Properties of LTI Systems

Jose Krause Perin

Stanford University

June 29, 2018

Last lecture

- A random process is an indexed collection of random variables
- A random process is strict-sense stationary (SSS) if all its finite-order statistics are time invariant. That's hard to verify in practice.
- A random process is wide-sense stationary (WSS) if its mean is constant and if its autocorrelation function only depends on the time difference.
- A random process is ergodic if its time averages are equal to its probability averages
- The Fourier transform of the autocorrelation function is called the power spectrum density (PSD). The PSD has units of W/Hz or dBm/Hz.
- When a random signal is filtered by an LTI system defined by $h[n] \leftrightarrow H(e^{j\omega})$, its autocorrelation function is filtered by an LTI system defined by $h[n] * h^*[-n]$, and its PSD is shaped by $|H(e^{j\omega})|^2$
- Random processes that have PSD constant over all frequencies are called white noise
- By the central limit theorem, the output of an LTI system to a random input is approximately Gaussian distributed

Today's lecture

Magnitude and Phase Response

Poles, Zeros, and the Frequency Response

All-Pass Systems

Minimum Phase Systems

Linear and Generalized Linear Phase Systems

Magnitude and phase response

Recall that complex exponentials are **eigenfunctions** of LTI systems

$$\mathrm{LTI}\{e^{j\omega n}\} = H(e^{j\omega})e^{j\omega n}$$

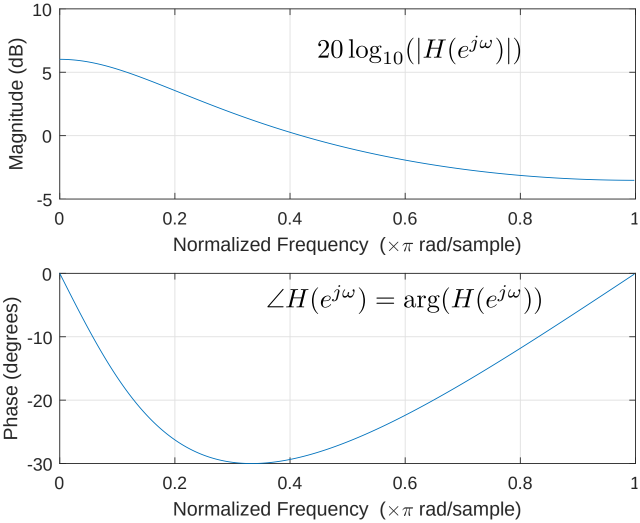
$H(e^{j\omega})$ is the corresponding **eigenvalue** of $e^{j\omega n}$
$H(e^{j\omega})$ tell us by how much the LTI system **scales** and **delays** $e^{j\omega n}$

$$H(e^{j\omega}) = \underbrace{|H(e^{j\omega})|}_{\text{Magnitude}} \exp(j \underbrace{\arg H(e^{j\omega})}_{\text{Phase}}) \qquad \text{(polar coordinates)}$$

Calculating the output $Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$

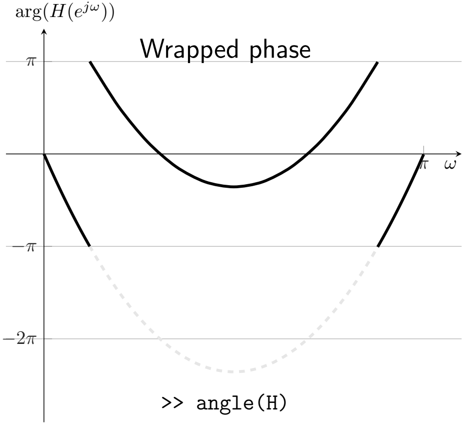
$$|Y(e^{j\omega})| = |H(e^{j\omega})| \cdot |X(e^{j\omega})| \qquad \text{(magnitudes multiply)}$$
$$\arg Y(e^{j\omega}) = \arg H(e^{j\omega}) + \arg X(e^{j\omega}) \qquad \text{(phases add)}$$

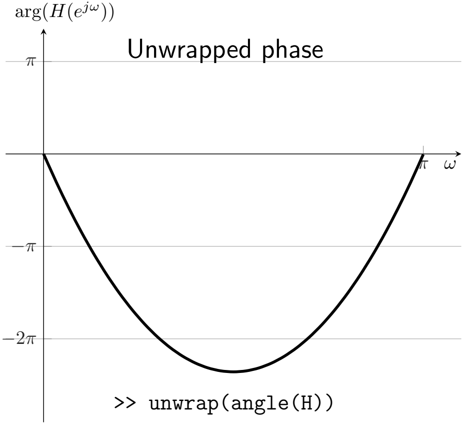
Magnitude and phase response



Phase unwrapping

Calculating the phase $\angle H(e^{j\omega})$ using $\arctan(\cdot)$ leads to discontinuities known as **phase wrapping**, since the image of $\arctan(\cdot)$ is $[-\pi, \pi]$.
Phase unwrapping corrects jumps of $\pm 2\pi$ in the phase response.





Group delay

Definition

$$\tau_g(\omega) = \mathrm{grd}\ H(e^{j\omega}) \equiv -\frac{d}{d\omega}\arg H(e^{j\omega}) \qquad \text{(group delay)}$$

Group delay measures by how much $e^{j\omega}$ is delayed by the LTI system.
In continuous-time, $\tau_g(\omega)$ has units of seconds. In discrete-time, $\tau_g(\omega)$ has units of samples.

Example

If a system has linear phase:

$$\arg H(e^{j\omega}) = -\omega n_d \qquad \textbf{(linear phase)}$$

Then, the group delay is constant:

$$\tau_g(\omega) = -\frac{d}{d\omega}(-\omega n_d) = n_d \qquad \textbf{(constant group delay)}$$

Conclusion: Linear-phase systems delay all frequencies equally.

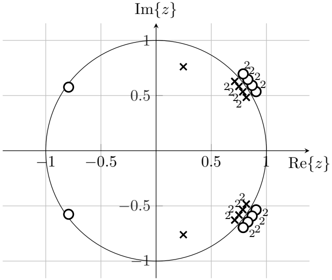
Effect of group delay

Consider the <u>causal</u> LTI system defined by the following z-transform

$$H(z) = \left(\frac{(1-0.98e^{j0.8\pi}z^{-1})(1-0.98e^{-j0.8\pi}z^{-1})}{(1-0.8e^{j0.4\pi}z^{-1})(1-0.8e^{-j0.4\pi}z^{-1})}\right)\prod_{k=1}^{4}\left(\frac{(c_k^*-z^{-1})(c_k-z^{-1})}{(1-c_kz^{-1})(1-c_k^*z^{-1})}\right)^2$$

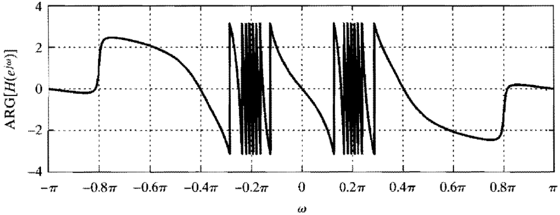
where $c_k = 0.95e^{j(0.15\pi+0.02\pi k)}$

