PROBLEM 1

Landmarks: $L_i = (L_x^i, L_y^i)$

States: $\begin{bmatrix} P_x \\ V_x \\ P_y \\ V_y \end{bmatrix}$

$\therefore$ We get,

Propagation model: $X_{t+1} = A_t X_t + \omega_t$

Measurement model: $y_t = C_t X_t + V_t$

(a) The velocities are constant and so we get eqs for each state propagation as:

$$P_i(t+1) = P_i(t) + V_i(t)\,\Delta t$$

for simplicity we define $\Delta t \in \mathbb{R}$ (time step) as constant and value equal to 1, which gives a representation

$$\underbrace{\begin{bmatrix} P_x(t+1) \\ V_x(t+1) \\ P_y(t+1) \\ V_y(t+1) \end{bmatrix}}_{X_{t+1}} = \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} P_x(t) \\ V_x(t) \\ P_y(t) \\ V_y(t) \end{bmatrix}}_{X_t} + \underbrace{\begin{bmatrix} 0 \\ \omega_x \\ 0 \\ \omega_y \end{bmatrix}}_{\omega_t}$$

$\Rightarrow$ A noise in position can be considered as an additional noise in velocity model:

$$\therefore \boxed{\begin{aligned} P_i[t+1] &= P_i[t] + v[t] \\ V_i[t+1] &= V_i[t] + \omega_i[t] \end{aligned}}$$

No noise in position model

$(\omega_i[t] = \omega_i \ \forall t)$

(b)

Euclidian distance based measurement model:

$$\begin{bmatrix} d^1 \\ d^2 \end{bmatrix} = \underbrace{\begin{bmatrix} \sqrt{(L^1_x - P_x)^2 + (L^1_y - P_y)^2} \\ \sqrt{(L^2_x - P_x)^2 + (L^2_y - P_y)^2} \end{bmatrix}}_{C_t \cdot X_t} + \underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{v_t}$$

(c)

From previous part we can see a non-linear measurement model that can be linearised using:

$$f(x+\delta x, y) = f(x,y) + \frac{\partial f(x,y)}{\partial x}\delta x$$

$\therefore$

$$\hat{C} = \begin{bmatrix} \frac{\partial r_1}{\partial P_x} & \frac{\partial r_1}{\partial V_x} & \frac{\partial r_1}{\partial P_y} & \frac{\partial r_1}{\partial V_y} \\[2ex] \frac{\partial r_2}{\partial P_x} & \frac{\partial r_2}{\partial V_x} & \frac{\partial r_2}{\partial P_y} & \frac{\partial r_2}{\partial V_y} \end{bmatrix}$$

$$\Rightarrow \underset{(2\times4)}{\hat{C}} = \begin{bmatrix} \frac{(P_x - L^1_x)}{\sqrt{(L^1_x - P_x)^2 + (L^1_y - P_y)^2}} & 0 & \frac{(P_y - L^1_y)}{\sqrt{(L^1_x - P_x)^2 + (L^1_y - P_y)^2}} & 0 \\[3ex] \frac{(P_x - L^2_x)}{\sqrt{(L^2_x - P_x)^2 + (L^2_y - P_y)^2}} & 0 & \frac{(P_y - L^2_y)}{\sqrt{(L^2_x - P_x)^2 + (L^2_y - P_y)^2}} & 0 \end{bmatrix}$$

(c)

Adding noise to the measurement model:

$$Y_t = \hat{C} X_t + V$$

(d)

Propagation

→ We are given: $X_t \sim N(\mu_t, \Sigma_t)$

$\omega_t \sim N(0, R_t)$

$\left[\begin{array}{l} A \in R^{4\times4} \\ \omega_t \in R^{4\times1} \end{array} \right]$

