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Math Review

for the Quantitative Reasoning measure of the *GRE*® General Test

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Overview

This Math Review will familiarize you with the mathematical skills and concepts that are important for solving problems and reasoning quantitatively on the Quantitative Reasoning measure of the *GRE*® General Test. The skills and concepts are in the areas of Arithmetic, Algebra, Geometry, and Data Analysis. The material covered includes many definitions, properties, and examples, as well as a set of exercises (with answers) at the end of each part. Note, however, that this review is not intended to be all-inclusive—the test may include some concepts that are not explicitly presented in this review.

If any material in this review seems especially unfamiliar or is covered too briefly, you may also wish to consult appropriate mathematics texts for more information. Another resource is the Khan Academy® page on the GRE website at www.ets.org/gre/khan, where you will find links to free instructional videos about concepts in this review.

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PART 1.

ARITHMETIC

The review of arithmetic begins with integers, fractions, and decimals and progresses to the set of real numbers. The basic arithmetic operations of addition, subtraction, multiplication, and division are discussed, along with exponents and roots. The review of arithmetic ends with the concepts of ratio and percent.

1.1 Integers

The **integers** are the numbers $1,\ 2,\ 3, \ldots,$ together with their negatives, $-1,\ -2,\ -3, \ldots,$ and 0. Thus, the set of integers is $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

The positive integers are greater than 0, the negative integers are less than 0, and 0 is neither positive nor negative. When integers are added, subtracted, or multiplied, the result is always an integer; division of integers is addressed below. The many elementary number facts for these operations, such as $7 + 8 = 15,\ 78 - 87 = -9,\ 7 - (-18) = 25,$ and $(7)(8) = 56,$ should be familiar to you; they are not reviewed here. Here are three general facts regarding multiplication of integers.

Fact 1: The product of two positive integers is a positive integer.

Fact 2: The product of two negative integers is a positive integer.

Fact 3: The product of a positive integer and a negative integer is a negative integer.

When integers are multiplied, each of the multiplied integers is called a **factor** or **divisor** of the resulting product. For example, $(2)(3)(10) = 60,$ so 2, 3, and 10 are factors of 60. The integers 4, 15, 5, and 12 are also factors of 60, since $(4)(15) = 60$ and $(5)(12) = 60.$ The positive factors of 60 are 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, and 60. The negatives of these integers are also factors of 60, since, for example, $(-2)(-30) = 60.$ There are no other factors of 60. We say that 60 is a **multiple** of each of its factors and that 60 is **divisible** by each of its divisors. Here are five more examples of factors and multiples.

Example 1.1.1: The positive factors of 100 are 1, 2, 4, 5, 10, 20, 25, 50, and 100.

Example 1.1.2: 25 is a multiple of only six integers: 1, 5, 25, and their negatives.

Example 1.1.3: The list of positive multiples of 25 has no end: $25,\ 50,\ 75,\ 100, \ldots$; likewise, every nonzero integer has infinitely many multiples.

Example 1.1.4: 1 is a factor of every integer; 1 is not a multiple of any integer except 1 and -1.

Example 1.1.5: 0 is a multiple of every integer; 0 is not a factor of any integer except 0.

The **least common multiple** of two nonzero integers *c* and *d* is the least positive integer that is a multiple of both *c* and *d*. For example, the least common multiple of 30 and 75 is 150. This is because the positive multiples of 30 are $30,\ 60,\ 90,\ 120,\ 150,\ 180,\ 210,\ 240,\ 270,\ 300,\ 330,\ 360,\ 390,\ 420,\ 450, \ldots,$ and the positive multiples of 75 are $75,\ 150,\ 225,\ 300,\ 375,\ 450, \ldots.$ Thus, the *common* positive multiples of 30 and 75 are $150, 300,\ 450, \ldots,$ and the least of these is 150.

The **greatest common divisor** (or **greatest common factor**) of two nonzero integers *c* and *d* is the greatest positive integer that is a divisor of both *c* and *d*. For example, the greatest common divisor of 30 and 75 is 15. This is because the positive divisors of 30 are 1, 2, 3, 5, 6, 10, 15, and 30, and the positive divisors of 75 are 1, 3, 5, 15, 25, and 75. Thus, the *common* positive divisors of 30 and 75 are 1, 3, 5, and 15, and the greatest of these is 15.

When an integer *c* is divided by an integer *d*, where *d* is a divisor of *c*, the result is always a divisor of *c*. For example, when 60 is divided by 6 (one of its divisors), the result is 10, which is another divisor of 60. If *d* is *not* a divisor of *c*, then the result can be viewed in three different ways. The result can be viewed as a fraction or as a decimal, both of which are discussed later, or the result can be viewed as a **quotient** with a **remainder**, where both are integers. Each view is useful, depending on the context. Fractions and decimals are useful when the result must be viewed as a single number, while quotients with remainders are useful for describing the result in terms of integers only.

Regarding quotients with remainders, consider the integer *c* and the positive integer *d*, where *d* is *not* a divisor of *c*; for example, the integers 19 and 7. When 19 is divided by 7, the result is greater than 2, since $(2)(7) < 19,$ but less than 3, since $19 < (3)(7).$

Because 19 is 5 more than $(2)(7)$, we say that the result of 19 divided by 7 is the quotient 2 with remainder 5, or simply 2 remainder 5. In general, when an integer *c* is divided by a positive integer *d*, you first find the greatest multiple of *d* that is less than or equal to *c*. That multiple of *d* can be expressed as the product *qd*, where *q* is the quotient. Then the remainder is equal to *c* minus that multiple of *d*, or $r = c - qd$, where *r* is the remainder. The remainder is always greater than or equal to 0 and less than *d*.

Here are four examples that illustrate a few different cases of division resulting in a quotient and remainder.

Example 1.1.6: 100 divided by 45 is 2 remainder 10, since the greatest multiple of 45 that is less than or equal to 100 is $(2)(45)$, or 90, which is 10 less than 100.

Example 1.1.7: 24 divided by 4 is 6 remainder 0, since the greatest multiple of 4 that is less than or equal to 24 is 24 itself, which is 0 less than 24. In general, the remainder is 0 if and only if *c* is divisible by *d*.

Example 1.1.8: 6 divided by 24 is 0 remainder 6, since the greatest multiple of 24 that is less than or equal to 6 is $(0)(24)$, or 0, which is 6 less than 6.

Example 1.1.9: -32 divided by 3 is -11 remainder 1, since the greatest multiple of 3 that is less than or equal to -32 is $(-11)(3)$, or -33, which is 1 less than -32.

Here are five more examples.

Example 1.1.10: 100 divided by 3 is 33 remainder 1, since $100 = (33)(3) + 1$.

Example 1.1.11: 100 divided by 25 is 4 remainder 0, since $100 = (4)(25) + 0$.

Example 1.1.12: 80 divided by 100 is 0 remainder 80, since $80 = (0)(100) + 80$.

Example 1.1.13: -13 divided by 5 is -3 remainder 2, since $-13 = (-3)(5) + 2$.

Example 1.1.14: -73 divided by 10 is -8 remainder 7, since $-73 = (-8)(10) + 7$.

If an integer is divisible by 2, it is called an **even integer**; otherwise, it is an **odd integer**. Note that when an odd integer is divided by 2, the remainder is always 1. The set of even integers is $\{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$, and the set of odd integers is $\{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$. Here are six useful facts regarding the sum and product of even and odd integers.

Fact 1: The sum of two even integers is an even integer.

Fact 2: The sum of two odd integers is an even integer.

Fact 3: The sum of an even integer and an odd integer is an odd integer.

Fact 4: The product of two even integers is an even integer.

Fact 5: The product of two odd integers is an odd integer.

Fact 6: The product of an even integer and an odd integer is an even integer.

A **prime number** is an integer greater than 1 that has only two positive divisors: 1 and itself. The first ten prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, and 29. The integer 14 is not a prime number, since it has four positive divisors: 1, 2, 7, and 14. The integer 1 is not a prime number, and the integer 2 is the only prime number that is even.

Every integer greater than 1 either is a prime number or can be uniquely expressed as a product of factors that are prime numbers, or **prime divisors**. Such an expression is called a **prime factorization**. Here are six examples of prime factorizations.

Example 1.1.15: $12 = (2)(2)(3) = \left(2^2\right)(3)$

Example 1.1.16: $14 = (2)(7)$

Example 1.1.17: $81 = (3)(3)(3)(3) = 3^4$

Example 1.1.18: $338 = (2)(13)(13) = (2)\left(13^2\right)$

Example 1.1.19: $800 = (2)(2)(2)(2)(2)(5)(5) = \left(2^5\right)\left(5^2\right)$

Example 1.1.20: $1{,}155 = (3)(5)(7)(11)$

An integer greater than 1 that is not a prime number is called a **composite number**. The first ten composite numbers are 4, 6, 8, 9, 10, 12, 14, 15, 16, and 18.

1.2 Fractions

A **fraction** is a number of the form $\frac{c}{d}$, where c and d are integers and $d \neq 0$. The integer c is called the **numerator** of the fraction, and d is called the **denominator**. For example, $\frac{-7}{5}$ is a fraction in which -7 is the numerator and 5 is the denominator. Such numbers are also called **rational numbers**. Note that every integer n is a rational number, because n is equal to the fraction $\frac{n}{1}$.

If both the numerator c and the denominator d, where $d \neq 0$, are multiplied by the same nonzero integer, the resulting fraction will be equivalent to $\frac{c}{d}$.

Example 1.2.1: Multiplying the numerator and denominator of the fraction $\frac{-7}{5}$ by 4 gives

$$\frac{-7}{5} = \frac{(-7)(4)}{(5)(4)} = \frac{-28}{20}$$

Multiplying the numerator and denominator of the fraction $\frac{-7}{5}$ by -1 gives

$$\frac{-7}{5} = \frac{(-7)(-1)}{(5)(-1)} = \frac{7}{-5}$$

For all integers c and d, the fractions $\frac{-c}{d}$, $\frac{c}{-d}$, and $-\frac{c}{d}$ are equivalent.

Example 1.2.2: $\frac{-7}{5} = \frac{7}{-5} = -\frac{7}{5}$

If both the numerator and denominator of a fraction have a common factor, then the numerator and denominator can be factored and the fraction can be reduced to an equivalent fraction.

Example 1.2.3: $\frac{40}{72} = \frac{(8)(5)}{(8)(9)} = \frac{5}{9}$

Adding and Subtracting Fractions

To add two fractions with the same denominator, you add the numerators and keep the same denominator.

Example 1.2.4: $-\frac{8}{11} + \frac{5}{11} = \frac{-8+5}{11} = \frac{-3}{11} = -\frac{3}{11}$

To add two fractions with different denominators, first find a **common denominator**, which is a common multiple of the two denominators. Then convert both fractions to equivalent fractions with the same denominator. Finally, add the numerators and keep the common denominator.

Example 1.2.5: To add the two fractions $\frac{1}{3}$ and $-\frac{2}{5}$, first note that 15 is a common denominator of the fractions.

Then convert the fractions to equivalent fractions with denominator 15 as follows.

$$\frac{1}{3} = \frac{1(5)}{3(5)} = \frac{5}{15} \text{ and } -\frac{2}{5} = -\frac{2(3)}{5(3)} = -\frac{6}{15}$$

Therefore, the two fractions can be added as follows.

$$\frac{1}{3} + \frac{-2}{5} = \frac{5}{15} + \frac{-6}{15} = \frac{5 + (-6)}{15} = -\frac{1}{15}$$

The same method applies to subtraction of fractions.

Multiplying and Dividing Fractions

To multiply two fractions, multiply the two numerators and multiply the two denominators. Here are two examples.

Example 1.2.6: $\left(\frac{10}{7}\right)\left(\frac{-1}{3}\right) = \frac{(10)(-1)}{(7)(3)} = \frac{-10}{21} = -\frac{10}{21}$

Example 1.2.7: $\left(\frac{8}{3}\right)\left(\frac{7}{3}\right) = \frac{56}{9}$

To divide one fraction by another, first **invert** the second fraction (that is, find its **reciprocal**), then multiply the first fraction by the inverted fraction. Here are two examples.

Example 1.2.8: $\frac{17}{8} \div \frac{3}{5} = \left(\frac{17}{8}\right)\left(\frac{5}{3}\right) = \frac{85}{24}$

Example 1.2.9: $\frac{\frac{3}{10}}{\frac{7}{13}} = \left(\frac{3}{10}\right)\left(\frac{13}{7}\right) = \frac{39}{70}$

Mixed Numbers

An expression such as $4\frac{3}{8}$ is called a **mixed number**. It consists of an integer part and a fraction part, where the fraction part has a value between 0 and 1; the mixed number $4\frac{3}{8}$ means $4 + \frac{3}{8}$.

To convert a mixed number to a fraction, convert the integer part to an equivalent fraction with the same denominator as the fraction, and then add it to the fraction part.

Example 1.2.10: To convert the mixed number $4\frac{3}{8}$ to a fraction, first convert the integer 4 to a fraction with denominator 8, as follows.

$$4 = \frac{4}{1} = \frac{4(8)}{1(8)} = \frac{32}{8}$$

Then add $\frac{3}{8}$ to it to get

$$4\frac{3}{8} = \frac{32}{8} + \frac{3}{8} = \frac{35}{8}$$

Fractional Expressions

Numbers of the form $\frac{c}{d}$, where either c or d is not an integer and $d \neq 0$, are called fractional expressions. Fractional expressions can be manipulated just like fractions. Here are two examples.

Example 1.2.11: Add the numbers $\frac{\pi}{2}$ and $\frac{\pi}{3}$.

Solution: Note that 6 is a common denominator of both numbers.

The number $\frac{\pi}{2}$ is equivalent to the number $\frac{3\pi}{6}$, and the number $\frac{\pi}{3}$ is equivalent to the number $\frac{2\pi}{6}$.

Therefore

$$\frac{\pi}{2} + \frac{\pi}{3} = \frac{3\pi}{6} + \frac{2\pi}{6} = \frac{5\pi}{6}$$

Example 1.2.12: Simplify the number $\dfrac{\frac{1}{\sqrt{2}}}{\frac{3}{\sqrt{5}}}$.

Solution: Note that the numerator of the number is $\frac{1}{\sqrt{2}}$ and the denominator of the number is $\frac{3}{\sqrt{5}}$. Note also that the reciprocal of the denominator is $\frac{\sqrt{5}}{3}$.

Therefore,

$$\frac{\frac{1}{\sqrt{2}}}{\frac{3}{\sqrt{5}}} = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{5}}{3}\right)$$

which can be simplified to $\frac{\sqrt{5}}{3\sqrt{2}}$.

Thus, the number $\dfrac{\frac{1}{\sqrt{2}}}{\frac{3}{\sqrt{5}}}$ simplifies to the number $\frac{\sqrt{5}}{3\sqrt{2}}$.

1.3 Exponents and Roots

<u>Exponents</u>

Exponents are used to denote the repeated multiplication of a number by itself; for example, $3^4 = (3)(3)(3)(3) = 81$ and $5^3 = (5)(5)(5) = 125$. In the expression 3^4, 3 is called the **base**, 4 is called the **exponent**, and we read the expression as "3 to the fourth power." Similarly, 5 to the third power is 125.

When the exponent is 2, we call the process **squaring**. Thus, 6 squared is 36; that is, $6^2 = (6)(6) = 36$. Similarly, 7 squared is 49; that is, $7^2 = (7)(7) = 49$.

When negative numbers are raised to powers, the result may be positive or negative; for example, $(-3)^2 = (-3)(-3) = 9$ and $(-3)^5 = (-3)(-3)(-3)(-3)(-3) = -243$. A negative number raised to an even power is always positive, and a negative number raised to an odd power is always negative. Note that $(-3)^2 = (-3)(-3) = 9$, but $-3^2 = -((3)(3)) = -9$. Exponents can also be negative or zero; such exponents are defined as follows.

> The exponent zero: For all nonzero numbers a, $a^0 = 1$. The expression 0^0 is undefined.
>
> Negative exponents: For all nonzero numbers a, $a^{-1} = \frac{1}{a}$, $a^{-2} = \frac{1}{a^2}$, $a^{-3} = \frac{1}{a^3}$, and so on. Note that $(a)\left(a^{-1}\right) = (a)\left(\frac{1}{a}\right) = 1$.

Roots

A **square root** of a nonnegative number n is a number r such that $r^2 = n$. For example, 4 is a square root of 16 because $4^2 = 16$. Another square root of 16 is -4, since $(-4)^2 = 16$. All positive numbers have two square roots, one positive and one negative. The only square root of 0 is 0. The expression consisting of the square root symbol $\sqrt{\ }$ placed over a nonnegative number denotes the *nonnegative* square root (or the positive square root if the number is greater than 0) of that nonnegative number. Therefore, $\sqrt{100} = 10$, $-\sqrt{100} = -10$, and $\sqrt{0} = 0$. Square roots of negative numbers are not defined in the real number system.

Here are four important rules regarding operations with square roots, where $a > 0$ and $b > 0$.

Rule 1: $\left(\sqrt{a}\right)^2 = a$

Example A: $\left(\sqrt{3}\right)^2 = 3$

Example B: $\left(\sqrt{\pi}\right)^2 = \pi$

Rule 2: $\sqrt{a^2} = a$

Example A: $\sqrt{4} = \sqrt{2^2} = 2$

Example B: $\sqrt{\pi^2} = \pi$

Rule 3: $\sqrt{a}\sqrt{b} = \sqrt{ab}$

Example A: $\sqrt{3}\sqrt{10} = \sqrt{30}$

Example B: $\sqrt{24} = \sqrt{4}\sqrt{6} = 2\sqrt{6}$

Rule 4: $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$

Example A: $\dfrac{\sqrt{5}}{\sqrt{15}} = \sqrt{\dfrac{5}{15}} = \sqrt{\dfrac{1}{3}}$

Example B: $\dfrac{\sqrt{18}}{\sqrt{2}} = \sqrt{\dfrac{18}{2}} = \sqrt{9} = 3$

A square root is a root of order 2. Higher order roots of a positive number n are defined similarly. For orders 3 and 4, the **cube root** of n, written as $\sqrt[3]{n}$, and **fourth root** of n, written as $\sqrt[4]{n}$, represent numbers such that when they are raised to the powers 3 and 4, respectively, the result is n. These roots obey rules similar to those above but with the exponent 2 replaced by 3 or 4 in the first two rules.

There are some notable differences between odd order roots and even order roots (in the real number system):

> For odd order roots, there is *exactly one* root for *every* number n, even when n is negative.
>
> For even order roots, there are *exactly two* roots for every *positive* number n and *no* roots for any *negative* number n.

For example, 8 has exactly one cube root, $\sqrt[3]{8} = 2$, but 8 has two fourth roots, $\sqrt[4]{8}$ and $-\sqrt[4]{8}$, whereas -8 has exactly one cube root, $\sqrt[3]{-8} = -2$, but -8 has no fourth root, since it is negative.

1.4 Decimals

The decimal number system is based on representing numbers using powers of 10. The place value of each digit corresponds to a power of 10. For example, the digits of the number 7,532.418 have the following place values.

Thousands		Hundreds	Tens	Ones or Units		Tenths	Hundredths	Thousandths
7	,	5	3	2	.	4	1	8

Arithmetic Figure 1

That is, the number 7,532.418 can be written as

$$7(1{,}000) + 5(100) + 3(10) + 2(1) + 4\left(\frac{1}{10}\right) + 1\left(\frac{1}{100}\right) + 8\left(\frac{1}{1{,}000}\right)$$

Alternatively, it can be written as

$$7\left(10^3\right)+5\left(10^2\right)+3\left(10^1\right)+2\left(10^0\right)+\ 4\left(10^{-1}\right)+1\left(10^{-2}\right)+8\left(10^{-3}\right)$$

If there are a finite number of digits to the right of the decimal point, converting a decimal to an equivalent fraction with integers in the numerator and denominator is a straightforward process. Since each place value is a power of 10, every decimal can be converted to an integer divided by a power of 10. Here are three examples.

Example 1.4.1: $2.3 = 2 + \frac{3}{10} = \frac{20}{10} + \frac{3}{10} = \frac{23}{10}$

Example 1.4.2: $90.17 = 90 + \frac{17}{100} = \frac{9{,}000 + 17}{100} = \frac{9{,}017}{100}$

Example 1.4.3: $0.612 = \frac{612}{1{,}000}$

Conversely, every fraction with integers in the numerator and denominator can be converted to an equivalent decimal by dividing the numerator by the denominator using long division (which is not in this review). The decimal that results from the long division will either **terminate**, as in $\frac{1}{4} = 0.25$ and $\frac{52}{25} = 2.08,$ or **repeat** without end, as in $\frac{1}{9} = 0.111...,$ $\frac{1}{22} = 0.0454545...,$ and $\frac{25}{12} = 2.08333....$ One way to indicate the repeating part of a decimal that repeats without end is to use a bar over the digits that repeat. Here are four examples of fractions converted to decimals.

Example 1.4.4: $\frac{3}{8} = 0.375$

Example 1.4.5: $\frac{259}{40} = 6.475$

Example 1.4.6: $-\frac{1}{3} = -0.\overline{3}$

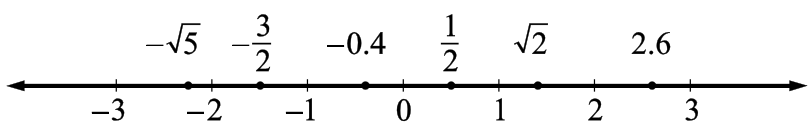
Example 1.4.7: $\frac{15}{14} = 1.0\overline{714285}$

Every fraction with integers in the numerator and denominator is equivalent to a decimal that either terminates or repeats. That is, every rational number can be expressed as a terminating or repeating decimal. The converse is also true; that is, every terminating or repeating decimal represents a rational number.

Not all decimals are terminating or repeating; for instance, the decimal that is equivalent to $\sqrt{2}$ is $1.41421356237...$, and it can be shown that this decimal does not terminate or repeat. Another example is $0.020220222022220222220...$, which has groups of consecutive 2s separated by a 0, where the number of 2s in each successive group increases by one. Since these two decimals do not terminate or repeat, they are not rational numbers. Such numbers are called **irrational numbers**.

1.5 Real Numbers

The set of **real numbers** consists of all rational numbers and all irrational numbers. The real numbers include all integers, fractions, and decimals. The set of real numbers can be represented by a number line called the **real number line**. Arithmetic Figure 2 below is a number line.



Arithmetic Figure 2

Every real number corresponds to a point on the number line, and every point on the number line corresponds to a real number. On the number line, all numbers to the left of 0 are negative and all numbers to the right of 0 are positive. As shown in Arithmetic Figure 2, the negative numbers -0.4, -1, $-\frac{3}{2}$, -2, $-\sqrt{5}$, and -3 are to the left of 0, and the positive numbers $\frac{1}{2}$, 1, $\sqrt{2}$, 2, 2.6, and 3 are to the right of 0. Only the number 0 is neither negative nor positive.

A real number x is **less than** a real number y if x is to the left of y on the number line, which is written as $x < y$. A real number y is **greater than** a real number x if y is to the

right of x on the number line, which is written as $y > x$. For example, the number line in Arithmetic Figure 2 shows the following three relationships.

Relationship 1: $-\sqrt{5} < -2$

Relationship 2: $\frac{1}{2} > 0$

Relationship 3: $1 < \sqrt{2} < 2$

A real number x is **less than or equal to** a real number y if x is to the left of, *or corresponds to the same point as,* y on the number line, which is written as $x \leq y$. A real number y is **greater than or equal to** a real number x if y is to the right of, *or corresponds to the same point as,* x on the number line, which is written as $y \geq x$.

To say that a real number x is between 2 and 3 on the number line means that $x > 2$ and $x < 3$, which can also be written as $2 < x < 3$. The set of all real numbers that are between 2 and 3 is called an **interval**, and $2 < x < 3$ is often used to represent that interval. Note that the endpoints of the interval, 2 and 3, are not included in the interval. Sometimes one or both of the endpoints are to be included in an interval. The following inequalities represent four types of intervals, depending on whether or not the endpoints are included.

Interval type 1: $2 < x < 3$

Interval type 2: $2 \leq x < 3$

Interval type 3: $2 < x \leq 3$

Interval type 4: $2 \leq x \leq 3$

There are also four types of intervals with only one endpoint, each of which consists of all real numbers to the right or to the left of the endpoint and include or do not include the endpoint. The following inequalities represent these types of intervals.

Interval type 1: $x < 4$

Interval type 2: $x \le 4$

Interval type 3: $x > 4$

Interval type 4: $x \ge 4$

The entire real number line is also considered to be an interval.

Absolute Value

The distance between a number x and 0 on the number line is called the **absolute value** of x, written as $|x|$. Therefore, $|3| = 3$ and $|-3| = 3$ because each of the numbers 3 and -3 is a distance of 3 from 0. Note that if x is positive, then $|x| = x$; if x is negative, then $|x| = -x$; and lastly, $|0| = 0$. It follows that the absolute value of any nonzero number is positive. Here are three examples.

Example 1.5.1: $|\sqrt{5}| = \sqrt{5}$

Example 1.5.2: $|-23| = -(-23) = 23$

Example 1.5.3: $|-10.2| = 10.2$

Properties of Real Numbers

Here are twelve general properties of real numbers that are used frequently. In each property, r, s, and t are real numbers.

Property 1: $r + s = s + r$ and $rs = sr$.

Example A: $8 + 2 = 2 + 8 = 10$

Example B: $(-3)(17) = (17)(-3) = -51$

Property 2: $(r + s) + t = r + (s + t)$ and $(rs)t = r(st)$.

Example A: $(7+3)+8 = 7+(3+8) = 18$

Example B: $\left(7\sqrt{2}\right)\sqrt{2} = 7\left(\sqrt{2}\sqrt{2}\right) = (7)(2) = 14$

Property 3: $r(s+t) = rs + rt$

Example: $5(3+16) = (5)(3) + (5)(16) = 95$

Property 4: $r + 0 = r,\ (r)(0) = 0,$ and $(r)(1) = r.$

Property 5: If $rs = 0,$ then either $r = 0$ or $s = 0$ or both.

Example: If $-2s = 0,$ then $s = 0.$

Property 6: Division by 0 is undefined.

Example A: $5 \div 0$ is undefined.

Example B: $\dfrac{-7}{0}$ is undefined.

Example C: $\dfrac{0}{0}$ is undefined.

Property 7: If both r and s are positive, then both $r + s$ and rs are positive.

Property 8: If both r and s are negative, then $r + s$ is negative and rs is positive.

Property 9: If r is positive and s is negative, then rs is negative.

Property 10: $|r + s| \le |r| + |s|.$ This is known as the **triangle inequality**.

Example: If $r = 5$ and $s = -2,$ then $|5 + (-2)| = |5 - 2| = |3| = 3$ and $|5| + |-2| = 5 + 2 = 7.$ Therefore, $|5 + (-2)| \le |5| + |-2|.$

Property 11: $|r||s| = |rs|$

Example: $|5||-2| = |(5)(-2)| = |-10| = 10$

Property 12: If $r > 1$, then $r^2 > r$. If $0 < s < 1$, then $s^2 < s$.

Example: $5^2 = 25 > 5$, but $\left(\frac{1}{5}\right)^2 = \frac{1}{25} < \frac{1}{5}$.

1.6 Ratio

The **ratio** of one quantity to another is a way to express their relative sizes, often in the form of a fraction, where the first quantity is the numerator and the second quantity is the denominator. Thus, if s and t are positive quantities, then the ratio of s to t can be written as the fraction $\frac{s}{t}$. The notation "s to t" and the notation "$s : t$" are also used to express this ratio. For example, if there are 2 apples and 3 oranges in a basket, we can say that the ratio of the number of apples to the number of oranges is $\frac{2}{3}$, or that it is 2 to 3, or that it is $2 : 3$. Like fractions, ratios can be reduced to lowest terms. For example, if there are 8 apples and 12 oranges in a basket, then the ratio of the number of apples to the number of oranges is still 2 to 3. Similarly, the ratio 9 to 12 is equivalent to the ratio 3 to 4.

If three or more positive quantities are being considered, say r, s, and t, then their relative sizes can also be expressed as a ratio with the notation "r to s to t." For example, if there are 5 apples, 30 pears, and 20 oranges in a basket, then the ratio of the number of apples to the number of pears to the number of oranges is 5 to 30 to 20. This ratio can be reduced to 1 to 6 to 4 by dividing each number by the greatest common divisor of 5, 30, and 20, which is 5.

A **proportion** is an equation relating two ratios; for example, $\frac{9}{12} = \frac{3}{4}$. To solve a problem involving ratios, you can often write a proportion and solve it by **cross multiplication**.

Example 1.6.1: To find a number x so that the ratio of x to 49 is the same as the ratio of 3 to 21, you can first write the following equation.

$$\frac{x}{49} = \frac{3}{21}$$

You can then cross multiply to get $21x = (3)(49)$, and finally you can solve for x to get $x = \frac{(3)(49)}{21} = 7$.

1.7 Percent

The term **percent** means *per hundred*, or *hundredths*. Percents are ratios that are often used to represent *parts of a whole*, where the whole is considered as having 100 parts. Percents can be converted to fraction or decimal equivalents. Here are three examples of percents.

Example 1.7.1: 1 percent means 1 part out of 100 parts. The fraction equivalent of 1 percent is $\frac{1}{100}$, and the decimal equivalent is 0.01.

Example 1.7.2: 32 percent means 32 parts out of 100 parts. The fraction equivalent of 32 percent is $\frac{32}{100}$, and the decimal equivalent is 0.32.

Example 1.7.3: 50 percent means 50 parts out of 100 parts. The fraction equivalent of 50 percent is $\frac{50}{100}$, and the decimal equivalent is 0.50.

Note that in the fraction equivalent, the *part* is the numerator of the fraction and the *whole* is the denominator. Percents are often written using the percent symbol, $\%$, instead of the word "percent." Here are five examples of percents written using the $\%$ symbol, along with their fraction and decimal equivalents.

Example 1.7.4: $100\% = \frac{100}{100} = 1$

Example 1.7.5: $12\% = \frac{12}{100} = 0.12$

Example 1.7.6: $8\% = \frac{8}{100} = 0.08$

Example 1.7.7: $10\% = \frac{10}{100} = 0.1$

Example 1.7.8: $0.3\% = \frac{0.3}{100} = 0.003$

Be careful not to confuse 0.01 with 0.01%. The percent symbol matters. For example, $0.01 = 1\%$ but $0.01\% = \frac{0.01}{100} = 0.0001$.

To compute a *percent*, given the *part* and the *whole*, first divide the part by the whole to get the decimal equivalent, then multiply the result by 100. The percent is that number followed by the word "percent" or the % symbol.

Example 1.7.9: If the whole is 20 and the part is 13, you can find the percent as follows.

$$\frac{part}{whole} = \frac{13}{20} = 0.65 = 65\%$$

Example 1.7.10: What percent of 150 is 12.9 ?

Solution: Here, the whole is 150 and the part is 12.9, so

$$\frac{part}{whole} = \frac{12.9}{150} = 0.086 = 8.6\%.$$

To find the *part* that is a certain *percent* of a *whole*, you can either multiply the *whole* by the decimal equivalent of the percent or set up a proportion to find the part.

Example 1.7.11: To find 30% of 350, you can multiply 350 by the decimal equivalent of 30%, or 0.3, as follows.

$$(350)(0.3) = 105$$

Alternatively, to use a proportion to find 30% of 350, you need to find the number of parts of 350 that yields the same ratio as 30 parts out of 100 parts. You want a number x that satisfies the proportion

$$\frac{part}{whole} = \frac{30}{100} \text{ or}$$

$$\frac{x}{350} = \frac{30}{100}$$

Solving for x yields $x = \frac{(30)(350)}{100} = 105,$ so 30% of 350 is 105.

Given the *percent* and the *part*, you can calculate the *whole*. To do this, either you can use the decimal equivalent of the percent or you can set up a proportion and solve it.

Example 1.7.12: 15 is 60% of what number?

Solution: Use the decimal equivalent of 60%. Because 60% of some number z is 15, multiply z by the decimal equivalent of 60%, or 0.6.

$$0.6z = 15$$

Now solve for z by dividing both sides of the equation by 0.6 as follows.

$$z = \frac{15}{0.6} = 25$$

Using a proportion, look for a number z such that

$$\frac{part}{whole} = \frac{60}{100} \text{ or}$$

$$\frac{15}{z} = \frac{60}{100}$$

Hence, $60z = (15)(100)$, and therefore, $z = \frac{(15)(100)}{60} = \frac{1{,}500}{60} = 25$. That is, 15 is 60% of 25.

Percents Greater than 100%

Although the discussion about percent so far assumes a context of a *part* and a *whole*, it is not necessary that the part be less than the whole. In general, the whole is called the **base** of the percent. When the numerator of a percent is greater than the base, the percent is greater than 100%.

Example 1.7.13: 15 is 300% of 5, since

$$\frac{15}{5} = \frac{300}{100}$$

Example 1.7.14: 250% of 16 is 40, since

$$\left(\frac{250}{100}\right)(16) = (2.5)(16) = 40$$

Note that the decimal equivalent of 300% is 3.0 and the decimal equivalent of 250% is 2.5.

Percent Increase, Percent Decrease, and Percent Change

When a quantity changes from an initial positive amount to another positive amount (for example, an employee's salary that is raised), you can compute the amount of change as a percent of the initial amount. This is called **percent change**. If a quantity increases from 600 to 750, then the base of the increase is the initial amount, 600, and the amount of the

increase is $750 - 600$, or 150. The **percent increase** is found by dividing the amount of increase by the base, as follows.

$$\frac{amount\ of\ increase}{base} = \frac{750 - 600}{600} = \frac{150}{600} = \frac{25}{100} = 25\%$$

We say the percent increase is 25%. Sometimes this computation is written as follows.

$$\left(\frac{750 - 600}{600}\right)(100\%) = \left(\frac{150}{600}\right)(100\%) = 25\%$$

If a quantity doubles in size, then the percent increase is 100%. For example, if a quantity increases from 150 to 300, then the percent increase is calculated as follows.

$$\frac{amount\ of\ increase}{base} = \frac{300 - 150}{150} = \frac{150}{150} = 100\%$$

If a quantity decreases from 500 to 400, calculate the **percent decrease** as follows.

$$\frac{amount\ of\ decrease}{base} = \frac{500 - 400}{500} = \frac{100}{500} = \frac{20}{100} = 20\%$$

The quantity decreased by 20%.

When computing a percent *increase*, the base is the *smaller* number. When computing a percent *decrease*, the base is the *larger* number. In either case, the base is the initial number, before the change.

Example 1.7.15: An investment in a mutual fund increased by 12% in a single day. If the value of the investment before the increase was \$1,300, what was the value after the increase?