It has the following pole-zero plot:

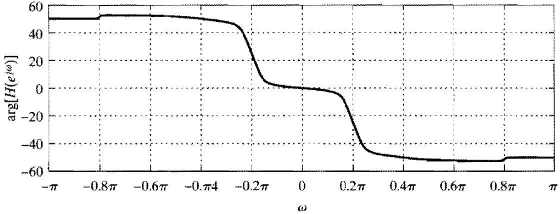


Effect of group delay

Wrapped and unwrapped phase response



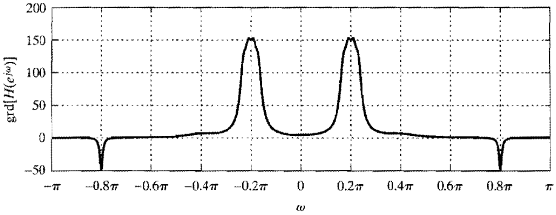
(a) Principle Value of Phase Response



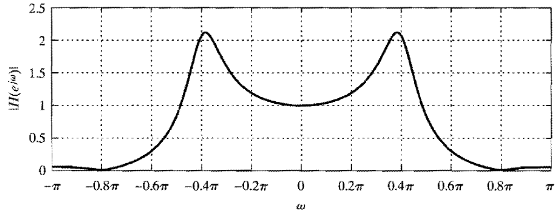
(b) Unwrapped Phase Response

Effect of group delay

Group delay and magnitude response



(a) Group delay of $H(z)$

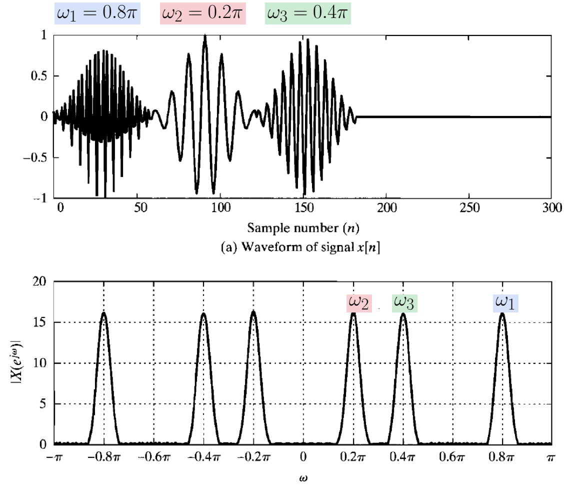


(b) Magnitude of Frequency Response

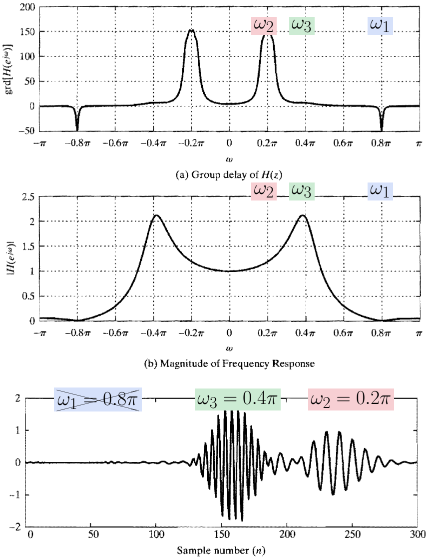
Effect of group delay

Consider the following input signal

Three sinusoidal pulses of frequencies $\omega_1 = 0.8\pi$, $\omega_2 = 0.2\pi$, and $\omega_3 = 0.4\pi$.



Effect of group delay



Comments:

- The sinusoidal pulse of frequency $\omega_1 = 0.8\pi$ was virtually eliminated, since $H(z)$ has a zero at frequency 0.8π.
- The sinusoidal pulse of frequency $\omega_2 = 0.2\pi$ had a gain of about 1.2, but note that the group delay at that frequency is 150 samples.
- The sinusoidal pulse of frequency $\omega_3 = 0.4\pi$ had its amplitude doubled, but remained centered around $n = 150$, since the group delay of $H(z)$ around 0.4π is negligible.
- Interestingly, the effect of the system was to switch the position of the two pulses in time. Note that now, the pulse of frequency $\omega_3 = 0.4\pi$ comes before the pulse of frequency $\omega_2 = 0.2\pi$.

Poles and zeros

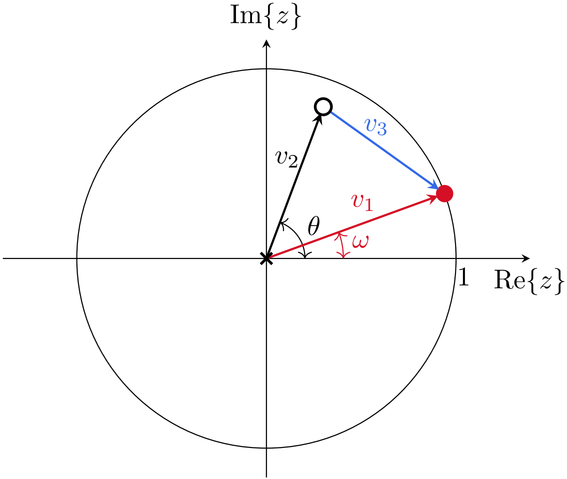
The **zeros** of $H(z)$ are the values of z for which $H(z) = 0$, while the **poles** of $H(z)$ are the values of z for which $H(z) = \infty$.
For **rational z-transforms** (ratio of two polynomials in z^{-1} or z), zeros and poles are the roots of the numerator and denominator polynomials, respectively.

$$
\begin{aligned}
H(z) = \frac{B(z)}{A(z)} &= \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}} \\
&= \frac{b_0}{a_0} z^{N-M} \frac{(z - z_1)(z - z_2)\ldots(z - z_M)}{(z - p_1)(z - p_2)\ldots(z - p_N)}
\end{aligned}
$$

- If the coefficients $\{b_0, \ldots, b_M\}, \{a_0, \ldots, a_N\}$ are real, the poles and zeros are either real or they appear in complex conjugate pairs
- $H(z)$ has **finite impulse response (FIR)** if $a_k = 0, k = 1, \ldots N$ i.e., all poles of $H(z)$ are at the origin.
- $H(z)$ has **infinite impulse response (IIR)** if $H(z)$ has poles away from the origin.

