STAT8003 Time Series Forecasting: Assignment1

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Problem 1

Are the following time series $\{Z_t\}$ weakly stationary?

(a) $Z_t = 5 + 2t + X_t$, where $\{X_t\}$ is a zero-mean stationary series with ACVF $\{\gamma_k\}$

(b) $Z_t = a_t + (a_{t-1})^{\theta_t}$, where $\{a_t\}$ and $\{\theta_t\}$ are two independently and identically distributed sequences, $\{a_t\}$ follows the standard normal distribution and $P(\theta_t = 1) = P(\theta_t = 2) = 0.5$.

(c) $e^{a_t} + 2a_{t-1}$,where $\{a_t\}$ are independent and identically distributed with the standard normal distribution.

Solution

(a)

$$\mu_t = E(Z_t) = E(5 + 2t + X_t) = 2t + 5 + E(X_t) = 2t + 5$$

μ_t is dependent on t. Thus, $\{Z_t\}$ is not stationary

(b)

$$\begin{aligned}
\mu_t = E(Z_t) = E(a_t + a_{t-1})P(\theta_t = 1) + E(a_t + a_{t-1}^2)P(\theta_t = 2) = 0.5 \\
\gamma(t,t) &= Cov(a_t + a_{t-1}^{\theta_t}, a_t + a_{t-1}^{\theta_t}) = Cov(a_t, a_t) + Cov(a_{t-1}^{\theta_t}, a_{t-1}^{\theta_t}) \\
&= Var(a_t) + Var[E(a_t^{\theta_t}|\theta_t)] + E[Var(a_t^{\theta_t}|\theta_t)] \\
&= 1 + 0.5 \times (1 - 0.5) + 1 \times 0.5 + 2 \times 0.5 = 2.75 \\
\gamma(t,t-1) &= Cov(a_t + a_{t-1}^{\theta_t}, a_{t-1} + a_{t-2}^{\theta_{t-1}}) = Cov(a_{t-1}^{\theta_t}, a_{t-1}) \\
&= E(a_t^{\theta_t+1}) - E(a_t^{\theta_t})E(a_t) = E(a_t^2)P(\theta_t = 1) + E(a_t^3)P(\theta_t = 2) - E(a_t^{\theta_t})E(a_t) \\
&= 1 \times 0.5 + 0 - 0 = 0.5 \\
\gamma(t,t-2) &= Cov(a_t + a_{t-1}^{\theta_t}, a_{t-2} + a_{t-3}^{\theta_t}) = 0
\end{aligned}$$

For $k \geq 2, \gamma(t, t-k) = 0$, it only denpends on the lag k. Thus, μ_t and γ_t are both independent of t, $\{Z_t\}$ is stationary.

(c)

$$\mu_t = E(e^{a_t} + 2a_{t-1}) = E(e^{a_t}) = e^{\frac{1}{2}}$$

$$\begin{aligned}
\gamma(t,t) = Cov(e^{a_t} + 2a_{t-1}, e^{a_t} + 2a_{t-1}) = Cov(e^{a_t}, e^{a_t}) + 4Cov(a_{t-1}, a_{t-1}) = e^2 - e + 4 \\
\gamma(t,t-1) &= Cov(e^{a_t} + 2a_{t-1}, e^{a_{t-1}} + 2a_{t-2}) = 2Cov(a_{t-1}, e^{a_t}) = 2e^{\frac{1}{2}} \\
\gamma(t,t-2) &= Cov(e^{a_t} + 2a_{t-1}, e^{a_{t-2}} + 2a_{t-3}) = 0
\end{aligned}$$

For $k \geq 2, \gamma(t, t-k) = 0$, it only denpends on the lag k, and μ_t is independent on t. Thus, $\{Z_t\}$ is stationary

Problem 2

Let $Zt = e^{a_t} - 2e^{a_{t-1}}$, where $\{a_t\}$ is sequence of i.i.d. normal random variables with mean zero and variance one.

(a) Is $\{Z_t\}$ stationary?

