Solutions to Laboratory Exercise 1

Matrix Math Operations

Problem 1.1 An investor holds two portfolios of assets (bonds and stocks) with the following number of shares allocated in each of the assets. These are represented in column vectors in the following form of (assets, number of shares).

$$\textbf{PortfolioA} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \textbf{PortfolioB} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

Find the total assets held in the two portfolios.

Sol: Create two vectors for the portfolios, and a summation to apply the arithmetic operation.

```
# 1.1
portfolioA = c(2, 4)
portfolioB = c(1, 8)
totalAssets = portfolioA + portfolioB
totalAssets
```

```
## [1]  3 12
```

Problem 1.2 From Problem 1.2, eliminate the bonds of portfolio **A** from the total assets held in the two portfolios.

Sol: Subtract the amount of bonds held in portfolioA from totalAssets by forming a new variable.

```
# 1.2
totalAssets1 = totalAssets - c(2, 0)
totalAssets1
```

```
## [1]  1 12
```

Problem 1.3 The prices of bonds and stocks for five different weeks are contained in the following matrix in the form of (weeks, assets).

$$\begin{bmatrix} 30 & 18 \\ 32 & 25 \\ 36 & 21 \\ 20 & 20 \\ 35 & 27 \end{bmatrix}$$

Suppose the investor holds two portfolios of assets (bonds and stocks) each with a different composition of the assets, represented in the following matrix (assets, portfolios).

$$\begin{bmatrix} 30 & 18 \\ 32 & 25 \end{bmatrix}$$

Determine the value of the two portfolios on each of the five weeks.

Sol: Create the price and quantity matrices and then apply the arithmetic operation to a new variable.

```
# 1.3
price = matrix(c(30, 32, 36, 20, 35,
                 18, 25, 21, 20, 27),
               nrow = 5, ncol = 2)
quantity = matrix(c(2, 4, 1, 8), nrow = 2, ncol = 2)
```

```r
valuePortfolios = price %*% quantity
valuePortfolios
```

```
##      [,1] [,2]
## [1,]  132  174
## [2,]  164  232
## [3,]  156  204
## [4,]  120  180
## [5,]  178  251
```

Problem 1.4 From Problem 1.1, what is the fraction of each asset in portfolio **A** from the total assets held in the two portfolios?

Sol: Element by element division

```r
# 1.4
portAshare = portfolioA / totalAssets
portAshare
```

```
## [1] 0.6666667 0.3333333
```

Problem 1.5 From Problem 1.1, what is the asset composition of portfolio **A** if we increase its size six times?

Sol: Element by element multiplication

```r
# 1.5
portfolioA * 6
```

```
## [1] 12 24
```

Problem 1.6 Solve for the following system of equations by using matrices:

$$3x + 4y - 6z - 9w = 15$$
$$2x - y + w = 2$$
$$y + z + w = 3$$
$$x + y + z = 1$$

Sol: Solve this system of linear equations by creating a matrix A that contains the numeric values and a vector B that contains the solutions to the equations. Since A is an n-by-n matrix and B is a column vector with n components, then vector $X = A^{-1}B$ is the solution to the system $AX = B$.

```r
# 1.6
A = matrix(c(3, 4, -6, -9,
             2, -1, 0, 1,
             0, 1, 1, 1,
             1, 1, -1, 0),
           nrow = 4, byrow = T)
A
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    3    4   -6   -9
## [2,]    2   -1    0    1
```

```
## [3,]    0    1    1    1
## [4,]    1    1   -1    0
```

```r
B = matrix(c(15, 2, 3, 1), nrow = 4)
B
```

```
##      [,1]
## [1,]   15
## [2,]    2
## [3,]    3
## [4,]    1
```

```r
X = solve(A) %*% B
X
```

```
##           [,1]
## [1,]  3.029412
## [2,]  1.676471
## [3,]  3.705882
## [4,] -2.382353
```