Poles, zeros, and the frequency response

Question: How do poles and zeros affect the frequency response?
Effect of a zero: $H(z) = 1 - re^{j\theta}z^{-1}$



Magnitude

$$H(z) = \frac{z - re^{j\theta}}{z}$$

$$|H(e^{j\omega})| = \left|\frac{e^{j\omega} - re^{j\theta}}{e^{j\omega}}\right|$$

$$= \left|\left|\frac{v_1 - v_2}{v_1}\right|\right|$$

$$= \frac{||v_3||}{||v_1||}$$

$$= ||v_3||$$

Poles, zeros, and the frequency response

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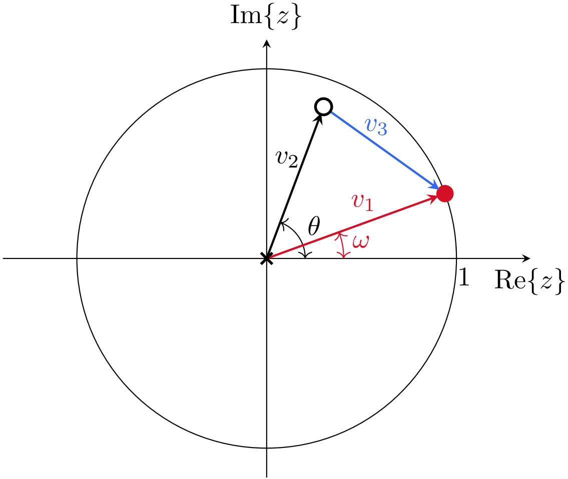
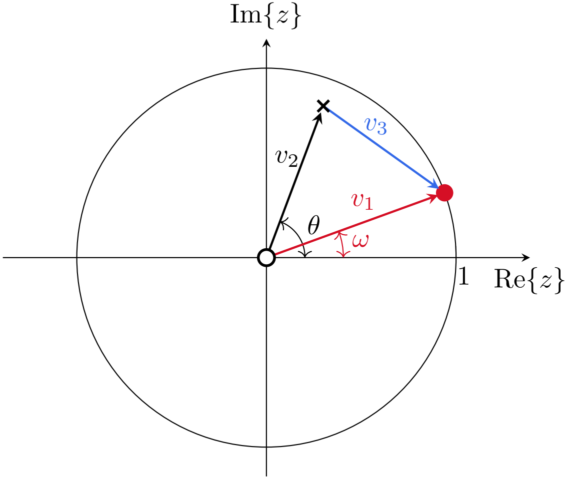


Phase

$$
\begin{aligned}
H(z) &= \frac{z - e^{j\theta}}{z} \\
\angle H(e^{j\omega}) &= \angle \frac{e^{j\omega} - re^{j\theta}}{e^{j\omega}} \\
&= \angle(e^{j\omega} - re^{j\theta}) \\
&- \angle e^{j\omega} \\
&= \angle(v_1 - v_2) - \angle v_1 \\
&= \angle v_3 - \omega
\end{aligned}
$$

Poles, zeros, and the frequency response

Effect of a pole: $H(z) = \dfrac{1}{1 - re^{j\theta} z^{-1}}$

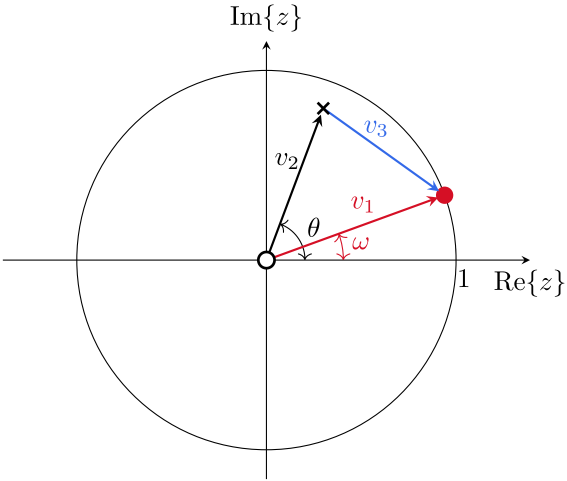


Magnitude

$$
\begin{aligned}
H(z) &= \frac{z}{z - re^{j\theta}} \\
|H(e^{j\omega})| &= \left| \frac{e^{j\omega}}{e^{j\omega} - re^{j\theta}} \right| \\
&= \left| \frac{v_1}{v_1 - v_2} \right| \\
&= \frac{||v_1||}{||v_3||} \\
&= \frac{1}{||v_3||}
\end{aligned}
$$

Poles, zeros, and the frequency response

Effect of a pole: $H(z) = \dfrac{1}{1 - re^{j\theta} z^{-1}}$



Phase

$$
\begin{aligned}
H(z) &= \frac{z}{z - re^{j\theta}} \\
\angle H(e^{j\omega}) &= \angle \frac{e^{j\omega}}{e^{j\omega} - re^{j\theta}} \\
&= -\angle(e^{j\omega} - re^{j\theta}) \\
&+ \angle e^{j\omega} \\
&= -\angle v_3 + \angle v_1 \\
&= -\angle v_3 + \omega
\end{aligned}
$$

Poles, zeros, and the frequency response

Summary of effect of poles and zeros

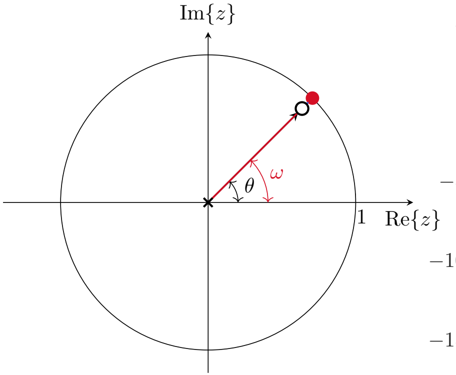
	Magnitude	Phase
Zero	$\|\|v_3\|\|$	$\angle v_3 - \omega$
Pole	$\dfrac{1}{\|\|v_3\|\|}$	$-\angle v_3 + \omega$

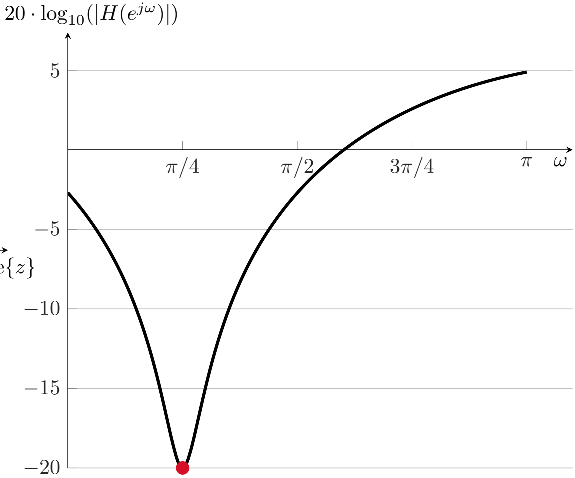
- As $e^{j\omega}$ approaches a zero, $|H(e^{j\omega})| \to 0$
- As $e^{j\omega}$ approaches a zero, $\angle|H(e^{j\omega})|$ decreases ($-\omega$ factor).
- At the zero there will be a negative-to-positive sign change of the phase response (**phase advance**) (must account effect of other zeros/poles)
- As $e^{j\omega}$ approaches a pole, $|H(e^{j\omega})| \to \infty$
- As $e^{j\omega}$ approaches a pole, $\angle|H(e^{j\omega})|$ increases ($+\omega$ factor)
- At the pole there will be a positive-to-negative sign change of the phase response (**phase lag**) (must account effect of other zeros/poles).