Solution: The percent increase is 12%. Therefore, the value of the increase is 12% of \$1,300, or, using the decimal equivalent, the increase is $(0.12)(\$1{,}300) = \$156$. Thus, the value of the investment after the change is

$$\$1{,}300 + \$156 = \$1{,}456$$

Because the final result is the sum of the initial investment (100% of \$1,300) and the increase (12% of \$1,300), the final result is $100\% + 12\% = 112\%$ of \$1,300. Thus, another way to get the final result is to multiply the value of the investment by the decimal equivalent of 112%, which is 1.12:

$$(\$1{,}300)(1.12) = \$1{,}456$$

A quantity may have several successive percent changes, where the base of each successive change is the result of the preceding percent change, as is the case in the following example.

Example 1.7.16: On September 1, 2013, the number of children enrolled in a certain preschool was 8% less than the number of children enrolled at the preschool on September 1, 2012. On September 1, 2014, the number of children enrolled in the preschool was 6% greater than the number of children enrolled in the preschool on September 1, 2013. By what percent did the number of students enrolled change from September 1, 2012 to September 1, 2014?

Solution: The *initial* base is the enrollment on September 1, 2012. The first percent change was the 8% decrease in the enrollment from September 1, 2012, to September 1, 2013. As a result of this decrease, the enrollment on September 1, 2013, was $(100 - 8)\%$, or 92%, of the enrollment on September 1, 2012. The decimal equivalent of 92% is 0.92.

So, if n represents the number of children enrolled on September 1, 2012, then the number of children enrolled on September 1, 2013, is equal to $0.92n$.

The *new* base is the enrollment on September 1, 2013, which is $0.92n$. The second percent change was the 6% increase in enrollment from September 1, 2013, to September 1, 2014. As a result of this increase, the enrollment on September 1, 2014, was $(100 + 6)\%$, or 106%, of the enrollment on September 1, 2013. The decimal equivalent of 106% is 1.06.

Thus, the number of children enrolled on September 1, 2014, was $(1.06)(0.92n)$, which is equal to $0.9752n$.

The percent equivalent of 0.9752 is 97.52%, which is 2.48% less than 100%. Thus, the percent change in the enrollment from September 1, 2012 to September 1, 2014, is a 2.48% decrease.

ARITHMETIC EXERCISES

Exercise 1. Evaluate the following.

(a) $15 - (6 - 4)(-2)$

(b) $(2 - 17) \div 5$

(c) $(60 \div 12) - (-7 + 4)$

(d) $3^4 - (-2)^3$

(e) $(-5)(-3) - 15$

(f) $(-2)^4 (15 - 18)^4$

(g) $(20 \div 5)^2 (-2 + 6)^3$

(h) $(-85)(0) - (-17)(3)$

Exercise 2. Evaluate the following.

(a) $\frac{1}{2} - \frac{1}{3} + \frac{1}{12}$

(b) $\left(\frac{3}{4} + \frac{1}{7}\right)\left(\frac{-2}{5}\right)$

(c) $\left(\frac{7}{8} - \frac{4}{5}\right)^2$

(d) $\left(\frac{3}{-8}\right) \div \left(\frac{27}{32}\right)$

Exercise 3. Which of the integers 312, 98, 112, and 144 are divisible by 8 ?

Exercise 4.

(a) What is the prime factorization of 372 ?

(b) What are the positive divisors of 372 ?

Exercise 5.

(a) What are the prime divisors of 100 ?

(b) What are the prime divisors of 144 ?

Exercise 6. Which of the integers 2, 9, 19, 29, 30, 37, 45, 49, 51, 83, 90, and 91 are prime numbers?

Exercise 7. What is the prime factorization of 585 ?

Exercise 8. Which of the following statements are true?

(a) $-5 < 3.1$

(b) $\sqrt{16} = 4$

(c) $7 \div 0 = 0$

(d) $0 < \left|-\frac{1}{7}\right|$

(e) $0.3 < \frac{1}{3}$

(f) $(-1)^{87} = -1$

(g) $\sqrt{(-3)^2} < 0$

(h) $\frac{21}{28} = \frac{3}{4}$

(i) $-|-23| = 23$

(j) $\frac{1}{2} > \frac{1}{17}$

(k) $\left(59^3\right)\left(59^2\right) = 59^6$

(l) $-\sqrt{25} < -4$

Exercise 9. Find the following.

(a) 40% of 15

(b) 150% of 48

(c) 0.6% of 800

(d) 15 is 30% of which number?

(e) 11 is what percent of 55 ?

Exercise 10. If a person's salary increases from \$200 per week to \$234 per week, what is the percent increase in the person's salary?

Exercise 11. If an athlete's weight decreases from 160 pounds to 152 pounds, what is the percent decrease in the athlete's weight?

Exercise 12. A particular stock is valued at \$40 per share. If the value increases by 20 percent and then decreases by 25 percent, what will be the value of the stock per share after the decrease?

Exercise 13. There are a total of 20 dogs and cats at a kennel. If the ratio of the number of dogs to the number of cats at the kennel is 3 to 2, how many cats are at the kennel?

Exercise 14. The integer c is even, and the integer d is odd. For each of the following integers, indicate whether the integer is even or odd.

(a) $c + 2d$

(b) $2c + d$

(c) cd

(d) c^d

(e) $(c + d)^2$

(f) $c^2 - d^2$

Exercise 15. When the positive integer n is divided by 3, the remainder is 2, and when n is divided by 5, the remainder is 1. What is the least possible value of n ?

ANSWERS TO ARITHMETIC EXERCISES

Exercise 1.

(a) 19

(b) -3

(c) 8

(d) 89

(e) 0

(f) 1,296

(g) 1,024

(h) 51

Exercise 2.

(a) $\frac{1}{4}$

(b) $-\frac{5}{14}$

(c) $\frac{9}{1,600}$

(d) $-\frac{4}{9}$

Exercise 3. 312, 112, and 144

Exercise 4.

(a) $372 = \left(2^2\right)(3)(31)$

(b) The positive divisors of 372 are 1, 2, 3, 4, 6, 12, 31, 62, 93, 124, 186, and 372.

Exercise 5.

(a) $100 = \left(2^2\right)\left(5^2\right)$, so the prime divisors are 2 and 5.

(b) $144 = \left(2^4\right)\left(3^2\right)$, so the prime divisors are 2 and 3.

Exercise 6. 2, 19, 29, 37, and 83

Exercise 7. $585 = \left(3^2\right)(5)(13)$

Exercise 8.

(a) True

(b) True

(c) False; division by 0 is undefined.

(d) True

(e) True

(f) True

(g) False; $\sqrt{(-3)^2} = \sqrt{9} = 3 > 0$

(h) True

(i) False; $-|-23| = -23$

(j) True

(k) False; $\left(59^3\right)\left(59^2\right) = 59^{3+2} = 59^5$

(l) True

Exercise 9.

(a) 6

(b) 72

(c) 4.8

(d) 50

(e) 20%

Exercise 10. 17%

Exercise 11. 5%

Exercise 12. $36 per share

Exercise 13. 8 cats

Exercise 14.

(a) $c + 2d$ is even.

(b) $2c + d$ is odd.

(c) cd is even.

(d) c^d is even.

(e) $(c + d)^2$ is odd.

(f) $c^2 - d^2$ is odd.

Exercise 15. 11

PART 2.

ALGEBRA

The review of algebra begins with algebraic expressions, equations, inequalities, and functions and then progresses to several examples of applying them to solve real-life word problems. The review of algebra ends with coordinate geometry and graphs of functions as other important algebraic tools for solving problems.

2.1 Algebraic Expressions

A **variable** is a letter that represents a quantity whose value is unknown. The letters x and y are often used as variables, although any symbol can be used. An **algebraic expression** has one or more variables and can be written as a single **term** or as a sum of terms. Here are four examples of algebraic expressions.

Example 2.1.1: $2x$

Example 2.1.2: $y - \frac{1}{4}$

Example 2.1.3: $w^3z + 5z^2 - z^2 + 6$

Example 2.1.4: $\frac{8}{n + p}$

In the examples above, $2x$ is a single term, $y - \frac{1}{4}$ has two terms, $w^3z + 5z^2 - z^2 + 6$ has four terms, and $\frac{8}{n + p}$ has one term.

In the expression $w^3z + 5z^2 - z^2 + 6$, the terms $5z^2$ and $-z^2$ are called **like terms** because they have the same variables, and the corresponding variables have the same exponents. A term that has no variable is called a **constant** term. A number that is multiplied by variables is called the **coefficient** of a term.

A **polynomial** is the sum of a finite number of terms in which each term is either a constant term or a product of a coefficient and one or more variables with positive integer exponents. The **degree** of each term is the sum of the exponents of the variables in the term. A variable that is written without an exponent has degree 1. The degree of a constant term is 0. The **degree of a polynomial** is the greatest degree of its terms.

Polynomials of degrees 2 and 3 are known as quadratic and cubic polynomials, respectively.

Example 2.1.5: The expression $4x^6 + 7x^5 - 3x + 2$ is a polynomial in one variable, x. The polynomial has four terms.

The first term is $4x^6$. The coefficient of this term is 4, and its degree is 6.

The second term is $7x^5$. The coefficient of this term is 7, and its degree is 5.

The third term is $-3x$. The coefficient of this term is -3, and its degree is 1.

The fourth term is 2. This term is a constant, and its degree is 0.

Example 2.1.6: The expression $2x^2 - 7xy^3 - 5$ is a polynomial in two variables, x and y. The polynomial has three terms.

The first term is $2x^2$. The coefficient of this term is 2, and its degree is 2.

The second term is $-7xy^3$. The coefficient of this term is -7; and, since the degree of x is 1 and the degree of y^3 is 3, the degree of the term $-7xy^3$ is 4.

The third term is -5, which is a constant term. The degree of this term is 0.

In this example, the degrees of the three terms are 2, 4, and 0, so the degree of the polynomial is 4.

Example 2.1.7: The expression $4x^3 - 12x^2 - x + 36$ is a cubic polynomial in one variable.

Operations with Algebraic Expressions

The same rules that govern operations with numbers apply to operations with algebraic expressions.

In an algebraic expression, like terms can be combined by simply adding their coefficients, as the following three examples show.

Example 2.1.8: $2x + 5x = 7x$

Example 2.1.9: $w^3z + 5z^2 - z^2 + 6 = w^3z + 4z^2 + 6$

Example 2.1.10: $3xy + 2x - xy - 3x = 2xy - x$

A number or variable that is a factor of each term in an algebraic expression can be factored out, as the following three examples show.

Example 2.1.11: $4x + 12 = 4(x + 3)$

Example 2.1.12: $15y^2 - 9y = 3y(5y - 3)$

Example 2.1.13: For values of x where it is defined, the algebraic expression $\frac{7x^2 + 14x}{2x + 4}$ can be simplified as follows.

First factor the numerator and the denominator to get $\frac{7x(x + 2)}{2(x + 2)}$.

Since $x + 2$ occurs as a factor in both the numerator and the denominator of the expression, canceling it out will give an equivalent fraction for all values of x for

which the expression is defined. Thus, for all values of x for which the expression is defined, the expression is equivalent to $\frac{7x}{2}$.

(A fraction is not defined when the denominator is equal to 0. The denominator of the original expression was $2(x+2)$, which is equal to 0 when $x = -2$, so the original expression is defined for all $x \neq -2$.)

To multiply two algebraic expressions, each term of the first expression is multiplied by each term of the second expression and the results are added, as the following example shows.

Example 2.1.14: Multiply $(x+2)(3x-7)$ as follows.

First multiply each term of the expression $x+2$ by each term of the expression $3x-7$ to get the expression $x(3x) + x(-7) + 2(3x) + 2(-7)$.

Then simplify each term to get $3x^2 - 7x + 6x - 14$.

Finally, combine like terms to get $3x^2 - x - 14$.

So you can conclude that $(x+2)(3x-7) = 3x^2 - x - 14$.

A statement of equality between two algebraic expressions that is true for all possible values of the variables involved is called an **identity**. Here are seven examples of identities.

Identity 1: $ca + cb = c(a+b)$

Identity 2: $ca - cb = c(a-b)$

Identity 3: $(a+b)^2 = a^2 + 2ab + b^2$

Identity 4: $(a-b)^2 = a^2 - 2ab + b^2$

Identity 5: $a^2 - b^2 = (a + b)(a - b)$

Identity 6: $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Identity 7: $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

Identities can be used to modify and simplify algebraic expressions, as in the following example.

Example 2.1.15: Simplify the algebraic expression $\frac{x^2 - 9}{4x - 12}$.

Solution: Use the identity $a^2 - b^2 = (a + b)(a - b)$ to factor the numerator of the expression, and use the identity $ca - cb = c(a - b)$ to factor the denominator of the expression to get

$$\frac{x^2 - 9}{4x - 12} = \frac{(x + 3)(x - 3)}{4(x - 3)}$$

Now, since $x - 3$ occurs as a factor in both the numerator and the denominator, it can be canceled out when $x - 3 \neq 0$, that is, when $x \neq 3$ (since the fraction is not defined when the denominator is 0). Therefore, for all $x \neq 3$, the expression is equivalent to $\frac{x + 3}{4}$.

2.2 Rules of Exponents

In the algebraic expression x^a, where x is raised to the power a, x is called the **base** and a is called the **exponent**. For all integers a and b and all positive numbers x, except $x = 1$, the following property holds: If $x^a = x^b$, then $a = b$.

Example: If $2^{3c+1} = 2^{10}$, then $3c + 1 = 10$, and therefore, $c = 3$.

Here are seven basic rules of exponents. In each rule, the bases x and y are nonzero real numbers and the exponents a and b are integers, unless stated otherwise.

Rule 1: $x^{-a} = \dfrac{1}{x^a}$

Example A: $4^{-3} = \dfrac{1}{4^3} = \dfrac{1}{64}$

Example B: $x^{-10} = \dfrac{1}{x^{10}}$

Example C: $\dfrac{1}{2^{-a}} = 2^a$

Rule 2: $\left(x^a\right)\left(x^b\right) = x^{a+b}$

Example A: $\left(3^2\right)\left(3^4\right) = 3^{2+4} = 3^6 = 729$

Example B: $\left(y^3\right)\left(y^{-1}\right) = y^2$

Rule 3: $\dfrac{x^a}{x^b} = x^{a-b} = \dfrac{1}{x^{b-a}}$

Example A: $\dfrac{5^7}{5^4} = 5^{7-4} = 5^3 = 125$

Example B: $\dfrac{t^3}{t^8} = t^{-5} = \dfrac{1}{t^5}$

Rule 4: $x^0 = 1$

Example A: $7^0 = 1$

Example B: $(-3)^0 = 1$

Note that 0^0 is not defined.

Rule 5: $\left(x^a\right)\left(y^a\right) = (xy)^a$

Example A: $\left(2^3\right)\left(3^3\right) = 6^3 = 216$

Example B: $(10z)^3 = 10^3 z^3 = 1{,}000z^3$

Rule 6: $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$

Example A: $\left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2} = \frac{9}{16}$

Example B: $\left(\frac{r}{4t}\right)^3 = \frac{r^3}{64t^3}$

Rule 7: $\left(x^a\right)^b = x^{ab}$

Example A: $\left(2^5\right)^2 = 2^{10} = 1{,}024$

Example B: $\left(3y^6\right)^2 = \left(3^2\right)\left(y^6\right)^2 = 9y^{12}$

The rules above are identities that are used to simplify expressions. Sometimes algebraic expressions look like they can be simplified in similar ways, but in fact they cannot. In order to avoid mistakes commonly made when dealing with exponents, keep the following six cases in mind.

Case 1: $\left(x^a\right)\left(y^b\right) \neq (xy)^{a+b}$

For example, $\left(2^4\right)\left(3^2\right) \neq (2 \times 3)^{4+2}$, since $\left(2^4\right)\left(3^2\right) = 144$ and $6^{4+2} = 6^6 = 46{,}656$.

Case 2: $\left(x^a\right)^b \neq x^a x^b$

Instead, $\left(x^a\right)^b = x^{ab}$ and $x^a x^b = x^{a+b}$; for example, $\left(4^2\right)^3 = 4^6$ and $4^2 4^3 = 4^5$.

Case 3: $(x + y)^a \neq x^a + y^a$.

In particular, note that $(x + y)^2 = x^2 + 2xy + y^2$; that is, the correct expansion contains the additional term $2xy$.

Case 4: $(-x)^2 \neq -x^2$

Instead, $(-x)^2 = x^2$. Note carefully where each negative sign appears.

Case 5: $\sqrt{x^2 + y^2} \neq x + y$

Case 6: $\dfrac{a}{x + y} \neq \dfrac{a}{x} + \dfrac{a}{y}$

But it *is* true that $\dfrac{x + y}{a} = \dfrac{x}{a} + \dfrac{y}{a}$.

2.3 Solving Linear Equations

An **equation** is a statement of equality between two mathematical expressions. If an equation involves one or more variables, the values of the variables that make the equation true are called the **solutions** of the equation. To **solve an equation** means to find the values of the variables that make the equation true, that is, the values that **satisfy the equation**. Two equations that have the same solutions are called **equivalent equations**. For example, $x + 1 = 2$ and $2x + 2 = 4$ are equivalent equations; both are true when $x = 1$ and are false otherwise. The general method for solving an equation is to find successively simpler equivalent equations so that the simplest equivalent equation makes the solutions obvious.

The following three rules are important for producing equivalent equations.

Rule 1: When the same constant is added to or subtracted from both sides of an equation, the equality is preserved and the new equation is equivalent to the original equation.

Rule 2: When both sides of an equation are multiplied or divided by the same nonzero constant, the equality is preserved and the new equation is equivalent to the original equation.

Rule 3: When an expression that occurs in an equation is replaced by an equivalent expression, the equality is preserved and the new equation is equivalent to the original equation.

Example: Since the expression $2(x+1)$ is equivalent to the expression $2x+2$, when the expression $2(x+1)$ occurs in an equation, it can be replaced by the expression $2x+2$, and the new equation will be equivalent to the original equation.

A **linear equation** is an equation involving one or more variables in which each term in the equation is either a constant term or a variable multiplied by a coefficient. None of the variables are multiplied together or raised to a power greater than 1. For example, $2x+1=7x$ and $10x-9y-z=3$ are linear equations, but $x+y^2=0$ and $xz=3$ are not.

Linear Equations in One Variable

To solve a linear equation in one variable, find successively simpler equivalent equations by combining like terms and applying the rules for producing simpler equivalent equations until the solution is obvious.

Example 2.3.1: Solve the equation $11x-4-8x=2(x+4)-2x$ as follows.

Combine like terms on the left side to get $3x-4=2(x+4)-2x$.

Replace $2(x+4)$ by $2x+8$ on the right side to get $3x-4=2x+8-2x$.

Combine like terms on the right side to get $3x - 4 = 8$.

Add 4 to both sides to get $3x = 12$.

Divide both sides by 3 to get $\frac{3x}{3} = \frac{12}{3}$.

Simplify to get $x = 4$.

You can always check your solution by substituting it into the original equation. If the resulting value of the right-hand side of the equation is equal to the resulting value of the left-hand side of the equation, your solution is correct.

Substituting the solution $x = 4$ into the left-hand side of the equation $11x - 4 - 8x = 2(x + 4) - 2x$ gives

$$11x - 4 - 8x = 11(4) - 4 - 8(4) = 44 - 4 - 32 = 8$$

Substituting the solution $x = 4$ into the right-hand side of the equation gives

$$2(x + 4) - 2x = 2(4 + 4) - 2(4) = 2(8) - 8 = 8$$

Since both substitutions give the same result, 8, the solution $x = 4$ is correct.

Note that it is possible for a linear equation to have no solutions. For example, the equation $2x + 3 = 2(7 + x)$ has no solution, since it is equivalent to the equation $3 = 14$, which is false. Also, it is possible that what looks to be a linear equation could turn out to be an identity when you try to solve it. For example, $3x - 6 = -3(2 - x)$ is true for all values of x, so it is an identity.

Linear Equations in Two Variables

A linear equation in two variables, x and y, can be written in the form

$$ax + by = c$$

where a, b, and c are real numbers and neither a nor b is equal to 0. For example, $3x + 2y = 8$ is a linear equation in two variables.

A solution of such an equation is an **ordered pair** of numbers (x, y) that makes the equation true when the values of x and y are substituted into the equation. For example, both the ordered pair $(2, 1)$ and the ordered pair $\left(-\frac{2}{3}, 5\right)$ are solutions of the equation $3x + 2y = 8,$ but the ordered pair $(1, 2)$ is not a solution. Every linear equation in two variables has infinitely many solutions.

A set of equations in two or more variables is called a **system of equations**, and the equations in the system are called **simultaneous equations**. To solve a system of equations in two variables, x and y, means to find ordered pairs of numbers (x, y) that satisfy all of the equations in the system. Similarly, to solve a system of equations in three variables, x, y, and z, means to find ordered triples of numbers (x, y, z) that satisfy all of the equations in the system. Solutions of systems with more than three variables are defined in a similar way.

Generally, systems of linear equations in two variables consist of two linear equations, each of which contains one or both of the variables. Often, such systems have a unique solution; that is, there is only one ordered pair of numbers that satisfies both equations in the system. However, it is possible that the system will not have any solutions, or that it will have infinitely many solutions.

There are two basic methods for solving systems of linear equations, by **substitution** or by **elimination**. In the substitution method, one equation is manipulated to express one variable in terms of the other. Then the expression is substituted in the other equation.

Example 2.3.2: Use substitution to solve the following system of two equations.

$$\begin{aligned} 4x + 3y &= 13 \\ x + 2y &= 2 \end{aligned}$$

Solution:

Part 1: You can solve for y as follows.

Express x in the second equation in terms of y as $x = 2 - 2y$.

Substitute $2 - 2y$ for x in the first equation to get $4(2 - 2y) + 3y = 13$.

Replace $4(2 - 2y)$ by $8 - 8y$ on the left side to get $8 - 8y + 3y = 13$.

Combine like terms to get $8 - 5y = 13$.

Solving for y gives $y = -1$.

Part 2: Now, you can use the fact that $y = -1$ to solve for x as follows.

Substitute -1 for y in the second equation to get $x + 2(-1) = 2$.

Solving for x gives $x = 4$.

Thus, the solution of the system is $x = 4$ and $y = -1$, or $(x, y) = (4, -1)$.

In the elimination method, the object is to make the coefficients of one variable the same in both equations so that one variable can be eliminated either by adding the equations together or by subtracting one from the other.

Example 2.3.3: Use elimination to solve the following system of two equations.

$$\begin{aligned} 4x + 3y &= 13 \\ x + 2y &= 2 \end{aligned}$$

(Note that this is the same system of equations that was solved by substitution in Example 2.3.2.)

Solution: Multiplying both sides of the second equation by 4 yields $4(x + 2y) = 4(2)$, or $4x + 8y = 8$.

Now you have two equations with the same coefficient of x.

$$4x + 3y = 13$$
$$4x + 8y = 8$$

If you subtract the equation $4x + 8y = 8$ from the equation $4x + 3y = 13$, the result is $-5y = 5$. Thus, $y = -1$, and substituting -1 for y in either of the original equations yields $x = 4$.

Again, the solution of the system is $x = 4$ and $y = -1$, or $(x, y) = (4, -1)$.

2.4 Solving Quadratic Equations

A **quadratic equation** in the variable x is an equation that can be written in the form

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers and $a \neq 0$. Quadratic equations have zero, one, or two real solutions.

<u>The Quadratic Formula</u>

One way to find solutions of a quadratic equation is to use the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where the notation $\pm$ is shorthand for indicating two solutions – one that uses the plus sign and the other that uses the minus sign.

Example 2.4.1: In the quadratic equation $2x^2 - x - 6 = 0$, we have $a = 2$, $b = -1$, and $c = -6$. Therefore, the quadratic formula yields

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-6)}}{2(2)}$$

When the expression under the square root sign is simplified, we get

$$x = \frac{-(-1) \pm \sqrt{49}}{2(2)}$$

which can be further simplified to

$$x = \frac{1 \pm \sqrt{49}}{4}$$

Finally, since $\sqrt{49} = 7,$ we get that

$$x = \frac{1 \pm 7}{4}$$

Hence this quadratic equation has two real solutions: $x = \frac{1+7}{4} = 2$ and $x = \frac{1-7}{4} = -\frac{3}{2}.$

Example 2.4.2: In the quadratic equation $x^2 + 4x + 4 = 0,$ we have $a = 1,$ $b = 4,$ and $c = 4.$ Therefore, the quadratic formula yields

$$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(4)}}{2(1)}$$

When the expression under the square root sign is simplified, we get

$$x = \frac{-4 \pm \sqrt{0}}{2(1)}$$

The expression under the square root sign is equal to 0 and $\sqrt{0} = 0$. Therefore, the expression can be simplified to

$$x = \frac{-4}{2(1)} = -2$$

Thus this quadratic equation has only one solution, $x = -2$.

Example 2.4.3: In the quadratic equation $x^2 + x + 5 = 0$, we have $a = 1$, $b = 1$, and $c = 5$. Therefore, the quadratic formula yields

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(5)}}{2(1)}$$

The expression under the square root sign is equal to -19. Since square roots of negative numbers are not real numbers, x is not a real number, and there is no real solution to this quadratic equation.

Solving Quadratic Equations by Factoring

Some quadratic equations can be solved more quickly by factoring.

Example 2.4.4: The quadratic equation $2x^2 - x - 6 = 0$ in Example 2.4.1 can be factored as $(2x + 3)(x - 2) = 0$. When a product is equal to 0, at least one of the factors must be equal to 0, so either $2x + 3 = 0$ or $x - 2 = 0$.

If $2x + 3 = 0$, then $2x = -3$ and $x = -\frac{3}{2}$.

If $x - 2 = 0$, then $x = 2$.

Thus the solutions are $-\frac{3}{2}$ and 2.

Example 2.4.5: The quadratic equation $5x^2 + 3x - 2 = 0$ can be factored as $(5x - 2)(x + 1) = 0$.

Therefore, either $5x - 2 = 0$ or $x + 1 = 0$.

If $5x - 2 = 0$, then $x = \frac{2}{5}$.

If $x + 1 = 0$, then $x = -1$.

Thus the solutions are $\frac{2}{5}$ and -1.

2.5 Solving Linear Inequalities

A mathematical statement that uses one of the following four inequality signs is called an **inequality**.

$<$ less than

$>$ greater than

$\leq$ less than or equal to

$\geq$ greater than or equal to

Inequalities can involve variables and are similar to equations, except that the two sides are related by one of the inequality signs instead of the equality sign used in equations. For example, the inequality $4x + 1 \leq 7$ is a linear inequality in one variable, which states that $4x + 1$ is less than or equal to 7. To **solve an inequality** means to find the set of all values of the variable that make the inequality true. This set of values is also known as the **solution set** of an inequality. Two inequalities that have the same solution set are called **equivalent inequalities**.

The procedure used to solve a linear inequality is similar to that used to solve a linear equation, which is to simplify the inequality by isolating the variable on one side of the inequality, using the following two rules.

Rule 1: When the same constant is added to or subtracted from both sides of an inequality, the direction of the inequality is preserved and the new inequality is equivalent to the original.

Rule 2: When both sides of the inequality are multiplied or divided by the same nonzero constant, the direction of the inequality is *preserved if the constant is positive* but the direction is *reversed if the constant is negative*. In either case, the new inequality is equivalent to the original.

Example 2.5.1: The inequality $-3x + 5 \leq 17$ can be solved as follows.

Subtract 5 from both sides to get $-3x \leq 12$.

Divide both sides by -3 and reverse the direction of the inequality to get $\frac{-3x}{-3} \geq \frac{12}{-3}$.

That is, $x \geq -4$.

Therefore, the solution set of $-3x + 5 \leq 17$ consists of all numbers greater than or equal to -4.

Example 2.5.2: The inequality $\frac{4x + 9}{11} < 5$ can be solved as follows.

Multiply both sides by 11 to get $4x + 9 < 55$.

Subtract 9 from both sides to get $4x < 46$.

Divide both sides by 4 to get $x < \frac{46}{4}$.

That is, $x < 11.5$.

Therefore, the solution set of the inequality $\frac{4x+9}{11} < 5$ consists of all numbers less than 11.5.

2.6 Functions

An algebraic expression in one variable can be used to define a **function** of that variable. Functions are usually denoted by letters such as *f*, *g*, and *h*. For example, the algebraic expression $3x + 5$ can be used to define a function *f* by

$$f(x) = 3x + 5$$

where $f(x)$ is called the value of *f* at *x* and is obtained by substituting the value of *x* in the expression above. For example, if $x = 1$ is substituted in the expression above, the result is $f(1) = 3(1) + 5 = 8$.

It might be helpful to think of a function *f* as a machine that takes an input, which is a value of the variable *x*, and produces the corresponding output, $f(x)$. For any function, each input *x* gives exactly one output $f(x)$. However, more than one value of *x* can give the same output $f(x)$. For example, if *g* is the function defined by $g(x) = x^2 - 2x + 3$, then $g(0) = 3$ and $g(2) = 3$.

The **domain** of a function is the set of all permissible inputs, that is, all permissible values of the variable *x*. For the functions *f* and *g* defined above, the domain is the set of all real numbers. Sometimes the domain of the function is given explicitly and is restricted to a specific set of values of *x*. For example, we can define the function *h* by $h(x) = x^2 - 4$ for $-2 \le x \le 2$. Without an explicit restriction, the domain is assumed to be the set of all values of *x* for which $f(x)$ is a real number.

Example 2.6.1: Let *f* be the function defined by $f(x) = \frac{2x}{x-6}$. In this case, *f* is not defined at $x = 6$, because $\frac{12}{0}$ is not defined. Hence, the domain of *f* consists of all real numbers except for 6.

Example 2.6.2: Let g be the function defined by $g(x) = x^3 + \sqrt{x+2} - 10$. In this case, $g(x)$ is not a real number if $x < -2$. Hence, the domain of g consists of all real numbers x such that $x \geq -2$.

Example 2.6.3: Let h be the function defined by $h(x) = |x|$, which is the distance between x and 0 on the number line (see Arithmetic, Section 1.5). The domain of h is the set of all real numbers. Also, $h(x) = h(-x)$ for all real numbers x, which reflects the property that on the number line the distance between x and 0 is the same as the distance between $-x$ and 0.

2.7 Applications

Translating verbal descriptions into algebraic expressions is an essential initial step in solving word problems. Three examples of verbal descriptions and their translations are given below.

Example 2.7.1: If the square of the number x is multiplied by 3 and then 10 is added to that product, the result can be represented algebraically by $3x^2 + 10$.

Example 2.7.2: If John's present salary s is increased by 14 percent, then his new salary can be represented algebraically by $1.14s$.

Example 2.7.3: If y gallons of syrup are to be distributed among 5 people so that one particular person gets 1 gallon and the rest of the syrup is divided equally among the remaining 4, then the number of gallons of syrup that each of the 4 people will get can be represented algebraically by $\frac{y-1}{4}$.

The remainder of this section gives examples of various applications.

<u>Average, Mixture, Rate, and Work Problems</u>

Example 2.7.4: Ellen has received the following scores on 3 exams: 82, 74, and 90. What score will Ellen need to receive on the next exam so that the average (arithmetic mean) score for the 4 exams will be 85 ?

Solution: Let x represent the score on Ellen's next exam. This initial step of assigning a variable to the quantity that is sought is an important beginning to solving the problem. Then in terms of x, the average of the 4 exam scores is

$$\frac{82 + 74 + 90 + x}{4}$$

which is supposed to equal 85. Now simplify the expression and set it equal to 85:

$$\frac{82 + 74 + 90 + x}{4} = \frac{246 + x}{4} = 85$$

Solving the resulting linear equation for x, you get $246 + x = 340,$ and so $x = 94.$

Therefore, Ellen will need to attain a score of 94 on the next exam.

Example 2.7.5: A mixture of 12 grams of vinegar and oil is 40 percent vinegar, where all of the measurements are by weight. How many grams of oil must be added to the mixture to produce a new mixture that is only 25 percent vinegar?

Solution: Let x represent the number of grams of oil to be added. Then the total number of grams of the new mixture will be $12 + x,$ and the total number of grams of vinegar in the new mixture will be $(0.40)(12).$ Since the new mixture must be 25 percent vinegar,

$$\frac{(0.40)(12)}{12 + x} = 0.25$$

Therefore, $(0.40)(12) = (12 + x)(0.25).$

Simplifying further gives $4.8 = 3 + 0.25x,$ so $1.8 = 0.25x,$ and $7.2 = x.$

Thus, 7.2 grams of oil must be added to produce a new mixture that is 25 percent vinegar.

Example 2.7.6: In a driving competition, Jeff and Dennis drove the same course at average speeds of 51 miles per hour and 54 miles per hour, respectively. If it took Jeff 40 minutes to drive the course, how long did it take Dennis?

Solution: Let x be the time, in minutes, that it took Dennis to drive the course. The distance d, in miles, is equal to the product of the rate r, in miles per hour, and the time t, in hours; that is,

$$d = rt$$

Note that since the rates are given in miles per *hour*, it is necessary to express the times in hours; for example, 40 minutes equals $\frac{40}{60}$ of an hour. Thus, the distance traveled by Jeff is the product of his speed and his time, $(51)\left(\frac{40}{60}\right)$ miles, and the distance traveled by Dennis is similarly represented by $(54)\left(\frac{x}{60}\right)$ miles.

Since the distances are equal, it follows that $(51)\left(\frac{40}{60}\right) = (54)\left(\frac{x}{60}\right)$.

From this equation it follows that $(51)(40) = 54x$ and $x = \frac{(51)(40)}{54} \approx 37.8$.

Thus, it took Dennis approximately 37.8 minutes to drive the course.

Example 2.7.7: A batch of computer parts consists of n identical parts, where n is a multiple of 60. Working alone at its constant rate, machine A takes 3 hours to produce a batch of computer parts. Working alone at its constant rate, machine B takes 2 hours to produce a batch of computer parts. How long will it take the two machines, working simultaneously at their respective constant rates, to produce a batch of computer parts?

Solution: Since machine A takes 3 hours to produce a batch, machine A can produce $\frac{1}{3}$ of the batch in 1 hour. Similarly, machine B can produce $\frac{1}{2}$ of the batch in 1 hour. If we let x represent the number of hours it takes both machines, working simultaneously, to produce the batch, then the two machines will produce $\frac{1}{x}$ of the batch in 1 hour. When

the two machines work together, adding their individual production rates, $\frac{1}{3}$ and $\frac{1}{2}$, gives their combined production rate $\frac{1}{x}$. Therefore, it follows that $\frac{1}{3} + \frac{1}{2} = \frac{1}{x}$.

