hat{\theta})(\theta^{*} - \hat{\theta}) + \Pi(x,\hat{\theta})\frac{d\hat{\theta}}{dt} \tag{5.5.28}$$

where

$$(A+bK) = \begin{bmatrix} 0 & 1 \\ b_1 & b_2 \end{bmatrix}$$

$$\Phi(x,\hat{\theta}) = \begin{bmatrix} L_{1f}h(x) \\ L_{1f}L_{\hat{f}}h(x) \end{bmatrix}$$

$$\Pi(x,\hat{\theta}) = \begin{bmatrix} 0 \\ \dfrac{\partial}{\partial\hat{\theta}}\left(L_{\hat{f}}h(x)\right) \end{bmatrix}$$

<u>Adaptive control</u>

Let's apply the adaptive law (5.3.18) to the system (5.5.30) and analyze stability of the adaptive control system using proposition 1. In this problem, it is assumed that the bounds of the uncertain true parameter are given by

$$0.9 \le \theta^{*} \le 1.1 \quad (\pm\ 10.0\ \%\ \text{error})$$

When we choose the constants $b_1 = -2.0$ and $b_2 = -3.0$, the positive definiteness matrix P, from equation (5.5.10), is

$$P = \begin{bmatrix} 1.4167 & 0.25 \\ 0.25 & 0.25 \end{bmatrix}$$

Consider the set $\Omega_c = \left\{z, \hat{\theta}\ ;\ V \le c\right\}$, where the Lyapunov function V is written

$$V = p_{11} z_1 + 2\, p_{12} z_1 z_2 + p_{22} z_2^2 + Q\,(\theta^* - \hat{\theta})^2 \qquad (5.5.31)$$

Now, we seek a positive constant c such that, for every $z, \hat{\theta} \in \Omega_c$, the system is linearizable with relative degree 2. As shown in the previous example, the maximum possible range of $\hat{\theta}$ for a given c has the following range from the equation: $Q\,(\theta^* - \hat{\theta})^2 \le c$

$$0.9 - \sqrt{c/Q} \le \hat{\theta} \le 1.1 + \sqrt{c/Q} \quad \text{(since it is given that } 0.9 \le \theta^* \le 1.1\text{)}$$

Define the set

$$S_{\hat{\theta}} \equiv \left\{\hat{\theta};\ 0.9 - \sqrt{c/Q} \le \hat{\theta} \le 1.1 + \sqrt{c/Q}\right\}$$
$$S_{\theta^*} \equiv \left\{\theta^*;\ 0.9 \le \theta^* \le 1.1\right\}$$

We want to find the constant c such that the set Ω_c never contains any points which satisfy the above condition 2 in (5.5.27) for every $\hat{\theta} \in S_{\hat{\theta}}$ and $\theta^* \in S_{\theta^*}$. This constant c can be found by the following way.

From condition (2) in (5.5.27) we have

$$x_1 = \frac{(m_{cf} - x_2 - m_{ce})\, \mu'(x_2, \hat{\theta})}{\sigma'(x_2)} - C_{dc} \qquad (5.5.32)$$

Consider the following equation:

$$p_{11} z_1 + 2\, p_{12} z_1 z_2 + p_{22} z_2^2 + Q\,(\theta^* - \hat{\theta})^2 \qquad (5.5.33)$$

Since $z = T(x, \hat{\theta})$, if x_1 in equation (5.5.33) is replaced by (5.5.32), then the resulting equation is a function of only $x_2, \hat{\theta}$. Let this function to be $\xi(x_2, \hat{\theta})$. For a given c, if the function $\xi(x_2, \hat{\theta}) \leq c$ for some x_2 and $\hat{\theta} \in S_{\hat{\theta}}$ and $\theta^* \in S_{\theta^*}$, then the set Ω_c contains the points which satisfy condition (2) in (5.5.27) (that is, the inequality $\xi(x_2, \hat{\theta}) \leq c$ has absolutely no solution if the set Ω_c never contain any points which satisfy the above condition 2 in (5.5.27).). In this way, we can find the possible maximum c.

Now, suppose that we choose Q:

$$Q = 20.0$$

In this example, using a numerical search method (program: BIOXIC); that is, for sufficiently large number of x_2 in the set Ω_c, $\hat{\theta} \in S_{\hat{\theta}}$, and $\theta^* \in S_{\theta^*}$, we check the condition $\xi(x_2, \hat{\theta}) \leq c$, we conclude that if $c \leq 0.85$, then the set Ω_c never conflict with condition (2) in (5.5.25).

As considered by Hoo and Kantor (1986), the uncontrollable region corresponds to very low methanol concentration and this uncontrollable condition may not happen in actual operation. Unfortunately, this cannot be proved mathematically. Therefore, we confine ourselves to only local properties of the adaptive control system. If we can assume rigorously that the actual operation is controllable in a large range, then the possible maximum value of the constant c will increase tremendously, and this results in a bigger domain of attraction for convergence.

<u>Simulation</u>

Suppose that the initial condition is

$$x_1(0) = -0.09, \ \ x_2(0) = 0.57 \text{ and } \hat{\theta}(0) = 1.0$$

With the given initial condition, the corresponding initial value of V is

$$V(0) = p_{11}z_1(0)^2 + 2\,p_{12}z_1(0)z_2(0) + p_{22}z_2(0)^2 + Q\,(\theta^* - \hat{\theta}(0))^2 \tag{5.5.34}$$

$$\leq 0.4857 + Q\left(\theta^* - \hat{\theta}(0)\right)^2 \Big|_{\theta^* = 1.1}$$

$$= 0.4857 + Q\,(0.01)$$

$$= 0.6857$$

Since this V(0) must be less than 0.85, we can see that the given system is linearizable with relative degree 2 in the set Ω_c, where c = 0.6857.

By the procedure as in the previous section (5.5.1), it can be easily checked that the matrix $\left[I - P(B + B^T)P\right]$ is positive definite in Ω_c, where c = 0.6857 (program : IPBPA2) and, by proposition 1, the response of the system (5.5.19) with the adaptive law (5.3.41) is bounded to the set Ω_c for every time $t \geq 0$ and moreover $\lim_{t \to \infty} z = 0$, which means that $\lim_{t \to \infty} y = 0$.

Fig. 5.10 shows the comparison between the adaptive and the nonadaptive control, where $x_1(0) = -0.09$, $x_2(0) = 0.57$, $Q = 20.0$, and it is assumed that true value of a_2 is 1.1. In this figure, we can see that the adaptive control results in good output regulation even though the nonadaptive control shows an offset (program: BIOMPM for the nonadaptive, BIOADA2 for the adaptive). Fig. 5.11 shows the estimated parameter $\hat{\theta}$. In this case also, the estimated parameter converges to its true value.

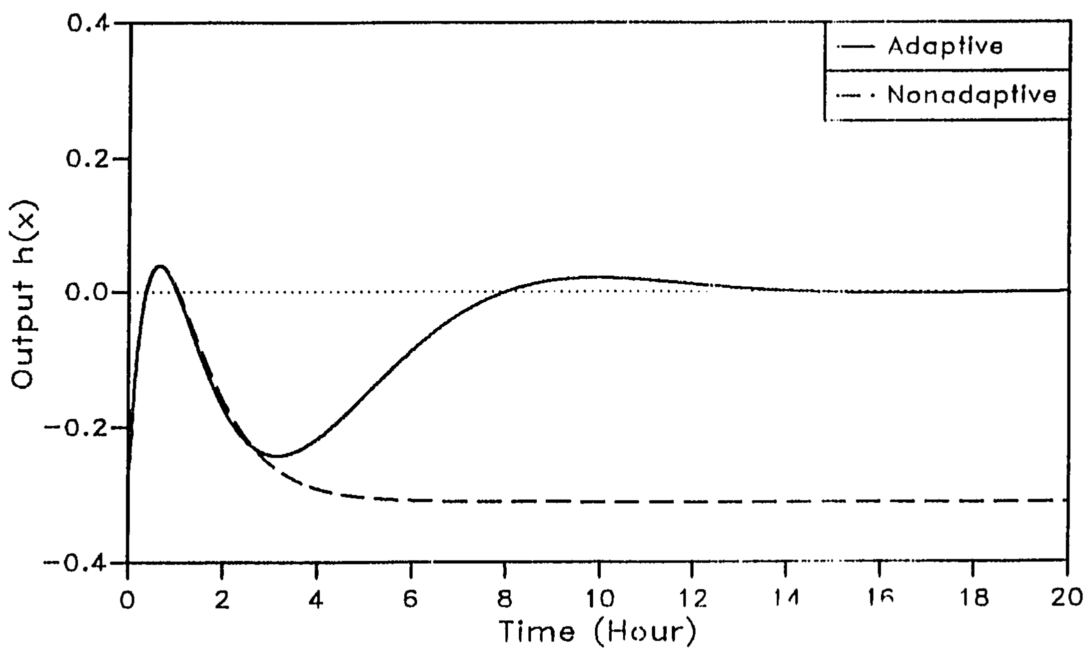

Fig. 5.10 Comparison between the adaptive and nonadaptive control response for uncertain parameter a_2, where

$x_1(0) = -0.09$, $x_2(0) = 0.57$, $\theta^* = 1.1$, $b_1 = -2$, $b_2 = -3$ and $Q = 20$.

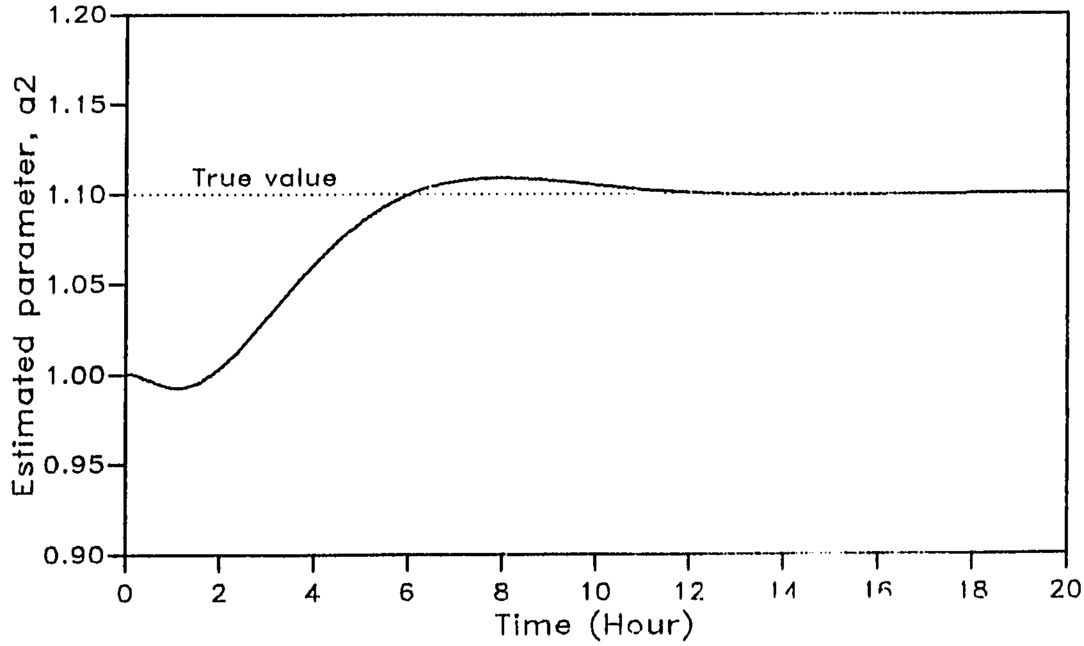

Fig. 5.11 Estimated parameter, a_2, where

$x_1(0) = -0.09$, $x_2(0) = 0.57$, $\theta^* = 1.1$, $b_1 = -2$, $b_2 = -3$ and $Q = 20$.

Case 2: Multiple Parametric Error: a_3 and a_6 are Uncertain.

In this section, we consider parametric uncertainties in both a_3 and a_6. Let us denote the true value of a_3 to be θ_1^* and a_6 to be θ_2^* which are uncertain, and the corresponding estimated values $\hat{\theta}_1$ and $\hat{\theta}_2$. And also $\bar{\theta} = \hat{\theta}(0) = [\hat{\theta}_1(0) \quad \hat{\theta}_2(0)]^T$ denote the nominal value of the uncertain parameters.

Mathematical model

In this case, the real plant and the mathematical model can be represented by

Real plant:

$$\dot{x} = f(x,\theta^*) + g(x,\theta^*)\, u$$

$$y = h(x)$$

Mathematical model:

$$\dot{x} = f(x,\hat{\theta}) + g(x,\hat{\theta})\, u$$

$$y = h(x)$$

where

$$f(x,\theta^*) = [f_1(x,\theta^*) \quad f_2(x,\theta^*)]^T$$

$$f_1(x,\theta^*) = \mu(x_2,\theta^*)\,(x_1+C_{de}) - D_e(x_1+C_{de})$$

$$f_2(x,\theta^*) = -\sigma(x_2,\theta^*)\,(x_1+C_{de}) + D_e(m_{cf} - x_2 - m_{ce})$$

$$f(x,\hat{\theta}) = [f_1(x,\hat{\theta}) \quad f_2(x,\hat{\theta})]^T$$

$$f_1(x,\hat{\theta}) = \mu(x_2,\hat{\theta})\,(x_1+C_{de}) - D_e(x_1+C_{de})$$

$$f_2(x,\hat{\theta}) = -\sigma(x_2,\hat{\theta})\,(x_1+C_{de}) + D_e(m_{cf} - x_2 - m_{ce})$$

$$\mu(x_2,\theta^*) = \frac{1}{K(x_2)}\left\{a_1(x_2+m_{ce})\left[a_2 + \theta_1^*(x_2+m_{ce})\right]\right\}$$

$$\mu(x_2,\hat{\theta}) = \frac{1}{K(x_2)}\left\{a_1(x_2+m_{ce})\left[a_2 + \hat{\theta}_1(x_2+m_{ce})\right]\right\}$$

$$\sigma(x_2, \theta^*) = \frac{1}{K(x_2)}\left\{ (x_2 + m_{ce})\left[a_4 + a_5(x_2 + m_{ce}) + \theta_2^*(x_2 + m_{ce})^2\right]\right\}$$

$$\sigma(x_2, \hat{\theta}) = \frac{1}{K(x_2)}\left\{ (x_2 + m_{ce})\left[a_4 + a_5(x_2 + m_{ce}) + \hat{\theta}_2(x_2 + m_{ce})^2\right]\right\}$$

$$g(x,\theta^*) = [g_1(x,\theta^*) \quad g_2(x,\theta^*)]^T$$

$$g_1(x,\theta^*) = - (x_1 + C_{de})$$

$$g_2(x,\theta^*) = m_{cf} - x_2 - m_{ce}$$

$$g(x, \hat{\theta}) = [g_1(x,\hat{\theta}) \quad g_2(x,\hat{\theta})]^T$$

$$g_1(x, \hat{\theta}) = g_1(x,\theta^*)$$

$$g_2(x, \hat{\theta}) = g_2(x,\theta^*)$$

From the above equations the vector fields f, $\hat{f}$, g and $\hat{g}$ are obvious. In this case, model-plant mismatch can be represented by

$$\Delta f(x, \theta^*, \hat{\theta}) = (\theta_1^* - \hat{\theta}_1)\ {}^1f(x) + (\theta_2^* - \hat{\theta}_2)\ {}^2f(x) \tag{5.5.35}$$

where $${}^1f(x) = \begin{bmatrix} {}^1f_1(x) \\ {}^1f_2(x) \end{bmatrix} = \begin{bmatrix} \frac{1}{K(x_2)}\left\{a_1\,(x_2 + m_{ce})^2\right\}(x_1 + C_{de}) \\ 0 \end{bmatrix}$$

$${}^2f(x) = \begin{bmatrix} {}^2f_1(x) \\ {}^2f_2(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{-1}{K(x_2)}(x_2 + m_{ce})^3\,(x_1 + C_{de}) \end{bmatrix}$$

and we can see that this model-plant mismatch does not satisfy matching condition.