(b) Calculate the ACF ρ_k with $k = 1$ if it is stationary

Solution

(a)

$$\mu_t = E(e^{a_t} - 2e^{a_{t-1}}) = E(e^{a_t}) - 2E(e^{a_{t-1}}) = -e^{\frac{1}{2}}$$

$$\begin{aligned}
\gamma(t,t) &= Cov(e^{a_t} - 2e^{a_{t-1}}, e^{a_t} - 2e^{a_{t-1}}) = Cov(e^{a_t}, e^{a_t}) + 4Cov(e^{a_{t-1}}, e^{a_{t-1}}) = 5(e^2 - e) \\
\gamma(t,t-1) &= Cov(e^{a_t} - 2e^{a_{t-1}}, e^{a_{t-1}} - 2e^{a_{t-2}}) = -2Cov(e^{a_{t-1}}, e^{a_{t-1}}) = -2(e^2 - e) \\
\gamma(t,t-2) &= Cov(e^{a_t} - 2e^{a_{t-1}}, e^{a_{t-2}} - 2e^{a_{t-3}}) = 0
\end{aligned}$$

For $k > 2, \gamma(t, t-k) = 0$, it only denpends on the lag k, and μ_t is independent on t. Thus, $\{Z_t\}$ is stationary.

(b) based on (a), when $k = 1$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = -0.4$$

Problem 3

Find the ACF of the following MA processes.

(a) $Z_t = a_t - 0.5a_{t-1}$

(b) $Z_t = a_t - a_{t-1} + 0.5a_{t-2}$

(c) $Z_t = a_t + 0.5a_{t-1} - a_{t-2} + 3a_{t-3}$

Solution

for a $MA(q)$ model

$$Z_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q}$$

Since it is stationary, we can calculate the ACVF of it.

$$\begin{aligned}
\gamma_0 &= Cov(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q}, a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q}) \\
&= Cov(a_t, a_t) + \theta_1^2 Cov(a_{t-1}, a_{t-1}) + \theta_2^2 Cov(a_{t-2}, a_{t-2}) + \cdots + \theta_q^2 Cov(a_{t-q}, a_{t-q}) \\
&= \sigma_a^2(1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2) \\
\gamma_1 &= Cov(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q}, a_{t-1} - \theta_1 a_{t-2} - \theta_2 a_{t-3} - \cdots - \theta_q a_{t-q-1}) \\
&= -\theta_1 Cov(a_{t-1}, a_{t-1}) + \theta_1\theta_2 Cov(a_{t-2}, a_{t-2}) + \cdots + \theta_{q-1}\theta_q Cov(a_{t-q}, a_{t-q}) \\
&= \sigma_a^2(-\theta_1 + \theta_1\theta_2 + \cdots + \theta_{q-1}\theta_q) \\
\gamma_2 &= Cov(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q}, a_{t-2} - \theta_1 a_{t-3} - \theta_2 a_{t-4} - \cdots - \theta_q a_{t-q-2}) \\
&= -\theta_2 Cov(a_{t-2}, a_{t-2}) + \theta_1\theta_3 Cov(a_{t-3}, a_{t-3}) + \cdots + \theta_{q-2}\theta_q Cov(a_{t-q}, a_{t-q}) \\
&= \sigma_a^2(-\theta_2 + \theta_1\theta_3 + \cdots + \theta_{q-2}\theta_q) \\
&\cdots \\
\gamma_k &= Cov(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q}, a_{t-k} - \theta_1 a_{t-k-1} - \theta_2 a_{t-k-2} - \cdots - \theta_q a_{t-q-k}) \\
&= -\theta_k Cov(a_{t-2}, a_{t-2}) + \theta_1\theta_{1+k} Cov(a_{t-3}, a_{t-3}) + \cdots + \theta_{q-k}\theta_q Cov(a_{t-q}, a_{t-q}) \\
&= \sigma_a^2(-\theta_k + \theta_1\theta_{k+1} + \cdots + \theta_{q-k}\theta_q)
\end{aligned}$$

For $k > q$, $\gamma_k = 0$. Thus, we can get ρ_k

$$\begin{cases} \rho_0 = 1 \\ \rho_1 = \dfrac{-\theta_1 + \theta_1\theta_2 + \cdots + \theta_{q-1}\theta_q}{1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2} \\ \rho_2 = \dfrac{-\theta_2 + \theta_1\theta_3 + \cdots + \theta_{q-2}\theta_q}{1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2} \\ \cdots \\ \rho_k = \dfrac{-\theta_k + \theta_1\theta_{k+1} + \cdots + \theta_{q-k}\theta_q}{1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2} & 2 < k \le q \\ \rho_k = 0 \qquad k > q \end{cases}$$