This equation is equivalent to $\frac{2}{6} + \frac{3}{6} = \frac{1}{x}$. So $\frac{5}{6} = \frac{1}{x}$ and $\frac{6}{5} = x$.

Thus, working together, the machines will take $\frac{6}{5}$ hours, or 1 hour 12 minutes, to produce a batch of computer parts.

Example 2.7.8: At a fruit stand, apples can be purchased for \$0.15 each and pears for \$0.20 each. At these rates, a bag of apples and pears was purchased for \$3.80. If the bag contained 21 pieces of fruit, how many of the pieces were pears?

Solution: If a represents the number of apples purchased and p represents the number of pears purchased, then the total cost of the fruit can be represented by the equation $0.15a + 0.20p = 3.80$, and the total number of pieces of fruit can be represented by the equation $a + p = 21$. Thus to answer the question, you need to solve the following system of equations.

$$\begin{aligned} 0.15a + 0.20p &= 3.80 \\ a + p &= 21 \end{aligned}$$

From the equation for the total number of fruit, $a = 21 - p$.

Substituting $21 - p$ for a in the equation for the total cost gives the equation

$$0.15(21 - p) + 0.20p = 3.80$$

So, $(0.15)(21) - 0.15p + 0.20p = 3.80$, which is equivalent to

$$3.15 - 0.15p + 0.20p = 3.80$$

Therefore $0.05p = 0.65$, and $p = 13$.

Thus, of the 21 pieces of fruit, 13 were pears.

Example 2.7.9: To produce a particular radio model, it costs a manufacturer \$30 per radio, and it is assumed that if 500 radios are produced, all of them will be sold. What must be the selling price per radio to ensure that the profit (revenue from the sales minus the total production cost) on the 500 radios is greater than \$8,200 ?

Solution: If the selling price per radio is y dollars, then the profit is $500(y-30)$ dollars.

Therefore, $500(y-30) > 8,200.$

Simplifying further gives $500y - 15,000 > 8,200,$ which simplifies to $500y > 23,200$ and then to $y > 46.4.$

Thus, the selling price must be greater than \$46.40 to ensure that the profit is greater than \$8,200.

Interest

Some applications involve computing **interest** earned on an investment during a specified time period. The interest can be computed as simple interest or compound interest.

Simple interest is based only on the initial deposit, which serves as the amount on which interest is computed, called the **principal**, for the entire time period. If the amount P is invested at a *simple annual interest rate of r percent*, then the value V of the investment at the end of t years is given by the formula

$$V = P\left(1 + \frac{rt}{100}\right)$$

In the case of **compound interest**, interest is added to the principal at regular time intervals, such as annually, quarterly, and monthly. Each time interest is added to the principal, the interest is said to be compounded. After each compounding, interest is earned on the new principal, which is the sum of the preceding principal and the interest just added. If the amount P is invested at an *annual interest rate of r percent*,

compounded annually, then the value V of the investment at the end of t years is given by the formula

$$V = P\left(1 + \frac{r}{100}\right)^t$$

If the amount P is invested at an *annual interest rate of r percent, compounded n times per year*, then the value V of the investment at the end of t years is given by the formula

$$V = P\left(1 + \frac{r}{100n}\right)^{nt}$$

Example 2.7.10: If \$10,000 is invested at a simple annual interest rate of 6 percent, what is the value of the investment after half a year?

Solution: According to the formula for simple interest, the value of the investment after $\frac{1}{2}$ year is

$$\$10,000\left(1 + 0.06\left(\frac{1}{2}\right)\right) = \$10,000(1.03) = \$10,300$$

Example 2.7.11: If an amount P is to be invested at an annual interest rate of 3.5 percent, compounded annually, what should be the value of P so that the value of the investment is \$1,000 at the end of 3 years? (Give your answer to the nearest dollar.)

Solution: According to the formula for 3.5 percent annual interest, compounded annually, the value of the investment after 3 years is

$$P(1 + 0.035)^3$$

and we set it to be equal to \$1,000 as follows.

$$P(1 + 0.035)^3 = \$1,000$$

To find the value of P, we divide both sides of the equation by $(1 + 0.035)^3$.

$$P = \frac{\$1,000}{(1 + 0.035)^3} \approx \$902$$

Thus, to the nearest dollar, \$902 should be invested.

Example 2.7.12: A college student expects to earn at least \$1,000 in interest on an initial investment of \$20,000. If the money is invested for one year at an annual interest rate of r percent, compounded quarterly, what is the least annual interest rate that would achieve the goal? (Give your answer to the nearest 0.1 percent.)

Solution: According to the formula for r percent annual interest, compounded quarterly, the value of the investment after 1 year is

$$\$20,000\left(1 + \frac{r}{400}\right)^4$$

By setting this value greater than or equal to \$21,000 and solving for r, we get

$$\$20,000\left(1 + \frac{r}{400}\right)^4 \geq \$21,000$$

which simplifies to

$$\left(1 + \frac{r}{400}\right)^4 \geq 1.05$$

We can use the fact that taking the positive fourth root of each side of an inequality preserves the direction of the inequality. It is also true that taking the positive square root or any other positive root of each side of an inequality preserves the direction of the inequality. Using this fact, take the positive fourth root of both sides of

$$\left(1+\frac{r}{400}\right)^4 \geq 1.05$$

to get

$$1+\frac{r}{400} \geq \sqrt[4]{1.05}$$

which simplifies to

$$r \geq 400\left(\sqrt[4]{1.05}-1\right)$$

To compute the positive fourth root of 1.05, we can use the fact that for any number $x \geq 0$, $\sqrt[4]{x} = \sqrt{\sqrt{x}}$.

This allows us to compute the positive fourth root of 1.05 by taking the positive square root of 1.05 and then taking the positive square root of the result.

Therefore we can conclude that

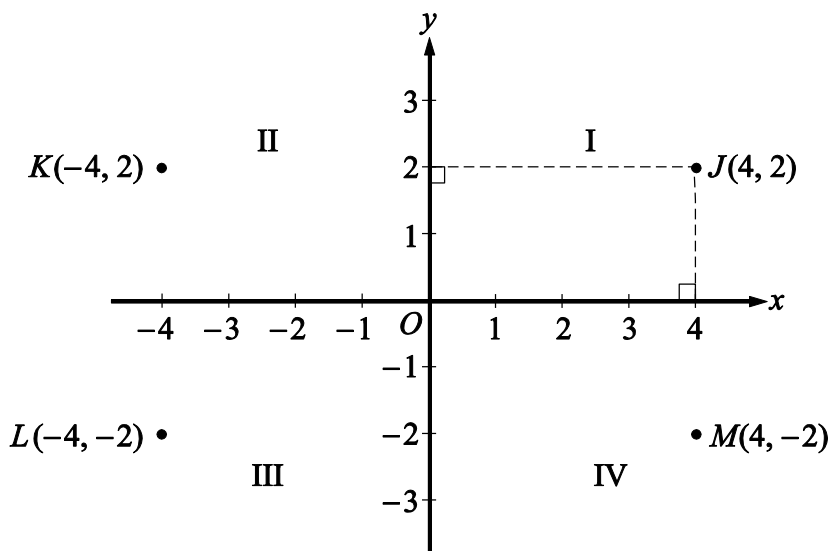
$$400\left(\sqrt[4]{1.05}-1\right) = 400\left(\sqrt{\sqrt{1.05}}-1\right) \approx 4.9$$

Since $r \geq 400\left(\sqrt[4]{1.05}-1\right)$ and $400\left(\sqrt[4]{1.05}-1\right)$, rounded to the nearest 0.1, is 4.9, the least interest rate is approximately 4.9 percent per year, compounded quarterly.

2.8 Coordinate Geometry

Two real number lines that are perpendicular to each other and that intersect at their respective zero points define a **rectangular coordinate system**, often called the ***xy*-coordinate system** or ***xy*-plane**. The horizontal number line is called the ***x*-axis** and the vertical number line is called the ***y*-axis**. The point where the two axes intersect is called the **origin**, denoted by *O*. The positive half of the *x*-axis is to the right of the origin, and the positive half of the *y*-axis is above the origin. The two axes divide the

plane into four regions called **quadrants**. The four quadrants are labeled I, II, III, and IV, as shown in Algebra Figure 1 below.



Algebra Figure 1

Each point J in the xy-plane can be identified with an ordered pair (x, y) of real numbers and is denoted by $J(x, y)$. The first number in the ordered pair is called the ***x*-coordinate**, and the second number is called the ***y*-coordinate**. A point with coordinates (x, y) is located $|x|$ units to the right of the y-axis if x is positive, or it is located $|x|$ units to the left of the y-axis if x is negative. Also, the point is located $|y|$ units above the x-axis if y is positive, or it is located $|y|$ units below the x-axis if y is negative. If $x = 0$, the point lies on the y-axis, and if $y = 0$, the point lies on the x-axis. The origin has coordinates $(0, 0)$. Unless otherwise noted, the units used on the x-axis and the y-axis are the same.

In Algebra Figure 1 above, the point $J(4, 2)$ is 4 units to the right of the y-axis and 2 units above the x-axis, the point $K(-4, 2)$ is 4 units to the left of the y-axis and 2 units above the x-axis, the point $L(-4, -2)$ is 4 units to the left of the y-axis and 2 units below the x-axis, and the point $M(4, -2)$ is 4 units to the right of the y-axis and 2 units below the x-axis.

Note that the three points $K(-4, 2)$, $L(-4, -2)$, and $M(4, -2)$ have the same coordinates as *J* except for the signs. These points are geometrically related to *J* as follows.

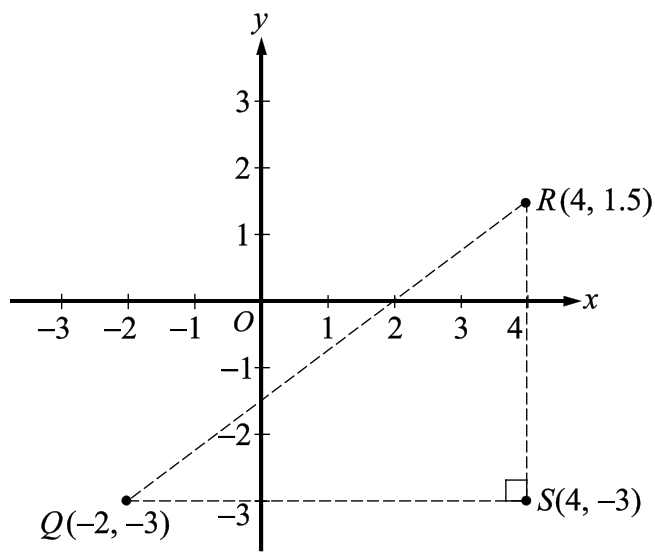
- *M* is the **reflection of *J* about the *x*-axis**, or *M* and *J* are **symmetric about the *x*-axis**.

- *K* is the **reflection of *J* about the *y*-axis**, or *K* and *J* are **symmetric about the *y*-axis**.

- *L* is the **reflection of *J* about the origin**, or *L* and *J* are **symmetric about the origin**.

Calculating the Distance Between Two Points

The distance between two points in the *xy*-plane can be found by using the Pythagorean theorem. For example, the distance between the two points $Q(-2, -3)$ and $R(4, 1.5)$ in Algebra Figure 2 below is the length of line segment *QR*. To find this length, construct a right triangle with hypotenuse *QR* by drawing a vertical line segment downward from *R* and a horizontal line segment rightward from *Q* until these two line segments intersect at the point $S(4, -3)$ forming a right angle, as shown in Algebra Figure 2. Then note that the horizontal side of the triangle has length $4 - (-2) = 6$ and the vertical side of the triangle has length $1.5 - (-3) = 4.5$.



Algebra Figure 2

Since line segment QR is the hypotenuse of the triangle, you can apply the Pythagorean theorem:

$$QR = \sqrt{6^2 + 4.5^2} = \sqrt{56.25} = 7.5$$

(For a discussion of right triangles and the Pythagorean theorem, see Geometry, Section 3.3.)

Graphing Linear Equations and Inequalities

Equations in two variables can be represented as graphs in the coordinate plane. In the xy-plane, the **graph of an equation** in the variables x and y is the set of all points whose ordered pairs (x, y) satisfy the equation.

The graph of a linear equation of the form $y = mx + b$ is a straight line in the xy-plane, where m is called the **slope** of the line and b is called the ***y*-intercept**.

The ***x*-intercepts** of a graph are the x-coordinates of the points at which the graph intersects the x-axis. Similarly, the ***y*-intercepts** of a graph are the y-coordinates of the points at which the graph intersects the y-axis. Sometimes the terms ***x*-intercept** and ***y*-intercept** refer to the actual intersection points.

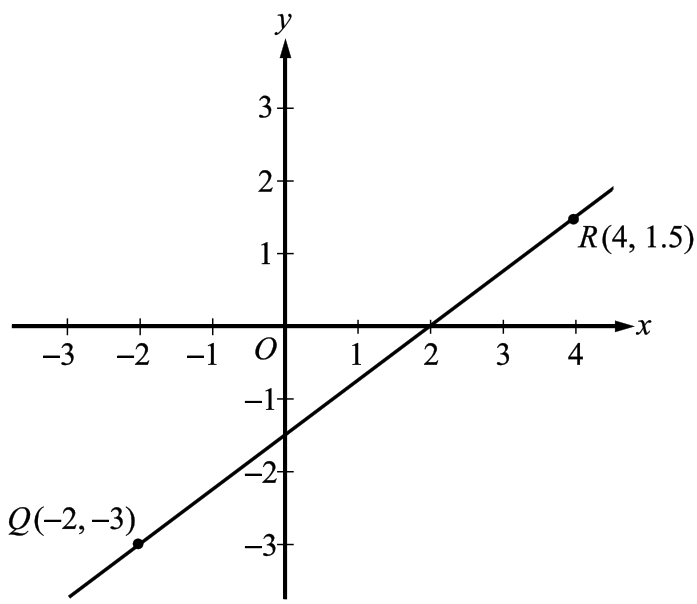
The slope of a line passing through two points $Q(x_1, y_1)$ and $R(x_2, y_2)$, where $x_1 \neq x_2$, is defined as

$$\frac{y_2 - y_1}{x_2 - x_1}$$

This ratio is often called "rise over run," where *rise* is the change in y when moving from Q to R and *run* is the change in x when moving from Q to R. A horizontal line has a slope of 0, since the rise is 0 for any two points on the line. Therefore, the equation of every horizontal line has the form $y = b$, where b is the y-intercept. The slope of a vertical line is not defined, since the run is 0. The equation of every vertical line has the form $x = a$, where a is the x-intercept.

Two lines are **parallel** if their slopes are equal. Two lines are **perpendicular** if their slopes are negative reciprocals of each other. For example, the line with equation $y = 2x + 5$ is perpendicular to the line with equation $y = -\frac{1}{2}x + 9$.

Example 2.8.1: Algebra Figure 3 below shows the graph of the line through the points $Q(-2, -3)$ and $R(4, 1.5)$.



Algebra Figure 3

In Algebra Figure 3 above, the slope of the line passing through the points

$Q(-2, -3)$ and $R(4, 1.5)$ is

$$\frac{1.5-(-3)}{4-(-2)} = \frac{4.5}{6} = 0.75$$

Line *QR* appears to intersect the *y*-axis close to the point $(0, -1.5)$, so the *y*-intercept of the line must be close to -1.5. To get the exact value of the *y*-intercept, substitute the coordinates of any point on the line into the equation $y = 0.75x + b$, and solve it for *b*.

For example, if you pick the point $Q(-2, -3)$ and substitute its coordinates into the equation, you get $-3 = (0.75)(-2) + b$.

Then adding $(0.75)(2)$ to both sides of the equation yields $b = -3 + (0.75)(2)$, or $b = -1.5$.

Therefore, the equation of line *QR* is $y = 0.75x - 1.5$.

You can see from the graph in Algebra Figure 3 that the x-intercept of line QR is 2, since QR passes through the point $(2, 0)$. More generally, you can find the x-intercept of a line by setting $y = 0$ in an equation of the line and solving it for x. So you can find the x-intercept of line QR by setting $y = 0$ in the equation $y = 0.75x - 1.5$ and solving it for x as follows.

Setting $y = 0$ in the equation $y = 0.75x - 1.5$ gives the equation $0 = 0.75x - 1.5$. Then adding 1.5 to both sides yields $1.5 = 0.75x$. Finally, dividing both sides by 0.75 yields $x = \frac{1.5}{0.75} = 2$.

Graphs of linear equations can be used to illustrate solutions of systems of linear equations and inequalities, as can be seen in Examples 2.8.2 and 2.8.3.

Example 2.8.2: Consider the following system of two linear equations in two variables.

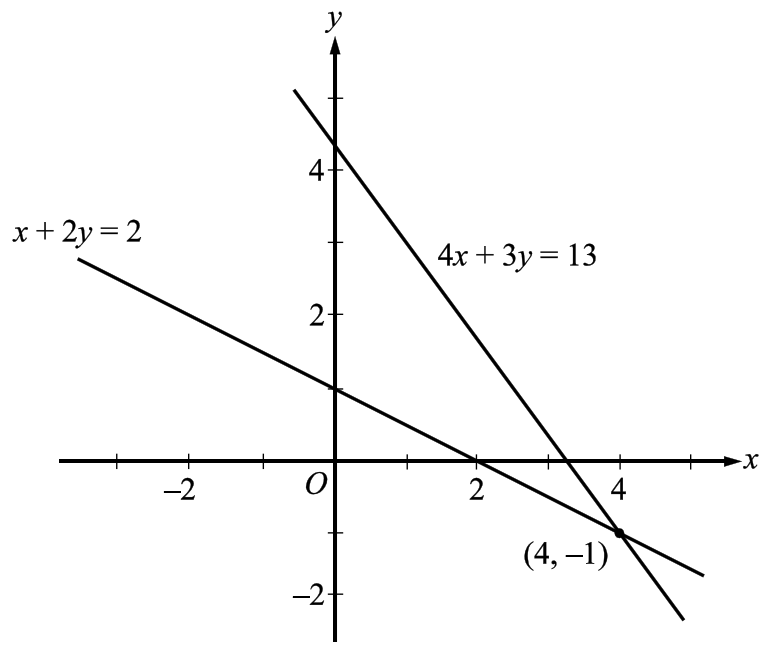
$$\begin{aligned} 4x + 3y &= 13 \\ x + 2y &= 2 \end{aligned}$$

(Note that this system was solved by substitution and by elimination in Section 2.3.)

Solving each equation for y in terms of x yields the following equivalent system of equations.

$$\begin{aligned} y &= -\frac{4}{3}x + \frac{13}{3} \\ y &= -\frac{1}{2}x + 1 \end{aligned}$$

Algebra Figure 4 below shows the graphs of the two equations in the xy-plane. The solution of the system of equations is the point at which the two graphs intersect, which is $(4, -1)$.



Algebra Figure 4

Example 2.8.3: Consider the following system of two linear inequalities.

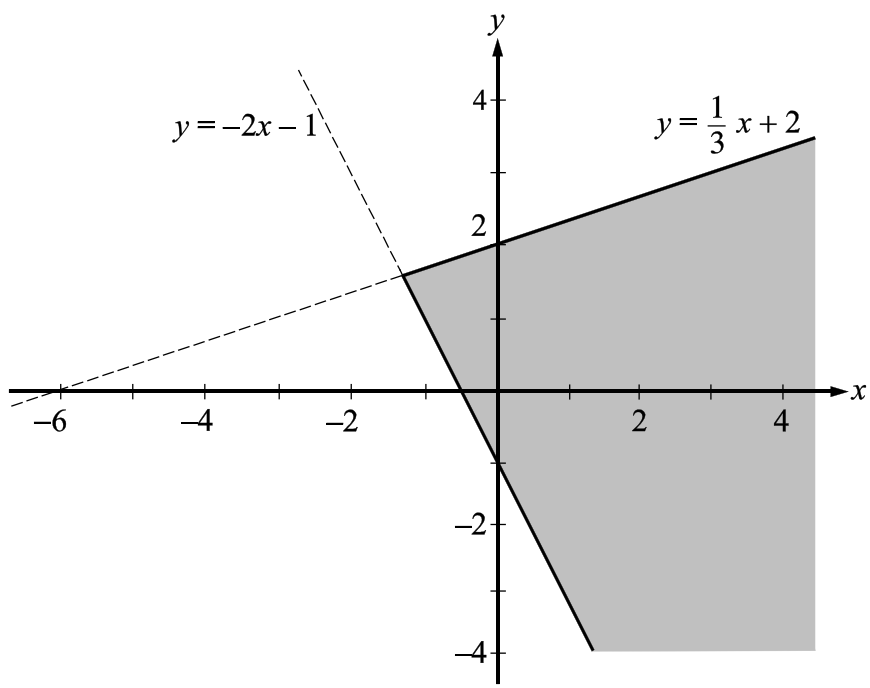
$$x - 3y \geq -6$$
$$2x + y \geq -1$$

Solving each inequality for y in terms of x yields the following equivalent system of inequalities.

$$y \leq \frac{1}{3}x + 2$$
$$y \geq -2x - 1$$

Each point (x, y) that satisfies the first inequality, $y \leq \frac{1}{3}x + 2,$ is either on the line $y = \frac{1}{3}x + 2$ or *below* the line because the y-coordinate is either equal to or *less than* $\frac{1}{3}x + 2.$ Therefore, the graph of $y \leq \frac{1}{3}x + 2$ consists of the line $y = \frac{1}{3}x + 2$ and the

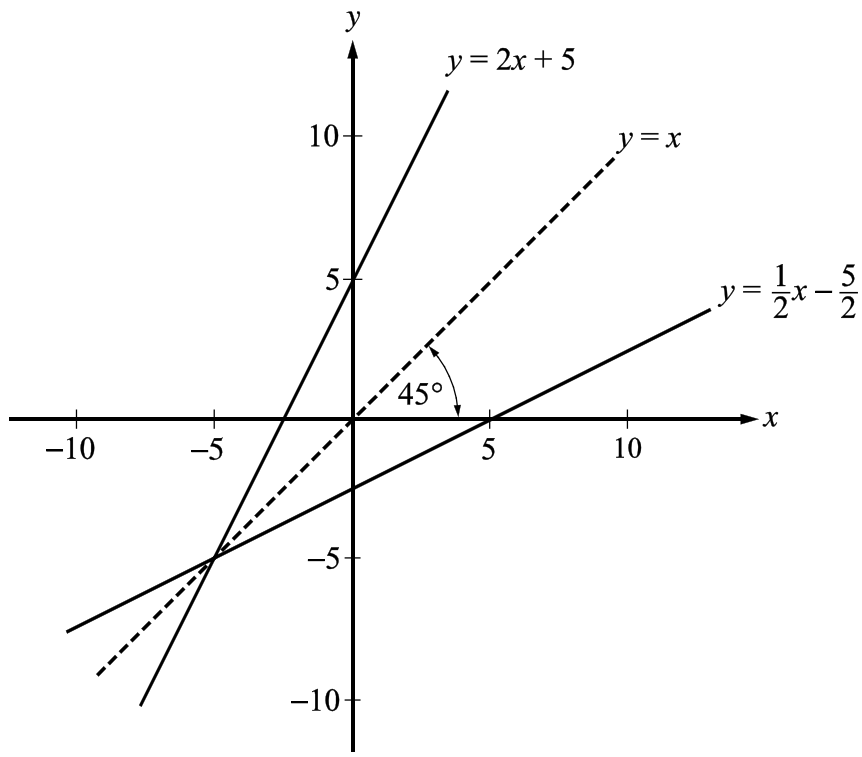
entire region below it. Similarly, the graph of $y \geq -2x - 1$ consists of the line $y = -2x - 1$ and the entire region *above* it. Thus, the solution set of the system of inequalities consists of all of the points that lie in the intersection of the two graphs described, which is represented by the shaded region shown in Algebra Figure 5 below, including the two half-lines that form the boundary of the shaded region.



Algebra Figure 5

Symmetry with respect to the x-axis, the y-axis, and the origin is mentioned earlier in this section. Another important symmetry is symmetry with respect to the line with equation $y = x$. The line $y = x$ passes through the origin, has a slope of 1, and makes a 45-degree angle with each axis. For any point with coordinates (a, b), the point with interchanged coordinates (b, a) is the reflection of (a, b) about the line $y = x$; that is, (a, b) and (b, a) are symmetric about the line $y = x$. It follows that interchanging x and y in the equation of any graph yields another graph that is the reflection of the original graph about the line $y = x$.

Example 2.8.4: Consider the line whose equation is $y = 2x + 5$. Interchanging x and y in the equation yields $x = 2y + 5$. Solving this equation for y yields $y = \frac{1}{2}x - \frac{5}{2}$. The line $y = 2x + 5$ and its reflection $y = \frac{1}{2}x - \frac{5}{2}$ are graphed in Algebra Figure 6 that follows.



Algebra Figure 6

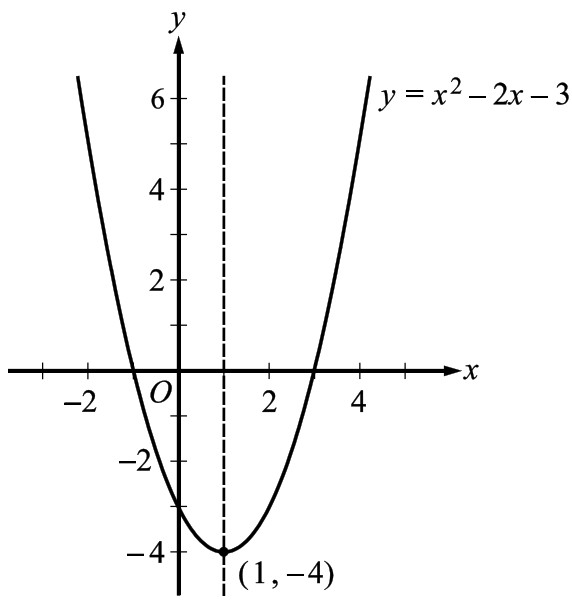
The line $y = x$ is a **line of symmetry** for the graphs of $y = 2x + 5$ and $y = \frac{1}{2}x - \frac{5}{2}$.

Graphing Quadratic Equations

The graph of a quadratic equation of the form $y = ax^2 + bx + c$, where a, b, and c are constants and $a \neq 0$, is a **parabola**. The x-intercepts of the parabola are the solutions of the equation $ax^2 + bx + c = 0$. If a is positive, the parabola opens upward and the **vertex** is its lowest point. If a is negative, the parabola opens downward and the vertex is

its highest point. Every parabola that is the graph of a quadratic equation of the form $y = ax^2 + bx + c$ is symmetric with itself about the vertical line that passes through its vertex. In particular, the two x-intercepts are equidistant from this line of symmetry.

Example 2.8.5: The quadratic equation $y = x^2 - 2x - 3$ has the graph shown in Algebra Figure 7 below.



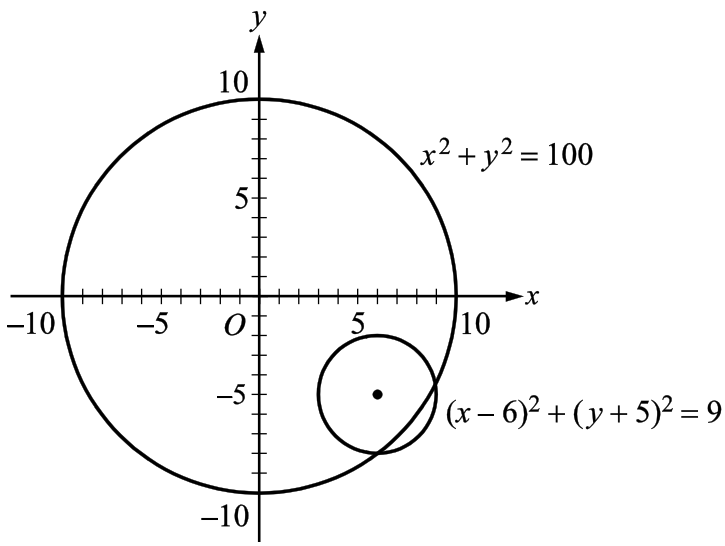
Algebra Figure 7

The graph indicates that the x-intercepts of the parabola are -1 and 3. The values of the x-intercepts can be confirmed by solving the quadratic equation $x^2 - 2x - 3 = 0$ to get $x = -1$ and $x = 3$. The point $(1, -4)$ is the vertex of the parabola, and the line $x = 1$ is its line of symmetry. The y-intercept is the y-coordinate of the point on the parabola at which $x = 0$, which is $y = 0^2 - 2(0) - 3 = -3$.

Graphing Circles

The graph of an equation of the form $(x - a)^2 + (y - b)^2 = r^2$ is a **circle** with its center at the point (a, b) and with radius $r > 0$.

Example 2.8.6: Algebra Figure 8 below shows the graph of two circles in the *xy*-plane. The larger of the two circles is centered at the origin and has radius 10, so its equation is $x^2 + y^2 = 100$. The smaller of the two circles has center $(6, -5)$ and radius 3, so its equation is $(x-6)^2 + (y+5)^2 = 9$.



Algebra Figure 8

2.9 Graphs of Functions

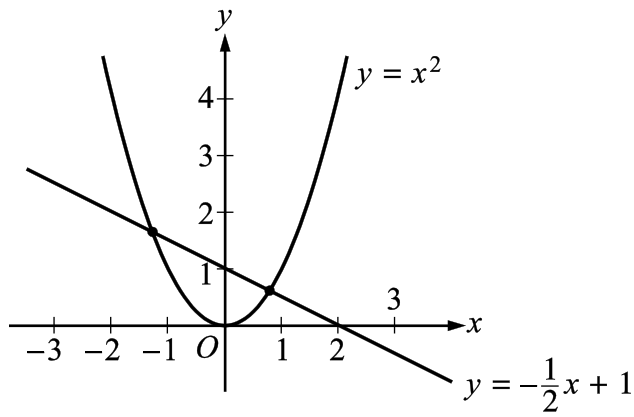
The coordinate plane can be used for graphing functions. To graph a function in the *xy*-plane, you represent each input x and its corresponding output $f(x)$ as a point (x, y), where $y = f(x)$. In other words, you use the *x*-axis for the input and the *y*-axis for the output.

Below are several examples of graphs of elementary functions.

Example 2.9.1: Consider the linear function defined by $f(x) = -\frac{1}{2}x + 1$. Its graph in the *xy*-plane is the line with the linear equation $y = -\frac{1}{2}x + 1$.

Example 2.9.2: Consider the quadratic function defined by $g(x) = x^2$. The graph of g is the parabola with the quadratic equation $y = x^2$.

The graph of both the linear equation $y = -\frac{1}{2}x + 1$ and the quadratic equation $y = x^2$ are shown in Algebra Figure 9 below.



Algebra Figure 9

Note that the graphs of f and g in Algebra Figure 9 above intersect at two points. These are the points at which $g(x) = f(x)$. We can find these points algebraically as follows.

Set $g(x) = f(x)$ and get $x^2 = -\frac{1}{2}x + 1$, which is equivalent to $x^2 + \frac{1}{2}x - 1 = 0$, or $2x^2 + x - 2 = 0$.

Then solve the equation $2x^2 + x - 2 = 0$ for x using the quadratic formula and get

$$x = \frac{-1 \pm \sqrt{1+16}}{4}$$

which represents the x-coordinates of the two solutions

$$x = \frac{-1+\sqrt{17}}{4} \approx 0.78 \text{ and } x = \frac{-1-\sqrt{17}}{4} \approx -1.28.$$

With these input values, the corresponding y-coordinates can be found using either f or g:

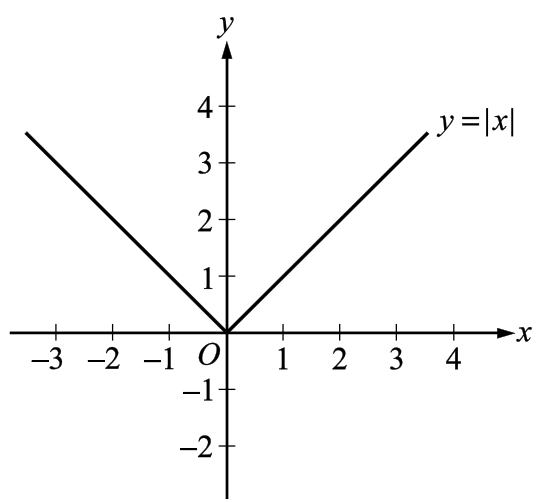
$$g\left(\frac{-1+\sqrt{17}}{4}\right) = \left(\frac{-1+\sqrt{17}}{4}\right)^2 \approx 0.61 \text{ and } g\left(\frac{-1-\sqrt{17}}{4}\right) = \left(\frac{-1-\sqrt{17}}{4}\right)^2 \approx 1.64.$$

Thus, the two intersection points can be approximated by $(0.78, 0.61)$ and $(-1.28, 1.64)$.

Example 2.9.3: Consider the absolute value function defined by $h(x) = |x|$. By using the definition of absolute value (see Arithmetic, Section 1.5), h can be expressed as a **piecewise-defined** function:

$$h(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

The graph of this function is V-shaped and consists of two linear pieces, $y = x$ and $y = -x$, joined at the origin, as shown in Algebra Figure 10 that follows.

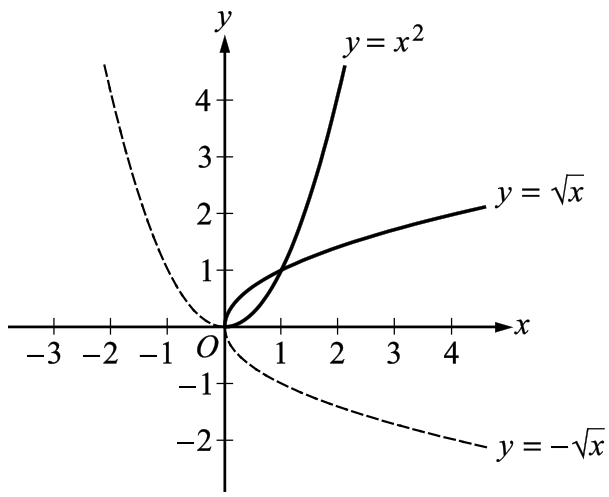


Algebra Figure 10

Example 2.9.4: Consider the positive square root function defined by $j(x) = \sqrt{x}$ for $x \geq 0$. The graph of this function is the upper half of a parabola lying on its side.

Also consider the negative square root function defined by $k(x) = -\sqrt{x}$ for $x \geq 0$. The graph of this function is the lower half of the parabola lying on its side.

The graphs of both of these functions, along with the graph of the parabola $y = x^2$, are shown in Algebra Figure 11 below.