Feedback Linearization

As shown in the previous case 1, the system is not linearizable with relative degree

2 if

$$\sigma'(x_2, \hat{\theta})\,(x_1 + C_{dc}) = (m_{cf} - x_2 - m_{ce})\,\mu'(x_2, \hat{\theta}) \tag{5.5.36}$$

where

$$\begin{aligned}\sigma'(x_2, \hat{\theta}) = {} & \frac{1}{K(x_2)}\Big\{a_4 + a_5(x_2 + m_{cc}) + \hat{\theta}_2(x_2 + m_{ce})^2 \\ & + (x_2 + m_{ce})\,(a_5 + 2\,\hat{\theta}_2(x_2 + m_{ce}))\Big\} \\ & - \frac{K'(x_2)}{K(x_2)^2}\Big\{(x_2 + m_{ce})\Big[a_4 + a_5(x_2 + m_{cc}) + \hat{\theta}_2(x_2 + m_{cc})^2\Big]\Big\}\end{aligned}$$

$$\begin{aligned}\mu'(x_2,\hat{\theta}) = {} & \frac{1}{K(x_2)}\Big[a_1\left(a_2 + \hat{\theta}_1(x_2 + m_{cc})\right) + a_1(x_2 + m_{cc})\hat{\theta}_1\Big] \\ & - \frac{K'(x_2)}{K(x_2)^2}\Big[a_1\,(x_2 + m_{cc})\left(a_2 + \hat{\theta}_1\,(x_2 + m_{ce})\right)\Big]\end{aligned}$$

As shown in the previous section, in the region where the above condition (5.5.36) is not satisfied, the coordinate transformation (5.5.26) and the linearizing state feedback (5.5.27) linearize the real plant.

Adaptive control

It is assumed that the parametric error bounds are given as follows:

$$-\,0.2244 \le \theta_1^{*} \le -\,0.1836 \qquad (\pm\ 10\ \%\ \text{error})$$

$$-\,0.7271 \le \theta_2^{*} \le -\,0.5949 \qquad (\pm\ 10\ \%\ \text{error})$$

When we choose the same constants $b_1 = -2.0$ and $b_2 = -\,3.0$ as the previous case 1, the positive matrix P is given by

$$P = \begin{bmatrix} 1.4167 & 0.25 \\ 0.25 & 0.25 \end{bmatrix}$$

Consider the set $\Omega_c = \left\{z, \hat{\theta}\,;\ V \le c\right\}$ where the Lyapunov function V is written

$$V = p_{11}z_1 + 2\,p_{12}z_1z_2 + p_{22}z_2^2 + (\theta^* - \hat{\theta})^T Q\,(\theta^* - \hat{\theta}) \tag{5.5.31}$$

Now, we seek a positive constant c such that, for every $z, \hat{\theta} \in \Omega_c$, the system is linearizable with relative degree 2. As shown in the previous example, the maximum possible range of $\hat{\theta}$, for a given constant c, can be found from the following equation:

$$(\theta^* - \hat{\theta})^T Q\,(\theta^* - \hat{\theta}) \leq c$$

For simplicity, if we choose the positive matrix Q to be diagonal:

$$Q = \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix}$$

then the maximum possible range of $\hat{\theta}$ can be found easily from the following equations:

$$q_{11}(\theta_1^* - \hat{\theta}_1)^2 \leq c$$

$$q_{22}(\theta_2^* - \hat{\theta}_2)^2 \leq c$$

From the above equations we can see that maximum possible range of the estimated parameters are, for each given c,

$$-\,0.2244 - \sqrt{c/q_{11}} \leq \hat{\theta}_1 \leq -\,0.1836 + \sqrt{c/q_{11}}$$

$$-\,0.7271 - \sqrt{c/q_{22}} \leq \hat{\theta}_2 \leq -\,0.5949 + \sqrt{c/q_{22}}$$

(remember that the possible range of θ^* is given.)

Define the set

$$S_{\hat{\theta}} \equiv \{\, \hat{\theta}_1, \hat{\theta}_2 \,;\, -\,0.2244 - \sqrt{c/q_{11}} \leq \hat{\theta}_1 \leq -\,0.1836 + \sqrt{c/q_{11}},$$
$$,\, -\,0.7271 - \sqrt{c/q_{22}} \leq \hat{\theta}_2 \leq -\,0.5949 + \sqrt{c/q_{22}} \,\}$$

$$S_{\theta^*} \equiv \{\, \theta_1^*, \theta_2^* \,;\, -\,0.2244 \leq \theta_1^* \leq -\,0.1836,,\, -\,0.7271 \leq \theta_2^* \leq -\,0.5949 \,\}$$

We want to find the constant c such that, for every $\hat{\theta} \in S_{\hat{\theta}}$ and $\theta^* \in S_{\theta^*}$, the set Ω_c never contains any points which satisfy condition (5.5.36). This constant c can be found in exactly same way as in the previous case 1.

Now, suppose that we choose the matrix Q as follows:

$$Q = \begin{bmatrix} 8.0 & 0.0 \\ 0.0 & 3.0 \end{bmatrix}$$

In this example for every $\hat{\theta} \in S_{\hat{\theta}}$ and $\theta^* \in S_{\theta^*}$, we can find that if $c \le 0.74$ (program: BIOXI36), then set Ω_c never conflict with the condition (5.5.36).

<u>Simulation</u>

Suppose that initial values of x_1, x_2, and the nominal value of θ^* are

$$x_1(0) = -0.09, \quad x_2(0) = 0.57, \quad \hat{\theta}_1(0) = -0.204 \text{ and } \hat{\theta}_2(0) = -0.661$$

Since

$$z_1 = h(x) = \frac{(m_{cf} - x_2 - m_{ce})}{(x_1 + C_{de})} - \frac{(m_{cf} - m_{ce})}{C_{de}}$$
$$z_2 = L_{\hat{f}} h(x) = -\frac{(m_{cf} - x_2 - m_{ce})}{(x_1 + C_{de})} \mu(x_2, \hat{\theta}) + \sigma(x_2, \hat{\theta})$$

Initial value of $z_1(0)$ and $z_2(0)$ are

$$z_1(0) = -0.272 \quad \text{and } z_2(0) = 1.536$$

Therefore, initial value of the Lyapunov function, defined in (5.5.31), is

$$\begin{aligned} V(0) &= p_{11}z_1(0)^2 + 2\,p_{12}z_1(0)z_2(0) + p_{22}z_2(0)^2 + (\theta^* - \hat{\theta}(0))^T Q\,(\theta^* - \hat{\theta}(0)) \\ &\le 0.4857 + q_{11}(\theta_1^* - \hat{\theta}_1(0))^2 + q_{22}(\theta_2^* - \hat{\theta}_2(0))^2 \Big|_{\substack{\theta_1^* = -0.1836 \\ \theta_2^* = -0.5949}} \\ &= 0.5021 \end{aligned}$$

Define a set

$$\Omega_c = \left\{ x, \hat{\theta}\,;\, V \le c, c = 0.5021 \right\}$$

This constant c is less than 0.74 so that the system is linearizable with relative degree 2. Now, let us apply the same adaptive law as (5.3.41) in the previous section:

$$\frac{d\hat{\theta}}{dt} = Q^{-1}\Phi^{T}(x, \hat{\theta})P\ T(x,\hat{\theta}) \qquad (5.5.37)$$

where

$$\Phi(x, \hat{\theta}) = \begin{bmatrix} L1_f h(x) & L2_f h(x) \\ L1_f L_{\hat{f}} h(x) & L2_f L_{\hat{f}} h(x) \end{bmatrix}$$

$$L1_f h(x) = \frac{-(m_{cf} - x_2 - m_{ce})}{(x_1 + C_{de})^2} \left[\frac{1}{K(x_2)} \left\{ a_1 (x_2 + m_{ce})^2 \right\} (x_1 + C_{de}) \right]$$

$$L2_f h(x) = \frac{1}{(x_1 + C_{de})} \left[\frac{1}{K(x_2)}\ (x_2 + m_{cc})^3\ (x_1 + C_{dc}) \right]$$

$$L1_f L_{\hat{f}} h(x) = \frac{\partial}{\partial x_1} \left(L_{\hat{f}} h(x) \right)\ {}^1 f_1(x)$$

$$L2_f L_{\hat{f}} h(x) = \frac{\partial}{\partial x_2} \left(L_{\hat{f}} h(x) \right)\ {}^2 f_2(x)$$

$$\frac{\partial}{\partial x_1} \left(L_{\hat{f}} h(x) \right) = \frac{(m_{cf} - x_2 - m_{ce})}{(x_1 + C_{de})^2} \cdot \left\{ \frac{1}{K(x_2)} \left[a_1 (x_2 + m_{ce}) \left(a_2 + \hat{\theta}_1 (x_2 + m_{ce}) \right) \right] \right\}$$

$$\frac{\partial}{\partial x_2} \left(L_{\hat{f}} h(x) \right) = \frac{1}{(x_1 + C_{de})} \left\{ \mu(x_2, \hat{\theta}) - (m_{cf} - x_2 - m_{ce})\ \mu'(x_2, \hat{\theta}) \right\} + \sigma'(x_2, \hat{\theta})$$

Now, we check the positive definiteness of the matrix $[I - P(B + B^T)P]$.

We can see easily

$$\Pi(x, \hat{\theta}) = \begin{bmatrix} 0 & 0 \\ \frac{\partial}{\partial \hat{\theta}_1} \left(L_{\hat{f}} h(x) \right) & \frac{\partial}{\partial \hat{\theta}_2} \left(L_{\hat{f}} h(x) \right) \end{bmatrix}$$

where $\dfrac{\partial}{\partial \hat{\theta}_1} \left(L_{\hat{f}} h(x) \right) = \dfrac{-(m_{cf} - x_2 - m_{ce})}{(x_1 + C_{de})\ K(x_2)} \left[a_1\ (x_2 + m_{ce})^2 \right]$

$$\frac{\partial}{\partial \hat{\theta}_2}\left(L_{\hat{f}} h(x)\right) = \frac{(x_2 + m_{ce})^3}{K(x_2)}$$

Therefore, we can find the following function:

$$B(x, \hat{\theta}) = \Phi(x, \hat{\theta})\, Q^{-1}\, \Pi^T(x, \hat{\theta})$$

By the same method as in the previous section (program: IPBPBIO), we can see that, for every x, $\hat{\theta} \in \Omega_c$, where c = 0.5021, matrix $\left[I - P(B + B^T)P\right]$ is positive definite in Ω_c. So, by proposition 1, the response of the adaptive control system is bounded to the set Ω_c, where c = 0.5021 for every time $t \geq 0$ and moreover $\lim_{t \to \infty} z = 0$.

Fig.5.12 shows the comparison between the adaptive and nonadaptive control, where it is assumed that true values $\theta_1^* = -0.1836$ and $\theta_2^* = -0.5949$, while the nominal values, i.e. initial values, are $\hat{\theta}_1(0) = -0.204$, $\hat{\theta}_2(0) = -0.661$ (program: BIOMPM for nonadaptive, BIOADP for adaptive). We can see that in this figure the system response of the adaptive control converges to the set point. Fig.5.13 shows the estimated parameters $\hat{\theta}_1$ and $\hat{\theta}_2$. Here, the estimated parameters do not converge to their true values. As explained earlier, the proposed adaptive control system does not guarantee that the estimated parameters converge to their true values.

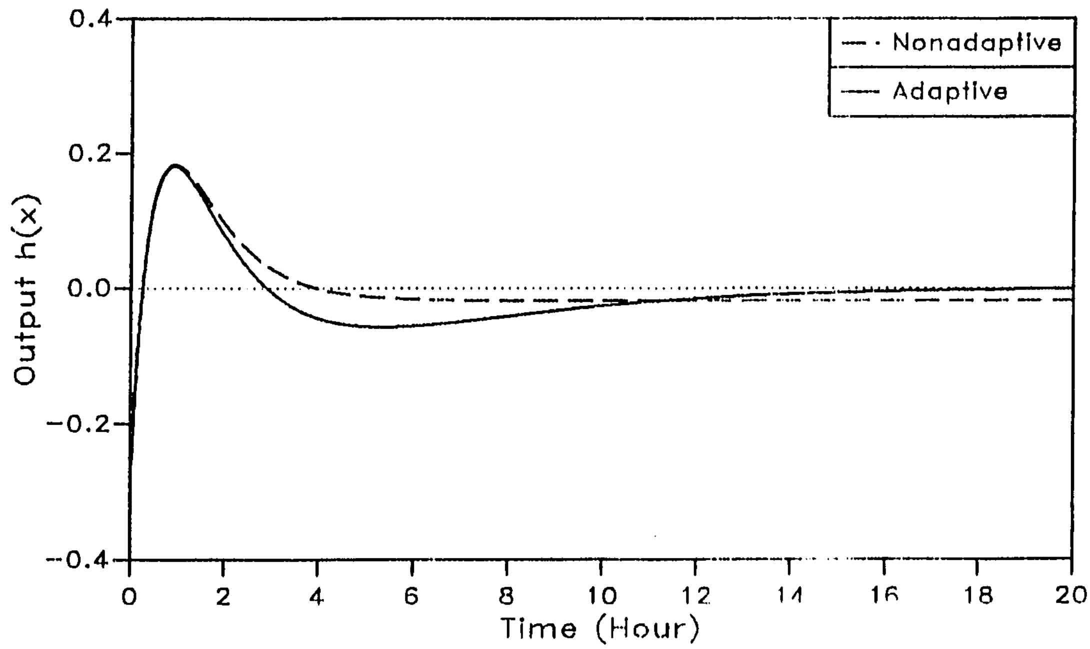

Fig. 5.12 Comparison between the adaptive and the nonadaptive control responses for uncertain parameters, a_3 and a_6, where $b_1 = -2.0$, $b_2 = -3.0$, $Q = \begin{bmatrix} 8.0 & 0.0 \\ 0.0 & 3.0 \end{bmatrix}$ with the initial condition:

$x_1(0) = -0.09$, $x_2(0) = 0.57$, $\hat{\theta}_1(0) = -0.204$, $\hat{\theta}_2(0) = -0.661$

(true parameter; $\theta_1^* = -0.1836$, $\theta_2^* = -0.5949$)

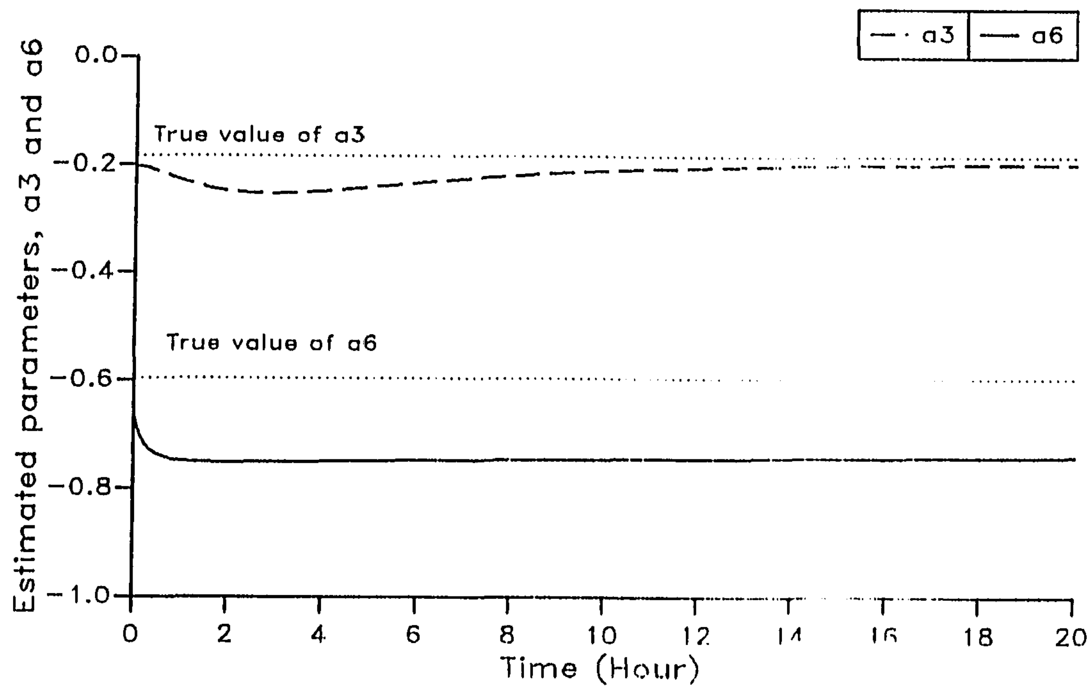

Fig. 5.13 Estimated parameters, a_3 and a_6, where $b_1 = -2.0$, $b_2 = -3.0$, $Q = \begin{bmatrix} 8.0 & 0.0 \\ 0.0 & 3.0 \end{bmatrix}$ with the initial condition:

$x_1(0) = -0.09$, $x_2(0) = 0.57$, $\hat{\theta}_1(0) = -0.204$, $\hat{\theta}_2(0) = -0.661$

(true parameter; $\theta_1^* = -0.1836$, $\theta_2^* = -0.5949$)

5.5.3. First Order Exothermic Reaction in a CSTR with Uncertainty in the Activation Energy

Consider the first order exothermic reaction in a CSTR, investigated in the previous section (5.5.1). In this section, we consider the case that the activation energy, E, has an error so that the dimensionless variable υ is uncertain. As we can see later, this parametric uncertainty does not satisfy linearity in the unknown parameters. In this case, model-plant mismatch caused by the error in the activation energy has been approximated so that it can be represented linearly in the unknown parameters. This approximation results in structural uncertainties, and this example shows that the proposed adaptive control algorithm is robust for some structural uncertainties.

