Number Theory I

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關於這堂課

- 先備知識
  - 基本語法
- 學習重點
  - 進位制
  - 質數測試
  - 因數分解
  - 篩法
  - 最大公因數
  - 模運算

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進位制

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ZeroJudge a034: 二進位制轉換

- 給定一個數字，請輸出他的二進位表示法

輸入:	輸出:
12	1100
19	10011
3	11

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ZeroJudge a034: 位元運算之進位篇

- 給定數字N，請輸出二進位計算N+1的進位次數
- 1< n <2147483647
- 最後一行的0代表輸入結束

輸入:
1
4
7
17
0

輸出:
1
0
3
1

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題單 - 進位制

- 二進位
  - zerojudge f672-數字轉二進位(問某數字二進位的第M位的值)
  - zerojudge a034: 二進位制轉換
  - zerojudge a414: 位元運算之進位篇
- 多進位
  - UVA 343 - What Base Is This?(給定兩個相同的數，輸出他們的進位制)
  - CodeForces 49B - Sum(問兩數相加最大可能是幾位數)

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質數測試
(Primality Test)

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質數測試

```
bool isPrime(int n) {
    if (n == 1) return false;
    for (int i = 2; i * i <= n; i++) {
        if (n % i == 0) return false;
    }
    return true;
}
```

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ZeroJudge a121: 質數又來囉

- 給定兩個數字a,b 請問a到b中有多少質數?
- $1 < a, b < 10^8$ 且$b-a<1000$

輸入:	輸出:
3 7	3
6 6	0
30 50	5

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題單 - 質數測試

- 判斷質數
  - zerojudge a121: 質數又來囉
  - zerojudge b513: 判斷質數-商競103
  - zerojudge a007: 判斷質數
  - zerojudge b552: 3.找質數
  - zerojudge d438: 10533 - Digit Primes
- 計算質數
  - CodeForces 26A(計算質數)

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因數分解
(Factorization)

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質因數分解

- 枚舉 i = 2 ~ sqrt(n),
  如果 n / i 可以整除,
  則持續 n /= i 將 n 中的質因數 i 都消掉
- 只有 1 個質因數會大於等於 sqrt(n)
- 180 = 2 x 2 x 3 x 3 x 5

```
vector<int> F(int n) {
  vector<int> ret;
  for (int i = 2; i * i <= n; i++) {
    if (n % i == 0) {
      while (n % i == 0) {
        ret.push_back(i);
        n /= i;
      }
    }
  }
  if (n > 1) ret.push_back(n);
  return ret;
}
```

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ZeroJudge a010: 因數分解

- 給定數字N，請輸出因數分解後的結果
- $1< n <10^9$

輸入1: 36 輸入2: 47	輸出2: 2^2 * 3^3 輸入2: 47

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找到所有因數

- 枚舉前 sqrt(n) 的正整數
- `i`
- `n / i`

```
vector<int> F(int n) {
  vector<int> ret;
  for (int i = 1; i * i <= n; i++) {
    if (n % i == 0) {
      if (i * i == n) {
        ret.push_back(i);
      } else {
        ret.push_back(i);
        ret.push_back(n / i);
      }
  }
  // if needed
  sort(ret.begin(), ret.end());
  return ret;
}
```

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LeetCode 1492. The kth Factor of n

- 輸出第 k 小的因數
- 1 <= k <= n <= 1000
- 找不到的話則輸出-1

Input: n = 12, k = 3
Output: 3

Input: n = 7, k = 2
Output: 7

Input: n = 4, k = 4
Output: -1

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計算因數個數

- 枚舉 i = 1 ~ sqrt(n)
- if i = sqrt(n):
  因數增加 1 個
- if i != sqrt(n):
  則會收集到 i 和 n/i 兩個不同的因數

```
int F(int n) {
  int cnt = 0;
  for (int i = 1; i * i <= n; i++) {
    if (n % i == 0) {
      if (i * i == n) {
        cnt += 1;
      } else {
        cnt += 2;
      }
    }
  }
  return cnt;
}
```

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計算因數個數

- $180 = 2 \times 2 \times 3 \times 3 \times 5$
  $= 2^2 \times 3^2 \times 5^1$
- 因數個數 =