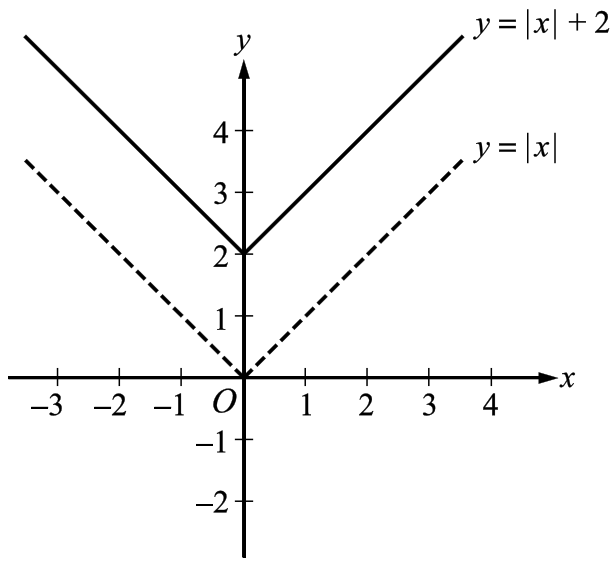
Algebra Figure 11

The graphs of $y = \sqrt{x}$ and $y = -\sqrt{x}$ are halves of a parabola because they are reflections of the right and left halves, respectively, of the parabola $y = x^2$ about the line $y = x$. This follows from squaring both sides of the two square root equations to get $y^2 = x$ and then interchanging x and y to get $y = x^2$.

Also note that $y = -\sqrt{x}$ is the reflection of $y = \sqrt{x}$ about the x-axis. In general, for any function h, the graph of $y = -h(x)$ is the **reflection** of the graph of $y = h(x)$ about the x-axis.

Example 2.9.5: Consider the function defined by $f(x) = |x| + 2$.

The graph of $f(x) = |x| + 2$ is the graph of $y = |x|$ shifted upward by 2 units, as shown in Algebra Figure 12 that follows.

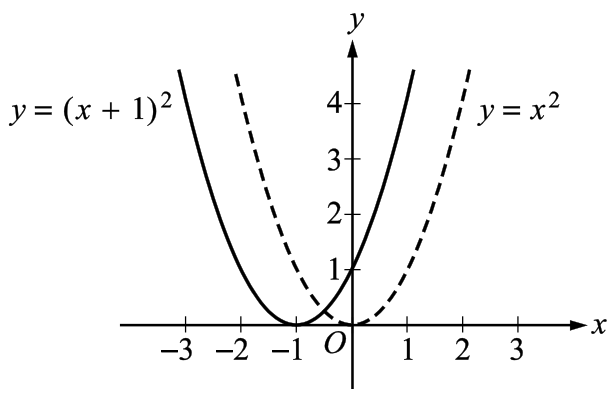


Algebra Figure 12

Similarly, the graph of the function $k(x) = |x| - 5$ is the graph of $y = |x|$ shifted downward by 5 units. (The graph of this function is not shown.)

Example 2.9.6: Consider the function defined by $g(x) = (x+1)^2$.

The graph of $g(x) = (x+1)^2$ is the graph of $y = x^2$ shifted to the left by 1 unit, as shown in Algebra Figure 13 below.



Algebra Figure 13

Similarly, the graph of the function $j(x) = (x - 4)^2$ is the graph of $y = x^2$ shifted to the right by 4 units. (The graph of this function is not shown.)

Note that in Example 2.9.5, the graph of the function $y = |x|$ was shifted upward and downward, and in Example 2.9.6, the graph of the function $y = x^2$ was shifted to the left and to the right. To double-check the direction of a shift, you can plot some corresponding values of the original function and the shifted function.

In general, for any function $h(x)$ and any positive number *c*, the following are true.

The graph of $h(x) + c$ is the graph of $h(x)$ **shifted upward** by *c* units.

The graph of $h(x) - c$ is the graph of $h(x)$ **shifted downward** by *c* units.

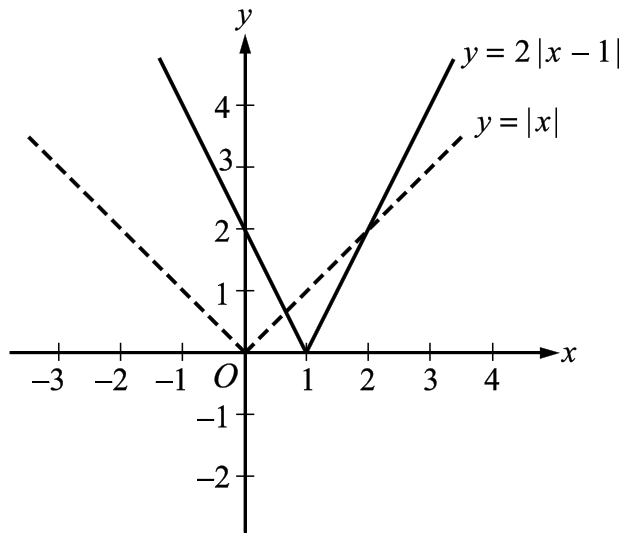
The graph of $h(x + c)$ is the graph of $h(x)$ **shifted to the left** by *c* units.

The graph of $h(x - c)$ is the graph of $h(x)$ **shifted to the right** by *c* units.

Example 2.9.7: Consider the functions defined by $f(x) = 2|x - 1|$ and $g(x) = -\frac{x^2}{4}$.

These functions are related to the absolute value function $|x|$ and the quadratic function x^2, respectively, in more complicated ways than in the preceding two examples.

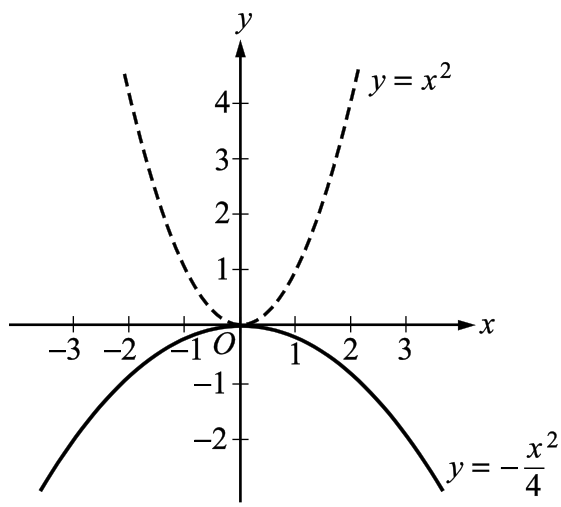
The graph of $f(x) = 2|x-1|$ is the graph of $y = |x|$ shifted to the right by 1 unit and then stretched, or dilated, vertically away from the x-axis by a factor of 2, as shown in Algebra Figure 14 below.



Algebra Figure 14

Similarly, the graph of the function $h(x) = \frac{1}{2}|x-1|$ is the graph of $y = |x|$ shifted to the right by 1 unit and then shrunk, or contracted, vertically toward the x-axis by a factor of $\frac{1}{2}$. (The graph of this function is not shown.)

The graph of $g(x) = -\frac{x^2}{4}$ is the graph of $y = x^2$ contracted vertically toward the x-axis by a factor of $\frac{1}{4}$ and then reflected in the x-axis, as shown in Algebra Figure 15 below.



Algebra Figure 15

In general, for any function $h(x)$ and any positive number c, the following are true.

The graph of $ch(x)$ is the graph of $h(x)$ **stretched vertically** by a factor of c if $c > 1$.

The graph of $ch(x)$ is the graph of $h(x)$ **shrunk vertically** by a factor of c if $0 < c < 1$.

ALGEBRA EXERCISES

Exercise 1. Find an algebraic expression to represent each of the following.

(a) The square of y is subtracted from 5, and the result is multiplied by 37.

(b) Three times x is squared, and the result is divided by 7.

(c) The product of $x + 4$ and y is added to 18.

Exercise 2. Simplify each of the following algebraic expressions.

(a) $3x^2 - 6 + x + 11 - x^2 + 5x$

(b) $3(5x - 1) - x + 4$

(c) $\dfrac{x^2 - 16}{x - 4}$, where $x \neq 4$

(d) $(2x + 5)(3x - 1)$

Exercise 3.

(a) What is the value of $f(x) = 3x^2 - 7x + 23$, when $x = -2$?

(b) What is the value of $h(x) = x^3 - 2x^2 + x - 2$, when $x = 2$?

(c) What is the value of $k(x) = \dfrac{5}{3}x - 7$ when $x = 0$?

Exercise 4. If the function g is defined for all nonzero numbers y by $g(y) = \frac{y}{|y|}$, find the value of each of the following.

(a) $g(2)$

(b) $g(-2)$

(c) $g(2) - g(-2)$

Exercise 5. Use the rules of exponents to simplify the following.

(a) $\left(n^5\right)\left(n^{-3}\right)$

(b) $\left(s^7\right)\left(t^7\right)$

(c) $\frac{r^{12}}{r^4}$

(d) $\left(\frac{2a}{b}\right)^5$

(e) $\left(w^5\right)^{-3}$

(f) $\left(5^0\right)\left(d^3\right)$

(g) $\frac{\left(x^{10}\right)\left(y^{-1}\right)}{\left(x^{-5}\right)\left(y^5\right)}$

(h) $\left(\frac{3x}{y}\right)^2 \div \left(\frac{1}{y}\right)^5$

Exercise 6. Solve each of the following equations for x.

(a) $5x - 7 = 28$

(b) $12 - 5x = x + 30$

(c) $5(x + 2) = 1 - 3x$

(d) $(x + 6)(2x - 1) = 0$

(e) $x^2 + 5x - 14 = 0$

(f) $x^2 - x - 1 = 0$

Exercise 7. Solve each of the following systems of equations for x and y.

(a) $\begin{aligned} x + y &= 24 \\ x - y &= 18 \end{aligned}$

(b) $\begin{aligned} 3x - y &= -5 \\ x + 2y &= 3 \end{aligned}$

(c) $\begin{aligned} 15x - 18 - 2y &= -3x + y \\ 10x + 7y + 20 &= 4x + 2 \end{aligned}$

Exercise 8. Solve each of the following inequalities for x.

(a) $-3x > 7 + x$

(b) $25x + 16 \geq 10 - x$

(c) $16 + x > 8x - 12$

Exercise 9. For a given two-digit positive integer, the tens digit is 5 more than the units digit. The sum of the digits is 11. Find the integer.

Exercise 10. If the ratio of $2x$ to $5y$ is 3 to 4, what is the ratio of x to y ?

Exercise 11.. Kathleen's weekly salary was increased by 8 percent to $712.80. What was her weekly salary before the increase?

Exercise 12. A theater sells children's tickets for half the adult ticket price. If 5 adult tickets and 8 children's tickets cost a total of $81, what is the cost of an adult ticket?

Exercise 13. Pat invested a total of $3,000. Part of the money was invested in a money market account that paid 10 percent simple annual interest, and the remainder of the money was invested in a fund that paid 8 percent simple annual interest. If the total interest earned at the end of the first year from these investments was $256, how much did Pat invest at 10 percent and how much at 8 percent?

Exercise 14. Two cars started from the same point and traveled on a straight course in opposite directions for 2 hours, at which time they were 208 miles apart. If one car traveled, on average, 8 miles per hour faster than the other car, what was the average speed of each car for the 2-hour trip?

Exercise 15. A group can charter a particular aircraft at a fixed total cost. If 36 people charter the aircraft rather than 40 people, then the cost per person is greater by $12.

(a) What is the fixed total cost to charter the aircraft?

(b) What is the cost per person if 40 people charter the aircraft?

Exercise 16. An antiques dealer bought c antique chairs for a total of x dollars. The dealer sold each chair for y dollars.

(a) Write an algebraic expression for the profit, P, earned from buying and selling the chairs.

(b) Write an algebraic expression for the profit per chair.

Exercise 17. Algebra Figure 16 that follows shows right triangle *PQR* in the *xy*-plane. Find the following.

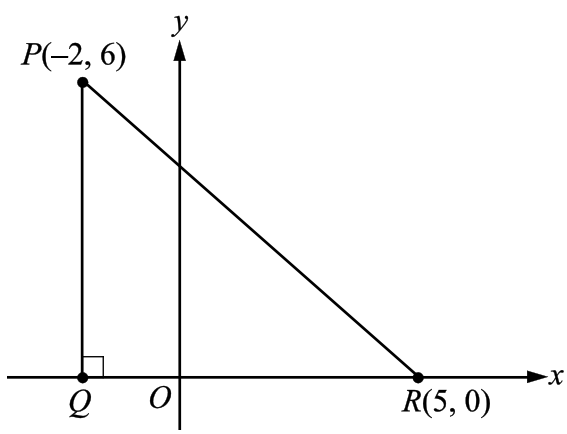
(a) The coordinates of point *Q*

(b) The lengths of line segment *PQ*, line segment *QR*, and line segment *PR*

(c) The perimeter of triangle *PQR*

(d) The area of triangle *PQR*

(e) The slope, *y*-intercept, and equation of the line passing through points *P* and *R*



Algebra Figure 16

Exercise 18. In the *xy*-plane, find the following.

(a) The slope and *y*-intercept of the line with equation $2y + x = 6$

(b) The equation of the line passing through the point $(3, 2)$ with *y*-intercept 1

(c) The *y*-intercept of a line with slope 3 that passes through the point $(-2, 1)$

(d) The *x*-intercepts of the graphs in parts (a), (b), and (c)

Exercise 19. For the parabola $y = x^2 - 4x - 12$ in the *xy*-plane, find the following.

(a) The *x*-intercepts

(b) The *y*-intercept

(c) The coordinates of the vertex

Exercise 20. For the circle $(x-1)^2 + (y+1)^2 = 20$ in the *xy*-plane, find the following.

(a) The coordinates of the center

(b) The radius

(c) The area

Exercise 21. For each of the following functions, give the domain and a description of the graph $y = f(x)$ in the *xy*-plane, including its shape, and the *x*- and *y*- intercepts.

(a) $f(x) = -4$

(b) $f(x) = 100 - 900x$

(c) $f(x) = 5 - (x+20)^2$

(d) $f(x) = \sqrt{x+2}$

(e) $f(x) = x + |x|$

ANSWERS TO ALGEBRA EXERCISES

Exercise 1.

(a) $37\left(5 - y^2\right)$, or $185 - 37y^2$

(b) $\dfrac{(3x)^2}{7}$, or $\dfrac{9x^2}{7}$

(c) $18 + (x + 4)(y)$, or $18 + xy + 4y$

Exercise 2.

(a) $2x^2 + 6x + 5$

(b) $14x + 1$

(c) $x + 4$

(d) $6x^2 + 13x - 5$

Exercise 3.

(a) 49

(b) 0

(c) -7

Exercise 4.

(a) 1

(b) -1

(c) 2

Exercise 5.

(a) n^2

(b) $(st)^7$

(c) r^8

(d) $\frac{32a^5}{b^5}$

(e) $\frac{1}{w^{15}}$

(f) d^3

(g) $\frac{x^{15}}{y^6}$

(h) $9x^2y^3$

Exercise 6.

(a) 7

(b) -3

(c) $-\frac{9}{8}$

(d) The two solutions are -6 and $\frac{1}{2}$.

(e) The two solutions are -7 and 2.

(f) The two solutions are $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$.

Exercise 7.

(a) $x = 21$ and $y = 3$

(b) $x = -1$ and $y = 2$

(c) $x = \frac{1}{2}$ and $y = -3$

Exercise 8.

(a) $x < -\frac{7}{4}$

(b) $x \geq -\frac{3}{13}$

(c) $x < 4$

Exercise 9. 83

Exercise 10. 15 to 8

Exercise 11. $660

Exercise 12. $9

Exercise 13. $800 at 10% and $2,200 at 8%

Exercise 14. 48 miles per hour and 56 miles per hour

Exercise 15.

(a) \$4,320

(b) \$108

Exercise 16.

(a) $P = cy - x$

(b) Profit per chair: $\frac{P}{c} = \frac{cy - x}{c} = y - \frac{x}{c}$

Exercise 17.

(a) The coordinates of point Q are $(-2, 0)$.

(b) The length of PQ is 6, the length of QR is 7, and the length of PR is $\sqrt{85}$.

(c) $13 + \sqrt{85}$

(d) 21

(e) Slope: $-\frac{6}{7}$; y-intercept: $\frac{30}{7}$

equation of line: $y = -\frac{6}{7}x + \frac{30}{7}$, or $7y + 6x = 30$

Exercise 18.

(a) Slope: $-\frac{1}{2}$; y-intercept: 3

(b) $y = \frac{x}{3} + 1$

(c) 7

(d) $6,\ -3,$ and $-\frac{7}{3}$

Exercise 19.

(a) $x = -2$ and $x = 6$

(b) $y = -12$

(c) $(2, -16)$

Exercise 20.

(a) $(1, -1)$

(b) $\sqrt{20}$

(c) 20π

Exercise 21.

(a) Domain: the set of all real numbers. The graph is a horizontal line with y-intercept -4 and no x-intercept.

(b) Domain: the set of all real numbers. The graph is a line with slope -900, y-intercept 100, and x-intercept $\frac{1}{9}$.

(c) Domain: the set of all real numbers. The graph is a parabola opening downward with vertex at $(-20, 5)$, line of symmetry $x = -20$, y-intercept -395, and x-intercepts $-20 \pm \sqrt{5}$.

(d) Domain: the set of numbers greater than or equal to -2. The graph is the upper half of a parabola opening to the right with vertex at $(-2, 0)$, x-intercept -2, and y-intercept $\sqrt{2}$.

(e) Domain: the set of all real numbers. The graph is two half-lines joined at the origin: one half-line is the negative x-axis and the other is a line starting at the origin with slope 2. Every nonpositive number is an x-intercept, and the y-intercept is 0. The function is equal to the following piecewise-defined function

$$f(x) = \begin{cases} 2x, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

PART 3.

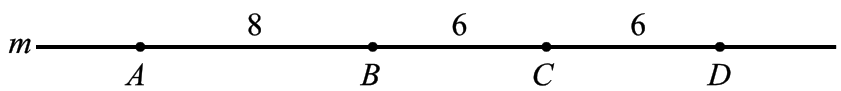
GEOMETRY

The review of geometry begins with lines and angles and progresses to other plane figures, such as polygons, triangles, quadrilaterals, and circles. The review of geometry ends with some basic three-dimensional figures. Coordinate geometry is covered in the Algebra part.

3.1 Lines and Angles

A **line** is understood to be a straight line that extends in both directions without ending. Given any two points on a line, a **line segment** is the part of the line that contains the two points and all the points between them. The two points are called **endpoints**. Line segments that have equal lengths are called **congruent line segments**. The point that divides a line segment into two congruent line segments is called the **midpoint** of the line segment.

In Geometry Figure 1 below, *A*, *B*, *C*, and *D* are points on line *m.*

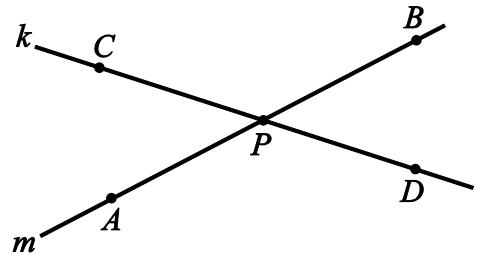


Geometry Figure 1

Line segment AB is the part of line *m* that consists of points *A* and *B* and all the points between *A* and *B*. According to Geometry Figure 1 above, the lengths of line segments AB, BC, and CD are 8, 6, and 6, respectively. Hence, line segments BC and CD are congruent. Since *C* is halfway between *B* and *D*, point *C* is the midpoint of line segment BD.

Sometimes the notation AB denotes line segment AB, and sometimes it denotes the **length** of line segment AB. The meaning of the notation can be determined from the context.

When two lines intersect at a point, they form four **angles**. Each angle has a **vertex** at the point of intersection of the two lines. For example, in Geometry Figure 2 below, lines k and m intersect at point P, forming the four angles APC, CPB, BPD, and DPA.

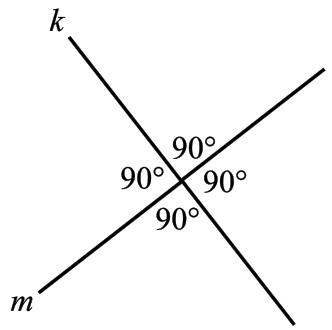


Geometry Figure 2

The first and third of the angles, that is, angles APC and BPD, are called **opposite angles**, also known as **vertical angles**. The second and fourth of the angles, that is, angles CPB and DPA, are also opposite angles. Opposite angles have equal measure, and angles that have equal measure are called **congruent angles**. Hence, opposite angles are congruent. The sum of the measures of the four angles is $360°$.

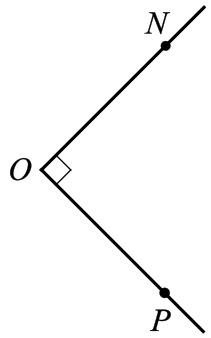
Sometimes the angle symbol $\angle$ is used instead of the word "angle." For example, angle APC can be written as $\angle APC$.

Two lines that intersect to form four congruent angles are called **perpendicular lines**. Each of the four angles has a measure of $90°$. An angle with a measure of $90°$ is called a **right angle**. Geometry Figure 3 below shows two lines, k and m, that are perpendicular, denoted by $k \perp m$.



Geometry Figure 3

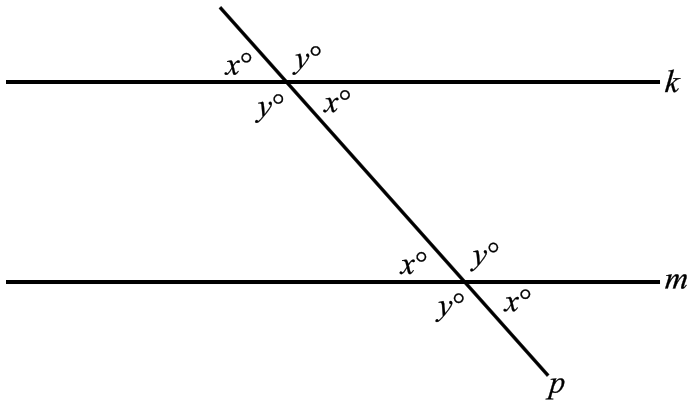
A common way to indicate that an angle is a right angle is to draw a small square at the vertex of the angle, as shown in Geometry Figure 4 below, where PON is a right angle.



Geometry Figure 4

An angle with measure less than $90°$ is called an **acute angle**, and an angle with measure between $90°$ and $180°$ is called an **obtuse angle**.

Two lines in the same plane that do not intersect are called **parallel lines**.
Geometry Figure 5 below shows two lines, k and m, that are parallel, denoted by $k \parallel m$.
The two lines are intersected by a third line, p, forming eight angles.

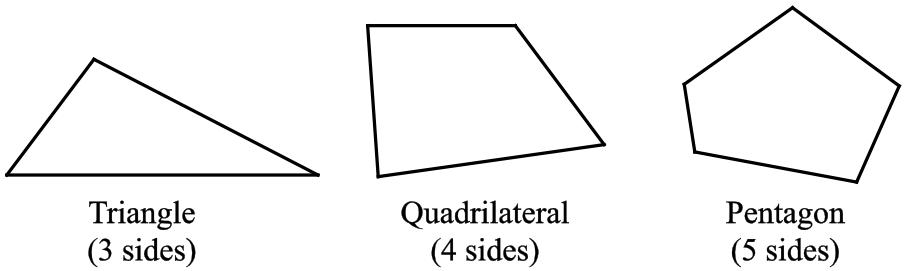


Geometry Figure 5

Note that four of the eight angles in Geometry Figure 5 have the measure $x°$, and the remaining four angles have the measure $y°$, where $x + y = 180$.

3.2 Polygons

A **polygon** is a closed figure formed by three or more line segments all of which are in the same plane. The line segments are called the **sides** of the polygon. Each side is joined to two other sides at its endpoints, and the endpoints are called **vertices**. In this discussion, the term "polygon" means "convex polygon," that is, a polygon in which the measure of each interior angle is less than 180°. Geometry Figure 6 below contains examples of a triangle, a quadrilateral, and a pentagon, all of which are convex.



Geometry Figure 6

The simplest polygon is a **triangle**. Note that a **quadrilateral** can be divided into 2 triangles by drawing a diagonal; and a **pentagon** can be divided into 3 triangles by

selecting one of the vertices and drawing 2 line segments connecting the selected vertex to the two nonadjacent vertices, as shown in Geometry Figure 7 below.

Geometry Figure 7

If a polygon has n sides, it can be divided into $n - 2$ triangles. Since the sum of the measures of the interior angles of a triangle is $180°$, it follows that the sum of the measures of the interior angles of an n-sided polygon is $(n - 2)(180°)$. For example, since a quadrilateral has 4 sides, the sum of the measures of the interior angles of a quadrilateral is $(4 - 2)(180°) = 360°$, and since a **hexagon** has 6 sides, the sum of the measures of the interior angles of a hexagon is $(6 - 2)(180°) = 720°$.

A polygon in which all sides are congruent and all interior angles are congruent is called a **regular polygon**. For example, since an **octagon** has 8 sides, the sum of the measures of the interior angles of an octagon is $(8 - 2)(180°) = 1{,}080°$. Therefore, in a **regular octagon** the measure of each angle is $\frac{1{,}080°}{8} = 135°$.

The **perimeter** of a polygon is the sum of the lengths of its sides. The **area** of a polygon refers to the area of the region enclosed by the polygon.

In the next two sections, we will look at some basic properties of triangles and quadrilaterals.

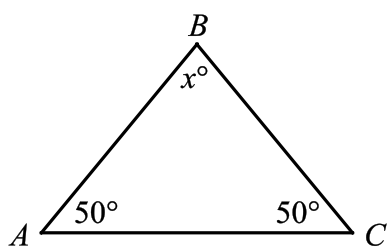
3.3 Triangles

Every triangle has three sides and three interior angles. The measures of the interior angles add up to $180°$. The length of each side must be less than the sum of the lengths of the other two sides. For example, the sides of a triangle could not have the lengths 4, 7, and 12 because 12 is greater than $4 + 7$.

The following are 3 types of special triangles.

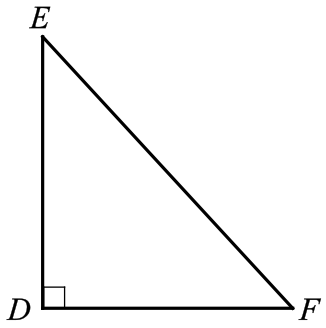
Type 1: A triangle with three congruent sides is called an **equilateral triangle**. The measures of the three interior angles of such a triangle are equal, and each measure is $60°$.

Type 2: A triangle with at least two congruent sides is called an **isosceles triangle**. If a triangle has two congruent sides, then the angles opposite the two congruent sides are congruent. The converse is also true. For example, in triangle ABC in Geometry Figure 8 below, the measure of angle A is $50°$, the measure of angle C is $50°$, and the measure of angle B is $x°$. Since both angle A and angle C have measure $50°$, it follows that the length of AB is equal to the length of BC. Also, since the sum of the three angles of a triangle is $180°$, it follows that $50 + 50 + x = 180$, and the measure of angle B is $80°$.



Geometry Figure 8

Type 3: A triangle with an interior right angle is called a **right triangle**. The side opposite the right angle is called the **hypotenuse**; the other two sides are called **legs**. For example, in right triangle DEF in Geometry Figure 9 below, side EF is the side opposite right angle D; therefore EF is the hypotenuse and DE and DF are legs.



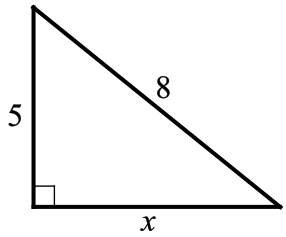
Geometry Figure 9

The Pythagorean Theorem

The **Pythagorean theorem** states that in a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs. Thus, for triangle DEF in Geometry Figure 9,

$$(EF)^2 = (DE)^2 + (DF)^2$$

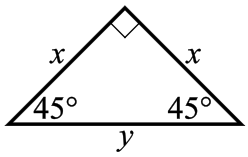
This relationship can be used to find the length of one side of a right triangle if the lengths of the other two sides are known. For example, consider a right triangle with hypotenuse of length 8, a leg of length 5, and another leg of unknown length x, as shown in Geometry Figure 10 below.



Geometry Figure 10

By the Pythagorean theorem, $8^2 = 5^2 + x^2$. Therefore, $64 = 25 + x^2$ and $39 = x^2$. Since $x^2 = 39$ and x must be positive, it follows that $x = \sqrt{39}$, or approximately 6.2.

The Pythagorean theorem can be used to determine the ratios of the lengths of the sides of two special right triangles. One special right triangle is an isosceles right triangle, as shown in Geometry Figure 11 below.



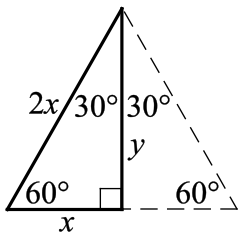
Geometry Figure 11

In Geometry Figure 11, the hypotenuse of the right triangle is of length *y,* both legs are of length *x*, and the angles opposite the legs are both 45-degree angles.

Applying the Pythagorean theorem to the isosceles right triangle in Geometry Figure 11 above shows that the lengths of its sides are in the ratio 1 to 1 to $\sqrt{2}$, as follows.

> By the Pythagorean theorem, $y^2 = x^2 + x^2$. Therefore $y^2 = 2x^2$ and $y = \sqrt{2}x$. So the lengths of the sides are in the ratio x to x to $\sqrt{2}x$, which is the same as the ratio 1 to 1 to $\sqrt{2}$.

The other special right triangle is a $30°\text{-}60°\text{-}90°$ right triangle, which is half of an equilateral triangle, as shown in Geometry Figure 12 below.



Geometry Figure 12

Note that the length of the horizontal side, x, is one-half the length of the hypotenuse, $2x$. Applying the Pythagorean theorem to the $30°\text{-}60°\text{-}90°$ right triangle shows that the lengths of its sides are in the ratio 1 to $\sqrt{3}$ to 2, as follows.

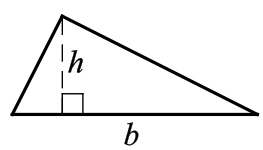
By the Pythagorean theorem, $x^2 + y^2 = (2x)^2$, which simplifies to $x^2 + y^2 = 4x^2$. Subtracting x^2 from both sides gives $y^2 = 4x^2 - x^2$ or $y^2 = 3x^2$. Therefore, $y = \sqrt{3}x$. Hence, the ratio of the lengths of the three sides of a $30°$-$60°$-$90°$ right triangle is x to $\sqrt{3}x$ to $2x$, which is the same as the ratio 1 to $\sqrt{3}$ to 2.

The Area of a Triangle

The **area** A of a triangle is given by the formula

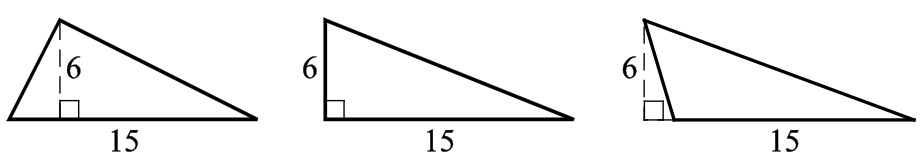
$$A = \frac{bh}{2}$$

where b is the length of a base, and h is the length of the corresponding height. Geometry Figure 13 below shows a triangle: the length of the horizontal base of the triangle is denoted by b and the length of the corresponding vertical height is denoted by h.



Geometry Figure 13

Any side of a triangle can be used as a base; the height that corresponds to the base is the perpendicular line segment from the opposite vertex to the base (or an extension of the base). Depending on the context, the term "base" can also refer to the *length* of a side of the triangle, and the term "height" can refer to the *length* of the perpendicular line segment from the opposite vertex to that side. The examples in Geometry Figure 14 below show three different configurations of a base and the corresponding height.



Geometry Figure 14

In all three triangles in Geometry Figure 14 above, the area is $\frac{(15)(6)}{2}$, or 45.

Congruent Triangles and Similar Triangles

Two triangles that have the same shape and size are called **congruent triangles**. More precisely, two triangles are congruent if their vertices can be matched up so that the corresponding angles and the corresponding sides are congruent.

By convention, the statement "triangles PQR and STU are congruent" does not just tell you that the two triangles are congruent, it also tells you what the corresponding parts of the two triangles are. In particular, because the letters in the name of the first triangle are given in the order PQR, and the letters in the name of the second triangle are given in the order STU, the statement tells you that angle P is congruent to angle S, angle Q is congruent to angle T, and angle R is congruent to angle U. It also tells you that sides PQ, QR, and PR in triangle PQR are congruent to sides ST, TU, and SU in triangle STU, respectively.

The following three propositions can be used to determine whether two triangles are congruent by comparing only some of their sides and angles.

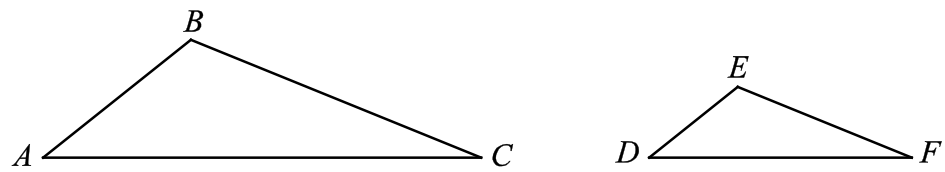
Proposition 1: If the three sides of one triangle are congruent to the three sides of another triangle, then the triangles are congruent. This proposition is called Side-Side-Side, or SSS, congruence.

Proposition 2: If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, then the triangles are congruent. This proposition is called Side-Angle-Side, or SAS, congruence.

Proposition 3: If two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, then the triangles are congruent. This proposition is called Angle-Side-Angle, or ASA, congruence. Note that if two angles of one triangle are congruent to two angles of another triangle, then the remaining angles are also congruent to each other, since the sum of the angle measures in any triangle is 180 degrees. Therefore, a similar proposition, called Angle-Angle-Side, or AAS, congruence, follows from ASA congruence.

Two triangles that have the same shape but not necessarily the same size are called **similar triangles**. More precisely, two triangles are similar if their vertices can be matched up so that the corresponding angles are congruent or, equivalently, the lengths of the corresponding sides have the same ratio, called the **scale factor of similarity**. For example, all $30°$-$60°$-$90°$ right triangles are similar triangles, though they may differ in size.

Geometry Figure 15 below shows two similar triangles, triangle ABC and triangle DEF.



Geometry Figure 15

As with the convention for congruent triangles, the letters in similar triangles ABC and DEF indicate their corresponding parts.

Since triangles ABC and DEF are similar, we have $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$. By cross multiplication, we can obtain other proportions, such as $\frac{AB}{BC} = \frac{DE}{EF}$.