(a) based on the conclusion above, the ACF of the given MA(1) model is

$$\begin{cases} \rho_0 = 1 \\ \rho_1 = \dfrac{-\theta_1}{1+\theta_1^2} = \dfrac{-0.5}{1+0.5^2} = -0.4 \\ \rho_k = 0 \qquad k \ge 2 \end{cases}$$

(b) based on the conclusion above, the ACF of the given MA(2) model is

$$\begin{cases} \rho_0 = 1 \\ \rho_1 = \dfrac{-\theta_1 + \theta_1\theta_2}{1+\theta_1^2+\theta_2^2} = \dfrac{-1 + 1 \times (-0.5)}{1 + 1^2 + (-0.5)^2} = -\dfrac{2}{3} \\ \rho_2 = \dfrac{-\theta_2}{1+\theta_1^2+\theta_2^2} = \dfrac{0.5}{1 + 1^2 + (-0.5)^2} = \dfrac{2}{9} \\ \rho_k = 0 \qquad k \ge 3 \end{cases}$$

(c) based on the conclusion above, the ACF of the given MA(3) model is

$$\begin{cases} \rho_0 = 1 \\ \rho_1 = \dfrac{-\theta_1 + \theta_1\theta_2 + \theta_2\theta_3}{1+\theta_1^2+\theta_2^2+\theta_3^2} = \dfrac{0.5 + (-0.5) \times 1 + 1 \times (-3)}{1 + (-0.5)^2 + 1^2 + (-3)^2} = -\dfrac{4}{15} \\ \rho_2 = \dfrac{-\theta_2 + \theta_1\theta_3}{1+\theta_1^2+\theta_2^2+\theta_3^2} = \dfrac{-1 + (-0.5) \times (-3)}{1 + (-0.5)^2 + 1^2 + (-3)^2} = \dfrac{2}{45} \\ \rho_3 = \dfrac{-\theta_3}{1+\theta_1^2+\theta_2^2+\theta_3^2} = \dfrac{3}{1 + (-0.5)^2 + 1^2 + (-3)^2} = \dfrac{4}{15} \\ \rho_k = 0 \qquad k \ge 4 \end{cases}$$

Problem 4

Consider AR(2) model, $Z_t = 0.9 + 0.2Z_{t-1} - 0.6Z_{t-2} + a_t$, where $\{a_t\} \sim WN(0, \sigma_a^2)$.

(a) is it stationary?

(b) find out the MA representation if possible.

(c) Calculate the autocovariance function γ_k with $k \ge 0$.

(d) Find out the variance for the sample mean $\overline{Z}_4 = (Z_1 + Z_2 + Z_3 + Z_4)/4$.

Solution

(a) the equation $Z_t = 0.9 + 0.2Z_{t-1} - 0.6Z_{t-2} + a_t$ can be written as

$$(1 - 0.2B + 0.6B^2)Z_t = 0.9 + a_t$$

Therefore, let

$$\phi(x) = 1 - 0.2x + 0.6x^2 = 0$$

By solving the equation, we can get

$$x = \frac{0.2 \pm \sqrt{(-0.2)^2 - 4 \times 0.6 \times 1}}{2}$$

since $|x| \approx 1.29 > 1$, the AR(2)model is stationary

(b) the MA representation of the given AR(2) model must have the form:

$$Z_t = \mu + \sum_{j=0}^{\infty} \psi_j a_{t-j}$$

then the equation $Z_t = 0.9 + 0.2Z_{t-1} - 0.6Z_{t-2} + a_t$ can be written as

$$\begin{aligned} \mu + \psi_0 a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \ldots = 0.9 + a_t + 0.2(\mu + \psi_0 a_{t-1} + \psi_1 a_{t-2} + \psi_2 a_{t-3} + \ldots) \\ - 0.6(\mu + \psi_0 a_{t-2} + \psi_1 a_{t-3} + \psi_2 a_{t-4} + \ldots) \end{aligned}$$

by comparing the coefficient of a_{t-j}, we can find

$$\begin{cases} \mu = 0.9 + 0.2\mu - 0.6\mu \quad \rightarrow \quad \mu = \dfrac{9}{14} \\ \psi_0 = 1 \\ \psi_1 = 0.2\psi_0 = 0.2 \\ \psi_2 = 0.2\psi_1 - 0.6\psi_0 = -0.56 \\ \ldots \\ \psi_k = 0.2\psi_{k-1} - 0.6\psi_{k-2} \qquad k \geq 3 \end{cases}$$