Mathematical model

Denote υ^* a true parameter, $\hat{\upsilon}(t)$ an estimated parameter, and $\bar{\upsilon} = \hat{\upsilon}(0)$ to be the nominal value of υ. In this case, it is also assumed that the parametric bound of υ^* is given. With some mathematical manipulation the real plant can be represented in the following dimensionless form:

Real plant :

$$\frac{dx_1}{d\tau} = (-x_1 + \alpha_c) - (x_1 - \alpha_c + 1)\exp\left(D - \frac{\upsilon^{*2}}{x_2\left(\frac{\upsilon^*}{\bar{\upsilon}}\right) - \alpha_T\left(\frac{\upsilon^*}{\bar{\upsilon}}\right) + \upsilon^*} \right) \tag{5.5.38}$$

$$\frac{dx_2}{d\tau} = (-x_2 + \alpha_T) - B(x_1 - \alpha_c + 1)\exp\left(D - \frac{\upsilon^{*2}}{x_2\left(\frac{\upsilon^*}{\bar{\upsilon}}\right) - \alpha_T\left(\frac{\upsilon^*}{\bar{\upsilon}}\right) + \upsilon^*} \right)$$

$$- \gamma\,(x_2 - \alpha_T + \alpha_W) + \gamma\, u$$

$$y = x_1$$

Note that in this equation dimensionless variables x_2, u, B, α_T and α_W are based on the value of $\bar{\upsilon}$, for example $x_2 = \frac{T - T^d}{\bar{\bar{T}}_0}\bar{\upsilon}$, $u = \frac{T_c - \bar{T}_c}{\bar{\bar{T}}_0}\bar{\upsilon}$, etc.

In the real plant, υ^* is an uncertain parameter and it is replaced by the estimated value, $\hat{\upsilon}$. So, the mathematical model can be written

Mathematical model :

$$\frac{dx_1}{d\tau} = (-x_1 + \alpha_c) - (x_1 - \alpha_c + 1)\exp\left(D - \frac{\hat{\upsilon}^2}{x_2\left(\frac{\hat{\upsilon}}{\bar{\bar{\upsilon}}}\right) - \alpha_T\left(\frac{\hat{\upsilon}}{\bar{\bar{\upsilon}}}\right) + \hat{\upsilon}}\right) \tag{5.5.39}$$

$$\frac{dx_2}{d\tau} = (-x_2 + \alpha_T) - B(x_1 - \alpha_c + 1)\exp\left(D - \frac{\hat{\upsilon}^2}{x_2\left(\frac{\hat{\upsilon}}{\bar{\bar{\upsilon}}}\right) - \alpha_T\left(\frac{\hat{\upsilon}}{\bar{\bar{\upsilon}}}\right) + \hat{\upsilon}}\right) - \gamma\,(x_2 - \alpha_T + \alpha_W) + \gamma\, u$$

$$y = x_1$$

For consistency in notation with the previous section (5.5.1), we will denote $\theta^* = \upsilon^*$, $\hat{\theta}(t) = \hat{\upsilon}(t)$ and $\bar{\theta} = \hat{\theta}(0) = \hat{\upsilon}(0)$.

Equation (5.5.38) and (5.5.39) can be written compactly

Real plant:

$$\dot{x} = f(x,\theta^*) + g(x,\theta^*)u \tag{5.5.40}$$

$$y = h(x)$$

Mathematical model:

$$\dot{x} = f(x,\hat{\theta}) + g(x,\hat{\theta})u \tag{5.5.41}$$

$$y = h(x)$$

where vector fields f, g for the vector functions $f(x,\theta^*)$, $g(x,\vartheta^*)$ and the vector fields $\hat{f}$, $\hat{g}$ for $f(x,\hat{\theta})$, $g(x,\hat{\theta})$ are obvious from (5.5.38) and (5.5.39). Clearly, the scalar field

$h(x) = x_1$.

Define a set S_θ as the set to which the uncertain true parameter θ^* belongs, for example, the set S_θ may be defined as $S_\theta = \{ \theta^* \mid \theta_{min} \leq \theta^* \leq \theta_{max} \}$, where θ_{max} and θ_{min} are the possible maximum and minimum value of θ^* respectively. In this example, we can see also that

$$f(0,\theta^*) \neq 0 \text{ for some } \theta^* \in S_\theta$$

From (5.5.40) and (5.5.41) the model-plant mismatch can be written

$$\Delta f(x,\theta^*,\hat{\theta}) = [\Delta f_1(x,\theta^*,\hat{\theta}) \quad \Delta f_2(x,\theta^*,\hat{\theta})]^T$$

where

$$\Delta f_1 = -(x_1-\alpha_c+1)\left[\exp\left(D - \frac{\upsilon^2}{x_2\left(\frac{\upsilon}{\bar{\upsilon}}\right) - \alpha_T\left(\frac{\upsilon}{\bar{\upsilon}}\right) + \upsilon}\right) - \exp\left(D - \frac{\hat{\upsilon}^2}{x_2\left(\frac{\hat{\upsilon}}{\bar{\upsilon}}\right) - \alpha_T\left(\frac{\hat{\upsilon}}{\bar{\upsilon}}\right) + \hat{\upsilon}}\right)\right]$$

$$\Delta f_2 = B \cdot \Delta f_1$$

From this equation we can see that $\Delta f \notin \text{span}\{g\}$; that is, it does not satisfy matching condition, and also it does not satisfy linearity in the unknown parameters.

We now find an approximate model-plant mismatch $\Delta\hat{f}(x, \theta^*, \hat{\theta})$ which satisfies linearity in the unknown parameters.

Define

$$A(x,\theta^*) = D - \frac{\theta^{*2}}{x_2\left(\frac{\theta^*}{\bar{\theta}}\right) - \alpha_T\left(\frac{\theta^*}{\bar{\theta}}\right) + \theta^*} \tag{5.5.42}$$

$$A(x,\hat{\theta}) = D - \frac{\hat{\theta}^2}{x_2\left(\frac{\hat{\theta}}{\bar{\theta}}\right) - \alpha_T\left(\frac{\hat{\theta}}{\bar{\theta}}\right) + \hat{\theta}} \tag{5.5.43}$$

Taking the Taylor series expansion of $\exp\left[A(x,\theta^*)\right]$ with respect to θ^* around the nominal value we have

$$\exp\left[A(x,\theta^*)\right] = \exp\left[A(x,\bar{\theta})\right] + \frac{\partial}{\partial\theta^*}\left\{\exp\left[A(x,\theta^*)\right]\right\}\Big|_{\theta^* = \bar{\theta}}(\theta^* - \bar{\theta}) + \cdots \qquad (5.5.44)$$

Similarly,

$$\exp\left[A(x,\hat{\theta})\right] = \exp\left[A(x,\bar{\theta})\right] + \frac{\partial}{\partial\hat{\theta}}\left\{\exp\left[A(x,\hat{\theta})\right]\right\}\Big|_{\hat{\theta} = \bar{\theta}}(\hat{\theta} - \bar{\theta}) + \cdots \qquad (5.5.45)$$

Since

$$\begin{aligned}\frac{\partial}{\partial\theta^*}\left\{\exp\left[A(x,\theta^*)\right]\right\}\Big|_{\theta^* = \bar{\theta}} &= \frac{\partial}{\partial\hat{\theta}}\left\{\exp\left[A(x,\hat{\theta})\right]\right\}\Big|_{\hat{\theta} = \bar{\theta}} \\ &= \exp\left[D - \frac{\bar{\theta}^2}{x_2 - \alpha_T + \bar{\theta}}\right]\frac{-\bar{\theta}}{x_2 - \alpha_T + \bar{\theta}}\end{aligned}$$

the model-plant mismatch can be approximated by

$$\begin{aligned}\Delta f_1(x, \theta^*, \hat{\theta}) &\cong -(x_1 - \alpha_c + 1)\exp\left[D - \frac{\bar{\theta}^2}{x_2 - \alpha_T + \bar{\theta}}\right]\frac{-\bar{\theta}}{x_2 - \alpha_T + \bar{\theta}}(\theta^* - \hat{\theta}) \\ \Delta f_2(x, \theta^*, \hat{\theta}) &\cong -B(x_1 - \alpha_c + 1)\exp\left[D - \frac{\bar{\theta}^2}{x_2 - \alpha_T + \bar{\theta}}\right]\frac{-\bar{\theta}}{x_2 - \alpha_T + \bar{\theta}}(\theta^* - \hat{\theta})\end{aligned}$$

(5.5.46)

Define

$${}^1f(x) = [\ {}^1f_1(x) \quad {}^1f_2(x)\ ]^T$$

where

$$\begin{aligned}{}^1f_1(x) &\equiv -(x_1-\alpha_c+1)\exp\left(D - \frac{\bar{\theta}^2}{x_2-\alpha_T+\bar{\theta}}\right)\left\{\frac{-\bar{\theta}}{x_2-\alpha_T+\bar{\theta}}\right\} \\ {}^1f_2(x) &\equiv -B(x_1-\alpha_c+1)\exp\left(D - \frac{\bar{\theta}^2}{x_2-\alpha_T+\bar{\theta}}\right)\left\{\frac{-\bar{\theta}}{x_2-\alpha_T+\bar{\theta}}\right\}\end{aligned}$$

So, approximated model-plant mismatch $\Delta\hat{f}\,(x, \theta^*, \hat{\theta})$ can be written

$$\Delta\hat{f}\,(x, \theta^*, \hat{\theta}) = [\Delta\hat{f}_1\,(x, \theta^*, \hat{\theta}) \quad \Delta\hat{f}_2\,(x, \theta^*, \hat{\theta})\,]^T$$

$$\text{where} \quad \Delta\hat{f}_1(x, \theta^*, \hat{\theta}) \equiv {}^1f_1(x)\,(\theta^* - \hat{\theta}\,)$$

$$\Delta\hat{f}_2(x, \theta^*, \hat{\theta}) \equiv {}^1f_2(x)\,(\theta^* - \hat{\theta}\,)$$

Finally, model-plant mismatch Δf can be represented by

$$\Delta f_1(x, \theta^*, \hat{\theta}) = \Delta\hat{f}_1(x, \theta^*, \hat{\theta}) + {}^1R_1(x, \theta^*, \hat{\theta})$$

$$\Delta f_2(x, \theta^*, \hat{\theta}) = \Delta\hat{f}_2(x, \theta^*, \hat{\theta}) + {}^1R_2(x, \theta^*, \hat{\theta})$$

Feedback Linearization

Now, we apply the linearizing coordinate transformation and state feedback based on the mathematical model. For this, it is necessary to find the relative degree of the mathematical model (5.5.39).

Since

$$L_{\hat{g}}h(x) = 0 \quad \text{and}$$

$$L_{\hat{g}}L_{\hat{f}}h(x) = \frac{\partial f_1(x,\hat{\theta})}{\partial x_2}\, g_2(x,\hat{\theta})$$

$$= -\,(x_1 - \alpha_c + 1)\ \exp\left(D - \frac{\hat{\theta}^2}{x_2\left(\frac{\hat{\theta}}{\theta}\right) - \alpha_T\left(\frac{\hat{\theta}}{\theta}\right) + \hat{\theta}}\right) \cdot \frac{\hat{\theta}^2\left(\frac{\hat{\theta}}{\theta}\right)}{\left[x_2\left(\frac{\hat{\theta}}{\theta}\right) - \alpha_T\left(\frac{\hat{\theta}}{\theta}\right) + \hat{\theta}\right]^2}\,\gamma$$

we can see that if all of the following conditions are satisfied, then the system is linearizable with relative degree 2

(1) $x_1 > \alpha_c - 1$

(2) $\alpha_T - \bar{\upsilon} < x_2 < \infty$

(3) $\hat{\upsilon} = \hat{\theta} \neq 0$

Define sets U and S_θ:

$$U = \left\{x;\ 1 - \alpha_c < x_1,\ \alpha_T - \bar{\upsilon} < x_2 < \infty\right\}$$

$$S_\theta = \left\{\hat{\theta};\ \hat{\theta} > 0\right\}$$

Then we can say that the mathematical model is linearizable with relative degree 2 for every $x, \hat{\theta} \in UxS_\theta$.

Applying the linearizing state feedback and the coordinate change, which is obvious from the previous section (5.5.1), to the real plant we then have

$$\dot{z} = (A + bK)\, z + \Phi(x, \hat{\theta})\, (\theta^* - \hat{\theta}) + \psi(x, \hat{\theta}) + \Pi(x, \hat{\theta}) \frac{d\hat{\theta}}{dt} \tag{5.5.47}$$

where

$$A + bK = \begin{bmatrix} 0 & 1 \\ b_1 & b_2 \end{bmatrix}$$

$$\Phi(x, \hat{\theta}) = \begin{bmatrix} L_{1f}h(x) \\ L_{1f}L_{\hat{f}}h(x) \end{bmatrix}$$

$$\psi(x, \hat{\theta}) = \begin{bmatrix} L_{1R}h(x) \\ L_{1R}L_{\hat{f}}h(x) \end{bmatrix}$$

$$\Pi(x, \hat{\theta}) = \begin{bmatrix} 0 \\ \dfrac{\partial}{\partial \hat{\theta}}(L_{\hat{f}}h(x)) \end{bmatrix}$$

$$L_{1f}h(x) = {}^1f_1(x)$$

$$L_{1f}L_{\hat{f}}h(x) = \frac{\partial f_1(x,\hat{\theta})}{\partial x_1}\, {}^1f_1(x) + \frac{\partial f_1(x,\hat{\theta})}{\partial x_2}\, {}^1f_2(x)$$

$$L_{1R}h(x) = {}^1R_1(x,\theta^*,\hat{\theta})$$

$$L_{1R}L_{\hat{f}}h(x) = \frac{\partial f_1(x,\hat{\theta})}{\partial x_1}\, {}^1R_1(x,\theta^*,\hat{\theta}) + \frac{\partial f_1(x,\hat{\theta})}{\partial x_2}\, {}^1R_2(x,\theta^*,\hat{\theta})$$

$$\frac{\partial f_1(x,\hat{\theta})}{\partial x_1} = -1 - \exp\left(D - \frac{\hat{\theta}^2}{x_2\left(\frac{\hat{\theta}}{\theta}\right) - \alpha_T\left(\frac{\hat{\theta}}{\theta}\right) + \hat{\theta}} \right)$$

$$\frac{\partial f_1(x,\hat{\theta})}{\partial x_2} = -(x_1 - \alpha_c + 1)\exp\left(D - \frac{\hat{\theta}^2}{x_2\left(\frac{\hat{\theta}}{\theta}\right) - \alpha_T\left(\frac{\hat{\theta}}{\theta}\right) + \hat{\theta}} \right) \frac{\hat{\theta}^2\left(\frac{\hat{\theta}}{\theta}\right)}{\left[x_2\left(\frac{\hat{\theta}}{\theta}\right) - \alpha_T\left(\frac{\hat{\theta}}{\theta}\right) + \hat{\theta}\right]^2}$$

$$\frac{\partial}{\partial \hat{\theta}}\left(L_{\hat{f}}h(x)\right) = (x_1 - \alpha_c + 1)\,\exp\left(D - \frac{\hat{\theta}^2}{x_2\left(\frac{\hat{\theta}}{\theta}\right) - \alpha_T\left(\frac{\hat{\theta}}{\theta}\right) + \hat{\theta}} \right) \cdot \frac{\hat{\theta}}{\left[x_2\left(\frac{\hat{\theta}}{\theta}\right) - \alpha_T\left(\frac{\hat{\theta}}{\theta}\right) + \hat{\theta}\right]}$$

For this system, we apply the following adaptive law, which is the same as (5.3.41):

$$\frac{d\hat{\theta}}{dt} = Q^{-1}\Phi^T(x, \hat{\theta})P\,z \tag{5.5.48}$$

Now, consider the adaptive control system (5.5.47) and (5.5.48).