  $(2 + 1) * (2 + 1) * (1 + 1)$

  $= 18$

	2	3	5
factor_cnt	2	2	1

```
vector<int> F(int n) {
  vector<int> factor_cnt;
  for (int i = 2; i * i <= n; i++) {
    int cnt = 0;
    if (n % i == 0) {
      while (n % i == 0) {
        cnt++;
        n /= i;
      }
    // 當前質因數有幾個
    factor_cnt.push_back(cnt);
  }
  if (n > 1) factor_cnt.push_back(1);
  int ans = 1;
  // 算出所有質因數組合
  for (int i = 0; i < factor_cnt.size(); i++)
{
    ans *= factor_cnt[i] + 1;
  }
  return ans;
}
```

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計算因數總和

- $180 = 2 \times 2 \times 3 \times 3 \times 5$
  $= 2^2 \times 3^2 \times 5^1$
- 因數總和

  $= (1 + 2 + 4) * (1 + 3 + 9) * (1 + 5)$

  $= 546$

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ZeroJudge c184: 盈虧互補

- 給定數字N, 請計算他的因數總和
- 如果總和等於自己 請輸出=n
- 如果總和的數其因數總和等於自己 請輸出因數總和
- 否則輸出0
- $2 \leq n \leq 10^7$, n=0代表結束

輸入:
6
220
12
0

輸出:
=6
284
0

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題單 -因數分解

- 因數分解
  - codeforces 230B (計算因數)
  - zerojudge a010: 因數分解
  - zerojudge a740: 质因数之和
  - zerojudge c184: 盈虧互補 (因數總和)
  - leetcode 1492. The kth Factor of n
  - 中女中d053: 2.質因數分解
- 變化題
  - zerojudge b687: 7. 坐好坐滿
  - leetcode 1390. Four Divisors

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篩法
(Sieve)

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Sieve 篩法

2	3	4	5	6	7	8
9	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	29
30	31	32	33	34	35	36
37	38	39	40	41	42	43
44	45	46	47	48	49	50

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Sieve 篩法

- 複雜度：O(N log N)

```
const int maxn = 2e5+5;
bool prime[maxn];

void sieve() {
    for (int i = 0; i < maxn; i++)  //初始化
        prime[i] = true;
    prime[0] = false;
    prime[1] = false;
    for (int i = 2; i < maxn; i++)
        if (prime[i])
            for (int j = i + i; j < maxn; j += i)
                prime[j] = false;
}
```

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Sieve 篩法

- 複雜度 : O(N)

```
int sieve[maxn];
vector<int> prime;

void linear_sieve() {
    for (int i = 2; i < maxn; i++) {
        if (!sieve[i]) prime.push_back(i);
        for (int j = 0; i * prime[j] < maxn; j++) {
            sieve[i * prime[j]] = true;
            if (i % prime[j] == 0) break;
        }
    }
}
```

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LeetCode 204. Count Primes

- 給定N 詢問1~N有多少質數？
- 0 <= n <= 5 * 10^6

輸入1:
14

輸入2:
1

輸出2:
6

輸入2:
0

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zerojudge e912: n! 的因數分解

- 輸入N，請對N!質因數分解
- 會有多行輸入，以EOF當作結束
- $0 <= n <= 10^4$

輸入1:
6
9

輸出:
6! = 2^4 * 3^2 * 5^1
9! = 2^7 * 3^4 * 5^1 * 7^1

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題單 - 篩法

- 篩法
  - CodeForces 26A(計算質數)
  - LeetCode 204. Count Primes
  - zerojudge e912: n! 的因數分解
  - zerojudge a569: 2-絕對遞增的質數子數列
  - zerojudge f426: 質數求和
  - CodeForces 102267B (需要線性篩法)
  - Zerojude d237: 質數合

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Property and Algorithm about GCD

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Property and Algorithm about GCD

- 基本性質
  - gcd(a, b) = gcd(b, a)
  - gcd(a, b, c) = gcd(gcd(a, b), c) = gcd(a, gcd(b, c))
- gcd(a, b) = gcd(b, a % b) if (b != 0)
- lcm(a, b) * gcd(a, b) = a * b