3.4 Quadrilaterals

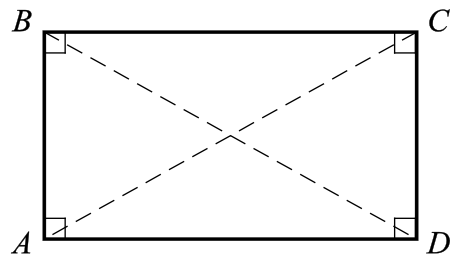
Every quadrilateral has four sides and four interior angles. The measures of the interior angles add up to $360°$.

<u>Special Types of Quadrilaterals</u>

The following are four special types of quadrilaterals.

Type 1: A quadrilateral with four right angles is called a **rectangle**. Opposite sides of a rectangle are parallel and congruent, and the two diagonals are also congruent.

Geometry Figure 16 below shows rectangle $ABCD$. In rectangle $ABCD$, opposite sides AD and BC are parallel and congruent, opposite sides AB and DC are parallel and congruent, and diagonal AC is congruent to diagonal BD.



Geometry Figure 16

Type 2: A rectangle with four congruent sides is called a **square**.

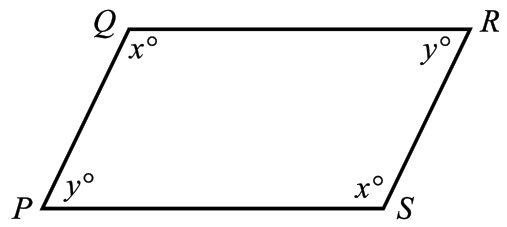
Type 3: A quadrilateral in which both pairs of opposite sides are parallel is called a **parallelogram**. In a parallelogram, opposite sides are congruent and opposite angles are congruent.

Note that all rectangles are parallelograms.

Geometry Figure 17 below shows parallelogram $PQRS$. In parallelogram $PQRS$:

Opposite sides PQ and SR are parallel and congruent.
Opposite sides QR and PS are parallel and congruent.
Opposite angles Q and S are congruent.
Opposite angles P and R are congruent.

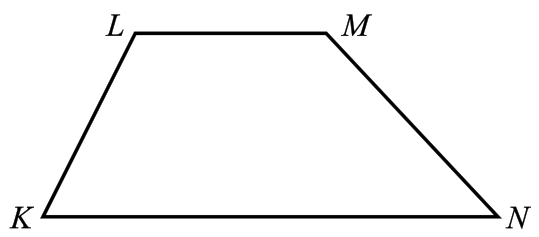
In the figure, angles Q and S are both labeled $x°$, and angles P and R are both labeled $y°$.



Geometry Figure 17

Type 4: A quadrilateral in which at least one pair of opposite sides is parallel is called a **trapezoid**. Two opposite, parallel sides of the trapezoid are called **bases** of the trapezoid.

Geometry Figure 18 below shows trapezoid $KLMN$. In trapezoid $KLMN$, horizontal side KN is parallel to horizontal side LM. Sides KN and LM are the bases of the trapezoid.



Geometry Figure 18

The Areas of the Special Types of Quadrilaterals

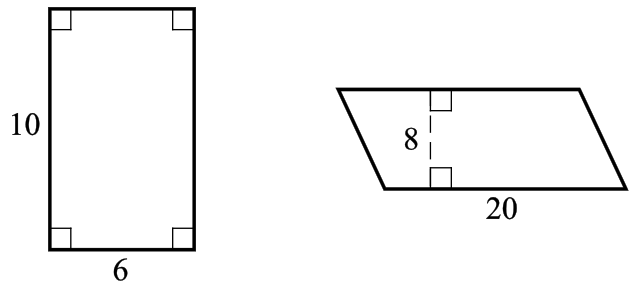
For all parallelograms, including rectangles and squares, the **area** A is given by the formula

$$A = bh$$

where b is the length of a base and h is the length of the corresponding height.

Any side of a parallelogram can be used as a base. The height corresponding to the base is the perpendicular line segment from any point on the side opposite the base to the base (or an extension of that base). Depending on the context, the term "base" can also refer to

the *length* of a side of the parallelogram, and the term "height" can refer to the *length* of the perpendicular line segment from that side to the opposite side. Examples of finding the areas of a rectangle and a parallelogram are shown in Geometry Figure 19 below.



$A = (6)(10) = 60$ $\qquad$ $A = (20)(8) = 160$

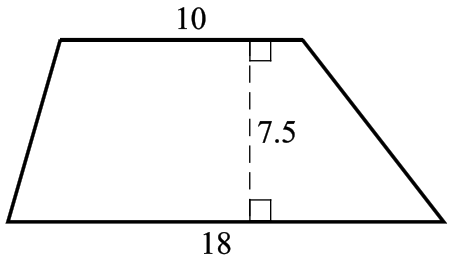
Geometry Figure 19

The **area** A of a trapezoid is given by the formula

$$A = \frac{1}{2}(b_1 + b_2)(h)$$

where b_1 and b_2 are the lengths of the bases of the trapezoid, and h is the corresponding height. For example, for the trapezoid in Geometry Figure 20 below with bases of length 10 and 18 and a height of 7.5, the area is

$$\frac{1}{2}(10 + 18)(7.5) = 105$$

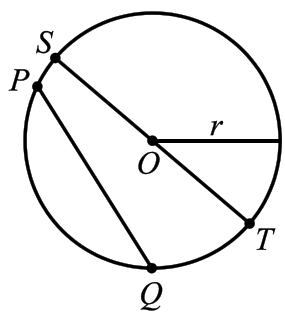


Geometry Figure 20

3.5 Circles

Given a point O in a plane and a positive number r, the set of points in the plane that are a distance of r units from O is called a **circle**. The point O is called the **center** of the circle and the distance r is called the **radius** of the circle. The **diameter** of the circle is twice the radius. Two circles with equal radii are called **congruent circles**.

Any line segment joining two points on the circle is called a **chord**. The terms "radius" and "diameter" can also refer to line segments: A **radius** is any line segment joining a point on the circle and the center of the circle, and a **diameter** is a chord that passes through the center of the circle. In Geometry Figure 21 below, O is the center of the circle, r is length of a radius, PQ is a chord, and ST is a diameter, as well as a chord.



Geometry Figure 21

The distance around a circle is called the **circumference** of the circle, which is analogous to the perimeter of a polygon. The ratio of the circumference C to the diameter d is the same for all circles and is denoted by the Greek letter π; that is,

$$\frac{C}{d} = \pi$$

The value of π is approximately 3.14 and can also be approximated by the fraction $\frac{22}{7}$.

If r is the radius of a circle, then $\frac{C}{d} = \frac{C}{2r} = \pi$, and so the circumference is related to the radius by the equation

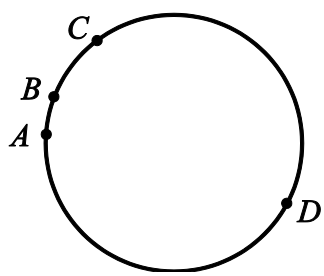
$$C = 2\pi r$$

For example, if a circle has a radius of 5.2, then its circumference is

$$(2)(\pi)(5.2) = (10.4)(\pi) \approx (10.4)(3.14)$$

which is approximately 32.7.

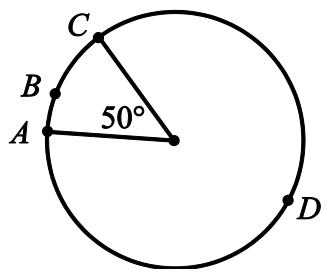
Given any two points on a circle, an **arc** is the part of the circle containing the two points and all the points between them. Two points on a circle are always the endpoints of two arcs. An arc is frequently identified by three points to avoid ambiguity. In Geometry Figure 22 below, there are four points on a circle. Going clockwise around the circle, the four points are *A*, *B*, *C*, and *D*. There are two different arcs between points *A* and *C*: arc ABC is the shorter arc between *A* and *C*, and arc ADC is the longer arc between *A* and *C*.



Geometry Figure 22

A **central angle** of a circle is an angle with its vertex at the center of the circle. The **measure of an arc** is the measure of its central angle, which is the angle formed by two radii that connect the center of the circle to the two endpoints of the arc. An entire circle is considered to be an arc with measure $360°$.

In Geometry Figure 23 below, there are four points on a circle: points *A*, *B*, *C*, and *D*. It is given that the radius of the circle is 5.



Geometry Figure 23

In Geometry Figure 23, the measure of the shorter arc between points A and C, that is arc ABC, is $50°$; and the measure of the longer arc, between points A and C, that is arc ADC, is $310°$.

To find the **length of an arc** of a circle, note that the ratio of the length of an arc to the circumference is equal to the ratio of the degree measure of the arc to $360°$. For example, since the radius of the circle in Geometry Figure 23 is 5, the circumference of the circle is 10π. Therefore,

$$\frac{\text{length of arc } ABC}{10\pi} = \frac{50}{360}$$

Multiplying both sides by 10π gives

$$\text{length of arc } ABC = \left(\frac{50}{360}\right)(10\pi)$$

Then, since

$$\left(\frac{50}{360}\right)(10\pi) = \frac{25\pi}{18} \approx \frac{(25)(3.14)}{18} \approx 4.4$$

it follows that the length of arc ABC is approximately 4.4.

The **area** of a circle with radius r is equal to πr^2. For example, since the radius of the circle in Geometry Figure 23 above is 5, the area of the circle is $\pi\left(5^2\right) = 25\pi$.

A **sector** of a circle is a region bounded by an arc of the circle and two radii. To find the **area of a sector**, note that the ratio of the area of a sector of a circle to the area of the entire circle is equal to the ratio of the degree measure of its arc to $360°$. For example, in the circle in Geometry Figure 23 the region bounded by arc ABC and the two radii is a sector with central angle $50°$, and the radius of the circle is 5. Therefore, if S represents the area of the sector with central angle $50°$, then

$$\frac{S}{25\pi} = \frac{50}{360}$$

Multiplying both sides by 25π gives

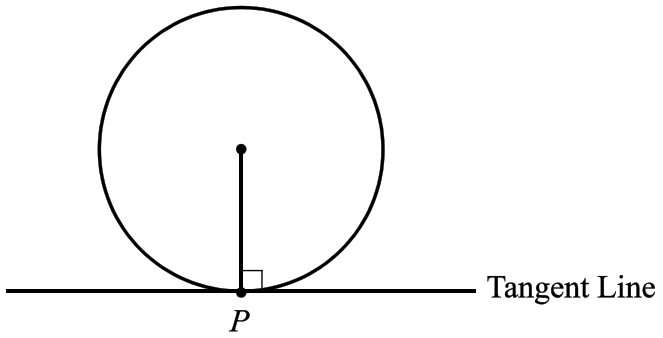
$$S = \left(\frac{50}{360}\right)(25\pi)$$

Then, since

$$\left(\frac{50}{360}\right)(25\pi) = \frac{125\pi}{36} \approx \frac{(125)(3.14)}{36} \approx 10.9$$

it follows that the area of the sector with central angle $50°$ is approximately 10.9.

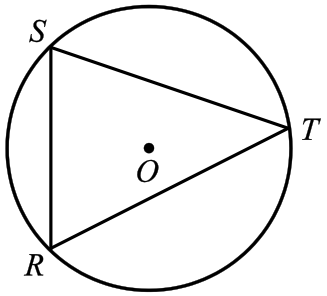
A **tangent** to a circle is a line that lies in the same plane as the circle and intersects the circle at exactly one point, called the **point of tangency**. If a line is tangent to a circle, then a radius drawn to the point of tangency is perpendicular to the tangent line. The converse is also true; that is, if a radius and a line intersect at a point on the circle and the line is perpendicular to the radius, then the line is a tangent to the circle at the point of intersection. Geometry Figure 24 below shows a circle, a line tangent to the circle at point *P*, and a radius drawn to point *P*.



Geometry Figure 24

A polygon is **inscribed** in a circle if all its vertices lie on the circle, or equivalently, the circle is **circumscribed** about the polygon.

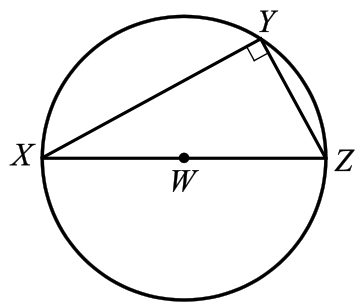
Geometry Figure 25 below shows triangle RST inscribed in a circle with center O. The center of the circle is inside the triangle.



Geometry Figure 25

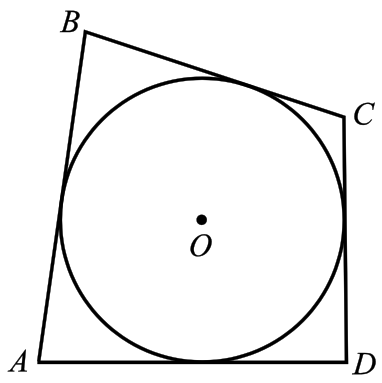
It is not always the case that if a triangle is inscribed in a circle, the center of the circle is inside the inscribed triangle. It is also possible for the center of the circle to be outside the inscribed triangle, or on one of the sides of inscribed triangle. Note that if the center of the circle is on one of the sides of the inscribed triangle, that side is a diameter of the circle.

If one side of an inscribed triangle is a diameter of the circle, then the triangle is a right triangle. Conversely, if an inscribed triangle is a right triangle, then one of its sides is a diameter of the circle. Geometry Figure 26 below shows right triangle XYZ inscribed in a circle with center W. In triangle XYZ, side XZ is a diameter of the circle and angle Y is a right angle.



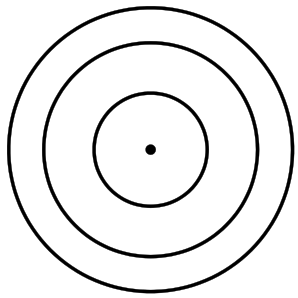
Geometry Figure 26

A polygon is circumscribed about a circle if each side of the polygon is tangent to the circle, or equivalently, the circle is inscribed in the polygon. Geometry Figure 27 below shows quadrilateral $ABCD$ circumscribed about a circle with center O.



Geometry Figure 27

Two or more circles with the same center are called **concentric circles**, as shown in Geometry Figure 28 below.

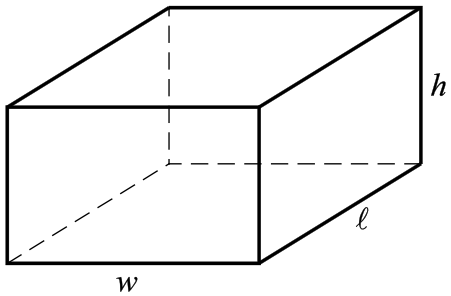


Geometry Figure 28

3.6 Three-Dimensional Figures

Basic three-dimensional figures include rectangular solids, cubes, cylinders, spheres, pyramids, and cones. In this section, we look at some properties of rectangular solids and right circular cylinders.

A **rectangular solid**, or **rectangular prism**, has 6 rectangular surfaces called **faces**, as shown in Geometry Figure 29 below. Adjacent faces are perpendicular to each other. Each line segment that is the intersection of two faces is called an **edge**, and each point at which the edges intersect is called a **vertex**. There are 12 edges and 8 vertices. The dimensions of a rectangular solid are the length ℓ, the width w, and the height h.



Geometry Figure 29

A rectangular solid with six square faces is called a **cube**, in which case $\ell = w = h$.

The **volume** V of a rectangular solid is the product of its three dimensions, or

$$V = \ell wh$$

The **surface area** A of a rectangular solid is the sum of the areas of the six faces, or

$$A = 2(\ell w + \ell h + wh)$$

For example, if a rectangular solid has length 8.5, width 5, and height 10, then its volume is

$$V = (8.5)(5)(10) = 425$$

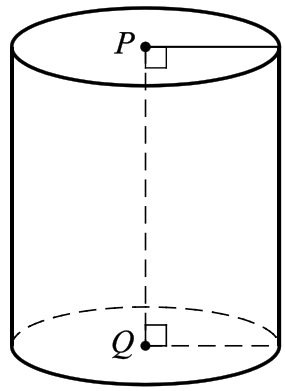
and its surface area is

$$A = 2((8.5)(5) + (8.5)(10) + (5)(10)) = 355$$

A **circular cylinder** consists of two bases that are congruent circles lying in parallel planes and a **lateral surface** made of all line segments that join points on the two circles and that are parallel to the line segment joining the centers of the two circles. The latter line segment is called the **axis** of the cylinder.

A **right circular cylinder** is a circular cylinder whose axis is perpendicular to its bases. The **height** of a right circular cylinder is the perpendicular distance between the two bases. Because the axis of a right circular cylinder is perpendicular to both bases, the height of a right circular cylinder is equal to the length of the axis.

The right circular cylinder shown in Geometry Figure 30 below has circular bases with centers P and Q. Line segment PQ is the axis of the cylinder and is perpendicular to both bases. The height of the cylinder is equal to the length of PQ.



Geometry Figure 30

The **volume** V of a right circular cylinder that has height h and a base with radius r is the product of the height and the area of the base, or

$$V = \pi r^2 h$$

The **surface area** A of a right circular cylinder is the sum of the areas of the two bases and the area of its lateral surface, or

$$A = 2\left(\pi r^2\right) + 2\pi rh$$

For example, if a right circular cylinder has height 6.5 and a base with radius 3, then its volume is

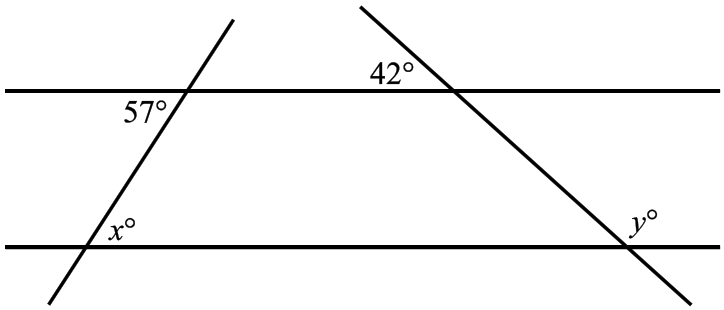
$$V = \pi\left(3^2\right)(6.5) = 58.5\pi$$

and its surface area is

$$A = 2\pi\left(3^2\right) + 2\pi(3)(6.5) = 57\pi$$

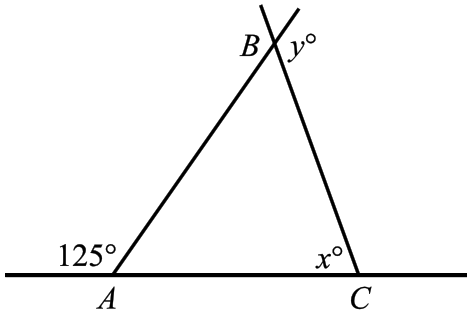
GEOMETRY EXERCISES

Exercise 1. In Geometry Figure 31 below, the two horizontal lines are parallel. Find the values of x and y.



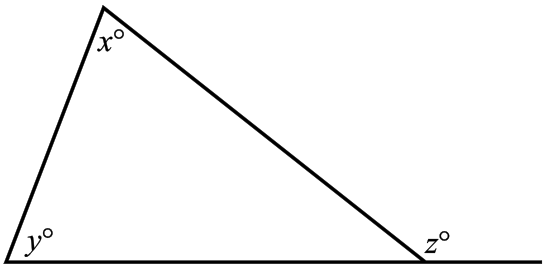
Geometry Figure 31

Exercise 2. In Geometry Figure 32 below, $AC = BC$. Find the values of x and y.



Geometry Figure 32

Exercise 3. In Geometry Figure 33 below, what is the relationship between x, y, and z ?



Geometry Figure 33

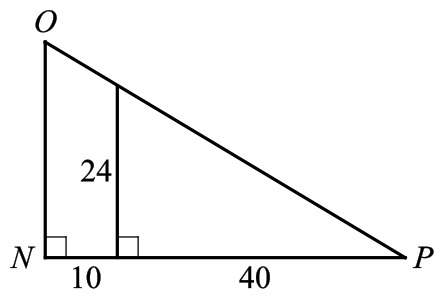
Exercise 4. What is the sum of the measures of the interior angles of a decagon (10-sided polygon) ?

Exercise 5. If the polygon in exercise 4 is regular, what is the measure of each interior angle?

Exercise 6. The lengths of two sides of an isosceles triangle are 15 and 22. What are the possible values of the perimeter?

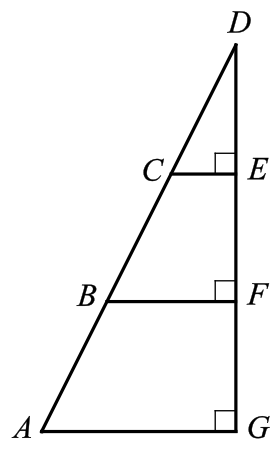
Exercise 7. Triangles PQR and XYZ are similar. If $PQ = 6$, $PR = 4$, and $XY = 9$, what is the length of side XZ ?

Exercise 8. What are the lengths of sides NO and OP of triangle NOP in Geometry Figure 34 below?



Geometry Figure 34

Exercise 9. In Geometry Figure 35 below, $AB = BC = CD$. If the area of triangle CDE is 42, what is the area of triangle ADG ?



Geometry Figure 35

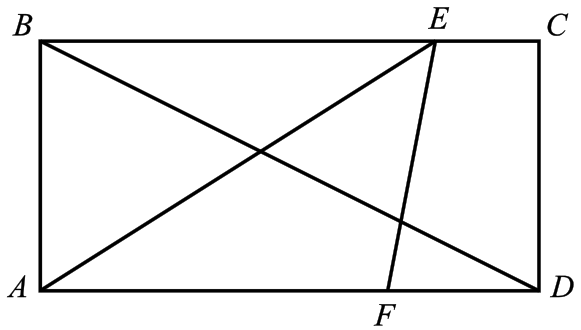
Exercise 10. In Geometry Figure 36 below, $ABCD$ is a rectangle, $AB = 5$, $AF = 7$, and $FD = 3$. Find the following.

(a) The area of rectangle $ABCD$

(b) The area of triangle AEF

(c) The length of diagonal BD

(d) The perimeter of rectangle $ABCD$



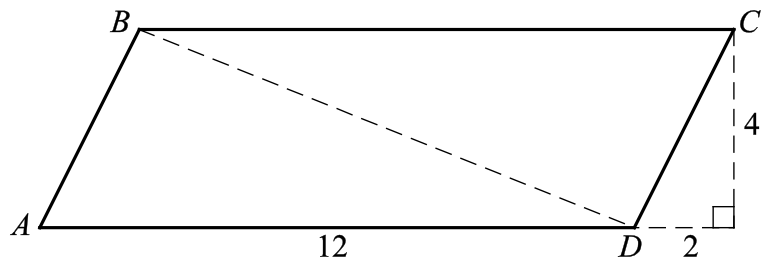
Geometry Figure 36

Exercise 11. In Geometry Figure 37 below, $ABCD$ is a parallelogram. Find the following.

(a) The area of $ABCD$

(b) The perimeter of $ABCD$

(c) The length of diagonal BD



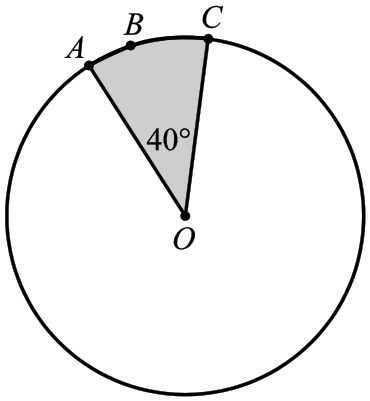
Geometry Figure 37

Exercise 12. In Geometry Figure 38 below, the circle with center O has radius 4. Find the following.

(a) The circumference of the circle

(b) The length of arc ABC

(c) The area of the shaded region



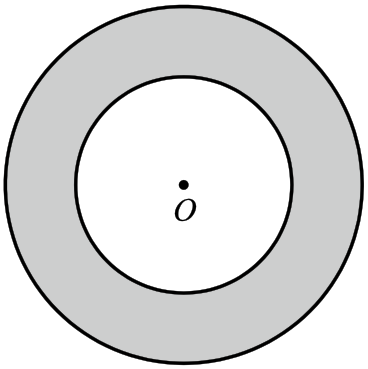
Geometry Figure 38

Exercise 13. Geometry Figure 39 below shows two concentric circles, each with center *O*. Given that the larger circle has radius 12 and the smaller circle has radius 7, find the following.

(a) The circumference of the larger circle

(b) The area of the smaller circle

(c) The area of the shaded region

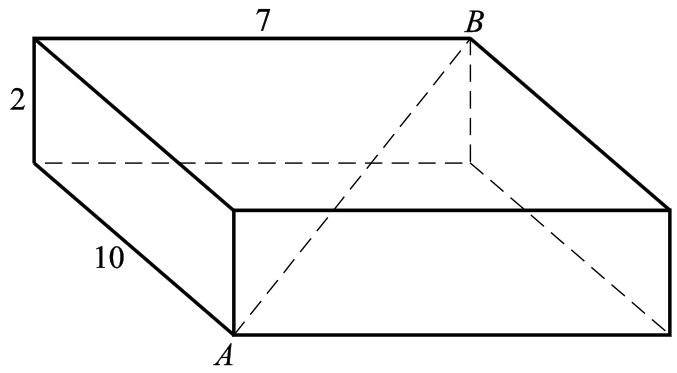


Geometry Figure 39

Exercise 14. For the rectangular solid in Geometry Figure 40 below, find the following.

(a) The surface area of the solid

(b) The length of diagonal AB



Geometry Figure 40

ANSWERS TO GEOMETRY EXERCISES

Exercise 1. $x = 57$ and $y = 138$

Exercise 2. $x = 70$ and $y = 125$

Exercise 3. $z = x + y$

Exercise 4. $1{,}440°$

Exercise 5. $144°$

Exercise 6. 52 and 59

Exercise 7. 6

Exercise 8. The length of side NO is 30 and the length of side OP is $10\sqrt{34}$.

Exercise 9. 378

Exercise 10.

(a) 50

(b) 17.5

(c) $5\sqrt{5}$

(d) 30

Exercise 11.

(a) 48

(b) $24 + 4\sqrt{5}$

(c) $2\sqrt{29}$

Exercise 12.

(a) 8π

(b) $\frac{8\pi}{9}$

(c) $\frac{16\pi}{9}$

Exercise 13.

(a) 24π

(b) 49π

(c) 95π

Exercise 14.

(a) 208

(b) $3\sqrt{17}$

PART 4.

DATA ANALYSIS

The review of data analysis begins with methods for presenting data, followed by counting methods and probability, and then progresses to distributions of data, random variables, and probability distributions. The review of data analysis ends with examples of data interpretation.

4.1 Methods for Presenting Data

Data can be organized and presented using a variety of methods. Tables are commonly used, and there are many graphical and numerical methods as well. In this section, we review tables and some common graphical methods for presenting and summarizing data.

In data analysis, a variable is any characteristic that can vary for a population of individuals or objects. Variables can be **quantitative**, or **numerical**, such as the age of individuals. Variables can also be **categorical**, or **nonnumerical**, such as the eye color of individuals.

Data are collected from a population by observing one or more variables. The **distribution of a variable**, or **distribution of data**, indicates how frequently different categorical or numerical data values are observed in the data.

Example 4.1.1: In a population of students in a sixth-grade classroom, a variable that can be observed is the height of each student. Note that the variable in this example is numerical.

Example 4.1.2: In a population of voters in a city's mayoral election, a variable that can be observed is the candidate that each voter voted for. Note that the variable in this example is nonnumerical.

The **frequency**, or **count**, of a particular category or numerical value is the number of times that the category or numerical value appears in the data. A **frequency distribution** is a table or graph that presents the categories or numerical values along with their corresponding frequencies. The **relative frequency** of a category or a numerical value is the corresponding frequency divided by the total number of data. Relative frequencies

may be expressed in terms of percents, fractions, or decimals. A **relative frequency distribution** is a table or graph that presents the relative frequencies of the categories or numerical values.

Tables

Tables are used to present a wide variety of data, including frequency distributions and relative frequency distributions. The rows and columns provide clear associations between categories and data. A frequency distribution is often presented as a 2-column table in which the categories or numerical values of the data are listed in the first column and the corresponding frequencies are listed in the second column. A relative frequency distribution table has the same layout but with relative frequencies instead of frequencies. When data include a large number of categories or numerical values, the categories or values are often grouped together in a smaller number of groups and the corresponding frequencies are given.

Example 4.1.3: A survey was taken to find the number of children in each of 25 families. A list of the 25 values collected in the survey follows.

1	2	0	4	1
3	3	1	2	0
4	5	2	3	2
3	2	4	1	2
3	0	2	3	1

Here are tables that present the resulting frequency distribution and relative frequency distribution of the data.

Frequency Distribution

Number of Children	Frequency
0	3
1	5
2	7
3	6
4	3
5	1
Total	25

Relative Frequency Distribution

Number of Children	Relative Frequency
0	12%
1	20%
2	28%
3	24%
4	12%
5	4%
Total	100%

Note that in the relative frequency distribution table the relative frequencies are expressed as percents and that the total for the relative frequencies is 100%. If the relative frequencies were expressed as decimals or fractions instead of percents, the total would be 1.

Example 4.1.4: Thirty students took a history test. Here is a list of the 30 scores on the test, from least to greatest.

62	63	68	70	72	72	72	75	76	76	76	76	78	78	82
82	85	85	85	85	85	86	87	88	91	91	92	95	97	100

The 30 students achieved 18 different scores on the test. Displaying the frequency distribution of this many different scores would make the frequency distribution table very large, so instead we group the scores into four groups: the scores from 61 to 70, the scores from 71 to 80, the scores from 81 to 90, and the scores from 91 to 100. Here is the frequency distribution of the scores with these groups.

Score	Frequency
61 to 70	4
71 to 80	10
81 to 90	10
91 to 100	6

In addition to being used to present frequency and relative frequency distributions, tables are used to display a wide variety of other data. Here are two examples.

Example 4.1.5: The following table shows the annual per capita income in a certain state, from 1930 to 1980.

Year	Annual Per Capita Income
1930	$656
1940	$680
1950	$1,717
1960	$2,437
1970	$4,198
1980	$10,291

Example 4.1.6: The following table shows the closest and farthest distance of the eight planets from the Sun, in millions of kilometers.

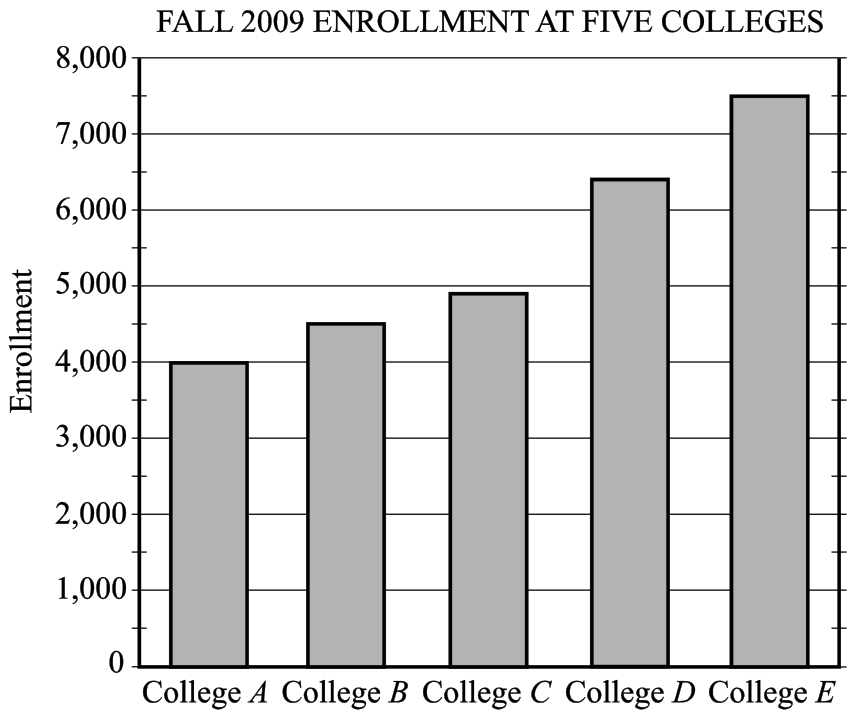
Planet	Closest Distance from the Sun (in millions of kilometers)	Farthest Distance from the Sun (in millions of kilometers)
Mercury	46	70
Venus	107	109
Earth	147	152
Mars	205	249
Jupiter	741	817
Saturn	1,350	1,510
Uranus	2,750	3,000
Neptune	4,450	4,550

Bar Graphs

A frequency distribution or relative frequency distribution of data collected from a population by observing one or more variables can be presented using a **bar graph**, or **bar chart**. In a bar graph, each of the data categories or numerical values is represented by a rectangular bar, and the height of each bar is proportional to the corresponding frequency or relative frequency. All of the bars are drawn with the same width, and the bars can be presented either vertically or horizontally. When data include a large number of different categories of numerical values, the categories or values are often grouped together in several groups and the corresponding frequencies or relative frequencies are given. Bar graphs enable comparisons across several categories more easily than tables do. For example, in a bar graph it is easy to identify the category with the greatest frequency by looking for the bar with the greatest height.

Here are two examples of frequency distributions presented as bar graphs.

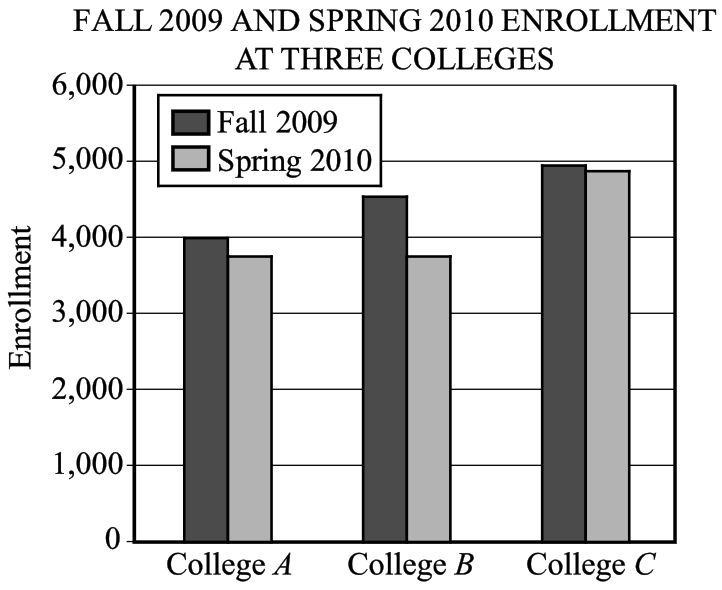
Example 4.1.7: Data Analysis Figure 1 is a bar graph with vertical bars. It shows the frequency distribution of one variable, fall 2009 enrollment. The variable is observed for five data categories, Colleges *A*, *B*, *C*, *D*, and *E*.