Define

$$\Theta(z, \hat{\theta}) = \Phi(x,\hat{\theta})$$

$$\Gamma(z, \hat{\theta}) = \Pi(x,\hat{\theta})$$

$$\mu(z, \hat{\theta}) = \psi\,(x, \hat{\theta})$$

Then the above system (5.5.47) can be written

$$\dot{z} = (A+bK)z + \Theta(z, \hat{\theta})(\theta^* - \hat{\theta}) + \mu(z, \hat{\theta}) + \Gamma(z, \hat{\theta})\frac{d\hat{\theta}}{dt} \tag{5.5.49}$$

To investigate the stability of this adaptive control system, we now consider the same Lyapunov candidate function (5.3.19) and a set Ω_c:

$$V = z^T P z + (\theta^* - \hat{\theta})^T Q (\theta^* - \hat{\theta})$$

$$\Omega_c \equiv \left\{ z, \phi;\ z^T P z + \phi^T Q \phi \le c \right\}$$

$$\text{where } \phi \equiv \theta^* - \hat{\theta}$$

The derivative of V along the trajectories is

$$\dot{V} = z^T\left[(A+bK)^T P + P(A+bK)\right] z + (\theta^* - \hat{\theta})^T \Theta^T(z, \hat{\theta}) P z + \mu^T(z, \hat{\theta}) P z$$
$$+ \left(\frac{d\hat{\theta}}{dt}\right)^T \Gamma^T(z, \hat{\theta})\, P z + z^T P \Theta(z, \hat{\theta})\, (\theta^* - \hat{\theta}) + z^T P \mu(z, \hat{\theta})$$
$$+ z^T P \Gamma(z, \hat{\theta}) \frac{d\hat{\theta}}{dt} - \left(\frac{d\hat{\theta}}{dt}\right)^T Q (\theta^* - \hat{\theta}) - (\theta^* - \hat{\theta})^T Q \frac{d\hat{\theta}}{dt} \tag{5.5.50}$$

Applying the adaptive law (5.5.48) to (5.5.50) we have

$$-\dot{V} = z^T\left[I - P\left(B(z,\hat{\theta}) + B^T(z, \hat{\theta})\right)P\right] z - 2\,\mu^T(z, \hat{\theta}) P z \tag{5.3.33}$$

$$\text{where } B(z, \hat{\theta}) = \Theta(z, \hat{\theta}) Q^{-1} \Gamma^T(z, \hat{\theta})$$

Then, similarly to Proposition 1, we can say that:

Proposition 3: Consider the adaptive control system (5.5.48) and (5.5.49). Suppose that

(1) There exists a constant $m_r \ge 0$ such that

$$\mu^T(z, \hat{\theta}) P z \le m_r\, z^T P z \qquad \text{for every } z, \phi \in \Omega_c$$

(2) The matrix $\left[I - P\left(B(z, \hat{\theta}) + B^T(z, \hat{\theta})\right)P - 2m_r P\right]$ is positive definite for every $z, \phi \in \Omega_c$

then, by LaSalle's theorem, if the initial condition of z and $\hat{\theta}$ is in the set Ω_c, then the solutions z(t) and $\hat{\theta}(t)$ are in the set Ω_c $\forall\, t \in [\,0, \infty)$ and moreover $\lim_{t \to \infty} z(t) = 0$. ///

The proof of the above Proposition 3 is almost the same as for Proposition 1.

In this Proportion 3, in order for the above condition (1) to be satisfied, the structural uncertainties resulting from the approximation must be cancelled at $z = 0$ because if they do not cancel at $z = 0$, the constant m_r goes to infinity. However, this condition is extremely restrictive. Therefore the above proportion 3 can be used for only very special cases.

In this example, however, we can see that even though the above condition (1) is not satisfied, the proposed adaptive control algorithm results in an excellent response.

Suppose that the true value of υ, θ^*, belongs to the set S_{θ^*} defined by

$$S_{\theta^*} = \left\{\theta^*;\ 32.9104 \leq \theta^* \leq 32.9704\right\}$$

Consider the Lyapunov function V and a set Ω_c such that

$$V = z^T P z + Q\,(\theta^* - \hat{\theta})^2$$

$$\Omega_c = \left\{z, \hat{\theta}\,;\ V \leq c\right\}$$

By the same argument as in the example in section (5.5.1), if $c \leq 0.269$, the set Ω_c is a proper subset UxS_θ.

Simulation

Suppose that initial values are

$$x_1(0) = 0.2, \quad x_2(0) = -1.0 \text{ and } \hat{\theta}(0) = 32.9404$$

With the chosen values of $b_1 = -2.1$ and $b_2 = -2.0$, the initial value of V is

$$V(0) = 0.0535 + Q(\theta^* - 32.9404)^2$$

Then, for every $\theta^* \in S_{\theta^*}$,

$$V(0) \leq 0.0535 + Q(\theta^{*} - 32.9404)^2\Big|_{\theta^* = 32.9704}$$
$$= 0.0535 + Q\,(9.0 \times 10^{-4})$$

Now, suppose that we pick up Q = 2.5. Then $V(0) \leq 0.0558$. Therefore when c = 0.0558, the set Ω_c is a proper subset UxS_θ.

Fig.5.14 shows the system response when Q = 2.5, where the true value of υ is assumed to be 32.9704 (program: X1ADN2). In this figure, we can see that the proposed adaptive control system results in good output regulation. Fig.5.15 shows the estimated parameter and in this case it converges to its true value. Fig.5.16 compares adaptive and the nonadaptive control under the same condition as Fig. 5.15. This result shows tremendous increase of performance when we use the adaptive control scheme. The nonadaptive approach results in a large offset.

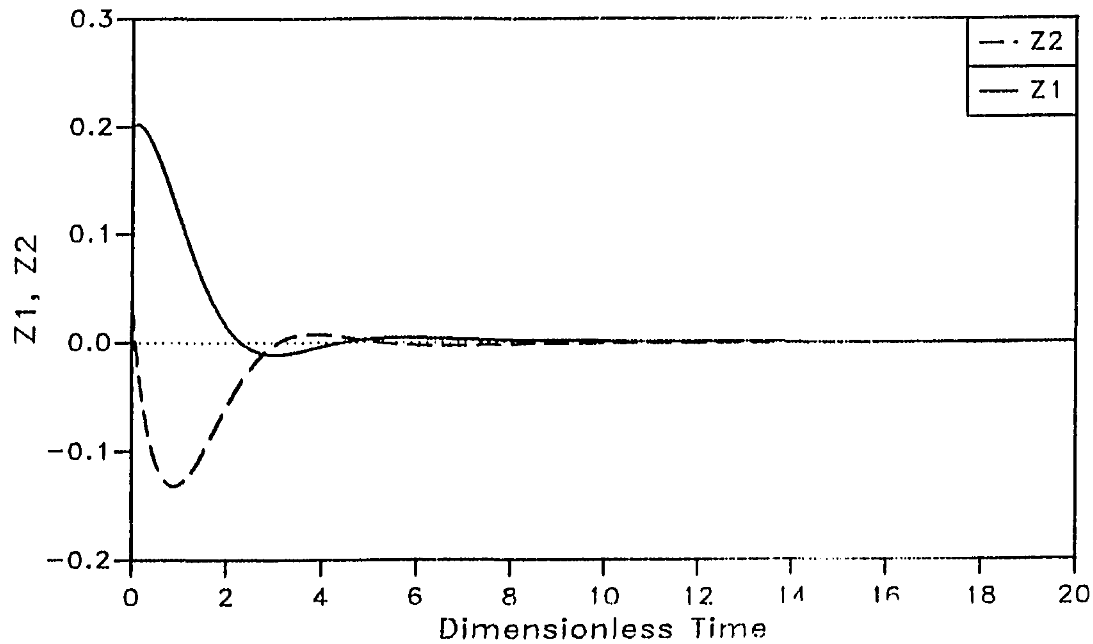

Fig. 5.14 Adaptive control response when model-plant mismatch does not satisfy linearity in the unknown parameters; $b_1 = -2.1$, $b_2 = -2.0$, $Q = 2.5$ and $x_1(0) = 0.2$, $x_2(0) = -1.0$, $\hat{\theta}(0) = 32.9404$ (true parameter, $\theta^* = 32.9704$)

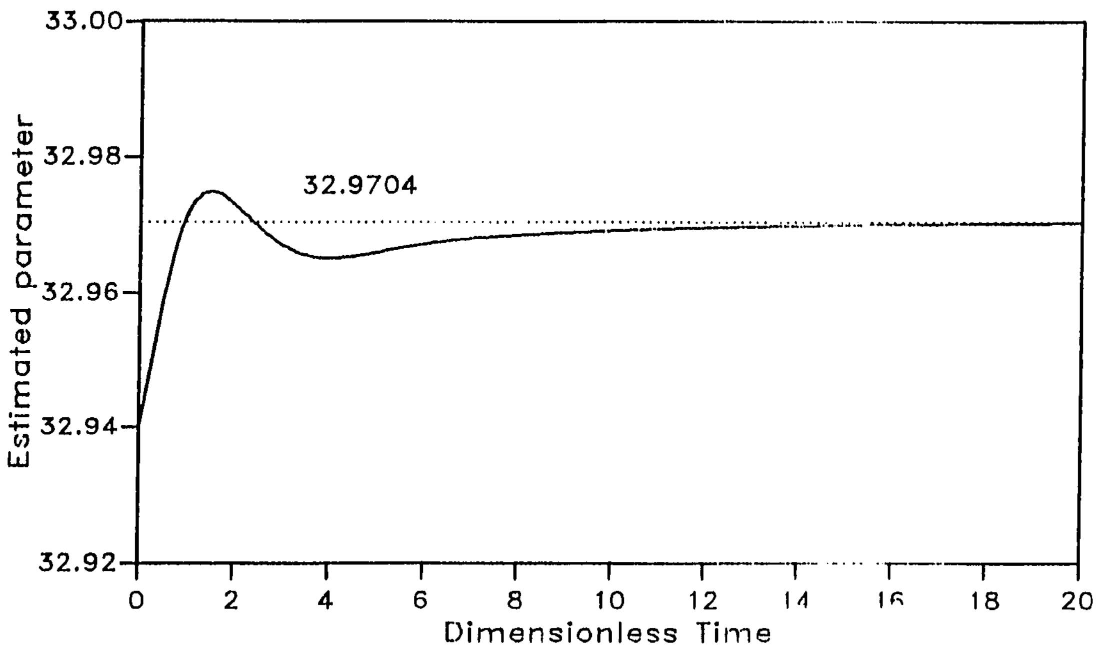

Fig. 5.15 Estimated parameter when parametric uncertainties do not satisfy linearity in the unknown parameters; $b_1 = -2.1$, $b_2 = -2.0$, $Q = 2.5$ and $x_1(0) = 0.2$, $x_2(0) = -1.0$, $\hat{\theta}(0) = 32.9404$ (true parameter, $\theta^* = 32.9704$)

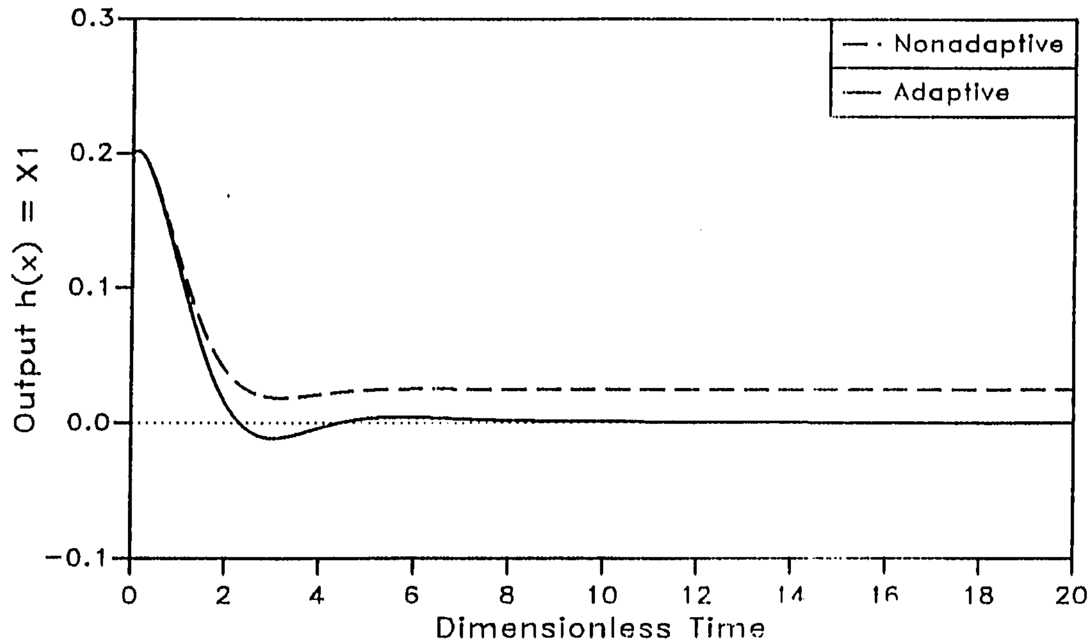

Fig. 5.16 Comparison between the adaptive and nonadaptive control when parametric uncertainties do not satisfy linearity in the unknown parameters;

$b_1 = -2.1$, $b_2 = -2.0$, $Q = 2.5$ and

$x_1(0) = 0.2$, $x_2(0) = -1.0$, $\hat{\theta}(0) = 32.9404$ (true parameter, $\theta^* = 32.9704$)

5.6. Application for Adaptive Output Tracking

Consider the same system (5.5.1) with the same parametric error in k_0 as section (5.5.1), where model-plant mismatch satisfies linearity in the unknown parameters. But, here, we want the output $y = h(x)$ to follow a given desired trajectory, $y_d(t)$.

Feedback Linearization

With the coordinate transformation (5.3.11) and the state feedback (5.4.1), the real plant (5.5.2) is transformed into the following equation:

$$\dot{e} = (A+bK)\, e + \Phi(x, \hat{\theta})\, (\theta^* - \hat{\theta}) + \Pi(x, \hat{\theta})\, \frac{d\hat{\theta}}{dt} \tag{5.6.1}$$

where $e = [e_1 \;\; e_2]^T$

$$e_1 = h(x) - y_d(t) = x_1 - y_d(t)$$

$$e_2 = L_{\hat{f}}\, h(x) - \dot{y}_d(t) = f_1(x, \hat{\theta}) - \frac{dy_d(t)}{dt}$$

and the other variables are defined in (5.3.7).