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Property and Algorithm about GCD

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ZeroJudge a024: 最大公因數(GCD)

- 給定兩個數字, 輸出他們的最大公因數

輸入:
12 15
1 100

輸出:
3
1

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題單 - GCD

- 基礎
  - zerojudge a024: 最大公因數(GCD)
  - Codeforces 664A Complicated GCD(考GCD性質)
  - zerojudge d693: 最小公倍數
  - zerojudge d256: 11388 - GCD LCM
- 變化
  - UVa 11827 - Maximum GCD
  - zerojudge e272: gcd(Fm,Fn)

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Modular Arithmetic

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Modular Arithmetic

- (a + b) mod m ≡ ((a mod m) + (b mod m)) mod m
- (a * b) mod m ≡ ((a mod m) * (b mod m)) mod m
- (a - b) mod m ≡ ((a mod m) - (b mod m)) mod m

很多程式在負數時 mod 的行為會不太一樣，
所以我們通常會把 (a - b) mod m 寫成 ((a - b) mod m + m) mod m
來保證在進行 mod m 運算時是正數

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輾轉相除法

- 求 gcd(276, 585)

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Extended GCD

- Bézout's 定理
  給 a, b ∈ Z, 必存在整數 x, y 使得 ax + by = gcd(a, b)
- 求 x, y

輾轉相除法
gcd(a, b) = gcd(b, a%b)

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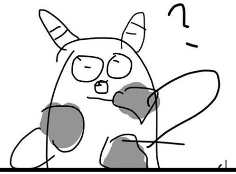
Extended GCD



輾轉相除法

gcd(a, b) = gcd(b, a%b)

- 給 a, b ∈ Z, 已知必存在整數 x, y 使得 ax + by = gcd(a, b), 求 x, y

$ax_1 + by_1 = gcd(a, b)$

$bx_2 + (a \% b)y_2 = gcd(b, a \% b) = gcd(a, b)$

$\Rightarrow ax_1 + by_1 = bx_2 + (a - [a / b] * b)y_2$

a mod b = a - [a / b] * b

$\Rightarrow x_1a + y_1b = y_2a + (x_2 - [a / b] * y_2)b$

$\Rightarrow x_1 = y_2, \quad y_1 = (x_2 - [a / b] * y_2)$

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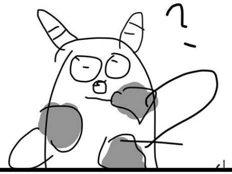
Extended GCD

- 276x + 585y = 3，求 x, y



輾轉相除法
gcd(a, b) = gcd(b, a%b)

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Extended GCD

- 給 a, b ∈ Z, 已知必存在整數 x, y 使得 ax + by = gcd(a, b), 求 x, y

```
LL extgcd(LL a, LL b, LL &x, LL &y) {
  if (!b) {
    x = 1;
    y = 0;
    return a;
  }
  LL res = extgcd(b, a % b, y, x);
  y -= a / b * x;
  return res;
}
```

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UVA 11417. GCD

- 給定N，請計算0~n之間所有數字的GCD總和
- 1<N<501

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zerojudge a289: Modular Multiplicative Inverse

- 給定a, n, 請輸出a對模數n的反元素
- 如果反元素不存在
- 0 <= a, n <= 10^8

輸入:
79 62
96 47
49 28

輸出:
11
24
No Inverse

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b836: kevin戀愛攻略系列題-2 說好的霸王花呢??

- 給定兩個數字n, m
- 請問1+(1+m)+(1+2m)+...+(1+km)是否可以等於n?
- 可以則輸出"Go Kevin!!", 否則"No Stop!!"
- (0 < n, m <= 2147483647)

輸入:
6
9

輸出:
Go Kevin!!

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題單 - 模運算

- 基礎題
    - uva 11417. GCD
    - zerojudge a058: MOD3
    - zerojudge b836: kevin戀愛攻略系列題-2 說好的霸王花呢??
- 模逆元
    - zerojudge a289: Modular Multiplicative Inverse
    - Kattis - Candy Distribution
- 變化題
    - uva 10229 - Modular Fibonacci
    - zerojudge d636: 大爆炸bomb

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