ECE368: Probabilistic Reasoning
Lab 3: Hidden Markov Model

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You should hand in: 1) A scanned .pdf version of this sheet with your answers (file size should be under 2 MB); 2) one Python file inference.py that contains your code. The files should be uploaded to Quercus.

1. (a) Write down the formulas of the forward-backward algorithm to compute the marginal distribution $p(\mathbf{z}_i|(\hat{x}_0, \hat{y}_0), \ldots, (\hat{x}_{N-1}, \hat{y}_{N-1}))$ for $i = 0, 1, \ldots, N-1$. Your answer should contain the initializations of the forward and backward messages, the recursion relations of the messages, and the computation of the marginal distribution based on the messages. (1 **pt**)

$\alpha(z_0) = P(z_0)\,P((\hat{x}_0, \hat{y}_0)|z_0)$, $\beta(z_{N-1}) = 1$ ← Initialization.

$\alpha(z_i) = P((\hat{x}_i, \hat{y}_i)|z_i) \sum_{z_{i-1}} \alpha(z_{i-1}) P(z_i|z_{i-1})$

$\beta(z_i) = \sum_{z_{i+1}} \beta(z_{i+1}) P((\hat{x}_{i+1}, y_{i+1})|z_{i+1}) P(z_{i+1}|z_i)$

← Recursive Relations.

$\gamma(z_n) = \alpha(z_n)\cdot\beta(z_n)$

$P(z_n|(\hat{x}_0, \hat{y}_0, \ldots, \hat{x}_{N-1}, \hat{y}_{N-1})) = \dfrac{\gamma(z_n)}{\sum_{z_n} \gamma(z_n)}$ ← Normalized Marginal Distribution.

(b) After you run the forward-backward algorithm on the data in test.txt, write down the obtained marginal distribution of the state at $i = 99$ (the last time step), i.e., $p(\mathbf{z}_{99}|(\hat{x}_0, \hat{y}_0), \ldots, (\hat{x}_{99}, \hat{y}_{99}))$. Only include states with non-zero probability in your answer. (2 **pt**)

$$P(z_{99}|(\hat{x}_0, \hat{y}_0), \ldots, (\hat{x}_{99}, \hat{y}_{99})) = \begin{cases} 0.8103 & z_{99} = (11, 0, \text{'stay'}) \\ 0.1796 & z_{99} = (11, 0, \text{'right'}) \\ 0.0101 & z_{99} = (10, 1, \text{'down'}) \\ 0, & \text{else.} \end{cases}$$

2. Modify your forward-backward algorithm so that it can handle missing observations. After you run the modified forward-backward algorithm on the data in test_missing.txt, write down the obtained marginal distribution of the state at $i = 30$, i.e., $p(\mathbf{z}_{30}|(\hat{x}_0, \hat{y}_0), \ldots, (\hat{x}_{99}, \hat{y}_{99}))$. Only include states with non-zero probability in your answer. (1 **pt**)

$$P(z_{30}|(\hat{x}_0, \hat{y}_0), \ldots, (\hat{x}_{99}, \hat{y}_{99})) = \begin{cases} 0.043 & z_{30} = (5, 7, \text{'right'}) \\ 0.043 & z_{30} = (5, 7, \text{'stay'}) \\ 0.913 & z_{30} = (6, 7, \text{'right'}) \\ 0, & \text{else} \end{cases}$$

3. (a) Write down the formulas of the Viterbi algorithm using $\mathbf{z}_i$ and $(\hat{x}_i, \hat{y}_i), i = 0, 1, \ldots, N-1$. Your answer should contain the initialization of the messages and the recursion of the messages in the Viterbi algorithm. (1 **pt**)

Initialization: $w_0(z_0) = \log\left(P((\hat{x}_0, \hat{y}_0)|z_0) \cdot P(z_0)\right)$.

Recursion: $w_{n+1}(z_{n+1}) = \log\left(P((\hat{x}_{n+1}, \hat{y}_{n+1})|z_{n+1})\right) + \underset{z_n}{\operatorname{argmax}} \left\{ \log\left(P(z_{n+1}|z_n)\right) + w_n(z_n) \right\}$.

(b) After you run the Viterbi algorithm on the data in test_missing.txt, write down the last 10 hidden states of the most likely sequence (i.e., $i = 90, 91, 92, \ldots, 99$) based on the MAP estimate. (3 **pt**)

z_{90} = (11, 5, 'down')
z_{91} = (11, 6, 'down')
z_{92} = (11, 7, 'down')
z_{93} = (11, 7, 'stay')
z_{94} = (11, 7, 'stay')
z_{95} = (10, 7, 'left')
z_{96} = (9, 7, 'left')
z_{97} = (8, 7, 'left')
z_{98} = (7, 7, 'left')
z_{99} = (6, 7, 'left').

4. Compute and compare the error probabilities of $\{\tilde{\mathbf{z}}_i\}$ and $\{\check{\mathbf{z}}_i\}$ using the data in test_missing.txt. The error probability of $\{\tilde{\mathbf{z}}_i\}$ is $\boxed{0.03}$. The error probability of $\{\check{\mathbf{z}}_i\}$ is $\boxed{0.02}$. (1 **pt**)

5. Is sequence $\{\check{\mathbf{z}}_i\}$ a valid sequence? If not, please find a small segment $\check{\mathbf{z}}_i, \check{\mathbf{z}}_{i+1}$ that violates the transition model for some time step i. You answer should specify the value of i as well as the corresponding states $\check{\mathbf{z}}_i, \check{\mathbf{z}}_{i+1}$. (1 **pt**)

z_{64} = (3, 7, 'stay') → z_{65} = (2, 7, 'stay') incorrect/actual.
→ z_{65} = (2, 7, 'left') correct

So, $i = 64$, and $\{\check{z}_i\}$ is not a valid sequence.