Adaptive control

Now, apply the following adaptive law to the above equation (5.6.1):

$$\frac{d\hat{\theta}}{dt} = Q^{-1}\Phi^T(x, \hat{\theta})\, P\, e \tag{5.6.2}$$

Suppose that $x_1(0) = 0.4$, $x_2(0) = -1.0$, and the nominal value of θ^*, $\hat{\theta}(0) = \exp(31.799)$, and it is given that the true value, θ^*, belongs to the set S_{θ^*} defined by

$$S_{\theta^*} = \left\{\theta^* \,;\, \exp(31.739) \le \theta^* \le \exp(31.859)\right\}.$$

We choose $b_1 = -2.1$ and $b_2 = -2.0$. Assume that the desired trajectory $y_d(t)$ has the following form:

$$y_d(t) = a \exp(-\tau_d t), \quad \text{where constant } a \geq 0 \text{ and } \tau_d \geq 0 \tag{5.6.3}$$

In this case, the Lyapunov candidate function V is

$$V = p_{11}e_1^2 + 2\, p_{12}e_1e_2 + p_{22}e_2^2 + Q\,(\theta^* - \hat{\theta})^2 \tag{5.6.4}$$

Now, we want to find the possible maximum c such that the set Ω_c, defined by

$$\Omega_c = \left\{ e, \hat{\theta};\ V \leq c \right\}$$

is a proper subset of UxS_θ for every time $t \geq 0$, where the set UxS_θ is defined in section (5.3). That is, for every $e, \hat{\theta} \in \Omega_c$, the system is linearizable with relative degree 2.

In this case, $e^TP\,e = c$ has exactly the same contour as shown in Fig. 5.2 for each given c in the e-coordinate system (instead of z-coordinate), and if $e_1 > -0.4953$, then $x_1 > -0.4953$ for any time $t \geq 0$, since $y_d(t)$ is always nonnegative. In the previous example, we found that if $c \leq 0.269$, then $x_1 > -0.4953$. Therefore, in this case also, if the constant $c \leq 0.269$, then the set Ω_c is a proper subset of UxS_θ.

When $a = 0.3$ and $\tau_d = 0.2$ in equation (5.3.6) with the given initial condition, initial value V(0) is

$$\begin{aligned} V(0) &\leq 1.25\,(0.4 - a)^2 + 0.476\,(0.4 - a)\,(-\,0.2452 + a\,\tau_d) \\ &\quad + 0.369\,(-\,0.2452 + a\,\tau_d)^2 + Q\,(1.595 \times 10^{25}) \\ &= 0.0164 + Q\,(1.595 \times 10^{25}) \end{aligned}$$

Suppose that we choose $Q = 0.2 \times 10^{-27}$ then $V(0) \leq c = 0.0196$, and this value of c is less than 0.269 so, for every $e, \hat{\theta} \in \Omega_c$; $c = 0.0196$, the system is linearizable with relative degree 2.

Next, in order to apply proposition 2, we must check whether the matrix

$\left[I - P\,(\,B(x, \hat{\theta}) + B^T(x, \hat{\theta})\,)\,P\right]$ is positive definite for every e, $\hat{\theta} \in \Omega_c$ (remember that in this case $\Delta g = 0$). This condition can be checked in the exactly same way as in the previous non-tracking cases for each fixed time. In this problem, the desired trajectory, $y_d(t)$, converges to a point so the above condition is checked along the desired trajectory. Using a computer program ' IPBP1TR' we can check that the above condition (5.6.5) is not violated with the previously given conditions, where by a similar argument as the previous chapter, when c = 0.0196, the estimated parameter, $\hat{\theta}$, belongs to the bound: $\exp(31.5613) \leq \hat{\theta} \leq \exp(31.9938)$.

Simulation

Fig.5.17 shows the response of the adaptive output tracking, where the desired output trajectory , $y_d(t) = 0.3 \exp(-\,0.2\,t)$. It is assumed that the true value of $\theta^* = \exp(31.859)$ even though the nominal value of θ^*, $\hat{\theta}(0) = \exp(31.799)$, (program: OUTEAD). In this figure we can see that the response of the adaptive control system approaches the given desired trajectory asymptotically.

As is well known, the desired output trajectory may be calculated using optimization theory. Therefore, sometimes, this adaptive output tracking may be used to get a smooth state feedback u, which is easily accessible in practice. For example, Fig.5.18 shows the state feedback u for each given desired output trajectory, $y_d(t) = 0.3 \exp(-\,\tau_d\,t)$, for each $\tau_d = 0.1, 0.2, 0.5$ and ∞. When $\tau_d = \infty$, there is no more tracking, i.e. $y_d(t) = 0$. In this figure we can see that, by adjusting the desired output trajectory, we can decrease the maximum magnitude of the linearizing state feedback.

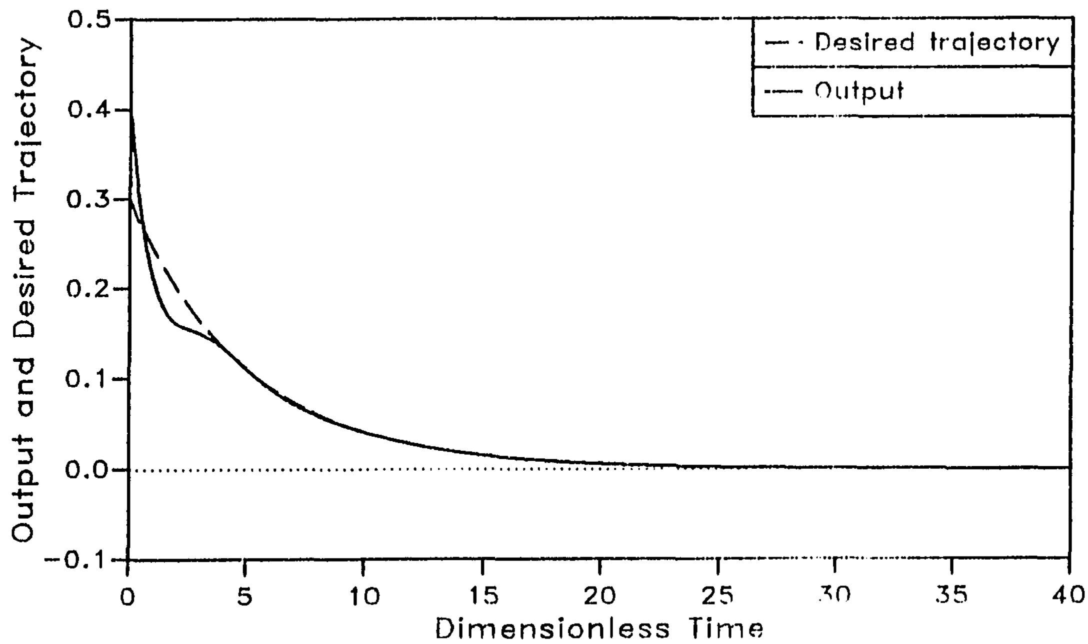

Fig. 5.17 Adaptive output tracking, where

$b_1 = -2.1$, $b_2 = -2.0$, $a = 0.3$, $\tau_d = 0.2$, $Q = 0.2 \times 10^{-27}$ and

$x_1(0) = 0.4$, $x_2(0) = -1.0$, $\hat{\theta}(0) = \exp(31.799)$ (true parameter $D^* = 31.859$)

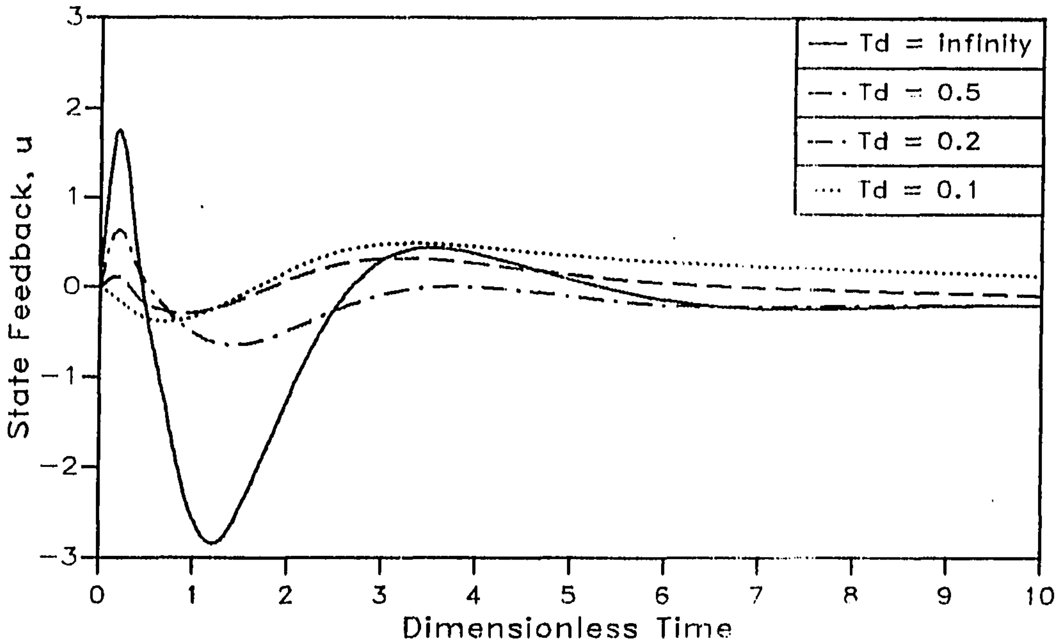

Fig. 5.18 State feedback for each τ_d = 0.1, 0.2, 0.5 and ∞, where

$b_1 = -2.1$, $b_2 = -2.0$, $a = 0.3$, $Q = 0.2 \times 10^{-27}$ and

$x_1(0) = 0.4$, $x_2(0) = -1.0$, $\hat{\theta}(0) = \exp(31.799)$ (true parameter $D^* = 31.859$)

5.7. Conclusion

We have considered parameter adaptive control for feedback linearizable systems, where parametric uncertainties can be represented linearly in the unknown parameters. The parametric uncertainties considered in this chapter do not necessarily require the following restrictive conditions: matching condition, the global Lipschitz continuity or the same equilibrium point for the mathematical model and the real plant for all possible uncertainties. The main feature of the adaptive control scheme proposed in this chapter is that the coordinate transformation and linearizing state feedback are functions of estimated parameters so that they are updated by the parameter estimation law constructed from the second method of Lyapunov. For this adaptive control system, we have found sufficient conditions for stability of the output regulation and tracking.

The developed adaptive control scheme has been applied to simulated chemical and biochemical reactors and has yielded good response. We have also shown the possibility that the proposed adaptive control scheme can be used to make the linearizing state feedback smooth by giving a proper output reference trajectory.

Moreover, it has been shown through a numerical simulation that the proposed adaptive control scheme may be applied to some cases where parametric uncertainties cannot be represented linearly in the unknown parameters. In this case, parametric uncertainties have been approximated so that they can be represented linearly in the unknown parameters. The proposed adaptive control is robust to certain structural uncertainties for the simulated example.

However, in this dissertation, we considered only the case that the system is feedback linearizable in the state space. To get more general results, it is necessary to consider input-output linearizable cases. Also, it may be necessary to develop a systematic approach to find a proper output reference trajectory within the given limits of manipulated variables. Such a study may be useful in many practical cases.

Chapter VI

Conclusion and Future Work

In this dissertation robustness of feedback linearization has been considered for parametric and structural uncertainties as well as unmodeled dynamics. Also, for parametric uncertainties, a nonlinear adaptive control of feedback linearizable processes has been developed.

The parametric or structural uncertainties considered in this dissertation need not satisfy the following restrictive conditions: matching condition, the global Lipschitz continuity or the same equilibrium point for the mathematical model and the real plant for all possible uncertainties. For this class of parametric or structural uncertainties, we have found sufficient conditions for boundedness and convergence of system trajectories.

This theoretical approach has been extended to the case of a nonlinear system which has unmodeled dynamics as well as parametric and structural uncertainties. For a high-order non-standard form of singularly perturbed system, dimensional reduction is also considered.

Our theoretical approach requires that unmodeled dynamics must be represented by a perturbed system which is linear in the state variables of unmodeled dynamics. For this kind of system, we have found sufficient conditions for boundedness and convergence of system trajectories when the linearizing state feedback based on a reduced dimensional model is applied to an uncertain real plant.

When model-plant mismatch does not satisfy the restrictive conditions mentioned above, feedback linearization does not guarantee asymptotic stability of the linearized system. For parametric uncertainties, an adaptive control approach can be a method to make feedback linearization robust.

In this dissertation we have proposed the adaptive control of feedback linearizable systems for parametric uncertainties which do not necessarily satisfy the above restrictive conditions. Moreover, we have demonstrated, by numerical simulation, that the proposed adaptive approach may be applied for some cases for which parametric uncertainties cannot be represented as linear in the unknown parameters.

The main feature of the proposed adaptive control scheme is that the coordinate transformation and state feedback are a function of estimated parameters so that they are updated by the parameter estimation law constructed from the second method of Lyapunov.

Besides the adaptive output regulation, we have also considered adaptive output tracking problems. We have shown that this approach can be used to decrease the maximum magnitude of the linearizing state feedback by choosing a proper output reference trajectory.

Feedback linearization may give an effective way to control many nonlinear chemical systems. However, in order to make this approach practically useful, a nonlinear state observer must be developed because feedback linearization requires all state variables even though input-output linearization is used. Unfortunately, a nonlinear state observer for a general class of nonlinear systems has not been developed yet. So the development of a nonlinear state observer is an important area for future study. However, as an alternative, we may use the already existing state estimators of linear systems. For example, the extended Kalman filter may be used.

As we indicate in this dissertation, robustness analysis of input-output linearization for a general case is very difficult. However, when the zero dynamics is asymptotically stable, it may be possible to analyze robustness using the central manifold theorem

[Isidori, 1989]. For a more general class of uncertainties, theoretical tools of stability analysis of nonlinear systems must be developed.

In this dissertation an adaptive approach is considered only for an exactly state-space linearizable system. To get more general results, it is necessary to consider input-output linearizable cases. It will be useful to extend the adaptive approach developed in this dissertation to the case in which zero dynamics is asymptotically stable, because our adaptive approach can be applied to a locally feedback linearizable system and also it does not increase the number of estimated parameters.

Also, it may be necessary to develope a systematic approach to find the linearizing state feedback when there exists a physical limit of manipulated variables. This study may be useful in many practical cases. We believe that, for some special cases, filtering of the state feedback with the concept of IMC (Internal Model Control) may be an effective approach. However, there is always difficulty in stability and robustness analysis of the general class of nonlinear systems.

Even though stability and robustness analysis of feedback linearization is very difficult, we believe this method can be very effective in practice when it is used with other well developed control theories such as IMC (Internal Model Control), optimal control theory, linear state observer, or adaptive control. As shown in this dissertation, feedback linearization produces a mildly nonlinear system from a highly nonlinear system. The effect of uncertainties is then not so severe.