Data Analysis Figure 1

From the graph, we can conclude that the college with the greatest fall 2009 enrollment was College *E* and the college with the least enrollment was College *A*. Also, we can estimate that the enrollment for College *D* was about 6,400.

Example 4.1.8: Data Analysis Figure 2 is a bar graph with vertical bars. It shows the frequency distributions of two variables, fall 2009 enrollment and spring 2010 enrollment. Both variables are observed for three data categories, Colleges *A*, *B*, and *C*.



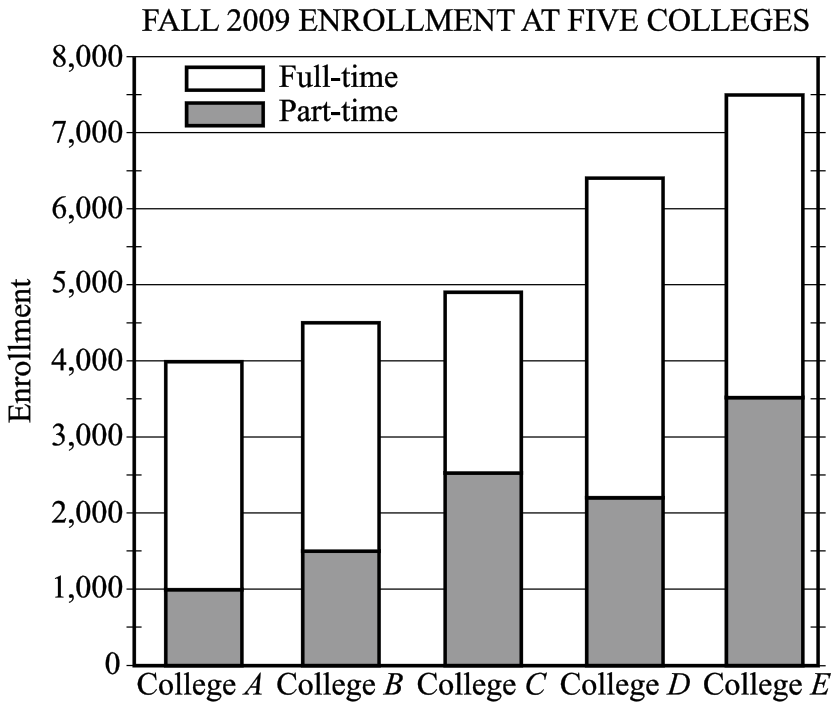
Data Analysis Figure 2

Observe that for all three colleges, the fall 2009 enrollment was greater than the spring 2010 enrollment. Also, the greatest decrease in the enrollment from fall 2009 to spring 2010 occurred at College *B*.

Segmented Bar Graphs

A **segmented bar graph**, or **stacked bar graph**, is similar to a regular bar graph except that in a segmented bar graph, each rectangular bar is divided, or segmented, into smaller rectangles that show how the variable is "separated" into other related variables. For example, rectangular bars representing enrollment can be divided into two smaller rectangles, one representing full-time enrollment and the other representing part-time enrollment, as shown in the following example.

Example 4.1.9: In Data Analysis Figure 3, the fall 2009 enrollment at the five colleges shown in Data Analysis Figure 1 is presented again, but this time each bar has been divided into two segments: one representing full-time enrollment and one representing part-time enrollment.



Data Analysis Figure 3

The total enrollment, the full-time enrollment, and the part-time enrollment at the five colleges can be estimated from the segmented bar graph in Data Analysis Figure 3. For example, for College *D*, the total enrollment was a little below 6,500, or approximately 6,400 students; the part-time enrollment was approximately 2,200; and the full-time enrollment was approximately $6{,}400 - 2{,}200$, or 4,200 students.

Although bar graphs are commonly used to compare frequencies, as in the examples above, they are sometimes used to compare numerical data that could be displayed in a table, such as temperatures, dollar amounts, percents, heights, and weights. Also, the categories sometimes are numerical in nature, such as years or other time intervals.

Histograms

When a list of data is large and contains many different values of a numerical variable, it is useful to organize the data by grouping the values into intervals, often called classes. To do this, divide the entire interval of values into smaller intervals of equal length and then count the values that fall into each interval. In this way, each interval has a frequency and a relative frequency. The intervals and their frequencies (or relative

frequencies) are often displayed in a **histogram**. Histograms are graphs of frequency distributions that are similar to bar graphs, but they must have a number line for the horizontal axis, which represents the numerical variable. Also, in a histogram, there are no regular spaces between the bars. Any spaces between bars in a histogram indicate that there are no data in the intervals represented by the spaces.

Example 4.5.1 in Section 4.5 illustrates a histogram with 50 bars.

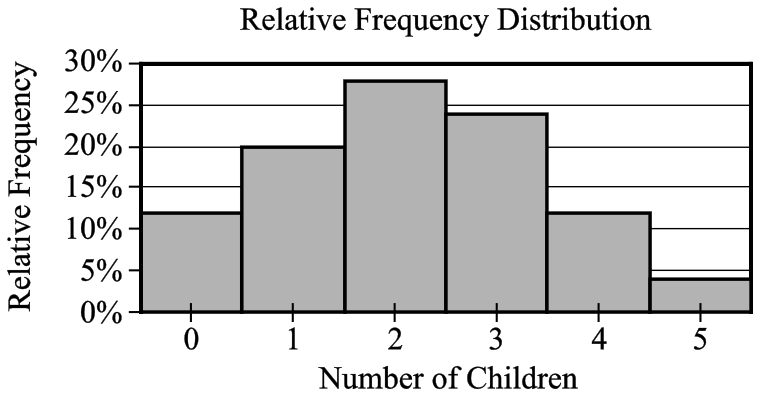
Numerical variables with just a few values can also be displayed using histograms, where the frequency or relative frequency of each value is represented by a bar centered over the value.

Example 4.1.10: In Example 4.1.3, the relative frequency distribution of the number of children of each of 25 families was presented as a 2-column table. For convenience, the table is repeated below.

Relative Frequency Distribution

Number of Children	Relative Frequency
0	12%
1	20%
2	28%
3	24%
4	12%
5	4%
Total	100%

This relative frequency distribution can also be displayed as a histogram, as shown in the following figure.



Data Analysis Figure 4

Histograms are useful for identifying the general shape of a distribution of data. Also evident are the "center" and degree of "spread" of the distribution, as well as high-frequency and low-frequency intervals. From the histogram in Data Analysis Figure 4 above, you can see that the distribution is shaped like a mound with one peak; that is, the data are frequent in the middle and sparse at both ends. The central values are 2 and 3, and the distribution is close to being symmetric about those values. Because the bars all have the same width, the area of each bar is proportional to the amount of data that the bar represents. Thus, the areas of the bars indicate where the data are concentrated and where they are not.

Finally, note that because each bar has a width of 1, the sum of the areas of the bars equals the sum of the relative frequencies, which is 100% or 1, depending on whether percents or decimals are used. This fact is central to the discussion of probability distributions in Section 4.5.

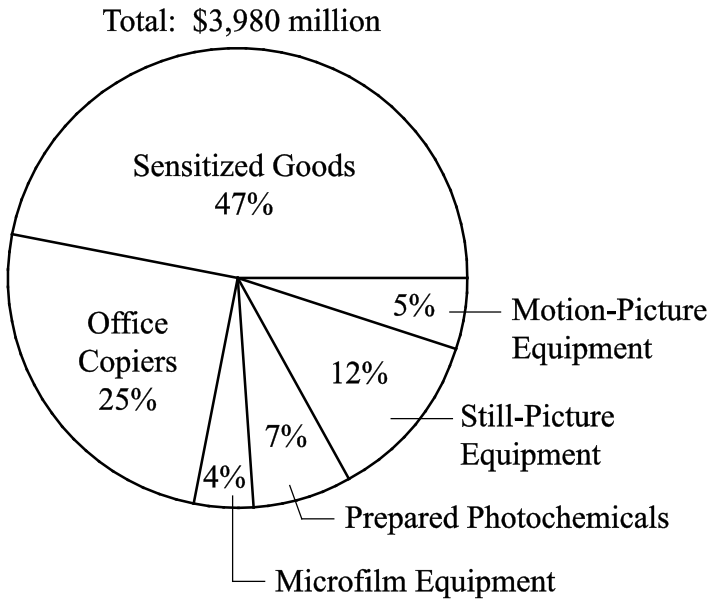
Circle Graphs

Circle graphs, often called **pie charts**, are used to represent data that have been separated into a small number of categories. They illustrate how a whole is separated into parts. The data are presented in a circle such that the area of the circle representing each category is proportional to the part of the whole that the category represents.

A circle graph may be used to represent a frequency distribution or a relative frequency distribution. More generally, a circle graph may represent any total amount that is distributed into a small number of categories, as in the following example.

Example 4.1.11:

UNITED STATES PRODUCTION OF PHOTOGRAPHIC
EQUIPMENT AND SUPPLIES IN 1971



Data Analysis Figure 5

From the graph you can see that Sensitized Goods is the category with the greatest dollar value.

Each part of a circle graph is called a **sector**. Because the area of each sector is proportional to the percent of the whole that the sector represents, the measure of the central angle of a sector is proportional to the percent of 360 degrees that the sector represents. For example, the measure of the central angle of the sector representing the category Prepared Photochemicals is 7 percent of 360 degrees, or 25.2 degrees.

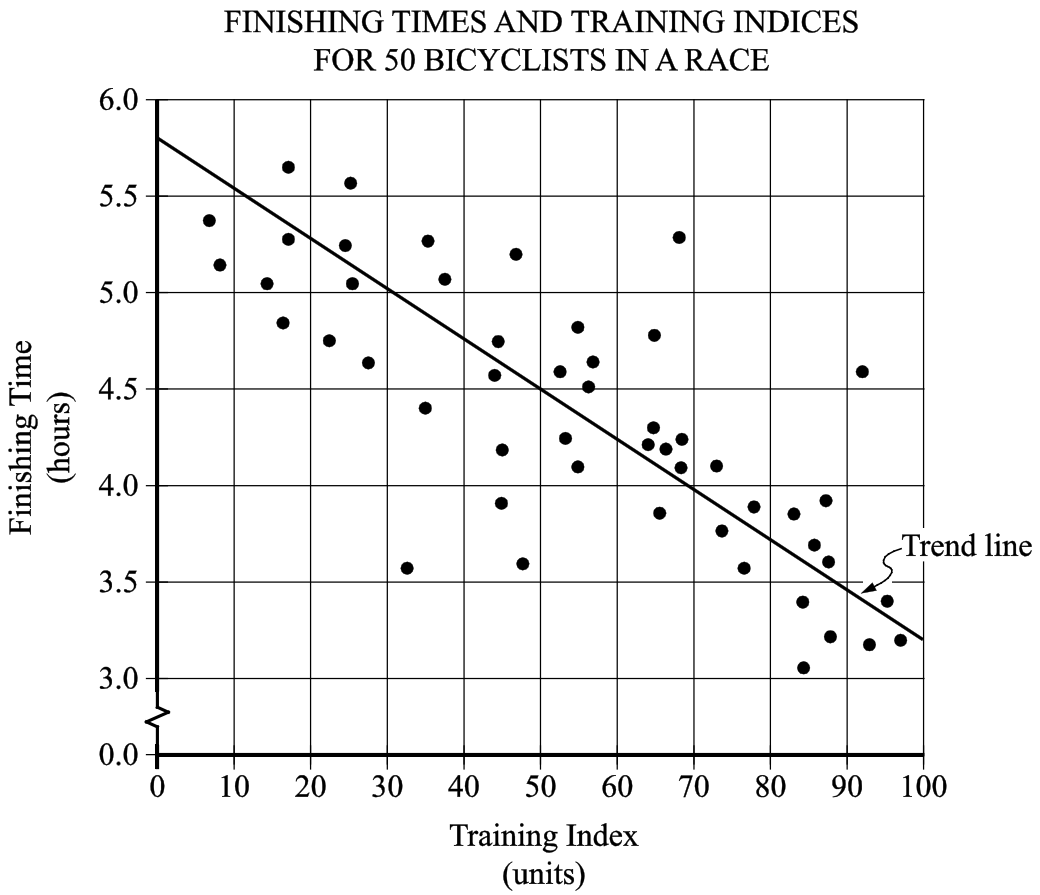
Scatterplots

A **scatterplot** is a type of graph that is useful for showing the relationship between two numerical variables whose values can be observed in a single population of individuals or objects. In a scatterplot, the values of one variable appear on the horizontal axis of a rectangular coordinate system and the values of the other variable appear on the vertical axis. For each individual or object in the data, an ordered pair of numbers is collected,

one number for each variable, and the pair is represented by a point in the coordinate system.

A scatterplot makes it possible to observe an overall pattern, or **trend**, in the relationship between the two variables. Also, the strength of the trend as well as striking deviations from the trend are evident. In many cases, a line or a curve that best represents the trend is also displayed in the graph and is used to make predictions about the population.

Example 4.1.12: A bicycle trainer studied 50 bicyclists to examine how the finishing time for a certain bicycle race was related to the amount of physical training each bicyclist did in the three months before the race. To measure the amount of training, the trainer developed a training index, measured in "units" and based on the intensity of each bicyclist's training. The data and the trend of the data, represented by a line, are displayed in the scatterplot in Data Analysis Figure 6 below.



Data Analysis Figure 6

When a trend line is included in the presentation of a scatterplot, you can observe how scattered or close the data are to the trend line, or to put it another way, how well the trend line fits the data. In the scatterplot in Data Analysis Figure 6 above, almost all of the data points are relatively close to the trend line. The scatterplot also shows that the finishing times generally decrease as the training indices increase.

The trend line can be used to make predictions. For example, it can be predicted, based on the trend line, that a bicyclist with a training index of 70 units will finish the race in approximately 4 hours. This value is obtained by noting that the vertical line at the training index of 70 units intersects the trend line very close to 4 hours.

Another prediction that can be made, based on the trend line, is the approximate number of minutes by which a bicyclist will lower his or her finishing time for each increase of 10 training index units. This prediction is derived from the ratio of the change in finishing time to the change in training index, or the slope of the trend line. Note that the slope is negative. To estimate the slope, estimate the coordinates of any two points on the line, for instance, the points at the extreme left and right ends of the line: $(0, 5.8)$ and $(100, 3.2)$.

The slope, which is measured in hours per unit, can be computed as follows:

$$\frac{3.2 - 5.8}{100 - 0} = \frac{-2.6}{100} = -0.026$$

The slope can be interpreted as follows: The finishing time is predicted to decrease 0.026 hour for every unit by which the training index increases. Since we want to know how much the finishing time decreases for an increase of *10 units*, we multiply the rate by 10 to get 0.26 hour per 10 units. To compute the decrease in *minutes* per 10 units, we multiply 0.26 by 60 to get approximately 16 minutes. Based on the trend line, it can be predicted that a bicyclist will decrease his or her finishing time by approximately 16 minutes for every increase of 10 training index units.

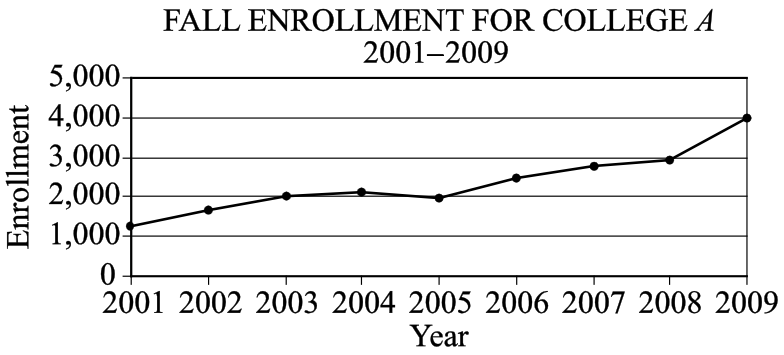
Line Graphs

A **line graph** is another type of graph that is useful for showing the relationship between two numerical variables, especially if one of the variables is time. A line graph uses a coordinate plane, where each data point represents a pair of values observed for the two numerical variables. There is at most one data point for each value on the horizontal axis,

similar to a function. The data points are in order from left to right, and consecutive data points are connected by a line segment.

When one of the variables is time, it is associated with the horizontal axis, which is labeled with regular time intervals. The data points may represent an interval of time, such as an entire day or year, or just an instant of time. Such a line graph is often called a **time series**.

Example 4.1.13:



Data Analysis Figure 7

The line graph shows that the greatest increase in fall enrollment between consecutive years was the change from 2008 to 2009. One way to determine this is by noting that the slope of the line segment joining the values for 2008 and 2009 is greater than the slopes of the line segments joining all other consecutive years. Another way to determine this is by noting that the increase in enrollment from 2008 to 2009 was greater than 1,000, but all other increases in enrollment were less than 1,000.

Although line graphs are commonly used to compare frequencies, as in Example 4.1.13 above, they can be used to compare any numerical data as the data change over time, such as temperatures, dollar amounts, percents, heights, and weights.

4.2 Numerical Methods for Describing Data

Data can be described numerically by various **statistics**, or **statistical measures**. These statistical measures are often grouped in three categories: measures of central tendency, measures of position, and measures of dispersion.

Measures of Central Tendency

Measures of **central tendency** indicate the "center" of the data along the number line and are usually reported as values that represent the data. There are three common measures of central tendency:

1. the **arithmetic mean**—usually called the **average** or simply the **mean**

2. the **median**

3. the **mode**

To calculate the **mean** of n numbers, take the sum of the n numbers and divide it by n.

Example 4.2.1: For the five numbers 6, 4, 7, 10, and 4, the mean is

$$\frac{6 + 4 + 7 + 10 + 4}{5} = \frac{31}{5} = 6.2$$

When several values are repeated in a list, it is helpful to think of the mean of the numbers as a **weighted mean** of only those values in the list that are *different*.

Example 4.2.2: Consider the following list of 16 numbers.

2, 4, 4, 5, 7, 7, 7, 7, 7, 7, 8, 8, 9, 9, 9, 9

There are only 6 different values in the list: 2, 4, 5, 7, 8, and 9. The mean of the numbers in the list can be computed as

$$\frac{1(2) + 2(4) + 1(5) + 6(7) + 2(8) + 4(9)}{1 + 2 + 1 + 6 + 2 + 4} = \frac{109}{16} = 6.8125$$

The number of times a value appears in the list, or the frequency, is called the **weight** of that value. So the mean of the 16 numbers is the weighted mean of the values 2, 4, 5, 7, 8, and 9, where the respective weights are 1, 2, 1, 6, 2, and 4. Note that the sum of the weights is the number of numbers in the list, 16.

The mean can be affected by just a few values that lie far above or below the rest of the data, because these values contribute directly to the sum of the data and therefore to the mean. By contrast, the **median** is a measure of central tendency that is fairly unaffected by unusually high or low values relative to the rest of the data.

To calculate the median of n numbers, first order the numbers from least to greatest. If n is odd, then the median is the middle number in the ordered list of numbers. If n is even, then there are *two* middle numbers, and the median is the average of these two numbers.

Example 4.2.3: The five numbers 6, 4, 7, 10, and 4 listed in increasing order are 4, 4, 6, 7, 10, so the median is 6, the middle number. Note that if the number 10 in the list is replaced by the number 24, the mean increases from 6.2 to

$$\frac{4 + 4 + 6 + 7 + 24}{5} = \frac{45}{5} = 9$$

but the median remains equal to 6. This example shows how the median is relatively unaffected by an unusually large value.

The median, as the "middle value" of an ordered list of numbers, divides the list into roughly two equal parts. However, if the median is equal to one of the data values and it is repeated in the list, then the numbers of data above and below the median may be rather different. For example, the median of the 16 numbers 2, 4, 4, 5, 7, 7, 7, 7, 7, 7, 8, 8, 9, 9, 9, 9 is 7, but four of the data are less than 7 and six of the data are greater than 7.

The **mode** of a list of numbers is the number that occurs most frequently in the list.

Example 4.2.4: The mode of the 6 numbers in the list 1, 3, 6, 4, 3, 5 is 3. A list of numbers may have more than one mode. For example, the list of 11 numbers 1, 2, 3, 3, 3, 5, 7, 10, 10, 10, 20 has two modes, 3 and 10.

Measures of Position

The three most basic **positions**, or locations, in a list of numerical data ordered from least to greatest are the beginning, the end, and the middle. It is useful here to label these as L for the least, G for the greatest, and M for the median. Aside from these, the most common measures of position are **quartiles** and **percentiles**. Like the median M, quartiles and percentiles are numbers that divide the data into roughly equal groups after the data have been ordered from the least value L to the greatest value G. There are three quartile numbers, called the **first quartile,** the **second quartile**, and the **third quartile**, that divide the data into four roughly equal groups; and there are 99 percentile numbers that divide the data into 100 roughly equal groups. As with the mean and median, the quartiles and percentiles may or may not themselves be values in the data.

In the following discussion of quartiles, the symbol Q_1 will be used to denote the first quartile, Q_2 will be used to denote the second quartile, and Q_3 will be used to denote the third quartile.

The numbers Q_1, Q_2, and Q_3 divide the data into 4 roughly equal groups as follows. After the data are listed in increasing order, the first group consists of the data from L to Q_1, the second group is from Q_1 to Q_2, the third group is from Q_2 to Q_3, and the fourth group is from Q_3 to G. Because the number of data may not be divisible by 4, there are various rules to determine the exact values of Q_1 and Q_3, and some statisticians use different rules, but in all cases Q_2 is equal to the median M. We use perhaps the most common rule for determining the values of Q_1 and Q_3. According to this rule, after the data are listed in increasing order, Q_1 is the median of the first half of the data in the ordered list and Q_3 is the median of the second half of the data in the ordered list, as illustrated in the following example.

Example 4.2.5: To find the quartiles for the list of 16 numbers 2, 4, 4, 5, 7, 7, 7, 7, 7, 7, 8, 8, 9, 9, 9, 9 (which are already listed in increasing order), first divide the numbers in

the list into two groups of 8 numbers each. The first group of 8 numbers is 2, 4, 4, 5, 7, 7, 7, 7, and the second group of 8 numbers is 7, 7, 8, 8, 9, 9, 9, 9, so that the second quartile, or median, is 7. To find the other quartiles, you can take each of the two smaller groups and find its median: the first quartile, Q_1, is 6 (the average of 5 and 7) and the third quartile, Q_3, is 8.5 (the average of 8 and 9).

In this example, note that the number 4 is in the lowest 25 percent of the distribution of data. There are different ways to describe this. We can say that 4 is below the first quartile, that is, below Q_1. We can also say that 4 is *in* the first quartile. The phrase "in a quartile" refers to being in one of the four groups determined by Q_1, Q_2, and Q_3.

Percentiles are mostly used for very large lists of numerical data ordered from least to greatest. Instead of dividing the data into four groups, the 99 percentiles $P_1, P_2, P_3, \ldots, P_{99}$ divide the data into 100 groups. Consequently, $Q_1 = P_{25}$, $M = Q_2 = P_{50}$, and $Q_3 = P_{75}$. Because the number of data in a list may not be divisible by 100, statisticians apply various rules to determine values of percentiles.

Measures of Dispersion

Measures of **dispersion** indicate the degree of spread of the data. The most common statistics used as measures of dispersion are the range, the interquartile range, and the standard deviation. These statistics measure the spread of the data in different ways.

The **range** of the numbers in a group of data is the difference between the greatest number G in the data and the least number L in the data; that is, $G - L$. For example, the range of the five numbers 11, 10, 5, 13, 21 is $21 - 5 = 16$.

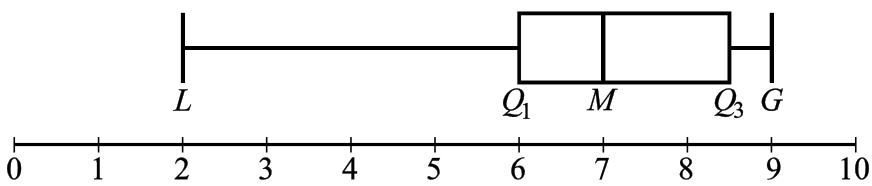
The simplicity of the range is useful in that it reflects the maximum spread of the data. Sometimes a data value is unusually small or unusually large in comparison with the rest of the data. Such data are called **outliers** because they lie so far out from the rest of the data. The range is directly affected by outliers.

A measure of dispersion that is not usually affected by outliers is the **interquartile range**. It is defined as the difference between the third quartile and the first quartile; that is, $Q_3 - Q_1$. Thus, the interquartile range measures the spread of the middle half of the data.

One way to summarize a group of numerical data and to illustrate its center and spread is to use the five numbers L, Q_1, Q_2, Q_3, and G. These five numbers can be plotted along a number line to show where the four quartile groups lie. Such plots are called **boxplots** or **box-and-whisker plots**, because a box is used to identify each of the two middle quartile groups of data, and "whiskers" extend outward from the boxes to the least and greatest values.

Example 4.2.6: In the list of 16 numbers 2, 4, 4, 5, 7, 7, 7, 7, 7, 7, 8, 8, 9, 9, 9, 9, the range is $9 - 2 = 7$, the first quartile, $Q_1 = 6$, and the third quartile, $Q_3 = 8.5$. So the interquartile range for the numbers in this list is $8.5 - 6 = 2.5$.

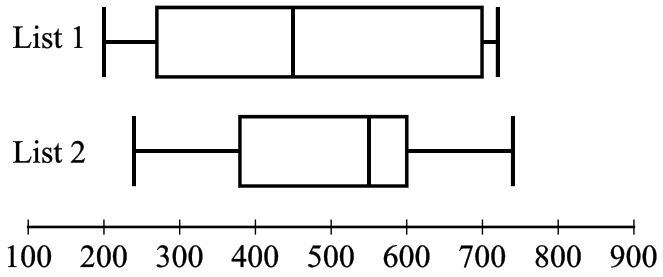
A boxplot for this list of 16 numbers is shown in the following figure.



Data Analysis Figure 8

There are a few variations in the way boxplots are drawn—the position of the ends of the boxes can vary slightly, and some boxplots identify outliers with certain symbols—but all boxplots show the center of the data at the median and illustrate the spread of the data in each of the four quartile groups. As such, boxplots are useful for comparing sets of data side by side.

Example 4.2.7: Two large lists of numerical data, list 1 and list 2, are summarized by the boxplots in the following figure.



Data Analysis Figure 9

Based on the boxplots, several different comparisons of the two lists can be made. First, the median of list 2, which is approximately 550, is greater than the median of list 1, which is approximately 450. Second, the two measures of spread, range and interquartile range, are greater for list 1 than for list 2. For list 1, these measures are approximately 520 and 430, respectively, and for list 2, they are approximately 500 and 220, respectively.

Unlike the range and the interquartile range, the **standard deviation** is a measure of spread that depends on each number in the list. Using the mean as the center of the data, the standard deviation takes into account how much each value differs from the mean and then takes a type of average of these differences. As a result, the more the data are spread away from the mean, the greater the standard deviation; and the more the data are clustered around the mean, the lesser the standard deviation.

The standard deviation of a group of numerical data is computed by

1. calculating the mean of the values,

2. finding the difference between the mean and each of the values,

3. squaring each of the differences,

4. finding the average of the squared differences, and

5. taking the nonnegative square root of the average of the squared differences.

Example 4.2.8: For the five data 0, 7, 8, 10, and 10, the standard deviation can be computed as follows. First, the mean of the data is 7, and the squared differences from the mean are

$$(7-0)^2,\ (7-7)^2,\ (7-8)^2,\ (7-10)^2,\ (7-10)^2$$

or 49, 0, 1, 9, 9. The average of the five squared differences is $\frac{68}{5}$, or 13.6, and the positive square root of 13.6 is approximately 3.7.

Note on terminology: The term "standard deviation" defined above is slightly different from another measure of dispersion, the **sample standard deviation**. The latter term is qualified with the word "sample" and is computed by dividing the sum of the squared differences by $n-1$ instead of n. The sample standard deviation is only slightly different from the standard deviation but is preferred for technical reasons for a sample of data that is taken from a larger population of data. Sometimes the standard deviation is called the **population standard deviation** to help distinguish it from the sample standard deviation.

Example 4.2.9: Six hundred applicants for several post office jobs were rated on a scale from 1 to 50 points. The ratings had a mean of 32.5 points and a standard deviation of 7.1 points. How many standard deviations above or below the mean is a rating of 48 points? A rating of 30 points? A rating of 20 points?

Solution: Let d be the standard deviation, so $d = 7.1$ points. Note that 1 standard deviation above the mean is

$$32.5 + d = 32.5 + 7.1 = 39.6$$

and 2 standard deviations above the mean is

$$32.5 + 2d = 32.5 + 2(7.1) = 46.7$$

Using the same reasoning, if 48 is r standard deviations above the mean, then

$$32.5 + rd = 32.5 + r(7.1) = 48$$

Solving the equation $32.5 + r(7.1) = 48$ for r gives

$$r = \frac{48 - 32.5}{7.1} = \frac{15.5}{7.1} \approx 2.2$$

Similarly, any rating of p points is $\frac{p - 32.5}{7.1}$ standard deviations from the mean.

The number of standard deviations that a rating of 30 is away from the mean is

$$\frac{30 - 32.5}{7.1} = \frac{-2.5}{7.1} \approx -0.4$$

where the negative sign indicates that the rating is 0.4 standard deviation *below* the mean.

The number of standard deviations that a rating of 20 is away from the mean is

$$\frac{20 - 32.5}{7.1} = \frac{-12.5}{7.1} \approx -1.8$$

where the negative sign indicates that the rating is 1.8 standard deviations *below* the mean.

To summarize:

1. 48 points is 15.5 points above the mean, or approximately 2.2 standard deviations above the mean.

2. 30 points is 2.5 points below the mean, or approximately 0.4 standard deviation below the mean.

3. 20 points is 12.5 points below the mean, or approximately 1.8 standard deviations below the mean.

One more instance, which may seem trivial, is important to note:

32.5 points is 0 points from the mean, or 0 standard deviations from the mean.

Example 4.2.9 above shows that for a group of data, each value can be located with respect to the mean by using the standard deviation as a ruler. The process of subtracting the mean from each value and then dividing the result by the standard deviation is called **standardization**. Standardization is a useful tool because for each data value, it provides a measure of position relative to the rest of the data independently of the variable for which the data was collected and the units of the variable.

Note that the standardized values $2.2,\ -0.4,$ and -1.8 from the last example are all between -3 and 3; that is, the corresponding ratings 48, 30, and 20 are all within 3 standard deviations of the mean. This is not surprising, based on the following fact about the standard deviation.

Fact: In *any group of data*, most of the data are within 3 standard deviations of the mean.

Thus, when *any group of data* are standardized, most of the data are transformed to an interval on the number line centered about 0 and extending from -3 to 3. The mean is always transformed to 0.

4.3 Counting Methods

When a set contains a small number of objects, it is easy to list the objects and count them one by one. When the set is too large to count that way, and when the objects are related in a patterned or systematic way, there are some useful techniques for counting the objects without actually listing them.

Sets and Lists

The term **set** has been used informally in this review to mean a collection of objects that have some property, whether it is the collection of all positive integers, all points in a circular region, or all students in a school that have studied French. The objects of a set are called **members** or **elements**. Some sets are **finite**, which means that their members can be completely counted. Finite sets can, in principle, have all of their members listed, using curly brackets, such as the set of even digits $\{0, 2, 4, 6, 8\}$. Sets that are not finite are called **infinite** sets, such as the set of all integers. A set that has no members is called the **empty set** and is denoted by the symbol $\varnothing$. A set with one or more members is called **nonempty**. If A and B are sets and all of the members of A are also members of B, then A is a **subset** of B. For example, $\{2, 8\}$ is a subset of $\{0, 2, 4, 6, 8\}$. Also, by convention, $\varnothing$ is a subset of every set.

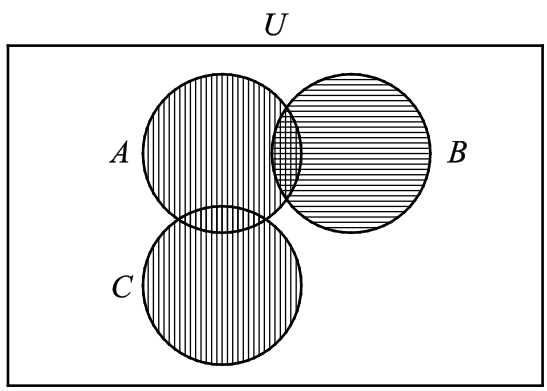
A **list** is like a finite set, having members that can all be listed, but with two differences. In a list, the members are ordered–that is, rearranging the members of a list makes it a different list. Thus, the terms "first element," "second element," and so on, make sense in a list. Also, elements can be repeated in a list and the repetitions matter. For example, the list 1, 2, 3, 2 and the list 1, 2, 2, 3 are different lists, each with four elements, and they are both different from the list 1, 2, 3, which has three elements.

In contrast to a list, when the elements of a set are given, repetitions are not counted as additional elements and the order of the elements does not matter. For example, the set $\{1, 2, 3, 2\}$ and the set $\{3, 1, 2\}$ are the same set, which has three elements. For any finite set S, the number of elements of S is denoted by $|S|$. Thus, if $S = \{6.2, -9, \pi, 0.01, 0\}$, then $|S| = 5$. Also, $|\varnothing| = 0$.

Sets can be formed from other sets. If S and T are sets, then the **intersection** of S and T is the set of all elements that are in both S and T and is denoted by $S \cap T$. The **union** of S and T is the set of all elements that are in either S or T or both and is denoted by $S \cup T$. If sets S and T have no elements in common, they are called **disjoint** or **mutually exclusive**.

A useful way to represent two or three sets and their possible intersections and unions is a **Venn diagram**. In a Venn diagram, sets are represented by circular regions that overlap if they have elements in common but do not overlap if they are disjoint. Sometimes the circular regions are drawn inside a rectangular region, which represents a **universal set**, of which all other sets involved are subsets.

Example 4.3.1: Data Analysis Figure 10 below is a Venn diagram using circular regions to represent the three sets A, B, and C. In the Venn diagram, the three circular regions are drawn in a rectangular region representing a universal set U.



Data Analysis Figure 10

The regions with vertical stripes represent the set $A \cup C$. The regions with horizontal stripes represent the set B. The region with both kinds of stripes represents the set $A \cap B$. The sets B and C are mutually exclusive, often written as $B \cap C = \varnothing$.