→ Process Model:

$$X_{t+1} = A_t X_t + \omega_t$$

$\therefore A_t X_t + \omega_t \sim N(A_t \mu_t, A_t \Sigma_t A_t^T + R_t)$ —①

$\therefore X_{t+1} \sim N(\mu_{t+1}, \Sigma_{t+1})$ —②

From ① & ②

$$\boxed{\mu_{t+1} = A_t \mu_t} \qquad \boxed{\Sigma_{t+1} = A_t \Sigma_t A_t^T + R_T}$$

$$\Rightarrow \mu_{t+1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mu_{px} \\ \mu_{vx} \\ \mu_{py} \\ \mu_{vy} \end{bmatrix} = \begin{bmatrix} \mu_{px} + \mu_{vx} \\ \mu_{vx} \\ \mu_{py} + \mu_{vy} \\ \mu_{vx} \end{bmatrix}$$

(d)

$$\Sigma_{t+1} = \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{A_t} \cdot \underbrace{\begin{bmatrix} \sigma_{px}^2 & 0 & 0 & 0 \\ 0 & \sigma_{vx}^2 & 0 & 0 \\ 0 & 0 & \sigma_{py}^2 & 0 \\ 0 & 0 & 0 & \sigma_{vy}^2 \end{bmatrix}}_{\Sigma_t} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{A_t^T}$$

$\therefore$ No noise in position

$$= \begin{bmatrix} \sigma_{px}^2+\sigma_{vx}^2 & \sigma_{vx}^2 & 0 & 0 \\ \sigma_{vx}^2 & \sigma_{vx}^2 & 0 & 0 \\ 0 & 0 & \sigma_{py}^2+\sigma_{vy}^2 & \sigma_{vy}^2 \\ 0 & 0 & \sigma_{vy}^2 & \sigma_{vy}^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \sigma_{wx}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_{wy}^2 \end{bmatrix}$$

$$\therefore \Sigma_{t+1} = \begin{bmatrix} \sigma_{px}^2+\sigma_{vx}^2 & \sigma_{vx}^2 & 0 & 0 \\ \sigma_{vx}^2 & \sigma_{vx}^2+\sigma_{wx}^2 & 0 & 0 \\ 0 & 0 & \sigma_{py}^2+\sigma_{vy}^2 & \sigma_{vy}^2 \\ 0 & 0 & \sigma_{vy}^2 & \sigma_{vy}^2+\sigma_{wy}^2 \end{bmatrix}$$

Measurement Model

$$y_t = \hat{C}_t x_t + v_t$$

$\hat{C}_t \in R^{2\times 4}$
$v_t \in R^{2\times 1}$

Given:

$$\left. \begin{array}{l} x_t \sim N(\hat{\mu}_t, \hat{\Sigma}_t) \\ v_t \sim N(0, \varphi_t) \end{array} \right\} \Leftrightarrow \hat{\Sigma}_t = \begin{bmatrix} \sigma_{px}^2 & 0 & 0 & 0 \\ 0 & \sigma_{vx}^2 & 0 & 0 \\ 0 & 0 & \sigma_{py}^2 & 0 \\ 0 & 0 & 0 & \sigma_{vy}^2 \end{bmatrix}$$

$$\therefore y_t \sim N(\mu_t, \Sigma_t)$$

$$\hat{\mu}_t = \begin{bmatrix} \hat{\mu}_{px} \\ \hat{\mu}_{vx} \\ \hat{\mu}_{py} \\ \hat{\mu}_{vy} \end{bmatrix} \qquad \varphi_t = \begin{bmatrix} \sigma_{v_1}^2 & 0 \\ 0 & \sigma_{v_2}^2 \end{bmatrix}$$

∴ we can define Kalman constant K_t:

$$K_t = \hat{\Sigma}_t \hat{C}_t^T (\hat{C}_t \hat{\Sigma}_t \hat{C}_t^T + Q_t)^{-1}$$

Then we can use this K_t to find updated values of μ_t & Σ_t as follows:

$K_t \in R^{4\times2}$
$\Sigma_t \in R^{4\times4}$
$\mu_t \in R^{4\times1}$

$$\boxed{\Sigma_t = (I - K_t \hat{C}_t) \hat{\Sigma}_t}$$

$$\boxed{\mu_t = \hat{\mu}_t + K_t (y_t - \underbrace{C_t(\hat{\mu}_t)}_{\text{Non-linear Func.}})}$$