Notation

A_R = cooling coil heat transfer area, cm^2

b_i = parameters of the new input

C = concentration, gmole / cc

C_A, C_B and C_C concentration of each component A, B, and C

C^d = desired operating point of concentration

C_A^d, C_B^d = desired operating point of component A, B

C_0 = input reactant concentration, gmole / cc

$\overline{C}_0$ = nominal value of C_0

C_{A0} = input reactant concentration, gmole / cc

C_p = heat capacity of reactor fluid, cal / gmole

c = equilibrium point in x-coordinate

d = equilibrium point in z-coordinate

d_c, d_T = inlet disturbances for concentration and temperature respectively

E = activation energy, cal / gmole

E_i = activation energy, i = 1, 2, 3, cal / gmole

f, g, $\hat{f}$, $\hat{g}$ = vector fields

h = scalar fields

K = feedback gain of the new input

k_0 = reaction constant, sec^{-1}

k_i = reaction constant, i = 1, 2, 3, sec^{-1}

M = Lipschitz constant

m = (incremental) Lipschitz constant

q = feed rate, cc / sec

R = gas constant, cal / gmole K

r = relative order

$T(\cdot)$ = transformation

T = temperature, K

T^d = desired operating point of temperature

T_c = coolant temperature, K

T_0 = input feed temperature, K

$\overline{T}_0$, $\overline{T}_c$ = nominal values of T_0 and T_c respectively

U = cooling coil heat transfer coefficient, cal / sec cm^2 K

u = input

u_0 = linearizing state feedback when there is no measurement error

V = reactor volume, cc

v = new input

x = vector of state variables

$\hat{x}$ = measured variable of x

y = output

z = vector of transformed state variables

Greek Letters

ΔH = heat of reaction, cal / gmole

ΔH_{AB} = heat of reaction $A \rightarrow B$, cal / gmole

ΔH_{BC} = heat of reaction $B \rightarrow C$, cal / gmole

Δx = measurement error in x

Δu = error in state feedbak error when there is measurement error, i.e. $u - u_0$

ρC_p = thermal capacity, cal / cc K

ρ = nonnegative integer

λ = eigenvalue

Λ = matrix in Jordan form

$\varphi(x)$ = added nonlinear term in the x - coordinate system

$\eta(z)$ = added nonlinear term in the z - coordinate system

$\overline{\sigma}$ = maximum singular value

$\underline{\sigma}$ = minimum singular value

Mathematical Symbols

o = composition

$\|\cdot\|$ = Euclidian norm or induced norm

$L_f^\rho h(x)$ = ρ-th order Lie derivative with respect to f

B_D = a set of D

$B_r(z = 0)$ = ball of radius of r centered z = 0

$B_r(z = d)$ = ball of radius of r centered z = d

$\mathbb{R}$ = set of real nomber

$\mathbb{R}^n$ = n-th dimensional Euclidian space

M^T = transpose of metrix M

$T^{-1}(\cdot)$ = inverse of $T(\cdot)$

$<\cdot,\cdot>$ = inner product

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Appendices

Appendix I

Calculation of the Jacobian of $\eta(z)$ in equation (3.4.14)

First let' find the inverse of $\frac{\partial T(x)}{\partial x}$.

$$\left[\frac{\partial T(x)}{\partial x}\right]^{-1} = \begin{bmatrix} \frac{\partial x_1}{\partial x_1} & \frac{\partial x_1}{\partial x_2} \\ \frac{\partial \hat{f}_1(x)}{\partial x_1} & \frac{\partial \hat{f}_1(x)}{\partial x_2} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ -1 - E(x_2,\hat{D}) & \frac{-\upsilon^2(x_1 - \alpha_c + 1)}{(x_2 - \alpha_T + \upsilon)^2} E(x_2,\hat{D}) \end{bmatrix}^{-1}$$

$$= -\frac{(x_2 - \alpha_T + \upsilon)^2}{\upsilon^2(x_1 - \alpha_c + 1) E(x_2,\hat{D})} \cdot \begin{bmatrix} \frac{-\upsilon^2(x_1 - \alpha_c + 1)}{(x_2 - \alpha_T + \upsilon)^2} E(x_2,\hat{D}) & 0 \\ 1 + E(x_2,\hat{D}) & 1 \end{bmatrix}$$

Let's calculate $\frac{\partial \varphi(x)}{\partial x}$.

since $\varphi_1(x) = \Delta f_1(x) = (x_1 - a_c + 1)\left[E(x_2, \hat{D}) - E(x_2, D)\right]$

$$\varphi_2(x) = \Delta A^*(x) = \frac{\partial \hat{f}_1(x)}{\partial x_1} \Delta f_1(x) + \frac{\partial \hat{f}_1(x)}{\partial x_2} \Delta f_2(x)$$

where $\dfrac{\partial\hat{f}_1(x)}{\partial x_1} = -1 - E(x_2, \hat{D})$

$$\frac{\partial\hat{f}_1(x)}{\partial x_2} = -\frac{\upsilon^2(x_1 - \alpha_c + 1)\,E(x_2, \hat{D})}{(x_2 - \alpha_T + \upsilon)^2}$$

$$\Delta f_2(x) = -B(x_1 - \alpha_c + 1)\left[E(x_2, D) - E(x_2, \hat{D})\right]$$

From the above equation

$$\frac{\partial\varphi_1(x)}{\partial x_1} = E(x_2, \hat{D}) - E(x_2, D)$$

$$\frac{\partial\varphi_1(x)}{\partial x_2} = \frac{\upsilon^2(x_1 - \alpha_c + 1)}{(x_2 - \alpha_T + \upsilon)^2}\left[E(x_2, \hat{D}) - E(x_2, D)\right]$$

Now $\dfrac{\partial\varphi_2(x)}{\partial x_1}$ is

$$\frac{\partial\varphi_2(x)}{\partial x_1} = \frac{\partial^2\hat{f}_1(x)}{\partial x_1^2}\Delta f_1(x) + \frac{\partial\hat{f}_1(x)}{\partial x_1}\frac{\partial\Delta f_1(x)}{\partial x_1} + \frac{\partial^2\hat{f}_1(x)}{\partial x_1\partial x_2}\Delta f_2(x) + \frac{\partial\hat{f}_1(x)}{\partial x_2}\frac{\partial\Delta f_2(x)}{\partial x_1}$$

where $\dfrac{\partial^2\hat{f}_1(x)}{\partial x_1^2} = 0$

$$\frac{\partial^2\hat{f}_1(x)}{\partial x_1\partial x_2} = -\frac{\upsilon^2\,E(x_2, \hat{D})}{(x_2 - \alpha_T + \upsilon)^2}$$

$$\frac{\partial\Delta f_2(x)}{\partial x_1} = -B\left[E(x_2, D) - E(x_2, \hat{D})\right]$$

so $$\frac{\partial\varphi_2(x)}{\partial x_1} = \left\{2\frac{\upsilon^2 B(x_1 - \alpha_c + 1)\;E(x_2, \hat{D})}{(x_2 - \alpha_T + \upsilon)^2} + 1 + E(x_2, \hat{D})\right\}\left[E(x_2, D) - E(x_2, \hat{D})\right]$$

and $$\frac{\partial\varphi_2(x)}{\partial x_2} = \frac{\partial^2\hat{f}_1(x)}{\partial x_1\partial x_2}\Delta f_1(x) + \frac{\partial\hat{f}_1(x)}{\partial x_2}\frac{\partial\Delta f_1(x)}{\partial x_2} + \frac{\partial^2\hat{f}_1(x)}{\partial x_2^2}\Delta f_2(x) + \frac{\partial\hat{f}_1(x)}{\partial x_2}\frac{\partial\Delta f_2(x)}{\partial x_2}$$

where $\dfrac{\partial^2\hat{f}_1(x)}{\partial x_2^2} = \dfrac{\upsilon^2(x_1 - \alpha_c + 1)\,E(x_2, \hat{D})}{(x_2 - \alpha_T + \upsilon)^3}\left\{2 - \dfrac{\upsilon^2}{(x_2 - \alpha_T + \upsilon)}\right\}$

$$\frac{\partial\Delta f_2(x)}{\partial x_1} = \frac{-B\upsilon^2(x_1 - \alpha_c + 1)}{(x_2 - \alpha_T + \upsilon)^2}\left[E(x_2, D) - E(x_2, \hat{D})\right]$$

Therefore

$$\frac{\partial \varphi_2(x)}{\partial x_2} = \frac{\upsilon^2(x_1 - \alpha_c + 1)\, E(x_2, \widehat{D})}{(x_2 - \alpha_T + \upsilon)^3}\left[E(x_2, D) - E(x_2, \widehat{D})\right]$$

$$\cdot \left\{1 + \frac{\left[1 + E(x_2, \widehat{D}\right]}{E(x_2, \widehat{D}} - \frac{2\,B\,(x_1 - \alpha_c + 1)}{(x_2 - \alpha_T + \upsilon)} + \frac{2\,B\upsilon^2(x_1 - \alpha_c + 1)}{(x_2 - \alpha_T + \upsilon)^2}\right\}$$

Appendix II

Bellman-Gronwall Inequality [Vidyasagar, 1978, p292]

Lemma: Suppose $c \geq 0$, $r(.)$ and $k(.)$ are nonnegative valued continuous functions, and suppose

$$r(t) \leq c + \int_0^t k(\tau)\, r(\tau)\, d\tau, \qquad \forall\, t \in [\,0, T]$$

Then

$$r(t) \leq c \exp\left[\int_0^t k(\tau)\, r(\tau)\, d\tau\right], \qquad \forall\, t \in [\,0, T] \qquad ///$$

Appendix III

(Local) Contraction Mapping Theorem [Vidyasagar, 1978, p78]

Theorem: Let X be a Banach space, and let B be a closed ball in X, i.e. a set of the form

$$B = \{ x : \| x - z \| \leq r \}$$

for some $z \in (X, \| . \|)$ and some $r \leq \infty$. Let $P : X \to X$ be an operator satisfying the following conditions

(i) P maps B into itself, i.e. $Px \in B$ whenever $x \in B$.

(ii) There is a constant $\rho < 1$ such that

$$\| Px - Py \| \leq \rho \| x - y \|, \quad \forall\, x, y \in B.$$

Then

(i) P has exactly one fixed point in B (call it x^*).

(ii) For any $x_0 \in B$, the sequence $(x_n)_1^\infty$ defined by

$$x_{n+1} = Px_n, \quad n \geq 0$$

converges to x^*. Moreover,

$$\| x_n - x^* \| \leq \frac{\rho^n}{1 - \rho} \| Px_0 - x_0 \|.$$

///

Appendix IV

Matching Condition [Gutman, 1979, Spong, 1986, Behtash, 1990]

Consider the following uncertain nonlinear plant

$$\dot{x} = f(x) + g(x)\,u \quad , x \in \mathbb{R}^n , u \in \mathbb{R}$$

where f and g are C^∞ (infinitely differentiable) – vector fields.

For this plant suppose that we have the following mathematical model

$$\dot{x} = \hat{f}(x) + \hat{g}(x)\,u \quad , x \in \mathbb{R}^n , u \in \mathbb{R}$$

where $\hat{f}$ and $\hat{g}$ are C^∞ vector fields.

Define model-plant mismatch Δf and Δg as follows:

$$\Delta f = f - \hat{f}$$

$$\Delta g = g - \hat{g}.$$

The above system is said to satisfy the matching condition if there exist smooth real-valued functions of x, d(x) and e(x) such that

$$\Delta f(x) = d(x)\, g(x) \tag{A.IV.1}$$

$$\Delta g(x) = e(x)\, g(x)$$

Example

Suppose that we have

$$\Delta f(x) = \begin{bmatrix} 0 \\ x^2 \end{bmatrix} \quad \text{and} \quad g(x) = \begin{bmatrix} 0 \\ c \end{bmatrix}$$

where c is a nonzero constant

It can be easily seen that if we choose

$$d(x) = \frac{x^2}{c}$$

then the above condition (A.IV.1) is satisfied. Therefore $\Delta f(x)$ satisfies the matching condition.

Now suppose that $\Delta f(x)$ has the following form

$$\Delta f(x) = \begin{bmatrix} c \\ \dot{x}^2 \end{bmatrix}$$

That is, the first element of vector function $\Delta f(x)$ is not zero. In this case there is no real-valued function d(x) which satisfies the above condition (A.IV.1). Therefore in this case $\Delta f(x)$ does not satisfy the matching condition. ///

Appendix V

Global Lipschitz Continuity [Vidyasagar, 1978]

(or **Global Lipschitz Condition**)

A real valued function f(t, x), where $x \in \mathbb{R}^n$ and $f: \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, satisfies a global Lipschitz continuity (condition) with respect to x if there exists a constant $M > 0$ such that

$$\| f(t, x_1) - f(t, x_2) \| \leq M \| x_1 - x_2 \|$$

for all x_1, x_2 in $\mathbb{R}^n$ and all $t \geq 0$.

If the above condition is satisfied only for all x_1, x_2 in a subset $D \subset \mathbb{R}^n$ then it is said that f(t, x) satisfies a local Lipschitz continuity (condition).

Appendix VI

Inverse Function Theorem [Boothby, 1975, p42]

Let W be an open subset of $\mathbb{R}^n$ and $T : W \rightarrow \mathbb{R}^n$, an infinitely differentiable mapping. Let DT denote the Jacobian matrix of the differential mapping T and DT(x) denote its value at x. If at a point x^0 in the set W the Jacobian $DT(x^0)$ is nonsingular, then there exists an open neighborhood U of x^0 in W such that $V = T(U)$ is open and $T : U \rightarrow V$ is a diffeomorphism (i.e., the mapping T is invertable, and T and the inverse of T are both infinitely differentiable mappings). If $x \in U$ and $z = T(x)$, then we have the following formula for the derivatives of the inverse of T, denoted by T^{-1}, at z

$$DT^{-1}(z) = [DT(x)]^{-1},$$

the term on the right denoting the inverse matrix to DT(x).