Conclusion: Zeros decrease the magnitude and introduce phase advance (negative group delay), while poles increase magnitude and introduce phase lag (positive group delay)

Poles, zeros, and the frequency response

Example: $H(z) = 1 - 0.9e^{j\pi/4}z^{-1}$





Poles, zeros, and the frequency response

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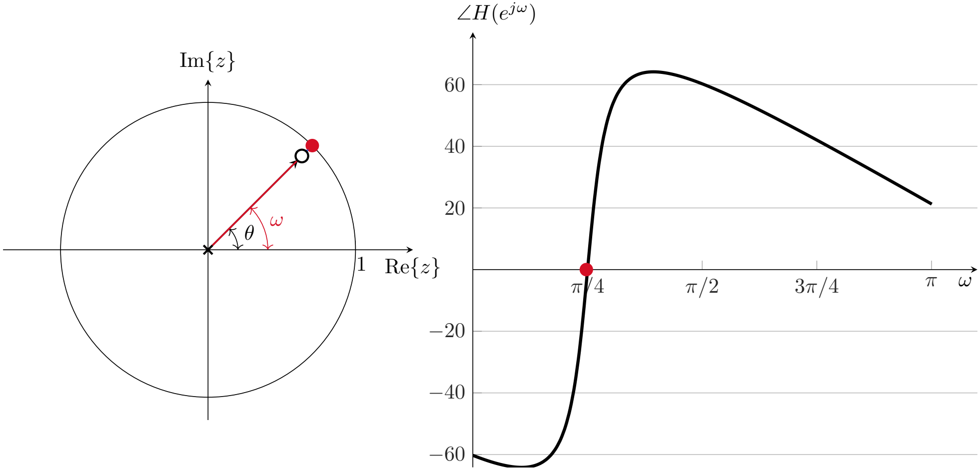
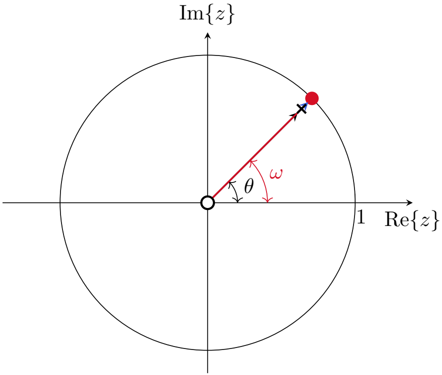
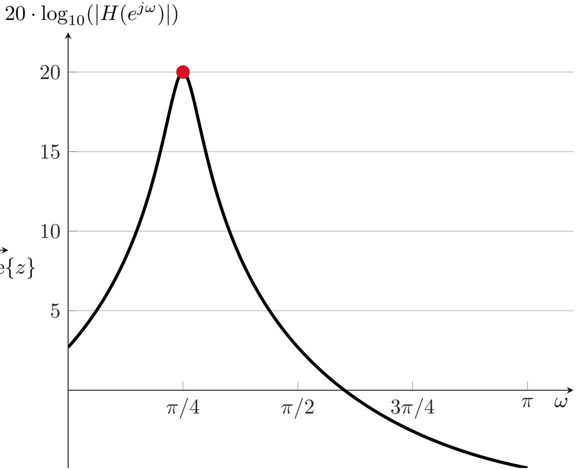


Poles, zeros, and the frequency response

Example: $H(z) = \dfrac{1}{1 - 0.9e^{j\pi/4}z^{-1}}$





Poles, zeros, and the frequency response

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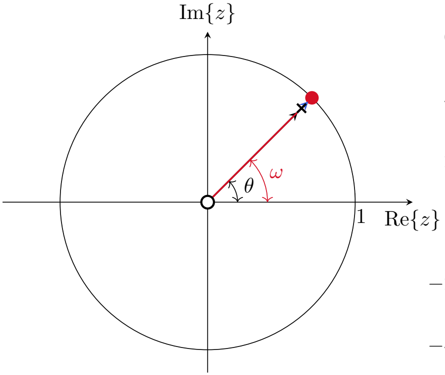
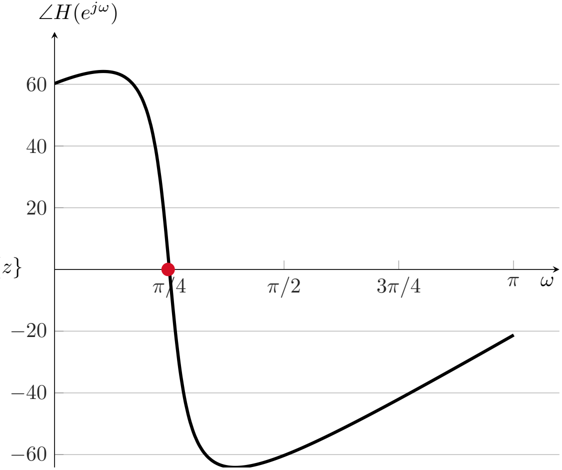
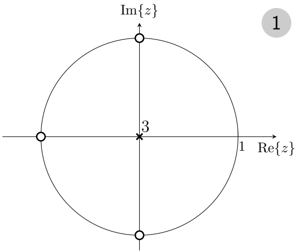
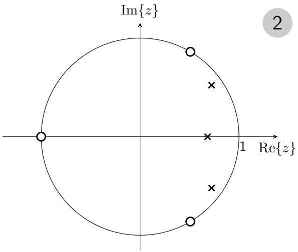
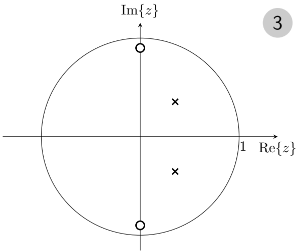


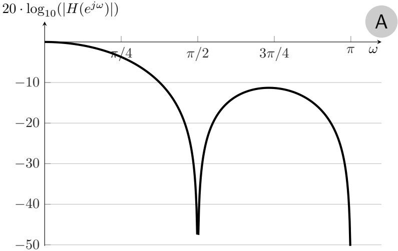


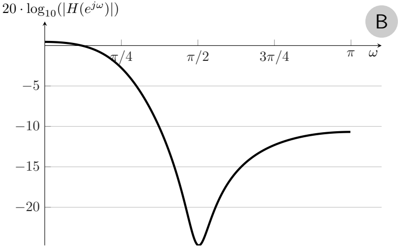
Match the pole-zero plots to the magnitude responses