The last example can be used to illustrate an elementary counting principle involving intersecting sets, called the **inclusion-exclusion principle**. This principle relates the numbers of elements in the union and intersection of two finite sets. The number of elements in the union of two sets equals the sum of their individual numbers of elements minus the number of elements in their intersection. If the sets in the example are finite, then we have for the union of A and B,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Because $A \cap B$ is a subset of both A and B, the subtraction is necessary to avoid counting the elements in $A \cap B$ twice. For the union of B and C, we have

$$|B \cup C| = |B| + |C|$$

because $B \cap C = \varnothing$.

Multiplication Principle

Suppose there are two choices to be made sequentially and that the second choice is independent of the first choice. Suppose also that there are k different possibilities for the first choice and m different possibilities for the second choice. The **multiplication principle** states that under those conditions, there are km different possibilities for the pair of choices.

For example, suppose that a meal is to be ordered from a restaurant menu and that the meal consists of one entrée and one dessert. If there are 5 entrées and 3 desserts on the menu, then there are $(5)(3) = 15$ different meals that can be ordered from the menu.

The multiplication principle applies in more complicated situations as well. If there are more than two independent choices to be made, then the number of different possible outcomes of all of the choices is the product of the numbers of possibilities for each choice.

Example 4.3.2: Suppose that a computer password consists of four characters such that the first character is one of the 10 digits from 0 to 9 and each of the next 3 characters is any one of the uppercase letters from the 26 letters of the English alphabet. How many different passwords are possible?

Solution: The description of the password allows repetitions of letters. Thus, there are 10 possible choices for the first character in the password and 26 possible choices for each of the next 3 characters in the password. Therefore, applying the multiplication principle, the number of possible passwords is $(10)(26)(26)(26) = 175,760$.

Note that if repetitions of letters are not allowed in the password, then the choices are not all independent, but a modification of the multiplication principle can still be applied. There are 10 possible choices for the first character in the password, 26 possible choices for the second character, 25 for the third character because the first letter cannot be repeated, and 24 for the fourth character because the first two letters cannot be repeated. Therefore, the number of possible passwords is $(10)(26)(25)(24) = 156,000$.

Example 4.3.3: Each time a coin is tossed, there are 2 possible outcomes—either it lands heads up or it lands tails up. Using this fact and the multiplication principle, you can conclude that if a coin is tossed 8 times, there are
$(2)(2)(2)(2)(2)(2)(2)(2) = 2^8 = 256$ possible outcomes.

Permutations and Factorials

Suppose you want to determine the number of different ways the 3 letters A, B, and C can be placed in order from 1st to 3rd. The following is a list of all the possible orders in which the letters can be placed.

ABC ACB BAC BCA CAB CBA

There are 6 possible orders for the 3 letters.

Now suppose you want to determine the number of different ways the 4 letters A, B, C, and D can be placed in order from 1st to 4th. Listing all of the orders for 4 letters is time-consuming, so it would be useful to be able to count the possible orders without listing them.

To order the 4 letters, one of the 4 letters must be placed first, one of the remaining 3 letters must be placed second, one of the remaining 2 letters must be placed third, and the

last remaining letter must be placed fourth. Therefore, applying the multiplication principle, there are $(4)(3)(2)(1)$, or 24, ways to order the 4 letters.

More generally, suppose n objects are to be ordered from 1st to nth, and we want to count the number of ways the objects can be ordered. There are n choices for the first object, $n-1$ choices for the second object, $n-2$ choices for the third object, and so on, until there is only 1 choice for the nth object. Thus, applying the multiplication principle, the number of ways to order the n objects is equal to the product

$$n(n-1)(n-2)\cdots(3)(2)(1)$$

Each order is called a **permutation**, and the product above is called the number of permutations of n objects.

Because products of the form $n(n-1)(n-2)\cdots(3)(2)(1)$ occur frequently when counting objects, a special symbol $n!$, called ***n*-factorial**, is used to denote this product.

For example,

$$1! = 1$$

$$2! = (2)(1)$$

$$3! = (3)(2)(1)$$

$$4! = (4)(3)(2)(1)$$

As a special definition, $0! = 1$.

Note that $n! = n(n-1)! = n(n-1)(n-2)! = n(n-1)(n-2)(n-3)!$ and so on.

Example 4.3.4: Suppose that 10 students are going on a bus trip, and each of the students will be assigned to one of the 10 available seats. Then the number of possible different seating arrangements of the students on the bus is

$$10! = (10)(9)(8)(7)(6)(5)(4)(3)(2)(1) = 3{,}628{,}800$$

Now suppose you want to determine the number of ways in which you can select 3 of the 5 letters A, B, C, D, and E and place them in order from 1st to 3rd. Reasoning as in the preceding examples, you find that there are $(5)(4)(3)$, or 60, ways to select and order them.

More generally, suppose that k objects will be selected from a set of n objects, where $k \le n$, and the k objects will be placed in order from 1st to kth. Then there are n choices for the first object, $n-1$ choices for the second object, $n-2$ choices for the third object, and so on, until there are $n-k+1$ choices for the kth object. Thus, applying the multiplication principle, the number of ways to select and order k objects from a set of n objects is $n(n-1)(n-2)...(n-k+1)$. It is useful to note that

$$n(n-1)(n-2)\cdots(n-k+1) = n(n-1)(n-2)\cdots(n-k+1)\frac{(n-k)!}{(n-k)!} = \frac{n!}{(n-k)!}$$

This expression represents the number of **permutations of *n* objects taken *k* at a time**– that is, the number of ways to select and order k objects out of n objects. This number is commonly denoted by the notation ${}_nP_k$.

Example 4.3.5: How many different 5-digit positive integers can be formed using the digits 1, 2, 3, 4, 5, 6, and 7 if none of the digits can occur more than once in the integer?

Solution: This example asks how many ways there are to order 5 integers chosen from a set of 7 integers. According to the counting principle above, there are

$(7)(6)(5)(4)(3) = 2{,}520$ ways to do this. Note that this is equal to $\frac{7!}{(7-5)!}$ $= \frac{(7)(6)(5)(4)(3)(2!)}{2!} = (7)(6)(5)(4)(3)$.

Combinations

Given the 5 letters A, B, C, D, and E, suppose that you want to determine the number of ways in which you can select 3 of the 5 letters, but unlike before, you do not want to count different orders for the 3 letters. The following is a list of all of the ways in which 3 of the 5 letters can be selected without regard to the order of the letters.

ABC ABD ABE ACD ACE
ADE BCD BCE BDE CDE

There are 10 ways of selecting the 3 letters without order. There is a relationship between selecting with order and selecting without order.

The number of ways to select 3 of the 5 letters without order, which is 10, *multiplied by* the number of ways to order the 3 letters, which is $3!$, or 6, *is equal to* the number of ways to select 3 of the 5 letters and order them, which is $\frac{5!}{2!} = 60$. In short,

$$\text{(number of ways to select without order)} \times \text{(number of ways to order)}$$

$$= \text{(number of ways to select with order)}$$

This relationship can also be rewritten as follows.

$$\text{(number of ways to select without order)} = \frac{\text{(number of ways to select with order)}}{\text{(number of ways to order)}}$$

For the example above, the number of ways to select without order is $\dfrac{\frac{5!}{2!}}{3!} = \dfrac{5!}{3!\,2!} = 10.$

More generally, suppose that k objects will be chosen from a set of n objects, where $k \le n,$ but that the k objects will not be put in order. The number of ways in which this can be done is called the number of **combinations of *n* objects taken *k* at a time** and is given by the formula $\dfrac{n!}{k!\,(n-k)!}$.

Another way to refer to the number of combinations of n objects taken k at a time is ***n* choose *k***, and two notations commonly used to denote this number are ${}_nC_k$ and $\binom{n}{k}$.

Example 4.3.6: Suppose you want to select a 3-person committee from a group of 9 students. How many ways are there to do this?

Solution: Since the 3 students on the committee are not ordered, you can use the formula for the combination of 9 objects taken 3 at a time, or "9 choose 3":

$$\frac{9!}{3!\,(9-3)!} = \frac{9!}{3!\,6!} = \frac{(9)(8)(7)}{(3)(2)(1)} = 84$$

Using the terminology of sets, given a set S consisting of n elements, n choose k is simply the number of subsets of S that consist of k elements.

The formula for n choose k, which is $\dfrac{n!}{k!\,(n-k)!}$ also holds when $k = 0$ and $k = n$.

Therefore

1. n choose 0 is $\frac{n!}{0!\,n!} = 1$.

(This reflects the fact that there is only one subset of S with 0 elements, namely the empty set).

2. n choose n is $\frac{n!}{n!\,0!} = 1$.

(This reflects the fact that there is only one subset of S with n elements, namely the set S itself).

Finally, note that n choose k is always equal to n choose $n-k$, because

$$\frac{n!}{(n-k)!\,(n-(n-k))!} = \frac{n!}{(n-k)!\,k!} = \frac{n!}{k!\,(n-k)!}$$

4.4 Probability

Probability is a way of describing uncertainty in numerical terms. In this section, we review some of the terminology used in elementary probability theory.

A **probability experiment**, also called a **random experiment**, is an experiment for which the result, or **outcome**, is uncertain. We assume that all of the possible outcomes of an experiment are known before the experiment is performed, but which outcome will actually occur is unknown. The set of all possible outcomes of a random experiment is called the **sample space**, and any particular set of outcomes is called an **event**. For example, consider a cube with faces numbered 1 to 6, called a 6-sided die. Rolling the die once is an experiment in which there are 6 possible outcomes: either 1, 2, 3, 4, 5, or 6 will appear on the top face. The sample space for this experiment is $\{1, 2, 3, 4, 5, 6\}$. Here are two examples of events for this experiment.

Event 1: Rolling the number 4. This event has only one outcome.

Event 2: Rolling an odd number. This even has three outcomes.

The **probability** of an event is a number from 0 to 1, inclusive, that indicates the likelihood that the event occurs when the experiment is performed. The greater the number, the more likely the event.

Example 4.4.1: Consider the following experiment. A box contains 15 pieces of paper, each of which has the name of one of the 15 students in a high school class consisting of 7 juniors and 8 seniors, all with different names. The instructor will shake the box for a while and then choose a piece of paper at random and read the name. Here the sample space is the set of 15 names. The assumption of **random selection** means that each of the names is **equally likely** to be selected. If this assumption is made, then the probability that any one particular name will be selected is equal to $\frac{1}{15}$.

For any event E, the probability that E occurs is often written as $P(E)$.

For the sample space in this example, $P(E)$, that is, the probability that event E occurs, is equal to

$$\frac{\text{the number of names in the event } E}{15}$$

If J is the event that the student selected is a junior, then $P(J) = \frac{7}{15}$.

In general, for a random experiment with a finite number of possible outcomes, if each outcome is equally likely to occur, then the probability that an event E occurs is defined by

$$P(E) = \frac{\text{the number of outcomes in the event } E}{\text{the number of possible outcomes in the experiment}}$$

In the case of rolling a 6-sided die, if the die is "fair," then the 6 outcomes are equally likely. So the probability of rolling a 4 is $\frac{1}{6}$, and the probability of rolling an odd number (that is, rolling a 1, 3, or 5) can be calculated as $\frac{3}{6} = \frac{1}{2}$.

The following are six general facts about probability.

Fact 1: If an event E is certain to occur, then $P(E) = 1$.

Fact 2: If an event E is certain *not* to occur, then $P(E) = 0$.

Fact 3: If an event E is possible but not certain to occur, then $0 < P(E) < 1$.

Fact 4: The probability that an event E will not occur is equal to $1 - P(E)$.

Fact 5: If E is an event, then the probability of E is the sum of the probabilities of the outcomes in E.

Fact 6: The sum of the probabilities of all possible outcomes of an experiment is 1.

If E and F are two events of an experiment, we consider two other events related to E and F.

Event 1: The event that both E and F occur, that is, outcomes in the set $E \cap F$

Event 2: The event that E or F, or both, occur, that is, outcomes in the set $E \cup F$

Events that cannot occur at the same time are said to be **mutually exclusive**. For example, if a 6-sided die is rolled once, the event of rolling an odd number and the event of rolling an even number are mutually exclusive. But rolling a 4 and rolling an even number are not mutually exclusive, since 4 is an outcome that is common to both events.

For events E and F, we have the following three rules.

Rule 1: $P(\text{either } E \text{ or } F, \text{ or both, occur}) = P(E) + P(F) - P(\text{both } E \text{ and } F \text{ occur})$, which is the inclusion-exclusion principle applied to probability.

Rule 2: If E and F are mutually exclusive, then $P(\text{both } E \text{ and } F \text{ occur}) = 0$, and therefore, $P(\text{either } E \text{ or } F, \text{ or both, occur}) = P(E) + P(F)$.

Rule 3: E and F are said to be **independent** if the occurrence of either event does not affect the occurrence of the other. If two events E and F are independent, then $P(\text{both } E \text{ and } F \text{ occur}) = P(E)P(F)$. For example, if a fair 6-sided die is rolled twice, the event E of rolling a 3 on the first roll and the event F of rolling a 3 on the second roll are independent, and the probability of rolling a 3 on both rolls is $P(E)P(F) = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{36}$. In this example, the experiment is actually "rolling the die twice," and each outcome is an ordered pair of results like "4 on the first roll and 1 on the second roll." But event E restricts only the first roll–to a 3–having no effect on the second roll; similarly, event F restricts only the second roll–to a 3–having no effect on the first roll.

Note that if $P(E) \neq 0$ and $P(F) \neq 0$, then events E and F cannot be both mutually exclusive and independent. For if E and F are independent, then $P(\text{both } E \text{ and } F \text{ occur}) = P(E)P(F) \neq 0$, but if E and F are mutually exclusive, then $P(\text{both } E \text{ and } F \text{ occur}) = 0$.

It is common to use the shorter notation "E and F" instead of "both E and F occur" and use "E or F" instead of "E or F or both occur." With this notation, we can restate the previous three rules as follows.

Rule 1: $P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)$

Rule 2: $P(E \text{ or } F) = P(E) + P(F)$ if E and F are mutually exclusive.

Rule 3: $P(E \text{ and } F) = P(E)P(F)$ if E and F are independent.

Example 4.4.2: If a fair 6-sided die is rolled once, let E be the event of rolling a 3 and let F be the event of rolling an odd number. These events are *not* independent. This is because rolling a 3 makes certain that the event of rolling an odd number occurs. Note that $P(E \text{ and } F) \neq P(E)P(F)$, since

$$P(E \text{ and } F) = P(E) = \frac{1}{6} \text{ and } P(E)P(F) = \left(\frac{1}{6}\right)\left(\frac{1}{2}\right) = \frac{1}{12}$$

Example 4.4.3: A 12-sided die, with faces numbered 1 to 12, is to be rolled once, and each of the 12 possible outcomes is equally likely to occur. The probability of rolling a 4 is $\frac{1}{12}$, so the probability of rolling a number that is *not* a 4 is $1 - \frac{1}{12} = \frac{11}{12}$.

The probability of rolling a number that is either a multiple of 5 (that is, rolling a 5 or a 10) or an odd number (that is, rolling a 1, 3, 5, 7, 9, or 11) is equal to

$$P(\text{multiple of 5}) + P(\text{odd}) - P(\text{multiple of 5 and odd}) = \frac{2}{12} + \frac{6}{12} - \frac{1}{12} = \frac{7}{12}$$

Another way to calculate this probability is to notice that rolling a number that is either a multiple of 5 or an odd number is the same as rolling one of the seven numbers 1, 3, 5, 7, 9, 10, and 11, which are equally likely outcomes. So by using the ratio formula to calculate the probability, the required probability is $\frac{7}{12}$.

Example 4.4.4: Consider an experiment with events A, B, and C for which $P(A) = 0.23,\ P(B) = 0.40,$ and $P(C) = 0.85.$

Suppose that events A and B are mutually exclusive and events B and C are independent. What is $P(A \text{ or } B)$ and $P(B \text{ or } C)$?

Solution: Since A and B are mutually exclusive,

$$P(A \text{ or } B) = P(A) + P(B) = 0.23 + 0.40 = 0.63$$

Since B and C are independent, $P(B \text{ and } C) = P(B)P(C)$. So

$$P(B \text{ or } C) = P(B) + P(C) - P(B \text{ and C}) = P(B) + P(C) - P(B)P(C)$$

Therefore,

$$P(B \text{ or } C) = 0.40 + 0.85 - (0.40)(0.85) = 1.25 - 0.34 = 0.91$$

Example 4.4.5: Suppose that there is a 6-sided die that is weighted in such a way that each time the die is rolled, the probabilities of rolling any of the numbers from 1 to 5 are all equal, but the probability of rolling a 6 is twice the probability of rolling a 1. When

you roll the die once, the 6 outcomes are *not equally likely*. What are the probabilities of the 6 outcomes?

Solution: Let p equal the probability of rolling a 1. Then each of the probabilities of rolling a 2, 3, 4, or 5 is equal to p, and the probability of rolling a 6 is equal to $2p$. Therefore, since the sum of the probabilities of all possible outcomes is 1, it follows that

$$1 = P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = p + p + p + p + p + 2p = 7p$$

So the probability of rolling each of the numbers from 1 to 5 is $\frac{1}{7}$, and the probability of rolling a 6 is $\frac{2}{7}$.

Example 4.4.6: Suppose that you roll the weighted 6-sided die from Example 4.4.5 twice. What is the probability that the first roll will be an odd number and the second roll will be an even number?

Solution: To calculate the probability that the first roll will be odd and the second roll will be even, note that these two events are independent. To calculate the probability that both occur, you must multiply the probabilities of the two independent events. First compute the individual probabilities.

$$P(\text{odd}) = P(1) + P(3) + P(5) = \frac{3}{7}$$

$$P(\text{even}) = P(2) + P(4) + P(6) = \frac{4}{7}$$

Then, $P(\text{first roll is odd and second roll is even}) = P(\text{odd})P(\text{even}) = \left(\frac{3}{7}\right)\left(\frac{4}{7}\right) = \frac{12}{49}$.

Two events that happen sequentially are not always independent. The occurrence of the first event may affect the occurrence of the second event. In this case, the probability that *both* events happen is equal to the probability that the first event happens multiplied by the probability that, *given that the first event has already happened*, the second event will happen as well.

Example 4.4.7: A box contains 5 orange disks, 4 red disks, and 1 blue disk. You are to select two disks at random and without replacement from the box. What is the probability that the first disk you select will be red and the second disk you select will be orange?

Solution: To solve, you need to calculate the following two probabilities and then multiply them.

1. The probability that the first disk selected from the box will be red

2. The probability that the second disk selected from the box will be orange, given that the first disk selected from the box is red

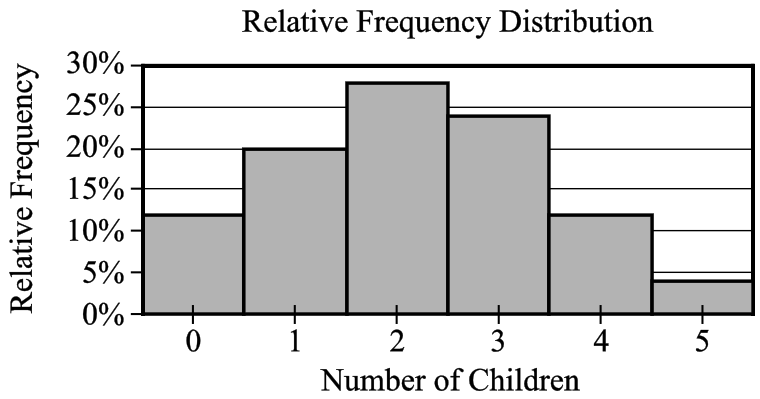
The probability that the first disk you select will be red is $\frac{4}{10} = \frac{2}{5}$. If the first disk you select is red, there will be 5 orange disks, 3 red disks, and 1 blue disk left in the box, for a total of 9 disks. Therefore, the probability that the second disk you select will be orange, given that the first disk you selected is red, is $\frac{5}{9}$. Multiply the two probabilities to get $\left(\frac{2}{5}\right)\left(\frac{5}{9}\right) = \frac{2}{9}$.

4.5 Distributions of Data, Random Variables, and Probability Distributions

In data analysis, variables whose values depend on chance play an important role in linking distributions of data to probability distributions. Such variables are called random variables. We begin with a review of distributions of data.

Distributions of Data

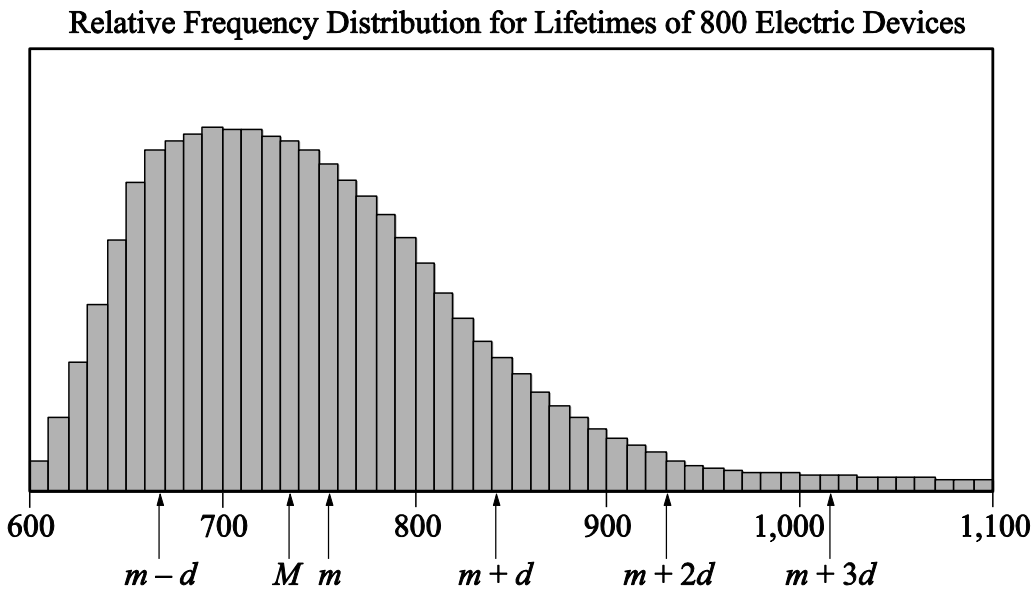
Recall that relative frequency distributions given in a table or histogram are a common way to show how numerical data are distributed. In a histogram, the areas of the bars indicate where the data are concentrated. The histogram of the relative frequency distribution of the number of children in each of 25 families in Data Analysis Figure 4 below illustrates a small group of data, with only 6 distinct data values and 25 data values altogether. (Note: This is the second occurrence of Data Analysis Figure 4, it was first encountered in Example 4.1.10.)



Data Analysis Figure 4 (repeated)

Many groups of data are much larger than 25 and have many more than 6 possible values, which are often measurements of quantities like length, money, or time.

Example 4.5.1: The lifetimes of 800 electric devices were measured. Because the lifetimes had many different values, the measurements were grouped into 50 intervals, or **classes**, of 10 hours each: 601 to 610 hours, 611 to 620 hours, and so on, up to 1,091 to 1,100 hours. The resulting relative frequency distribution, as a histogram, has 50 thin bars and many different bar heights, as shown in Data Analysis Figure 11 below.



Data Analysis Figure 11

In the histogram, the median is represented by M, the mean is represented by m, and the standard deviation is represented by d.

According to the graph:

- A data value 1 standard deviation below the mean, represented by $m - d$, is between 660 and 670.
- The median, represented by M, is between 730 and 740.
- The mean, represented by m, is between 750 and 760.
- A data value 1 standard deviation above the mean, represented by $m + d$, is between 840 and 850.
- A data value 2 standard deviations above the mean, represented by $m + 2d$, is approximately 930.

A data value 3 standard deviations above the mean, represented by $m + 3d$, is between 1,010 and 1,020.

The standard deviation marks show how most of the data are within 3 standard deviations of the mean, that is, between the numbers $m - 3d$ and $m + 3d$. Note that $m + 3d$ is shown in the figure, but $m - 3d$ is not.

The tops of the bars of the relative frequency distribution in the histogram in Data Analysis Figure 11 have a relatively smooth appearance and begin to look like a curve. In general, histograms that represent very large data sets grouped into many classes have a relatively smooth appearance. Consequently, the distribution can be modeled by a smooth curve that is close to the tops of the bars. Such a model retains the shape of the distribution but is independent of classes.

Recall from Example 4.1.10 that the sum of the areas of the bars of a relative frequency histogram is 1. Although the units on the horizontal axis of a histogram vary from one data set to another, the vertical scale can be adjusted (stretched or shrunk) so that the sum of the areas of the bars is 1. With this vertical scale adjustment, the area under the curve that models the distribution is also 1. This model curve is called a **distribution curve**, but it has other names as well, including **density curve** and **frequency curve**.

The purpose of the distribution curve is to give a good illustration of a large distribution of numerical data that does not depend on specific classes. To achieve this, the main property of a distribution curve is that the area under the curve in any vertical slice, just like a histogram bar, represents the proportion of the data that lies in the corresponding interval on the horizontal axis, which is at the base of the slice.

Random Variables

When analyzing data, it is common to choose a value of the data at random and consider that choice as a random experiment, as introduced in Section 4.4. Then, the probabilities of events involving the randomly chosen value may be determined. Given a distribution of data, a variable, say X, may be used to represent a randomly chosen value from the distribution. Such a variable X is an example of a **random variable**, which is a variable whose value is a numerical outcome of a random experiment.

Example 4.5.2: In Example 4.1.3, data consisting of numbers of children in each of 25 families was summarized in a frequency distribution table. For convenience, the frequency distribution table is repeated below.

Frequency Distribution

Number of Children	Frequency
0	3
1	5
2	7
3	6
4	3
5	1
Total	25

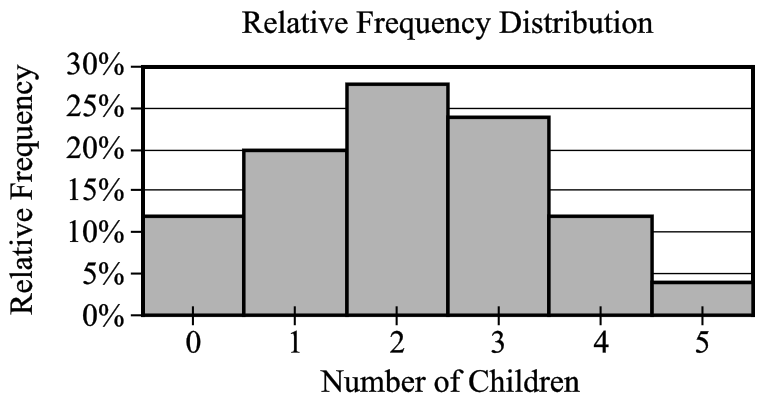
Let X be the random variable representing the number of children in a randomly chosen family among the 25 families. What is the probability that $X = 3$? That $X > 3$? That X is less than the mean of the distribution?

Solution: To determine the probability that $X = 3$, realize that this is the same as determining the probability that a family with 3 children will be chosen.

Since there are 6 families with 3 children and each of the 25 families is equally likely to be chosen, the probability that a family with 3 children will be chosen is $\frac{6}{25}$. That is,

$X = 3$ is an event, and its probability is $P(X = 3) = \frac{6}{25}$, or 0.24. It is common to use the shorter notation $P(3)$ instead of $P(X = 3)$, so you could write $P(3) = 0.24$.

Note that in the histogram shown in Data Analysis Figure 4 below, the area of the bar corresponding to $X = 3$ as a proportion of the combined areas of all of the bars is equal to this probability. This indicates how probability is related to area in a histogram for a relative frequency distribution. (Note: This is the third occurrence of Data Analysis Figure 4, it was previously encountered in Example 4.1.10 and at the beginning of this section.)



Data Analysis Figure 4 (repeated)

To determine the probability that $X > 3$, notice that the event $X > 3$ is the same as the event "$X = 4$ or $X = 5$". Because $X = 4$ and $X = 5$ are mutually exclusive events, we can use the rules of probability from Section 4.4.

$$P(X > 3) = P(4) + P(5) = \frac{3}{25} + \frac{1}{25} = 0.12 + 0.04 = 0.16$$

To determine the probability that X is less than the mean of the distribution, first compute the mean of the distribution as follows.

$$\frac{0(3)+1(5)+2(7)+3(6)+4(3)+5(1)}{25}=\frac{54}{25}=2.16$$

Then, calculate the probability that X is less than the mean of the distribution (that is, the probability that X is less than 2.16).

$$P(X<2.16)=P(0)+P(1)+P(2)=\frac{3}{25}+\frac{5}{25}+\frac{7}{25}=\frac{15}{25}=0.6$$

The following table shows all 6 possible values of X and their probabilities. This table is called the **probability distribution** of the random variable X.

X	$P(X)$
0	0.12
1	0.20
2	0.28
3	0.24
4	0.12
5	0.04

Note that the probabilities are simply the relative frequencies of the 6 possible values expressed as decimals instead of percents. The following statement indicates a fundamental link between data distributions and probability distributions.

Statement: For a random variable that represents a randomly chosen value from a distribution of data, the probability distribution of the random variable is the same as the relative frequency distribution of the data.

Because the probability distribution and the relative frequency distribution are essentially the same, the probability distribution can be represented by a histogram. Also, all of the descriptive statistics–such as mean, median, and standard deviation–that apply to the distribution of data also apply to the probability distribution. For example, we say that the probability distribution above has a mean of 2.16, a median of 2, and a standard deviation of about 1.3, since the 25 data values have these statistics, as you can check.

These statistics are similarly defined for the random variable X above. Thus, we would say that the **mean of the random variable X** is 2.16. Another name for the mean of a random variable is **expected value**. So we would also say that the expected value of X is 2.16.

Note that the mean of X is equal to

$$\frac{0(3) + 1(5) + 2(7) + 3(6) + 4(3) + 5(1)}{25}$$

which can also be expressed as

$$0\left(\frac{3}{25}\right) + 1\left(\frac{5}{25}\right) + 2\left(\frac{7}{25}\right) + 3\left(\frac{6}{25}\right) + 4\left(\frac{3}{25}\right) + 5\left(\frac{1}{25}\right)$$

which is the same as

$$0P(0) + 1P(1) + 2P(2) + 3P(3) + 4P(4) + 5P(5)$$

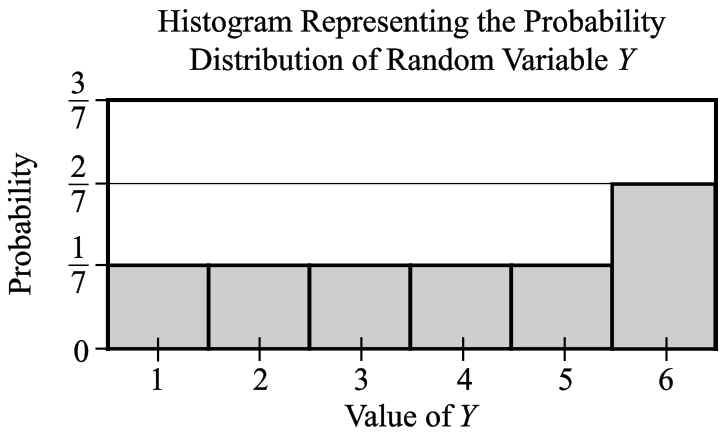
Therefore, the mean of the random variable X is the sum of the products $XP(X)$ for all values of X, that is, the sum of each value of X multiplied by its corresponding probability $P(X)$.

The preceding example involves a common type of random variable–one that represents a randomly chosen value from a distribution of data. However, the concept of a random variable is more general. A random variable can be any quantity whose value is the result of a random experiment. The possible values of the random variable are the same as the outcomes of the experiment. So any random experiment with numerical outcomes naturally has a random variable associated with it, as in the following example.

Example 4.5.3: Let Y represent the outcome of the experiment of rolling the weighted 6-sided die in Example 4.4.5. (In that example, the probabilities of rolling any of the numbers from 1 to 5 are all equal, but the probability of rolling a 6 is twice the probability of rolling a 1.) Then Y is a random variable with 6 possible values, the numbers 1 through 6. Each of the six values of Y has a probability. The probability distribution of the random variable Y is shown below, first in a table, then as a histogram.

Table Representing the Probability Distribution of Random Variable Y

Y	$P(Y)$
1	$\frac{1}{7}$
2	$\frac{1}{7}$
3	$\frac{1}{7}$
4	$\frac{1}{7}$
5	$\frac{1}{7}$
6	$\frac{2}{7}$



Data Analysis Figure 12

The mean, or expected value, of Y can be computed as

$$P(1) + 2P(2) + 3P(3) + 4P(4) + 5P(5) + 6P(6)$$

which is equal to

$$\left(\frac{1}{7}\right) + 2\left(\frac{1}{7}\right) + 3\left(\frac{1}{7}\right) + 4\left(\frac{1}{7}\right) + 5\left(\frac{1}{7}\right) + 6\left(\frac{2}{7}\right)$$

This sum simplifies to $\frac{1}{7} + \frac{2}{7} + \frac{3}{7} + \frac{4}{7} + \frac{5}{7} + \frac{12}{7}$, or $\frac{27}{7}$, which is approximately 3.86.

Both of the random variables X and Y above are examples of **discrete random variables** because their values consist of discrete points on a number line.

A basic fact about probability from Section 4.4 is that the sum of the probabilities of all possible outcomes of an experiment is 1, which can be confirmed by adding all of the probabilities in each of the probability distributions for the random variables X and Y above. Also, the sum of the areas of the bars in a histogram for the probability distribution of a random variable is 1. This fact is related to the following fundamental link between the areas of the bars of a histogram and the probabilities of a discrete random variable.

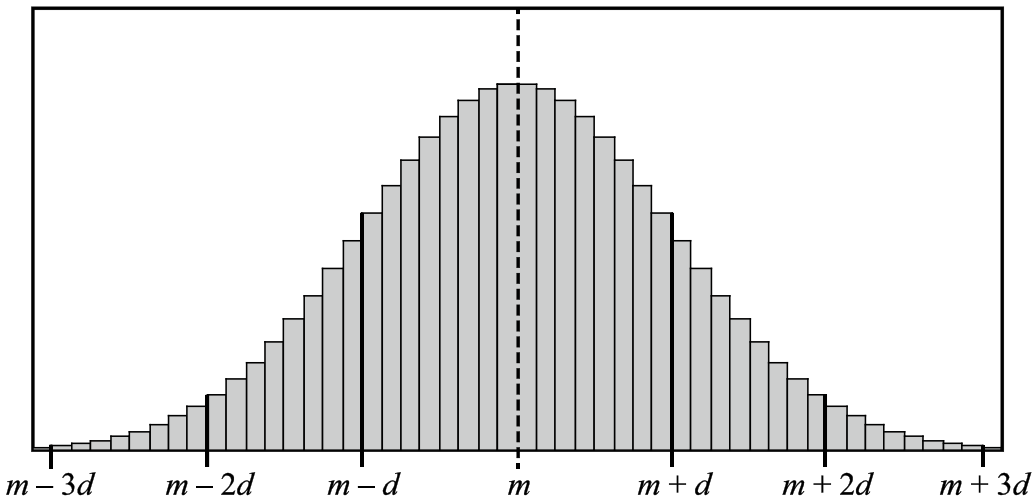
> Fundamental Link: In a histogram representing the probability distribution of a random variable, the area of each bar is proportional to the probability represented by the bar.

