STAT 598Z: Homework 4 Solution

March 22, 2012

Problem 1 Derive a scheme for drawing samples uniformly from the unit disk. Generate 3,000 samples using your scheme and plot them.

Answer See attached code.

Problem 1a Let $(X_1, Y_1), \ldots, (X_{3000}, Y_{3000})$ denote samples generated from **Problem 1**, where X_i and Y_i denote the x-coordinate and y-coordinate of the i-th sample respectively.

Answer See attached code.

Problem 1b Define $Z_i = X_i/Y_i$. Plot the histogram of Z_i's. Show that Z_i's follow standard Cauchy distribution.

Answer Recall that the ratio of two normal distributions is Cauchy distribution. Also, recall from Box-Muller transform that if you sample the radius r from exponential distribution and sample the angle θ uniformly, then $x = r\cos\theta$ and $y = r\sin\theta$ are two independent normal random variables. Therefore, $\frac{x}{y} = \frac{\cos\theta}{\sin\theta}$ follows standard Cauchy distribution. You sampled the radius differently from the Box-Muller transform, but the radius cancels out when you take a ratio.

Problem 2 Let $X \sim Beta(1, \beta)$, and let $Y \sim X^{\frac{1}{\gamma}}$. Recall that the Beta distribution is given by:

$$Beta(x|\alpha, \beta) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1},$$

where $B(\cdot, \cdot)$ is the beta function.

- Derive the density function of Y.
- Derive the cumulative distribution function of Y.
- Generate 3,000 independent samples from the distribution of Y using $\alpha = 5, \beta = 5$, and plot their histogram.

Answer The density function of $X \sim Beta(1, \beta)$ is:

$$f(x|\beta) = \beta(1-x)^{\beta-1}. \tag{1}$$

Using change of variable $y = x^{\frac{1}{\gamma}}$,

$$f(y|\beta, \gamma) = \gamma \beta y^{\gamma - 1}(1 - y^{\gamma})^{\beta - 1}. \tag{2}$$

By simple calculus it is easy to derive the cumulative distribution function:

$$F(y|1, \beta, \gamma) = 1 - (1 - y^{\gamma})^{\beta}. \tag{3}$$

To sample from this distribution, use the inverse transform, i.e., solve the following equation in y:

$$u = 1 - (1 - y^{\gamma})^{\beta}. \tag{4}$$

By simple manipulation,

$$y = (1 - (1 - u)^{1/\beta})^{1/\gamma}. \tag{5}$$