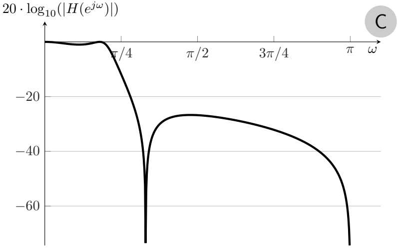




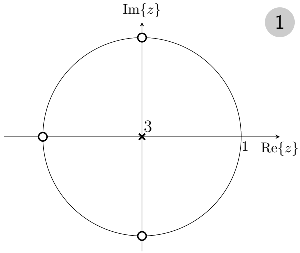


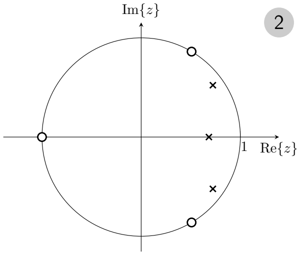


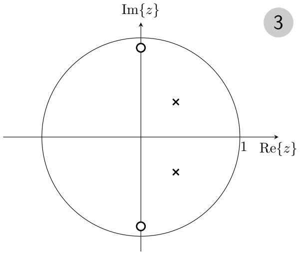


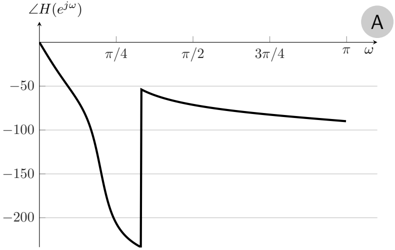


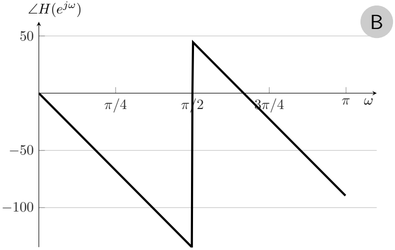
Match the pole-zero plots to the phase responses

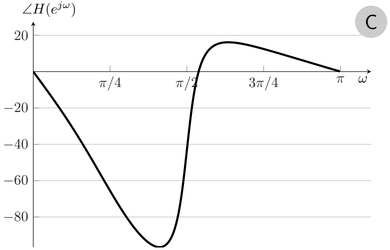




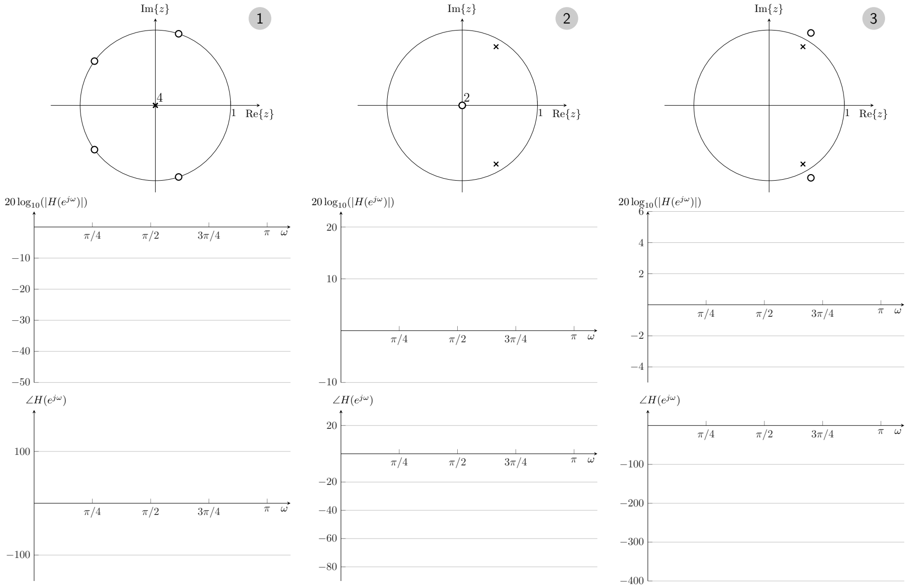








Sketch the magnitude and phase responses



Important classes of LTI systems

- All-pass systems
- Minimum-phase systems
- Linear-phase systems
- Generalize linear-phase systems

All-pass systems

Definition

An all-pass system has magnitude response independent of ω:

$$|H(e^{j\omega})| = A = \text{Constant}, \forall\omega$$

This condition is satisfied by systems of the form

$$H_{ap}(z) = \frac{z^{-1} - e_k^*}{1 - e_k z^{-1}} = e_k^* \frac{z - 1/e_k^*}{z - e_k}$$

$$\text{pole at } e_k \iff \text{zero at } 1/e_k^*$$

For every pole inside the unit circle e_k, there's a zero outside the unit circle at the **conjugate reciprocal** $1/e_k^*$ location. More generally,

$$H_{ap}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

d_k are real poles and e_k, e_k^* are complex conjugate poles.

Causal all-pass systems properties

1. The unwrapped phase is always non-positive (it may be zero)

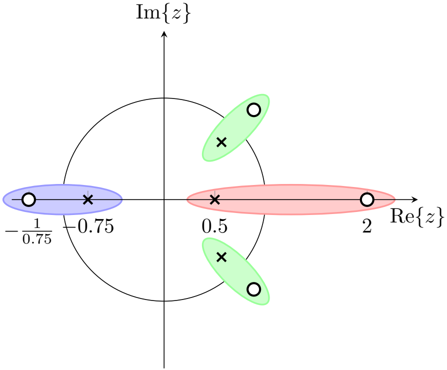
$$\arg(H_{ap}(e^{j\omega})) \leq 0, \qquad 0 \leq \omega \leq \pi$$

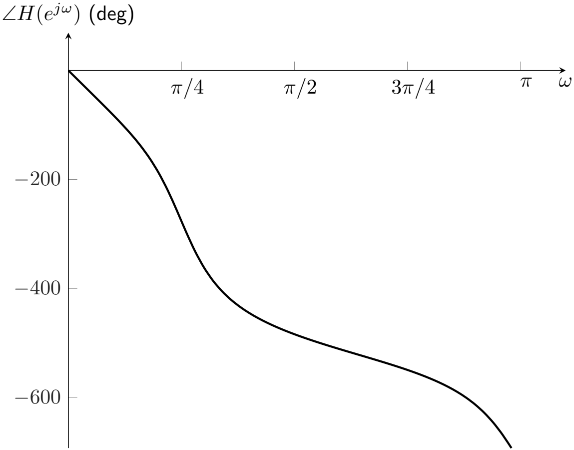
2. As a consequence of the above property, the group delay is always non-negative

$$\tau_g = -\frac{d}{d\omega} \arg(H_{ap}(e^{j\omega})) \geq 0, \qquad 0 \leq \omega \leq \pi$$

All-pass systems: example

$$H_{ap}(z) = \frac{z^{-1}+0.75}{1+0.75z^{-1}} \, \frac{z^{-1}-0.5}{1-0.5z^{-1}} \, \frac{(z^{-1}-0.8e^{-j\pi/4})(z^{-1}-0.8e^{j\pi/4})}{(1-0.8e^{-j\pi/4}z^{-1})(1-0.8e^{j\pi/4}z^{-1})}$$





Minimum phase systems

Definition

A causal and stable system is **minimum phase** if all its zeros are inside the unit circle.

As a consequence, if a system $H(z)$ is **minimum phase**, then the system $H^{-1}(z)$ (its inverse) is also causal, stable, and minimum phase.