Problem 1.7 Solve the equations $AAXX = BB$, where

$$\mathbf{AA} = \begin{bmatrix} 13 & -8 & -3 \\ -8 & 10 & -1 \\ -3 & -1 & 11 \end{bmatrix} \quad \mathbf{BB} = \begin{bmatrix} 20 \\ -5 \\ 0 \end{bmatrix} \quad \mathbf{XX} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Sol: This problem is solved in a similar fashion to Problem 1.6. Since AA is an n-by-n matrix and BB is a column vector with n components, then $XX = AA^{-1}BB$ is the solution to the system $AAXX = BB$.

```r
# 1.7
AA = matrix(c(13, -8, -3,
              -8, 10, -1,
              -3, -1, 11),
            nrow = 3, byrow = T)
AA
```

```
##      [,1] [,2] [,3]
## [1,]   13   -8   -3
## [2,]   -8   10   -1
## [3,]   -3   -1   11
```

```r
BB = matrix(c(20, -5, 0), nrow = 3)
BB
```

```
##      [,1]
## [1,]   20
## [2,]   -5
## [3,]    0
```

```r
XX = solve(AA) %*% BB
XX
```

```
##      [,1]
## [1,]    3
## [2,]    2
## [3,]    1
```

Problem 1.8 There are two securities in a portfolio, bonds and stocks, which provide annual cash payments of \$100 and \$60 per unit, based on today's state of the economy. If the economy slows down, their payments would be \$100 and \$20. An investor holds 20 units of bonds and 10 units of stocks. The investor's receipts equal cash payments time units. The payments he will receive from the portfolio for each possible economic state are \$2,600 (if the economy remains flat) and \$2,200 (if the economy slows down).

(a) Formulate the problem as two linear equations of the form

$$a_1 x + a_2 y = b$$

where a_1, a_2 and b are constants and x and y are variables.

Sol: In the next period, there are two possible states: a flat state and a down state. For each state of the economy we must formulate an equation. Let

a_1 = cash payments of bonds
a_2 = cash payments of stocks
x = units of bonds
y = units of stocks
b = total receipts

Then for the flat state of the economy over the next period the equation is:

\$100(20) +\$60(10) = \$2,600

And for the down state of the economy over the next period the equation is:

\$100(20) + \$20(10) = \$2,200

(b) Represent this system with matrix algebra as an equation involving three matrices in the form $AAAXXX = BBB$.

Sol: For each possible state, AAA = the matrix of prices, XXX = the vector of weights (or units), and BBB = the vector of receipts.

```r
# 1.8 (b)
AAA = matrix(c(100, 60, 100, 20),
             nrow = 2, byrow = T)
AAA
```

```
##      [,1] [,2]
## [1,]  100   60
## [2,]  100   20
```

```r
XXX = matrix(c(20, 10), nrow = 2)
XXX
```

```
##      [,1]
## [1,]   20
## [2,]   10
```

```r
BBB = matrix(c(2600, 2000), nrow = 2)
BBB
```

```
##      [,1]
## [1,] 2600
## [2,] 2000
```

(c) Given these portfolio characteristics, how many units of each security should the investor hold to receive \$8,000 if the economy remains flat and \$6,000 if the economy slows down? Show the two equivalent solutions to this system.

Sol: In this case we are given the matrix of prices and the vector of receipts, but we must solve for the vector of weights (or units) by multiplying the inverse of the matrix of prices by the vector of weights.

```r
# 1.8 (c)
AAAA = AAA
AAAA
```

```
##      [,1] [,2]
## [1,]  100   60
## [2,]  100   20
```

```r
BBBB = matrix(c(8000, 6000), nrow = 2)
BBBB
```

```
##      [,1]
## [1,] 8000
## [2,] 6000
```

```r
XXXX = solve(AAAA) %*% BBBB
XXXX
```

```
##      [,1]
## [1,]   50
## [2,]   50
```