Appendix VII

Computer Programs

```
*      PROGRAM : EIGEN
*
*      CALCULATE CONSTANTS ALFA AND K SUCH THAT
*              NORM OF EXP(AT).LE.ALFA*EXP(-KT)
*              (ASSUMPTION 1.2 IN THEOREM 1)
*         USING ALAMOS SUBROUTINE
*
*      DEFINITION OF VARIABLES
*        B1 AND B2 = ASSIGNED CONSTANTS (EQ. 3.4.6 P58)
*        RK = CONSTANT K
*
       DIMENSION A(2,2),WORK(4)
       COMPLEX X(2,2),S(2),E(2),U(2,2),V(2,2)
*
       WRITE(7,17)
    17 FORMAT(5X,'B1',8X,'B2',8X,'RK',8X,'ALFA',6X,'ALFA/K')
*
*      CONSTANTS B1 AND B2
       DO 20 B1=-10.,-4.0,0.5
       DO 20 B2=-10.,-0.9,0.1
*
*      CALCULATE ABS(MAX. EIGEN VALUE)==K
*      PARAMETERS FOR THE SUBROUTINE SGEEV IN ALAMOS
       LDA=2
       N=2
       LDV=2
       JOB=1
*
*      INPUT A MATRIX
*      ______________

       A(1,1)=0.0
       A(1,2)=1.0
       A(2,1)=B1
       A(2,2)=B2
*
       CALL SGEEV(A,LDA,N,E,V,LDV,WORK,JOB,INFO)
*
       RK=MAX(REAL(E(1)),REAL(E(2)))
       RK=ABS(RK)
       IF (RK.EQ.0.) THEN
       GO TO 20
       ENDIF
*
*      COMPUTE THE SINGULAR VALUES OF THE COMPLEX MATRIX
*      USING THE LOS ALAMOS PACKAGE
*
       DO 10 I=1,2
       DO 10 J=1,2
   10  X(I,J)=V(I,J)
*
*      PARAMETERS FOR SUBROUTINE CSVDC
       LDX=2
       N=2
       LDU=2
       LDV=2
       JOB=00
```

```
*
      CALL CSVDC(X,LDX,N,2,S,E,U,LDU,V,LDV,WORK,00,INFO)
*
      RS2=REAL(S(2))
      IF (RS2.EQ.0.) THEN
      GO TO 20
      ENDIF
      ALFA=REAL(S(1))/REAL(S(2))
      ALPK=ALFA/RK
      IF(ALPK.LT.2.5) THEN
      WRITE(7,37) B1,B2,RK,ALFA,ALPK
   37 FORMAT(/,3X,5(G11.4,3X))
      ENDIF
   20 CONTINUE
      STOP
      END
```

```
*     PROGRAM : OUTX1RO
*
*     SIMULATION OF NONLINEAR CHEM. REACTOR
*     RESPONSE OF THE CLOSED LOOP WITH STATE FEEDBACK
*
*     MODEL PLANT MISMATCHWITH +- 2% ERROR IN K0
*     SISO SYSTEM
*     C--CONTROLED VARIABLE, TC--- MANUPLATE VARIABLE
*------------------------------------------------------------
*
*     DEFINITION OF VARIABLES
*     B1, B2 = ELEMENTS OF VECTOR K IN V = KZ
*     ALFAC, ALFAT, ALFAW, RQ, RMU, D, B, GAMA
*                 = DIMENSIONLESS VARIABLES IN TABLE 3.2
*
*     DALFAC, DALFAT, DRQ, DRMU, DD DGAMA = NOMINAL VALUES
*          OF ALFAC, ALFAT, RQ, RMU, D, DGAMA, RESPECTIVELY
*     U = STATE FEEDBACK
*
      IMPLICIT REAL*8(A-H,O-Z)
      EXTERNAL DERIV
      DIMENSION Y(2),YDOT(2),RWORK(44),IWORK(22),ATOL(2)
      COMMON U,B1,B2
      COMMON DALFAC,DRQ,DD,DRMU,DALFAT

*     SYSTEM PARAMETERS
*
*     --------------------------
*     INITIAL CONDITIONS
      Y(1)=-0.104
      Y(2)=0.540
*
C     CONSTANTS B1 AND B2
      B1=-2.1
      B2=-2.0
*     --------------------------

C     LSODE PARAMETERS
      NEQ=2
      T=0.
      ITOL=2
      RTOL=1.0D-8
      ATOL(1)=1.0D-15
      ATOL(2)=1.0D-10
      ITASK=1
      ISTATE=1
      IOPT=0
      LRW=44
      LIW=22
      MF=22
C
C     INCREASE THE CALCULATION STEP OF LSODE
      DO 19 I=5,10
      RWORK(I)=0.0D0
      IWORK(I)=0
   19 CONTINUE
      IWORK(6)=500000
```

```
C
*     COORDINATE TRANSFORMATION Z2
*
      DRMU=32.94043687
      DD=31.799
      DALFAT=-1.20339588
      DALFAC=0.5047
      DRQ=1.0
*
      Z2=(-Y(1)+DALFAC)*DRQ-(Y(1)-DALFAC+1.)*
     &        DEXP(DD-(DRMU**2/(Y(2)+DRMU-DALFAT)))
*
      WRITE(6,16) T,Y(1),Y(2)

*     CONVERSION INTO REAL CONCENTRATION AND TEMPERATURE
*
      COBAR=8.016D-03
      CD=3.97D-03
      TOBAR=383.3
      TD=397.3
      CON=Y(1)*COBAR+CD
      TEM=Y(2)*TOBAR/DRMU+TD
      WRITE(7,17) CON, TEM

*     SAMPLING STEP H
      H=0.1
      FH=200.*H
      DO 100 TOUT=H,FH,H

      CALL LSODE(DERIV,NEQ,Y,T,TOUT,ITOL,RTOL,ATOL,ITASK,ISTATE,
     &           IOPT,RWORK,LRW,IWORK,LIW,JAC,MF)

*     COORDINATE TRANSFORMATION Z2
*
      Z2=(-Y(1)+DALFAC)*DRQ-(Y(1)-DALFAC+1.)*
     &        DEXP(DD-(DRMU**2/(Y(2)+DRMU-DALFAT)))
*
C     WRITE(6,16) T,Y(1),Y(2)

*     CONVERSION INTO REAL CONCENTRATION AND TEMPERATURE
*
      COBAR=8.016D-03
      CD=3.97D-03
      TOBAR=383.3
      TD=397.3
      CON=Y(1)*COBAR+CD
      TEM=Y(2)*TOBAR/DRMU+TD
      WRITE(7,17) CON, TEM

      WRITE(7,17) Y(1),Y(2)
   16 FORMAT(10X,G12.5,3X,G12.5,3X,G12.5,3X,G12.5)
   17 FORMAT(10X,2(G12.5,3X))

      ISTATE=3
  100 CONTINUE

      STOP
```

```
      END

      SUBROUTINE DERIV(NEQ,T,Y,YDOT)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION Y(2),YDOT(2)
      COMMON U,B1,B2
      COMMON DALFAC,DRQ,DD,DRMU,DALFAT

*     DIMENSIONLESS VARIABLES
*     PLANT--------------------------------------

      D=31.8188             ]  +2% ERROR IN KO
      RMU=32.94043687       ] NOMINAL VALUE
      B=-7.6476
      GAMA=0.459316287
      ALFAT=-1.20339588
      ALFAC=0.5047
      ALFAW=4.580551225
      RQ=1.

*     DISTURBANCES
      DT=0.000D0
      DC=0.000D0

*     MODEL--------------------------------------

      DRMU=32.9404
      DB=B
      DD=31.7990
      DGAMA=GAMA
      DALFAT=ALFAT
      DALFAC=ALFAC
      DALFAW=ALFAW
      DRQ=1.

*     EQUATION 1, PLANT ----------------------------

      YDOT(1)=(-Y(1)+ALFAC)*RQ-(Y(1)-ALFAC+1.)*
     &        DEXP(D-(RMU**2/(Y(2)+RMU-ALFAT)))+DC*RQ

*     !---------------------------------------------
*     !      STATE FEEDBACK U BASED ON THE MODEL     !
*     !______________________________________________!

*     MODEL FUNCTION F1(X)
      F1=(-Y(1)+DALFAC)*DRQ-(Y(1)-DALFAC+1.)*
     &        DEXP(DD-(DRMU**2/(Y(2)+DRMU-DALFAT)))

*     MODEL FUNCTION F2(X)
      F2=(-Y(2)+DALFAT)*DRQ-DB*(Y(1)-DALFAC+1.)*
     &    DEXP(DD-(DRMU**2/(Y(2)+DRMU-DALFAT)))-
     &    DGAMA*(DALFAW-DALFAT+Y(2))

*     PARTIAL DERIVATIVE OF MODEL F1(X) W.R.T X1
      F1X1=-DRQ-DEXP(DD-(DRMU**2/(Y(2)-DALFAT+DRMU)))

*     PARTIAL DERIVATIVE OF MODEL F1(X) W. R. T X2
```

```
      F1X2=(-(DRMU**2)*(Y(1)-DALFAC+1.)/
     &     ((Y(2)-DALFAT+DRMU)**2))*
     &     DEXP(DD-(DRMU**2/(Y(2)-DALFAT+DRMU)))

      U=(1./(F1X2*DGAMA))
     &         *(B1*Y(1)+B2*F1-(F1X1*F1+F1X2*F2))

*     EQUATION2, PLANT---------------------------

      YDOT(2)=(-Y(2)+ALFAT)*RQ-B*(Y(1)-ALFAC+1.)
     &      *DEXP(D-(RMU**2/(Y(2)+RMU-ALFAT)))
     &      -GAMA*(ALFAW-ALFAT+Y(2))+GAMA*U+RQ*DT

      RETURN
      END
```

```
*     PROGRAM : LIPSCH1
*
*     FIND LIPSCHITZ CONSTANT AND DELAT-ETA
*
      IMPLICIT REAL *8(A-H,O-Z)
      DIMENSION X(3)
      DIMENSION A(2,2), WORK(2), S(2),E(2),U(2,2),V(2,2)
      COMMON /PHYSIC/ RQ,ALFAC,RMU,ALFAT,DM,ALFAW,B,GAMA
      COMMON /RADIUS/ RAD

*     DEFINITION OF VARIABLES
*     S(1) = MAXIMUM SINGULAR VALUE
*     REMM = LIPSCHITZ CONSTANT
*     DELETA=CONSTANT DELTA-ETA ( IN TABLE 3.3)
*     RAD = RADIUS OF THE BALL BR
*     RQ, ALFAC, RMU, ALFAT, D, ALFAW, B, GAMA
*        = PHYSICAL PARAMETERS OF CSTR
*          ( DEFINED IN TABLE 3.2)
*     X3 = POSSIBLE VALUE OF D
*     REK = K SUCH THAT NORM OF EXP(AT). LE. ALFA EXP(-KT)
*     REALFA= ALFA

C  PHYSICAL PARAMETRS OF CSTR
      RQ=1.
      ALFAC=0.5047
      RMU=32.9404
      ALFAT=-1.2031
      DM=31.7990          ] NOMINAL VALUE OF D
      ALFAW=4.5805
      B=-7.6476
      GAMA=0.4593

C-------------------------------------
      RAD=0.25
C-------------------------------------

      WRITE(6,26)
  26  FORMAT (3X,'S(1)',3X,'X(1)',3X,'X(2)',3X,'X(3)')

      X1MIN=-0.4953          ] POSSIBLE MINIMUM OF X1
      X2MIN=-1.5
      X3MIN=31.77880         ] -2% ERROR

      X1MAX=0.5047           ] POSSIBLE MAXIMUM OF X1
      X2MAX=1.5
      X3MAX=31.8188          ] +2% ERROR

      X1INC=(X1MAX-X1MIN)/90.
      X2INC=(X2MAX-X2MIN)/90.
      X3INC=(X3MAX-X3MIN)/80.

*  CALCULATE THE DELTAETA
      CALL ETA0(X3MAX,DELETA)

      REK=1.000        ] K
      REALFA=2.5660    ] ALFA
```

```
*  COMPARED SLOPE M==REM
*        ACTUAL SLOPE M MUST BE LESS THAN REM
      REM=REK/REALFA-DELETA/RAD
*
      REMM=REM-1.D-5

      DO 10 X1=X1MIN,X1MAX,X1INC
      DO 10 X2=X2MIN,X2MAX,X2INC
      DO 10 X3=X3MIN,X3MAX,X3INC

      X(1)=X1
      X(2)=X2
      X(3)=X3

      CALL CONST(X,CON)
      IF(CON.GE.0.0) THEN
         A(1,1)=F11(X)
         A(1,2)=F12(X)
         A(2,1)=F21(X)
         A(2,2)=F22(X)

*        COMPUTES THE SINGULAR VALUES OF GIVEN A
*        LOS ALAMOS SUBROUTINE, WHICH IS ACTUALLY LINPACK.
*
         CALL DSVDC(A,2,2,2,S,E,U,2,V,2,WORK,11,INFO)

      XXXX=0.1100                ] S(1) IS LESS THAN XXXX
      IF(S(1).GE.XXXX) THEN
C
*        WHERE S(1) IS MAXIMUM SINGULAR VALUE == L2-NORM
         WRITE(6,16) S(1),X(1),X(2),X(3)
  16     FORMAT(2X,4(G13.6,2X))
      ENDIF
      ENDIF
  10  CONTINUE

      WRITE(6,36) RAD, DELETA,REMM,S(1)
  36  FORMAT(2X,'RAD=',2X,G12.5,'DELETA=',2X,G12.5,2X
     &          ,'REMM=',G12.5,/,2X,'S(1)=', 1X, G12.5)
      WRITE(6,46) A(1,1), A(1,2), A(2,1), A(2,2)
  46  FORMAT(2X,'A=',2X,4(G12.5,1X))
      STOP
      END

      FUNCTION F11(X)
      IMPLICIT REAL *8(A-H,O-Z)
      DIMENSION X(3)
      COMMON /PHYSIC/ RQ,ALFAC,RMU,ALFAT,DM,ALFAW,B,GAMA

      ED=DEXP(X(3)-(RMU**2/(X(2)-ALFAT+RMU)))
      EDM=DEXP(DM-(RMU**2/(X(2)-ALFAT+RMU)))

      F11=(ED-EDM)/EDM
      RETURN
      END

      FUNCTION F12(X)
```

```
      IMPLICIT REAL *8(A-H,O-Z)
      DIMENSION X(3)
      COMMON /PHYSIC/ RQ,ALFAC,RMU,ALFAT,DM,ALFAW,B,GAMA

      ED=DEXP(X(3)-(RMU**2/(X(2)-ALFAT+RMU)))
      EDM=DEXP(DM-(RMU**2/(X(2)-ALFAT+RMU)))

      F12=(ED-EDM)/EDM
      RETURN
      END

      FUNCTION F21(X)
      IMPLICIT REAL *8(A-H,O-Z)
      DIMENSION X(3)
      COMMON /PHYSIC/ RQ,ALFAC,RMU,ALFAT,DM,ALFAW,B,GAMA

      ED=DEXP(X(3)-(RMU**2/(X(2)-ALFAT+RMU)))
      EDM=DEXP(DM-(RMU**2/(X(2)-ALFAT+RMU)))
      X1AC=(X(1)-ALFAC+1.)
      X2AT=(X(2)-ALFAT+RMU)
      FF1=2.*RMU**2*B*X1AC/(X2AT**2)
      FF2=2.*B*(X1AC/X2AT)*(EDM+1.)
      FF3=(1.+EDM)**2/EDM

      F21=(-FF3+FF2-FF1)*(ED-EDM)
      RETURN
      END

      FUNCTION F22(X)
      IMPLICIT REAL *8(A-H,O-Z)
      DIMENSION X(3)
      COMMON /PHYSIC/ RQ,ALFAC,RMU,ALFAT,DM,ALFAW,B,GAMA

      ED=DEXP(X(3)-(RMU**2/(X(2)-ALFAT+RMU)))
      EDM=DEXP(DM-(RMU**2/(X(2)-ALFAT+RMU)))
      X1AC=(X(1)-ALFAC+1.)
      X2AT=(X(2)-ALFAT+RMU)

      GG1=2.*RMU**2*B*X1AC/(X2AT**2)
      GG2=2.*B*(X1AC/X2AT)
      GG3=(1.+EDM)/EDM

      F22=(-1.-GG3+GG2-GG1)*(ED-EDM)
      RETURN
      END

      SUBROUTINE CONST(X,CON)
      IMPLICIT REAL *8(A-H,O-Z)
      DIMENSION X(3)
      COMMON /PHYSIC/ RQ,ALFAC,RMU,ALFAT,DM,ALFAW,B,GAMA
      COMMON /RADIUS/ RAD

      F1XM=(-X(1)+ALFAC)*RQ-(X(1)-ALFAC+1.)*
     &   DEXP(DM-(RMU**2/(X(2)-ALFAT+RMU)))

      CON=RAD**2-X(1)**2-F1XM**2
```

```
      RETURN
      END
      SUBROUTINE ETA0(X3MAX,DELETA)

      IMPLICIT REAL *8(A-H,O-Z)
      COMMON /PHYSIC/ RQ,ALFAC,RMU,ALFAT,DM,ALFAW,B,GAMA

      D=X3MAX
      ED0=DEXP(D-(RMU**2/(-ALFAT+RMU)))
      EDM0=DEXP(DM-(RMU**2/(-ALFAT+RMU)))

      THETA1=(1.-ALFAC)*(EDM0-ED0)
      AT1=((1.-ALFAC)*RMU**2)/((RMU-ALFAT)**2)
      THETA2=(-1.-EDM0)*THETA1+AT1*EDM0*B
     &                         *(1.-ALFAC)*(ED0-EDM0)