Properties of minimum phase systems

1. **Minimum phase-lag property** (the reason they are called minimum phase):
   Proof: Since for any $H(z)$ we can write $H(z) = H_{min}(z)H_{ap}(z)$. The phase response is

$$\begin{aligned}
\arg H(z) &= \arg H_{min}(z) + \arg H_{ap}(z) && \text{(phases add)} \\
\arg H(z) &\leq \arg H_{min}(z) && \text{(from properties of all-pass systems: } \arg H_{ap}(z) \leq 0) \\
-\arg H(z) &\geq -\arg H_{min}(z) && \text{(phase lag } \equiv -\arg(\cdot))
\end{aligned}$$

   Conclusion: the systems $H(z)$ and $H_{min}(z)$ have the same magnitude response, but $H_{min}(z)$ has the minimum phase-lag response.
2. **Minimum group-delay property**: similarly to the property above.

$$\begin{aligned}
\text{grd } H(z) &= \text{grd } H_{min}(z) + \text{grd } H_{ap}(z) && \text{(group delays add)} \\
\text{grd } H(z) &\geq \text{grd } H_{min}(z) && \text{(from properties of all-pass systems: grd } H_{ap}(z) \leq 0)
\end{aligned}$$

Properties of minimum phase systems

3. **Minimum energy-delay property**: The N first samples of the impulse response of a minimum phase system $h_{min}[m] \leftrightarrow H_{min}(e^{j\omega})$ have more energy than the N first samples of any other system $h[m] \leftrightarrow H(e^{j\omega})$ with same magnitude response. Formally, If $|H_{min}(e^{j\omega})| = |H(e^{j\omega})|$, then

$$\sum_{m=0}^{N-1} |h[m]|^2 \leq \sum_{m=0}^{N-1} |h_{min}[m]|^2, \text{for any } N$$

For proof, see problem 5.71 of the textbook.

Minimum-phase/all-pass decomposition

Any rational system function $H(z)$ can be uniquely decomposed into a cascade of minimum phase system and an all-pass system:

$$H(z) = H_{min}(z)H_{ap}(z)$$

Proof sketch:
Suppose that $H(z)$ has one zero outside the unit circle $1/c^*$. We can remove $1/c^*$ from $H(z)$ by simply writing

$$\begin{aligned} H(z) &= H_1(z)(z^{-1} - c^*) \\ &= \underbrace{H_1(z)(1 - cz^{-1})}_{\text{minimum phase}} \underbrace{\frac{z^{-1} - c^*}{1 - cz^{-1}}}_{\text{all-pass}} \end{aligned}$$

Since $|c| < 1$, the factor $H_1(z)(1 - cz^{-1})$ is also minimum phase and it differs from $H(z)$ only in that the zero of $H(z)$ that was outside the unit circle at $z = 1/c^*$ is reflected inside the unit circle to the conjugate reciprocal location $z = c$.
This proof can be easily extended if $H(z)$ has more than one zero outside the unit circle.

Algorithm for minimum-phase/all-pass decomposition

Given a non-minimum phase system $H(z)$, we wish to find a minimum phase system $H_{min}(z)$ and an all-pass system $H_{ap}(z)$ such that

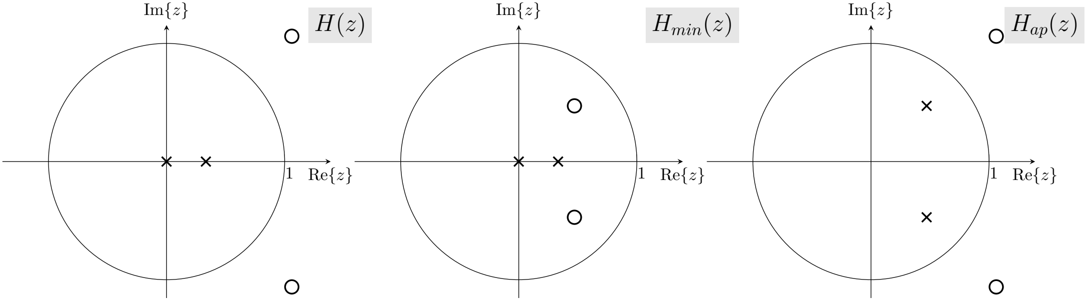
$$H(z) = H_{min}(z) H_{ap}(z)$$

1. Write $H_1(z)$, which contains all zeros and poles of $H(z)$ that are inside the unit circle.
2. For each zero $1/c^*$ of $H(z)$ outside the unit circle, add a zero to $H_1(z)$ at the conjugate reciprocal location c. And add the term $\frac{z^{-1}-c^*}{1-cz^{-1}}$ to the all-pass system.
3. By convention, we make $|H_{ap}(z)| = 1$, so the gain of $H_{min}(z)$ at zero frequency must be equal to the gain of $H(z)$.

Example of minimum-phase/all-pass decomposition

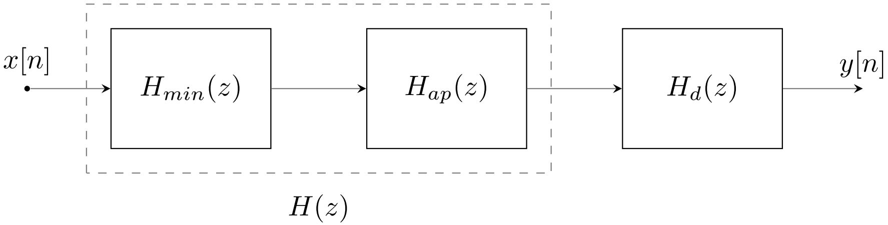
Consider the non-minimum phase system

$$H(z) = \frac{(1 + \frac{3}{2}e^{j\pi/4}z^{-1})(1 + \frac{3}{2}e^{-j\pi/4}z^{-1})}{1 - \frac{1}{3}z^{-1}}$$



Implications of minimum-phase/all-pass decomposition

Suppose we want to design $H_d(z)$ to compensate for the effects of $H(z)$.



Ideally, we'd like to have $H_d(z) = H^{-1}(z)$. But what if $H^{-1}(z)$ is unstable?

$$H_d(z) = H_{min}^{-1}(z)$$

Design $H_d(z)$ to compensate for the minimum phase system. $H_{min}^{-1}(z)$ is guaranteed to be stable.
This choice of $H_d(z)$ compensates for the magnitude, but there'll still be an uncompensated phase distortion introduced by $H_{ap}(z)$. *Oftentimes* that's good enough.

Relation between magnitude and phase plots

Question: given the magnitude response $|H(e^{j\omega})|$ of a causal system, what can we tell about its phase response $\arg H(e^{j\omega})$?

For minimum phase systems, the magnitude response perfectly describes the phase response.

Kramers-Kronig relations

For any real and causal system defined by $h(t) \leftrightarrow H(j\Omega) = H_r(j\Omega) + jH_i(j\Omega)$, where $H_r(j\Omega) = \mathrm{Re}\{H(j\Omega)\}$ and $H_i(j\Omega) = \mathrm{Im}\{H(j\Omega)\}$,
the **Kramers-Kronig relations** establish that

$$H(j\Omega) = H_r(j\Omega) - j\mathcal{H}\{H_r(j\Omega)\},$$

where $\mathcal{H}\{\cdot\}$ is the **Hilbert transform**.
Conclusion: The imaginary part of $H(j\Omega)$ is determined from the Hilbert transform of the real part. Knowing just the real part is sufficient to completely specify the system, and the imaginary part is "redundant" information.