If the die in Example 4.4.5 were a fair die instead of weighted, then the probability of each of the outcomes would be $\frac{1}{6}$, and consequently, each of the bars in the histogram of the probability distribution would have the same height. Such a flat histogram indicates a **uniform distribution**, since the probability is distributed uniformly over all possible outcomes.

The Normal Distribution

Many natural processes yield data that have a relative frequency distribution shaped somewhat like a bell, as in the distribution with mean m and standard deviation d in Data Analysis Figure 13 below.

Approximately Normal Relative Frequency Distribution



Data Analysis Figure 13

Such data are said to be **approximately normally distributed** and have the following four properties.

Property 1: The mean, median, and mode are all nearly equal.

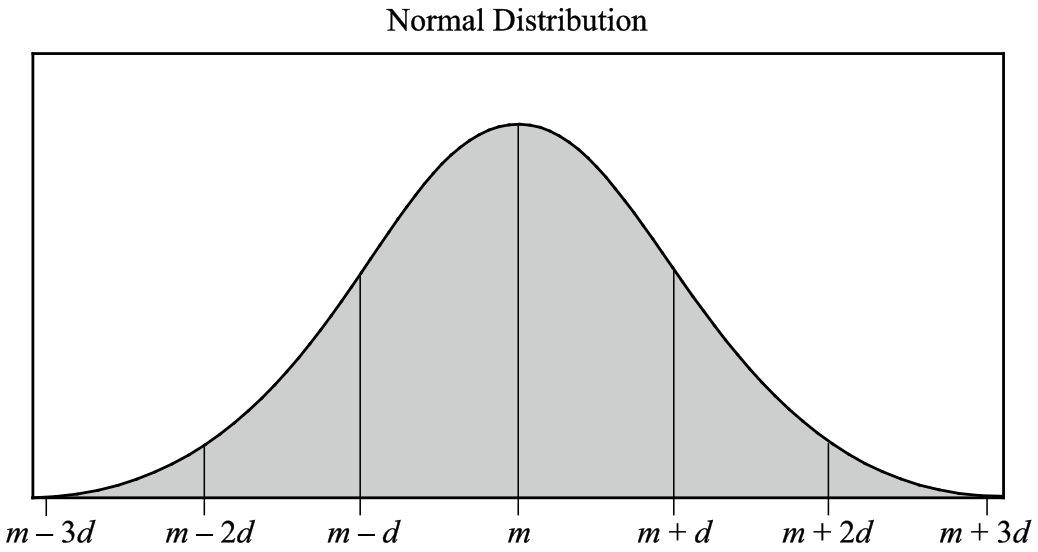
Property 2: The data are grouped fairly symmetrically about the mean.

Property 3: About two-thirds of the data are within 1 standard deviation of the mean.

Property 4: Almost all of the data are within 2 standard deviations of the mean.

As stated above, you can always associate a random variable X with a distribution of data by letting X be a randomly chosen value from the distribution. If X is such a random variable for the distribution in Data Analysis Figure 13, we say that X is approximately normally distributed.

As described in the example about the lifetimes of 800 electric devices, relative frequency distributions are often approximated using a smooth curve–a distribution curve or density curve–for the tops of the bars in the histogram. The region below such a curve represents a distribution called a **continuous probability distribution**. There are many different continuous probability distributions, but the most important one is the **normal distribution**, which has a bell-shaped curve like the one shown in Data Analysis Figure 14 below.

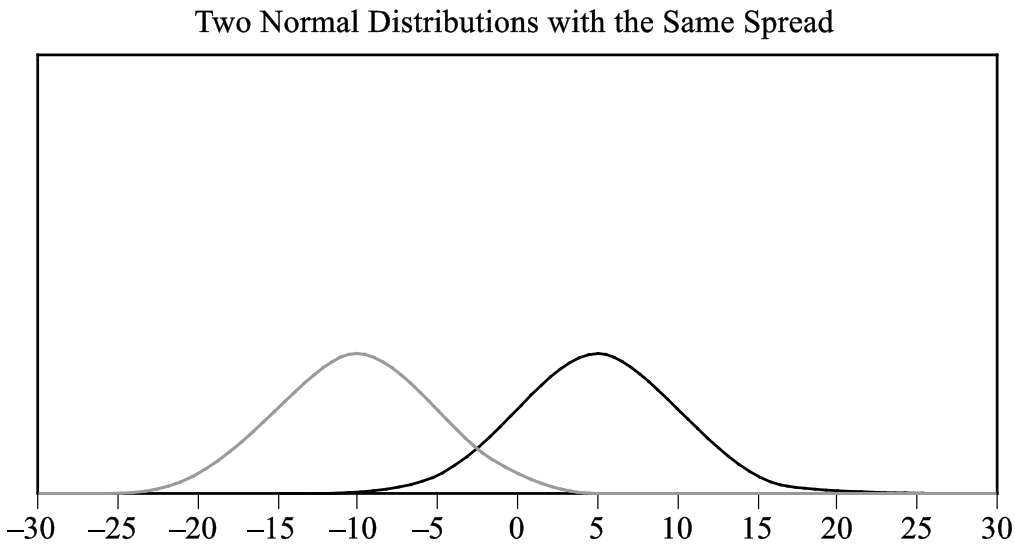


Data Analysis Figure 14

Just as a data distribution has a mean and standard deviation, the normal probability distribution has a mean and standard deviation. Also, the properties listed above for the approximately normal distribution of data hold for the normal distribution, except that the mean, median, and mode are exactly the same and the distribution is perfectly symmetric about the mean.

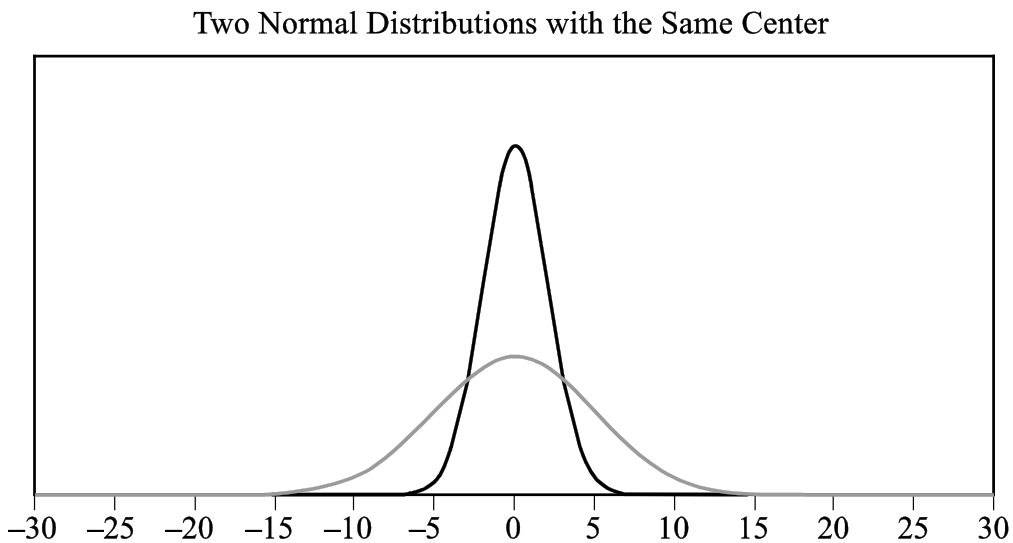
A normal distribution, though always shaped like a bell, can be centered around any mean and can be spread out to a greater or lesser degree, depending on the standard deviation. The less the standard deviation, the less spread out the curve is; that is to say, at the mean the curve is higher and as you move away from the mean in either direction it drops down toward the horizontal axis faster.

In Data Analysis Figure 15 below there are two normal distributions that have different centers, -10 and 5, respectively, but the spread of the distributions is the same. The two distributions have the same shape, so one can be shifted horizontally onto the other.



Data Analysis Figure 15

In Data Analysis Figure 16 below there are two normal distributions that have different spreads, but the same center. The mean of both distributions is 0. One of the distributions is high and spread narrowly about the mean; and the other is low and spread widely about the mean. The standard deviation of the high, narrow distribution is less than the standard deviation of the low, wide distribution.



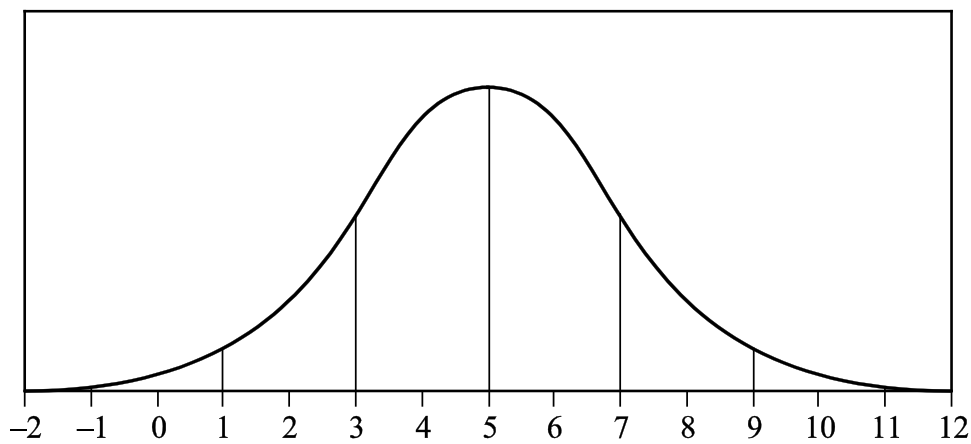
Data Analysis Figure 16

As mentioned earlier, areas of the bars in a histogram for a discrete random variable correspond to probabilities for the values of the random variable; the sum of the areas is 1 and the sum of the probabilities is 1. This is also true for a continuous probability distribution: the area of the region under the curve is 1, and the areas of vertical slices of the region, like the areas of the bars of a histogram, are equal to probabilities of a random variable associated with the distribution. Such a random variable is called a **continuous random variable**, and it plays the same role as a random variable that represents a randomly chosen value from a distribution of data. The main difference is that we seldom consider the event in which a continuous random variable is equal to a single value like $X = 3$; rather, we consider events that are described by intervals of values such as $1 < X < 3$ and $X > 10$. Such events correspond to vertical slices under a continuous probability distribution, and the areas of the vertical slices are the probabilities of the corresponding events. (Consequently, the probability of an event such as $X = 3$ would correspond to the area of a line segment, which is 0.)

Example 4.5.4: If W is a random variable that is normally distributed with a mean of 5 and a standard deviation of 2, what is $P(W > 5)$? Approximately what is $P(3 < W < 7)$? Which of the four numbers 0.5, 0.1, 0.05, or 0.01 is the best estimate of $P(W < -1)$?

Solution: Data Analysis Figure 17 below is a graph of a normal distribution with a mean of 5 and a standard deviation of 2.

The numbers 3 and 7 are 1 standard deviation away from the mean, the numbers 1 and 9 are 2 standard deviations away from the mean, and the numbers -1 and 11 are 3 standard deviations away from the mean.



Data Analysis Figure 17

Since the mean of the distribution is 5, and the distribution is symmetric about the mean, the event $W > 5$ corresponds to exactly half of the area under the normal distribution. So $P(W > 5) = \frac{1}{2}$.

For the event $3 < W < 7$, note that since the standard deviation of the distribution is 2, the values 3 and 7 are one standard deviation below and above the mean, respectively. Since about two-thirds of the area is within one standard deviation of the mean, $P(3 < W < 7)$ is approximately $\frac{2}{3}$.

For the event $W < -1$, note that -1 is 3 standard deviations below the mean. Since the graph makes it fairly clear that the area of the region under the normal curve to the left of -1 is much less than 5 percent of all of the area, the best of the four estimates given for $P(W < -1)$ is 0.01.

The **standard normal distribution** is a normal distribution with a mean of 0 and standard deviation equal to 1. To transform a normal distribution with a mean of *m* and a standard deviation of *d* to a standard normal distribution, you standardize the values; that is, you subtract *m* from any observed value of the normal distribution and then divide the result by *d*.

Very precise values for probabilities associated with normal distributions can be computed using calculators, computers, or statistical tables for the standard normal distribution. In the preceding example, more precise values for $P(3 < W < 7)$ and $P(W < -1)$ are 0.683 and 0.0013, respectively. Such calculations are beyond the scope of this review.

4.6 Data Interpretation Examples

Example 4.6.1: This example is based on the following table.

DISTRIBUTION OF CUSTOMER COMPLAINTS RECEIVED
BY AIRLINE *P*, 2003 and 2004

Category	2003	2004
Flight problem	20.0%	22.1%
Baggage	18.3	21.8
Customer service	13.1	11.3
Reservation and ticketing	5.8	5.6
Credit	1.0	0.8
Special passenger accommodation	0.9	0.9
Other	40.9	37.5
Total	100.0%	100.0%
Total number of complaints	22,998	13,278

(a) Approximately how many complaints concerning credit were received by Airline *P* in 2003 ?

(b) By approximately what percent did the total number of complaints decrease from 2003 to 2004 ?

(c) Based on the information in the table, which of the following three statements are true?

Statement 1: In each of the years 2003 and 2004, complaints about flight problems, baggage, and customer service together accounted for more than 50 percent of all customer complaints received by Airline *P*.

Statement 2: The number of special passenger accommodation complaints was unchanged from 2003 to 2004.

Statement 3: From 2003 to 2004, the number of flight problem complaints increased by more than 2 percent.

Solutions:

(a) According to the table, in 2003, 1 percent of the total number of complaints concerned credit. Therefore, the number of complaints concerning credit is equal to 1 percent of 22,998. By converting 1 percent to its decimal equivalent, you obtain that the number of complaints in 2003 is equal to $(0.01)(22,998)$, or about 230.

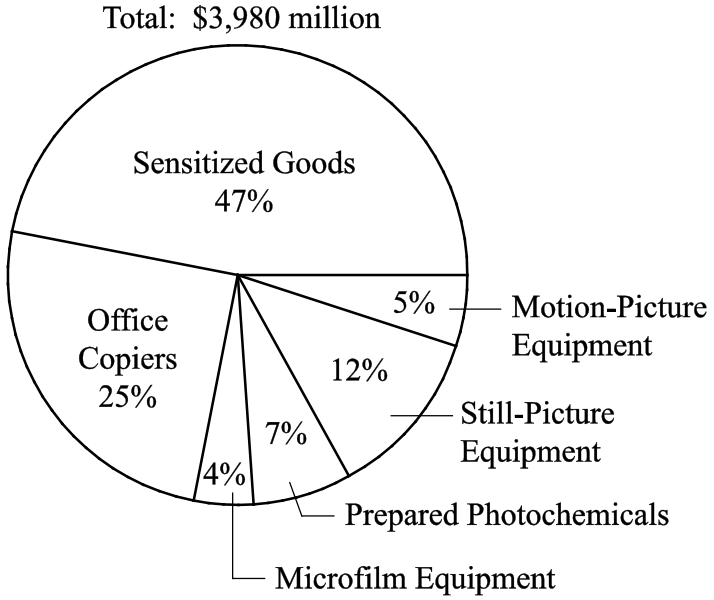
(b) The decrease in the total number of complaints from 2003 to 2004 was $22,998 - 13,278$, or 9,720. Therefore, the percent decrease was $\left(\frac{9,720}{22,998}\right)(100\%)$, or approximately 42 percent.

(c) Since $20.0+18.3+13.1$ and $22.1+21.8+11.3$ are both greater than 50, statement 1 is true. For statement 2, the *percent* of special passenger accommodation complaints *did* remain the same from 2003 to 2004, but the number of such complaints decreased because the total number of complaints decreased. Thus, statement 2 is false. For

statement 3, the *percents* shown in the table for flight problems do in fact increase by more than 2 percentage points, but the bases of the percents are different. The total number of complaints in 2004 was much lower than the total number of complaints in 2003, and clearly 20 percent of 22,998 is greater than 22.1 percent of 13,278. So, the number of flight problem complaints actually decreased from 2003 to 2004, and statement 3 is false.

Example 4.6.2: This example is based on the following circle graph.

UNITED STATES PRODUCTION OF PHOTOGRAPHIC
EQUIPMENT AND SUPPLIES IN 1971



Data Analysis Figure 18

(a) Approximately what was the ratio of the value of sensitized goods to the value of still picture equipment produced in 1971 in the United States?

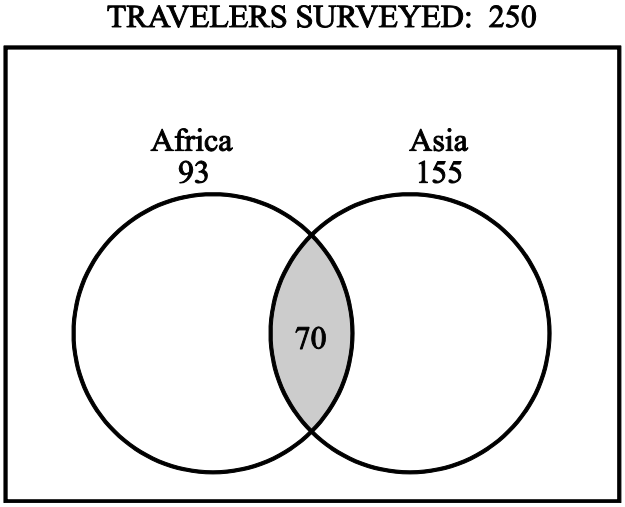
(b) If the value of office copiers produced in 1971 was 30 percent greater than the corresponding value in 1970, what was the value of office copiers produced in 1970 ?

Solutions:

(a) The ratio of the value of sensitized goods to the value of still picture equipment is equal to the ratio of the corresponding percents shown because the percents have the same base, which is the total value. Therefore, the ratio is 47 to 12, or approximately 4 to 1.

(b) The value of office copiers produced in 1971 was 0.25 times \$3,980 million, or \$995 million. Therefore, if the corresponding value in 1970 was x million dollars, then $1.3x = 995$ million. Solving for x yields $x = \frac{995}{1.3} \approx 765$, so the value of office copiers produced in 1970 was approximately \$765 million.

Example 4.6.3: In a survey of 250 European travelers, 93 have traveled to Africa, 155 have traveled to Asia, and of these two groups, 70 have traveled to both continents, as illustrated in the Venn diagram below.



Data Analysis Figure 19

(a) How many of the travelers surveyed have traveled to Africa but *not* to Asia?

(b) How many of the travelers surveyed have traveled to *at least one* of the two continents of Africa and Asia?

(c) How many of the travelers surveyed have traveled *neither* to Africa *nor* to Asia?

Solutions:

In the Venn diagram in Data Analysis Figure 19, the rectangular region represents the set of all travelers surveyed; the two circular regions represent the two sets of travelers to Africa and Asia, respectively; and the shaded region represents the subset of those who have traveled to both continents.

(a) The travelers surveyed who have traveled to Africa but not to Asia are represented in the Venn diagram by *the part of the left circle that is not shaded*. This suggests that the answer can be found by taking the shaded part away from the leftmost circle, in effect, subtracting the 70 from the 93, to get 23 travelers who have traveled to Africa, but not to Asia.

(b) The travelers surveyed who have traveled to at least one of the two continents of Africa and Asia are represented in the Venn diagram by that part of the rectangle that is *in at least one of the two circles*. This suggests adding the two numbers 93 and 155. But the 70 travelers who have traveled to both continents would be counted twice in the sum $93 + 155$. To correct the double counting, subtract 70 from the sum so that these 70 travelers are counted only once:

$$93 + 155 - 70 = 178$$

(c) The travelers surveyed who have traveled neither to Africa nor to Asia are represented in the Venn diagram by the part of the rectangle that is not in either circle. Let N be the number of these travelers. Note that the entire rectangular region has two main non overlapping parts: the part outside the circles and the part inside the circles. The first part represents N travelers and the second part represents $93 + 155 - 70 = 178$ travelers (from part b). Therefore, $250 = N + 178$, and solving for N yields $N = 250 - 178 = 72$.

DATA ANALYSIS EXERCISES

Exercise 1. The daily temperatures, in degrees Fahrenheit, for 10 days in May were 61, 62, 65, 65, 65, 68, 74, 74, 75, and 77.

(a) Find the mean, median, mode, and range of the temperatures.

(b) If each day had been 7 degrees warmer, what would have been the mean, median, mode, and range of those 10 temperatures?

Exercise 2. The numbers of passengers on 9 airline flights were 22, 33, 21, 28, 22, 31, 44, 50, and 19. The standard deviation of these 9 numbers is approximately equal to 10.22.

(a) Find the mean, median, mode, range, and interquartile range of the 9 numbers.

(b) If each flight had had 3 times as many passengers, what would have been the mean, median, mode, range, interquartile range, and standard deviation of the 9 numbers?

(c) If each flight had had 2 fewer passengers, what would have been the interquartile range and standard deviation of the 9 numbers?

Exercise 3. A group of 20 values has a mean of 85 and a median of 80. A different group of 30 values has a mean of 75 and a median of 72.

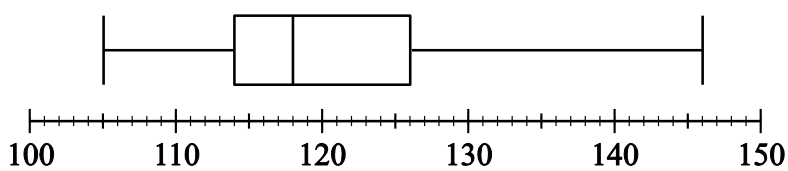
(a) What is the mean of the 50 values?

(b) What is the median of the 50 values?

Exercise 4. Find the mean and median of the values of the random variable X, whose relative frequency distribution is given in the following table.

X	Relative Frequency
0	0.18
1	0.33
2	0.10
3	0.06
4	0.33

Exercise 5. Eight hundred insects were weighed, and the resulting measurements, in milligrams, are summarized in the following boxplot.



Data Analysis Figure 20

(a) What are the range, the three quartiles, and the interquartile range of the measurements?

(b) If the 80th percentile of the measurements is 130 milligrams, about how many measurements are between 126 milligrams and 130 milligrams?

Exercise 6. In how many different ways can the letters in the word STUDY be ordered?

Exercise 7. Martha invited 4 friends to go with her to the movies. There are 120 different ways in which they can sit together in a row of 5 seats, one person per seat. In how many of those ways is Martha sitting in the middle seat?

Exercise 8. How many 3-digit positive integers are odd and do not contain the digit 5 ?

Exercise 9. From a box of 10 lightbulbs, you are to remove 4. How many different sets of 4 lightbulbs could you remove?

Exercise 10. A talent contest has 8 contestants. Judges must award prizes for first, second, and third places, with no ties.

(a) In how many different ways can the judges award the 3 prizes?

(b) How many different groups of 3 people can get prizes?

Exercise 11. If an integer is randomly selected from all positive 2-digit integers, what is the probability that the integer chosen has

(a) a 4 in the tens place?

(b) at least one 4 in the tens place or the units place?

(c) no 4 in either place?

Exercise 12. In a box of 10 electrical parts, 2 are defective.

(a) If you choose one part at random from the box, what is the probability that it is *not* defective?

(b) If you choose two parts at random from the box, without replacement, what is the probability that both are defective?

Exercise 13. A certain college has 8,978 full-time students, some of whom live on campus and some of whom live off campus.

The following table shows the distribution of the 8,978 full-time students, by class and living arrangement.

	Freshmen	Sophomores	Juniors	Seniors
Live on campus	1,812	1,236	950	542
Live off campus	625	908	1,282	1,623

(a) If one full-time student is selected at random, what is the probability that the student who is chosen will not be a freshman?

(b) If one full-time student who lives off campus is selected at random, what is the probability that the student will be a senior?

(c) If one full-time student who is a freshman or sophomore is selected at random, what is the probability that the student will be a student who lives on campus?

Exercise 14. Let A, B, C, and D be events for which

$P(A \text{ or } B) = 0.6,\ P(A) = 0.2,\ P(C \text{ or } D) = 0.6,$ and $P(C) = 0.5.$

The events A and B are mutually exclusive, and the events C and D are independent.

(a) Find $P(B)$.

(b) Find $P(D)$.

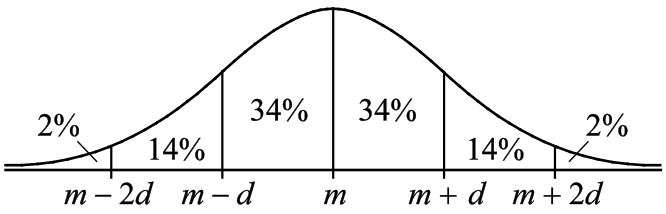
Exercise 15. Lin and Mark each attempt independently to decode a message. If the probability that Lin will decode the message is 0.80 and the probability that Mark will decode the message is 0.70, find the probability that

(a) both will decode the message

(b) at least one of them will decode the message

(c) neither of them will decode the message

Exercise 16. Data Analysis Figure 21 below shows the graph of a normal distribution with mean m and standard deviation d, including approximate percents of the distribution corresponding to the six regions shown.



Data Analysis Figure 21

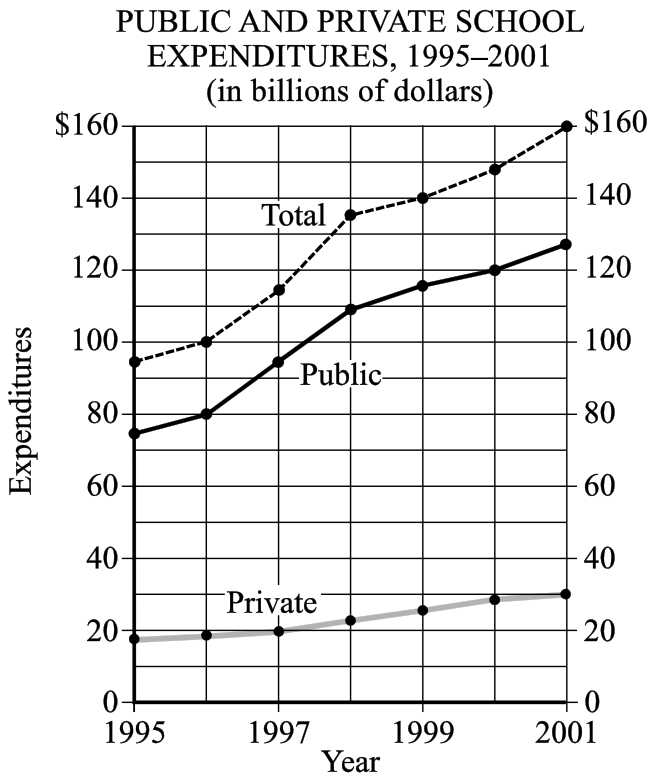
Suppose the heights of a population of 3,000 adult penguins are approximately normally distributed with a mean of 65 centimeters and a standard deviation of 5 centimeters.

(a) Approximately how many of the adult penguins are between 65 centimeters and 75 centimeters tall?

(b) If an adult penguin is chosen at random from the population, approximately what is the probability that the penguin's height will be less than 60 centimeters? Give your answer to the nearest 0.05.

_____

Exercise 17. This exercise is based on the following graph.



Data Analysis Figure 22

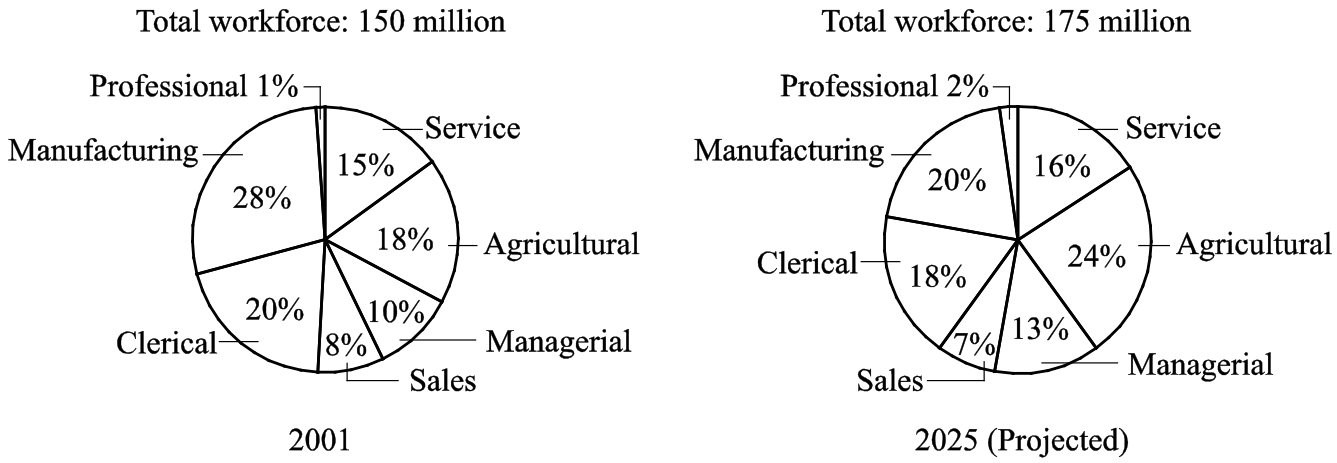
(a) For which year did total expenditures increase the most from the year before?

(b) For 2001, private school expenditures were what percent of total expenditures? Give your answer to the nearest percent.

_____

Exercise 18. This exercise is based on the following data.

DISTRIBUTION OF WORKFORCE BY OCCUPATIONAL CATEGORY
FOR REGION *Y* IN 2001 AND PROJECTED FOR 2025



Data Analysis Figure 23

(a) In 2001, how many categories each comprised more than 25 million workers?

(b) What is the ratio of the number of workers in the Agricultural category in 2001 to the projected number of such workers in 2025 ?

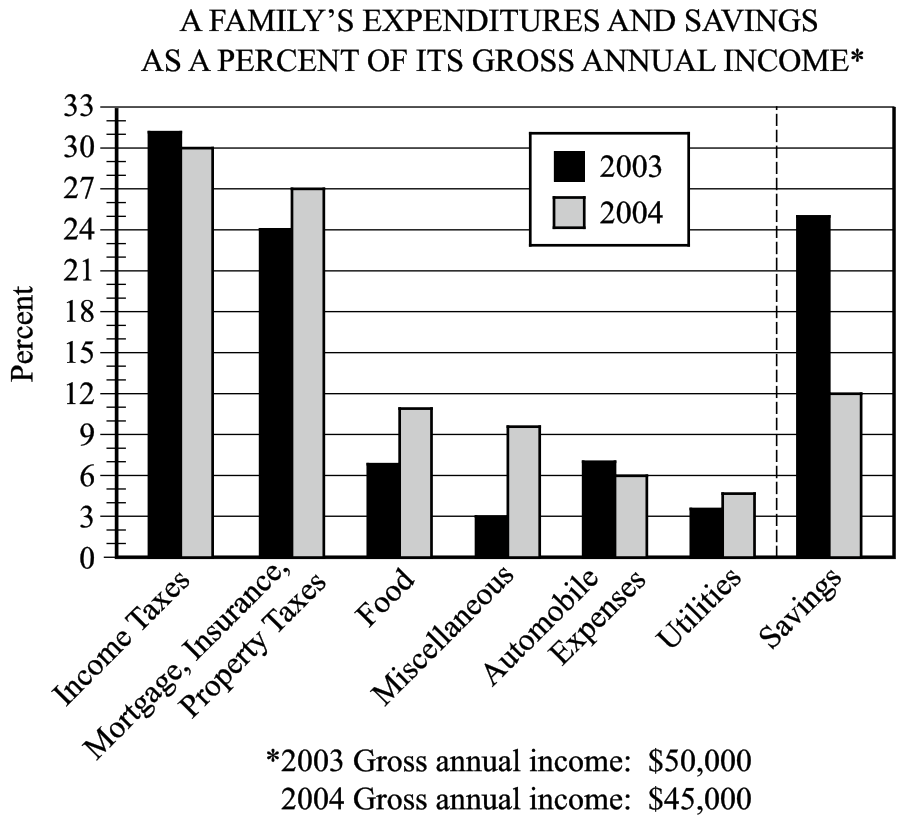
(c) From 2001 to 2025, there is a projected increase in the number of workers in which of the following three categories?

Category 1: Sales

Category 2: Service

Category 3: Clerical

Exercise 19. This exercise is based on the following data.



Data Analysis Figure 24

(a) In 2003 the family used a total of 49 percent of its gross annual income for two of the categories listed. What was the total amount of the family's income used for those same categories in 2004 ?

(b) Of the seven categories listed, which category of expenditure had the greatest percent increase from 2003 to 2004 ?

ANSWERS TO DATA ANALYSIS EXERCISES

Exercise 1. In degrees Fahrenheit, the statistics are

(a) mean = 68.6, median = 66.5, mode = 65, range = 16

(b) mean = 75.6, median = 73.5, mode = 72, range = 16

Exercise 2.

(a) mean = 30, median = 28, mode = 22, range = 31, interquartile range = 17

(b) mean = 90, median = 84, mode = 66, range = 93, interquartile range = 51

standard deviation $\approx 3(10.22) = 30.66$

(c) interquartile range = 17, standard deviation $\approx$ 10.22

Exercise 3.

(a) mean = 79

(b) The median cannot be determined from the information given.

Exercise 4. mean = 2.03, median = 1

Exercise 5.

(a) range = 41, $Q_1 = 114$, $Q_2 = 118$, $Q_3 = 126$, interquartile range = 12

(b) 40 measurements

Exercise 6. 5! = 120

Exercise 7. 24

Exercise 8. 288

Exercise 9. 210

Exercise 10.

(a) 336

(b) 56

Exercise 11.

(a) $\frac{1}{9}$

(b) $\frac{1}{5}$

(c) $\frac{4}{5}$

Exercise 12.

(a) $\frac{4}{5}$

(b) $\frac{1}{45}$

Exercise 13.

(a) $\frac{6,541}{8,978}$

(b) $\frac{1,623}{4,438}$

(c) $\frac{3,048}{4,581}$

Exercise 14.

(a) 0.4

(b) 0.2

Exercise 15.

(a) 0.56

(b) 0.94

(c) 0.06

Exercise 16.

(a) 1,440

(b) 0.15

Exercise 17.

(a) 1998

(b) 19%

Exercise 18.

(a) Three

(b) 9 to 14, or $\frac{9}{14}$

(c) Categories 1, 2, and 3

Exercise 19.

(a) $17,550

(b) Miscellaneous