      DELETA=DSQRT(THETA1**2+THETA2**2)

      RETURN
      END
```

```
*      PROGRAM : CHCONT2
*      DRAW THE CONTOUR FOR EACH !!Z!!
*
*    DRAWING PROCEDURE;
*    1. FROM THE UNSORTED DATA (ORIGINAL DATA)
*         FIND THE EQUATION OF Y
*    2. BY USING 'CHCONT2' CALCULATE THE CONTOUR DATA
*         IN X-COORNINATE
*    3. TRANSFORM X-COORDINATE INTO C AND T COORDINATE
*
*------------------------------------------------------------

*     DEFINITION OF VARIABLES
*     Z1 AND Z2 = COORDINATE TRANSFORMATION
*     X1 AND X2 = ORIGINAL COORDINATE SYSTEM
*     ZCONT = RADIUS OF THE BALL; THAT IS, !!Z!! .LE. ZCONT
*     X1MIN. X2MIN = POSSIBLE MINIMUM VALUE OF X1 AND X2
*                    IN THE BALL BR, WHERE RADIUS = ZCONT
*     X1MAX. X2MAX = POSSIBLE MAXIMUM VALUE OF X1 AND X2
*                    IN THE BALL BR, WHERE RADIUS = ZCONT

*  CONTOUR RADIUS, ZCONT
      ZCONT=0.042

      X1MIN=-0.2
      X1MAX=0.2
      X1INC=(X1MAX-X1MIN)/200.

      X2MIN=-2.0
      X2MAX=1.0
      X2INC=(X2MAX-X2MIN)/200.

      DO 10 X1=X1MIN,X1MAX,X1INC
      DO 10 X2=X2MIN,X2MAX,X2INC

      Y=-(0.23/0.050)*X1-0.002                ] FOR !!Z!! - 0.042
*
      IF(X2.GE.Y)GOTO 10

      CALL CONT(X1,X2,Z1,Z2)
      ZNORM=SQRT(Z1**2+Z2**2)

      IF(ABS(ZNORM-ZCONT).LE.1.0E-03) THEN
      WRITE(6,17) X1,X2
      ENDIF
   10 CONTINUE

      X3INC=-X1INC
      DO 20 X1=X1MAX,X1MIN,X3INC
      DO 20 X2=X2MIN,X2MAX,X2INC

      Y=-(0.23/0.050)*X1-0.002                ] FOR !!Z!! = 0.042
      IF(X2.LE.Y)GOTO 20

      CALL CONT(X1,X2,Z1,Z2)
      ZNORM=SQRT(Z1**2+Z2**2)
```

```
      IF(ABS(ZNORM-ZCONT).LE.1.0E-03) THEN
      WRITE(6,18) X1,X2
      ENDIF
   20 CONTINUE
   17 FORMAT(2X,2(G12.3,1X))
   18 FORMAT(2X,2(G12.3,1X))
      STOP
      END

      SUBROUTINE CONT(X1,X2,Z1,Z2)

*     MODEL PARAMETERS
      RMU=32.9404
      B=-7.6476
      D=31.799
      GAMA=0.4593
      ALFAT=-1.2031
      ALFAC=0.5047
      ALFAW=4.5805
      RQ=1.

      Z1=X1

      F1=(-X1+ALFAC)*RQ-(X1-ALFAC+1.)*
     &        EXP(D-(RMU**2/(X2+RMU-ALFAT)))

      Z2=F1
      RETURN
      END
```

```
*      PROGRAM : OUTX1ME
*      SIMULATION OF NONLINEAR CHEM. REACTOR
*      RESPONSE OF THE CLOSED LOOP WITH STATE FEEDBACK
*
*      MEASURMENT ERROR IN X1
*      SISO SYSTEM
*      C--CONTROLED VARIABLE, TC--- MANUPLATE VARIABLE
*----------------------------------------------------------

*      DEFINITION OF VARIABLES
*      X1M= MEASURED X1
*      Y(1)= TRUE VALUE OF STATE VARIABLE X1
*      THE OTHER VARIABLES ARE DEFINED IN
*        THE PROGRAM 'OUTX1RO'.

       IMPLICIT REAL*8(A-H,O-Z)
       EXTERNAL DERIV
       DIMENSION Y(2),YDOT(2),RWORK(44),IWORK(22),ATOL(2)
       COMMON U,B1,B2

*      SYSTEM PARAMETERS
*
*      -------------------------
*      INITIAL CONDITIONS
       Y(1)=0.081
       Y(2)=-0.31
       U=0.

*      CONSTANT B1 AND B2
       B1=-4.0
       B2=-2.7

C      LSODE PARAMETERS
       NEQ=2
       T=0.
       ITOL=2
       RTOL=1.0D-6
       ATOL(1)=1.0D-15
       ATOL(2)=1.0D-10
       ITASK=1
       ISTATE=1
       IOPT=0
       LRW=44
       LIW=22
       MF=22

C      INCREASE THE CALCULATION STEP FOR LSODE
       DO 19 I=5,10
       RWORK(I)=0.0D0
       IWORK(I)=0
   19  CONTINUE
       IWORK(6)=500000

       WRITE(6,26) T,Y(1),Y(2)

*  CONVERSION INTO REAL CONCENTRATION AND TEMPERATURE
```

```
      COBAR=8.016D-03
      CD=3.97D-03
      TOBAR=383.3
      TD=397.3
      DRMU=32.9404
      CON=Y(1)*COBAR+CD
      TEM=Y(2)*TOBAR/DRMU+TD

      WRITE(7,17) CON, TEM
   26 FORMAT(10X,G12.5,2X,',',1X,G12.5,2X
     &          ,',',G12.5,2X,',',G12.5)

C     SAMPLING STEP H
      H=0.1
      FH=200.*H      ] NUMBER OF CALCULATION
      DO 100 TOUT=H,FH,H

      CALL LSODE(DERIV,NEQ,Y,T,TOUT,ITOL,RTOL,ATOL,ITASK,ISTATE,
     &          IOPT,RWORK,LRW,IWORK,LIW,JAC,MF)
      WRITE(6,16) T,Y(1),Y(2)

*  CONVERSION INTO REAL CONCENTRATION AND TEMPERATURE

      COBAR=8.016D-03
      CD=3.97D-03
      TOBAR=383.3
      TD=397.3
      DRMU=32.9404
      CON=Y(1)*COBAR+CD
      TEM=Y(2)*TOBAR/DRMU+TD

      WRITE(7,17) CON, TEM
   16 FORMAT(10X,G12.5,2X,',',1X,G12.5,2X
                ,',',G12.5,2X,',',G12.5)
   17 FORMAT(10X,2(G12.5,2X,',',1X))

      ISTATE=3
  100 CONTINUE

      STOP
      END

      SUBROUTINE DERIV(NEQ,T,Y,YDOT)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION Y(2),YDOT(2)
      COMMON U, B1,B2

C     DIMENSIONLESS VARIABLES
C     PLANT

      RMU=32.9404
      B=-7.6476
      D=31.799
      GAMA=0.4593
      ALFAT=-1.2031
      ALFAC=0.5047
      ALFAW=4.5805
```

```
      RQ=1.

C     DISTURBANCES
      DT=0.000D-00
      DC=0.000D-00


C     EQUATION 1

      YDOT(1)=(-Y(1)+ALFAC)*RQ-(Y(1)-ALFAC+1.)*
     &        DEXP(D-(RMU**2/(Y(2)+RMU-ALFAT)))+DC*RQ

C     !---------------------------------------------!
C     !              STATE FEEDBACK U               !
C     !_____________________________________________!

*     MEASURMENT ERROR IN X1
      DELX1=SIN(T)*0.01
      X1M=Y(1)+DELX1

C     FUNCTION F1(X)
      F1=(-X1M+ALFAC)*RQ-(X1M-ALFAC+1.)*
     &        DEXP(D-(RMU**2/(Y(2)+RMU-ALFAT)))

C     FUNCTION F2(X)
      F2=(-Y(2)+ALFAT)*RQ-B*(X1M-ALFAC+1.)*
     &    DEXP(D-(RMU**2/(Y(2)+RMU-ALFAT)))-
     &    GAMA*(ALFAW-ALFAT+Y(2))

C     PARTIAL DERIVATIVE OF F1(X) W.R.T X1
      F1X1=-RQ-DEXP(D-(RMU**2/(Y(2)-ALFAT+RMU)))

C     PARTIAL DERIVATIVE OF F1(X) W. R. T X2
      F1X2=(-(RMU**2)*(X1M-ALFAC+1.)/((Y(2)-ALFAT+RMU)**2))
     &    *DEXP(D-(RMU**2/(Y(2)-ALFAT+RMU)))

      U=(1./(F1X2*GAMA))*(B1*X1M+B2*F1-(F1X1*F1+F1X2*F2))

C     EQUATION2

      YDOT(2)=(-Y(2)+ALFAT)*RQ-B*(Y(1)-ALFAC+1.)*
     &    DEXP(D-(RMU**2/(Y(2)+RMU-ALFAT)))-
     &    GAMA*(ALFAW-ALFAT+Y(2))+GAMA*U+RQ*DT

      RETURN
      END
```

```
*    PROGRAM : LIP2V2
*
*    FIND MW AND DELTAW DEFINED IN CHAPTER 4 (P117)
*
*    EXOTHERMIC REACTION IN CSTR
*    FOR THE REACTION A===B----C
*    CONTROLLED VARIABLE = CONVERSION OF A
*    MANIPULATE VARIABLE = TEMPERATURE OF THE JACKET
*
*    DEFINITION OF VARIABLES
*    RMW = MW
*    RAD = RADIUS OF BALL BR
*    D1MIN=POSSIBLE MINIMUM VALUE OF D1
*    D1MAX=POSSIBLE MAXIMUM VALUE OF D1

      IMPLICIT REAL *8(A-H,O-Z)
      DIMENSION X(2)
*
      COMMON /PHYSC1/ ALFAA,ALFAB,ALFAT,D1M,ALFAW,GAMA
      COMMON /PHYSC2/ D1,D2,D3,RMU1,RMU2,RMU3,B
      COMMON /RADIUS/ RAD
*
*   PHYSICAL PARAMETRS OF CSTR
*    GROUP PHYSC1
      ALFAA=0.5042
      ALFAB=0.9975
      ALFAT=-1.2006
      D1M=31.7990           ] NOMINAL VALUE OF D1
      ALFAW=4.5805
      GAMA=0.4593

*   GROUP PHYSC2
      D2=41.5202
      D3=37.0973
      RMU1=32.9404
      RMU2=47.5160
      RMU3=32.9404
      B=-7.6476
*
*   RADIUS OF BALL = RAD
      RAD=0.2
*
*   FIRST FIND THE CONSTANT, DELTAW
*
*   POSSIBLE RANGE OF D1
*
      D1MIN=31.789
      D1MAX=31.809
      D1INC=(D1MAX-D1MIN)/100.
*
      X(1)=0.
      X(2)=0.
      DELTAW=0.
      DO 15 D1=D1MIN,D1MAX,D1INC
      DELW=ABS((-ALFAA+1.)*ED1(X)
     &      +(ALFAB-1.)*(1.+ED2(X)+ED3(X)))
```

```
         IF(DELW.GE.DELTAW) THEN
          DELTAW=DELW
         ENDIF
   15  CONTINUE
*
       WRITE(6,16) RAD,DELTAW
   16  FORMAT(2X,'RAD=',1X,G12.5,2X,'DELTAW=',1X,G12.5)
*
*  NOW, FIND THE CONSTANT, RMW

*  POSSIBLE RANGE OF X AND D1
*
       X1MIN=-0.30     ] LOWER BOUND OF X1 WHEN RAD = 0.20
       X1MAX=0.30      ] UPPER BOUND OF X1 WHEN RAD = 0.20
       X2MIN=-1.3      ] LOWER BOUND OF X2 WHEN RAD = 0.20
       X2MAX=1.3       ] UPPER BOUND OF X2 WHEN RAD = 0.20
*
       D1MIN=31.789
       D1MAX=31.809
*
       X1INC=(X1MAX-X1MIN)/90.
       X2INC=(X2MAX-X2MIN)/90.
       D1INC=(D1MAX-D1MIN)/20.
*
       RMW=0.0
*
       DO 10 X1=X1MIN,X1MAX,X1INC
       DO 10 X2=X2MIN,X2MAX,X2INC
       DO 10 D1=D1MIN,D1MAX,D1INC

       X(1)=X1
       X(2)=X2
*
       CALL CONST(X,CON)
         IF(CON.GE.0.0) THEN           ] IF---A
         RNU=ABS((X1-ALFAA+1.)*ED1(X)
     &           +(ALFAB-1.)*(1.+ED2(X)+ED3(X)))
         F1M=(-X1+ALFAA)-(X1-ALFAA+1.)*ED1M(X)
     &                 +(1.-ALFAB)*ED2(X)
         RDE=DSQRT(X1**2+F1M**2)
         S=(RNU-DELTAW)/RDE
*
           IF(S.GE.RMW) THEN           ]IF---B
           RMW=S

           ENDIF                          ]ENDIF---B
         ENDIF                         ]ENDIF---A
   10  CONTINUE

       WRITE(6,36) RAD, RMW
   36  FORMAT(2X,'RAD=',2X,G12.5,2X,'RMW=',G12.5)

       STOP
       END
       SUBROUTINE CONST(X,CON)
*---------------------------
*
```

```
*  CHECK X1 AND X2 ARE IN THE BALL BR
      IMPLICIT REAL *8(A-H,O-Z)
      DIMENSION X(2)
      COMMON /PHYSC1/ ALFAA,ALFAB,ALFAT,D1M,ALFAW,GAMA
      COMMON /PHYSC2/ D1,D2,D3,RMU1,RMU2,RMU3,B
      COMMON /RADIUS/ RAD

      F1XM=(-X(1)+ALFAA)-(X(1)-ALFAA+1.)*ED1M(X)
     &     +(1.-ALFAB)*ED2(X)
      CON=RAD**2-X(1)**2-F1XM**2

      RETURN
      END
      FUNCTION ED1M(X)
*----------------------
*
      IMPLICIT REAL *8(A-H,O-Z)
      DIMENSION X(2)
      COMMON /PHYSC1/ ALFAA,ALFAB,ALFAT,D1M,ALFAW,GAMA
      COMMON /PHYSC2/ D1,D2,D3,RMU1,RMU2,RMU3,B
      COMMON /RADIUS/ RAD
*
      ED1M=DEXP(D1M-RMU1*RMU1/(X(2)-ALFAT+RMU1))
      RETURN
      END

      FUNCTION ED1(X)
*---------------------
*
      IMPLICIT REAL *8(A-H,O-Z)
      DIMENSION X(2)
      COMMON /PHYSC1/ ALFAA,ALFAB,ALFAT,D1M,ALFAW,GAMA
      COMMON /PHYSC2/ D1,D2,D3,RMU1,RMU2,RMU3,B
      COMMON /RADIUS/ RAD
*
      ED1=DEXP(D1-RMU1*RMU1/(X(2)-ALFAT+RMU1))
      RETURN
      END

      FUNCTION ED2(X)
*--------------------
*
      IMPLICIT REAL *8(A-H,O-Z)
      DIMENSION X(2)
      COMMON /PHYSC1/ ALFAA,ALFAB,ALFAT,D1M,ALFAW,GAMA
      COMMON /PHYSC2/ D1,D2,D3,RMU1,RMU2,RMU3,B
      COMMON /RADIUS/ RAD
*
      ED2=DEXP(D2-RMU1*RMU2/(X(2)-ALFAT+RMU1))
      RETURN
      END

      FUNCTION ED3(X)
*-----------------------
*
      IMPLICIT REAL *8(A-H,O-Z)
      DIMENSION X(2)
```

```
      COMMON /PHYSC1/ ALFAA,ALFAB,ALFAT,D1M,ALFAW,GAMA
      COMMON /PHYSC2/ D1,D2,D3,RMU1,RMU2,RMU3,B
      COMMON /RADIUS/ RAD
*
      ED3=DEXP(D3-RMU1*RMU3/(X(2)-ALFAT+RMU1))
      RETURN
      END
```

Vita

Weon Ho Kim was born in Seoul, Korea on July 28, 1959; the eldest son of Chung Rye Kang and Kwang Un Kim. He received a B.S. degree in chemical engineering from Hanyang University in Seoul, Korea in 1981 and a M.S. degree in Chemical Engineering from Seoul National University in 1983. After serving in Korean Army, he worked for Chon Engineering Company and for Korea Advanced Energy Research Institute, Korea. In November, 1984 he married Ae Ja Woo. In August, 1986 he entered Louisiana State University as a candidate for the degree of Doctor of Philosophy.

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Title of Dissertation: Feedback Linearization of Nonlinear Systems: Robustness and Adaptive Control

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Date of Examination:

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