- $H_r(j\Omega)$ is the Fourier transform of the even part of the impulse response, and $H_i(j\Omega)$ is the Fourier transform of the odd part of the impulse response (recall even/odd decomposition from lecture 1)
- **Causality check**: compare the imaginary part of the frequency response with the Hilbert transform of the real part.
- All physical systems are causal, so the Kramers-Kronig relations apply to all physical LTI systems.

The Hilbert transform

The Hilbert transform of a signal $x(t)$ is defined as

$$\mathcal{H}\{x(t)\} = h_{HT}(t) * x(t),$$

where

$$h_{HT}(t) = \begin{cases} \frac{1}{\pi t}, & t \neq 0 \\ 0, & t = 0 \end{cases}$$

$$\mathcal{F}\{h_{HT}(t)\} = H_{HT}(\Omega) = -j\text{sign}(\Omega) = \begin{cases} -j, & \Omega > 0 \\ 0, & \Omega = 0 \\ j, & \Omega < 0 \end{cases}$$

More generally, the Hilbert transform is an operation over a function of some variable. If that function is $H(j\Omega)$ we would have

$$\mathcal{H}\{H(j\Omega)\} = h_{HT}(\Omega) * H(j\Omega)$$

Relating magnitude and phase responses

$$H(e^{j\omega}) = |H(e^{j\omega})| \exp(j \arg(H(e^{j\omega})))$$

$$\ln H(e^{j\omega}) = \underbrace{\ln |H(e^{j\omega})|}_{\text{real}} + j \underbrace{\arg(H(e^{j\omega}))}_{\text{imaginary}} \qquad \text{(taking } \ln \text{ of both sides)}$$

Now we could use the Kramers-Kronig relation to obtain

$$\arg(H(e^{j\omega})) = -\mathcal{H}\{\ln |H(e^{j\omega})|\}$$

The Kramers-Kronig relations only hold if $\ln H(e^{j\omega})$ is causal. It turns out that if $H(e^{j\omega})$ is minimum phase, then $\ln H(e^{j\omega})$ is causal and we can apply the Kramers-Kronig relation.
Conclusion: The phase response of a minimum phase system is related to the log-magnitude by the Hilbert transform.

Recall from the minimum phase/all-pass decomposition

$$H(e^{j\omega}) = H_{min}(e^{j\omega})H_{ap}(e^{j\omega})$$

$$|H(e^{j\omega})| = |H_{min}(e^{j\omega})|\,\cancelto{1}{|H_{ap}(e^{j\omega})|}$$

Therefore, if we apply the above procedure to a non-minimum phase system $H(z)$, we will recover the phase of its minimum phase component $H_{min}(z)$.

Linear-phase systems

Definition

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{-j\omega\alpha}$$

The phase is a linear function of ω: $\arg H(e^{j\omega}) = -\alpha\omega$. Consequently, the group delay is constant: $\mathrm{grd}\ H(e^{j\omega}) = \alpha$.

This definition of linear-phase systems fails to include simple cases. **Example:** When $H(e^{j\omega})$ is purely real, changes of sign in $H(e^{j\omega})$ cause jumps of $\pm\pi$ in $\mathrm{grd}\ H(e^{j\omega})$. Hence, those systems are not strictly linear phase.

Generalized linear phase systems

Definition

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{-j\omega\alpha + j\beta}$$

a term β is added to account for constant phase changes. The phase function is now *affine*.

$$\arg H(e^{j\omega}) = \beta - \alpha\omega, 0 < \omega < \pi \qquad \text{(phase)}$$

$$\text{grd } H(e^{j\omega}) = \alpha \qquad \text{(group delay)}$$

Ideal delay

Consider the system with frequency response

$$H(e^{j\omega}) = e^{-j\alpha\omega}, 0 \leq \omega \leq \pi \qquad \text{(ideal delay)}$$

If α is integer:

$$h[n] = \delta[n - \alpha] \qquad \text{(from DTFT delay property)}$$

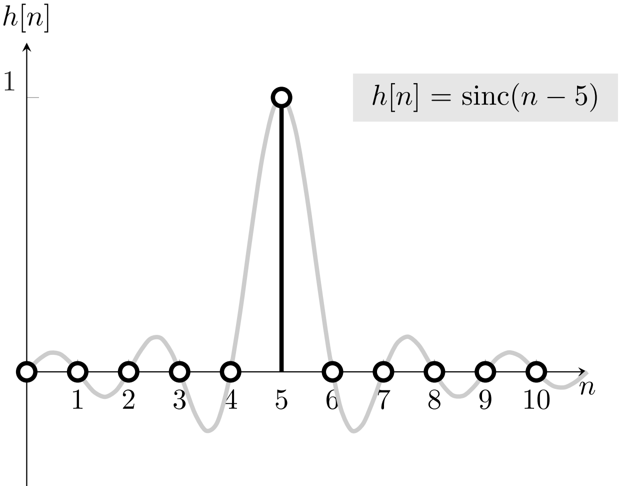
If α is not integer:

$$h[n] = \frac{\sin \pi(n - \alpha)}{\pi(n - \alpha)} = \operatorname{sinc}(n - \alpha)$$

What does not integer delay mean?

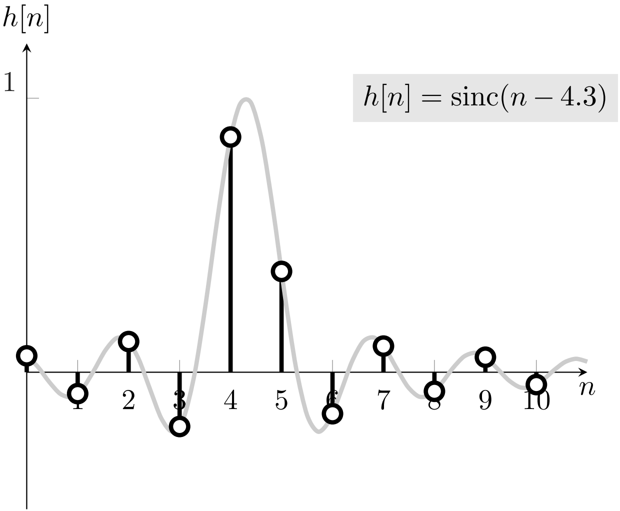
Ideal delay

If α is integer, delay corresponds to shifting samples.



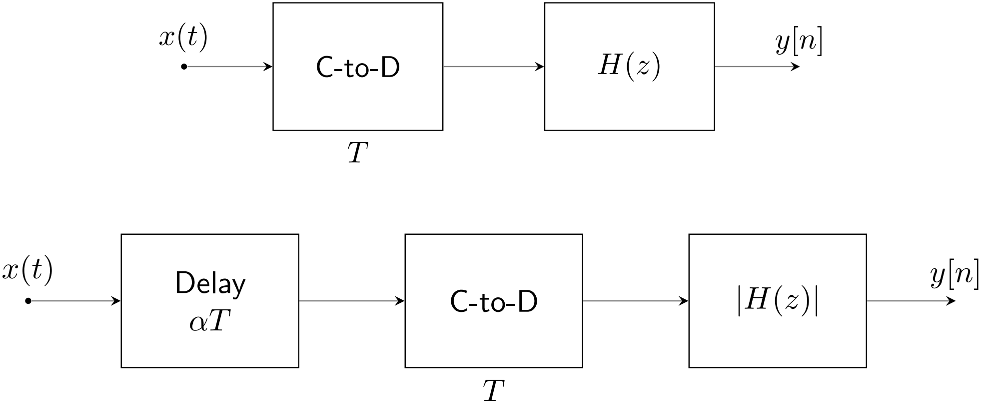
Ideal delay

If α is not integer, delay corresponds to *resampling*. Other samples are used to interpolate signal to obtain the value at the non-integer instants.



Interpretation of non-integer delay

These two systems are equivalent.



Causal FIR Linear-phase systems

Types I & II: even symmetry

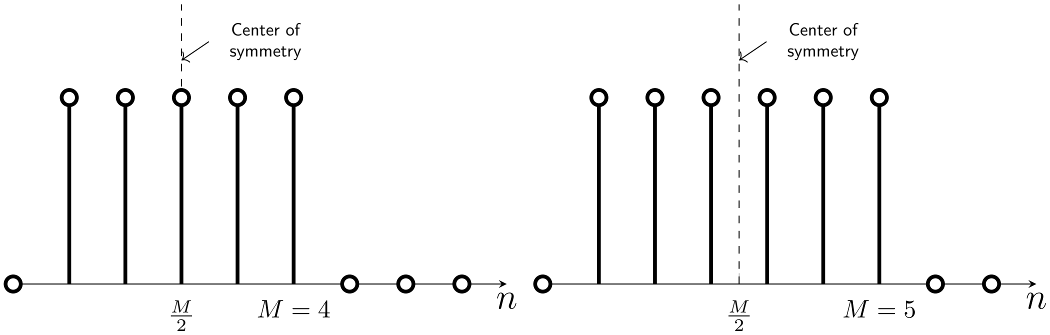
$$h[M-n] = h[n], 0 \leq n \leq M \qquad \text{(even symmetry)}$$

$$H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2} \qquad \text{(Delay of } M/2\text{)}$$

$$A_e(e^{j\omega}) = A_e(e^{-j\omega}) \qquad \text{(due to even symmetry)}$$

Type I: M even, integer delay

Type II: M odd, half sample delay



FIR Linear-phase systems

Types III & IV: odd symmetry

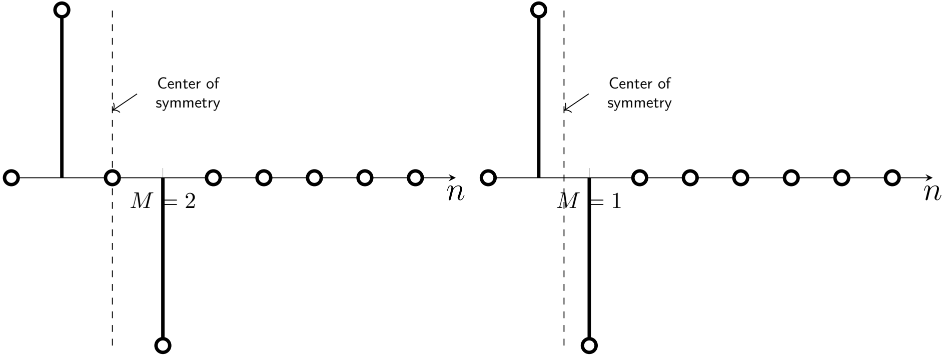
$$h[M-n] = -h[n], 0 \le n \le M \qquad \text{(odd symmetry)}$$

$$H(e^{j\omega}) = jA_o(e^{j\omega})e^{-j\omega M/2} \qquad \text{(Delay of } M/2\text{)}$$

$$A_o(e^{j\omega}) = -A_o(e^{-j\omega}) \qquad \text{(due to odd symmetry)}$$

Type III: M even, integer delay

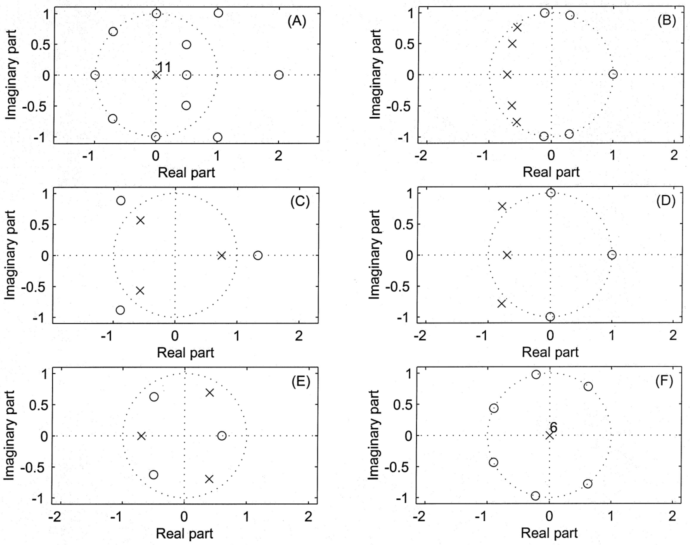
Type IV: M odd, half sample delay



Summary of linear phase FIR systems

	Even symmetry	Odd symmetry
M even	Type I	Type III zeros at $z = \pm 1$
M odd	Type II zeros at $z = -1$	Type IV zeros at $z = +1$
	$H(z^{-1}) = z^M H(z)$	$H(z^{-1}) = -z^M H(z)$

Question from midterm 2014



1. Which systems are FIR?
2. Which ones are stable?
3. Which ones are *likely* all-pass?
4. Which ones are minimum-phase?
5. Which ones are *likely* linear phase systems?
6. Which ones could be Type II lowpass linear phase systems?
7. Which ones likely have highpass frequency response?

Summary

- The frequency response of a system tell us how much each frequency was scaled (magnitude response), and delayed (phase response) by the system.
- Poles increase magnitude and introduce phase lag (positive group delay)
- Zeros decrease the magnitude and introduce phase lead (negative group delay)
- All-pass systems have constant magnitude response. For each pole at e_k, there will be a zero at $1/e_k^*$ (conjugate reciprocal)
- Minimum phase systems have all zeros inside the unit circle. Hence, its inverse is stable.
- Any system can be decomposed into a cascade of a minimum phase system and an all-pass system
- For minimum phase systems, the phase response is given by the Hilbert transform of the log-magnitude response.
- The phase response of a generalized linear phase systems is an affine function
- FIR systems are linear phase as long as their impulse response is symmetric
- Linear phase rational IIR systems do not exist