Thus, the MA representation of the given AR(2) model is

$$Z_t = \mu + \sum_{j=0}^{\infty} \psi_j a_{t-j}$$

where the values of μ and ψ_j follow the above equation

(c) Given $Z_t = 0.9 + 0.2Z_{t-1} - 0.6Z_{t-2} + a_t$, it is obvious that when $k \geq 1$

$$\begin{aligned} Cov(Z_t, Z_{t-k}) &= Cov(0.9 + 0.2Z_{t-1} - 0.6Z_{t-2} + a_t, Z_{t-k}) \\ \rightarrow \gamma_k &= 0.2\gamma_{k-1} - 0.6\gamma_{k-2} \\ \rightarrow \rho_k &= 0.2\rho_{k-1} - 0.6\rho_{k-2} \end{aligned}$$

by using this conclusion, ρ_k can be easily computed

Problem 4 continued on next page...

$$\begin{cases} \rho_0 = 1 \\ \rho_1 = 0.2\rho_0 - 0.6\rho_1 \quad \rightarrow \quad \rho_1 = \dfrac{1}{8} \\ \rho_2 = 0.2\rho_1 - 0.6\rho_0 = -\dfrac{23}{40} \\ \dots \\ \rho_k = 0.2\rho_{k-1} - 0.6\rho_{k-2} \qquad k \geq 3 \end{cases}$$

for γ_0, based on the equation $Z_t = 0.9 + 0.2Z_{t-1} - 0.6Z_{t-2} + a_t$,

$$\begin{aligned} Cov(Z_t, Z_t) &= Cov(0.9 + 0.2Z_{t-1} - 0.6Z_{t-2} + a_t, Z_t) \\ &\rightarrow \gamma_0 = 0.2\gamma_1 - 0.6\gamma_2 + Cov(a_t, Z_t) \\ &\rightarrow \gamma_0 = 0.2\rho_1\gamma_0 - 0.6\rho_2\gamma_2 + \sigma_a^2 \end{aligned}$$

Then we can get the ACVF of the AR(2) model, since μ and ρ_k are all known

$$\begin{cases} \gamma_0 = \dfrac{\sigma_a^2}{1 - 0.2\rho_1 + 0.6\rho_2} \\ \gamma_k = \rho_k\gamma_0 \qquad k \geq 1 \end{cases}$$

(d)

$$\begin{aligned} Var(\overline{Z}_4) = Cov(\overline{Z}_4, \overline{Z}_4) &= Cov(\frac{Z_1 + Z_2 + Z_3 + Z_4}{4}, \frac{Z_1 + Z_2 + Z_3 + Z_4}{4}) \\ &= \frac{1}{16}\sum_{i=1}^{4}\sum_{j=1}^{4} Cov(Z_i, Z_j) \\ &= \frac{1}{16}(4\gamma_0 + 6\gamma_1 + 4\gamma_2 + 2\gamma_3) \end{aligned}$$

Since γ_k is known based on (c), then the Variance of $\overline{Z}_4$ can be calculated by this equation. The final outcome Is:

$$Var(\overline{Z}_4) = \frac{23}{112}\sigma_a^2$$

Problem 5

Consider an ARMA(2,2) model

$$Z_t = 0.5Z_{t-1} + 0.25Z_{t-2} + a_t - 1.5a_{t-1} + 0.75a_{t-2}$$

where $\{a_t\}$ is the white noise with mean zero and variance σ_a^2

(a) calculate the expectation $E(Z_t)$

(b) Is is stationary? Is it invertible?

(c) Find out the MA and AR representation if thet exist

Solution

(a) the model can be written as

$$(1 - 0.5B - 0.25B^2)Z_t = (1 - 1.5B + 0.75B^2)a_t$$

Problem 5 continued on next page...

Therefore,

$$\phi(x) = 1 - 0.5x - 0.25x^2 \qquad \theta(x) = 1 - 1.5x + 0.75x^2$$

$$\begin{cases} \phi(x) = 0 \quad \rightarrow \quad x = \dfrac{0.5 \pm \sqrt{(-0.5)^2 - 4 \times (-0.25) \times 1}}{2 \times (-0.25)} \quad \rightarrow \quad |x| = 3.24 \ or \ 1.24 > 1 \\ \theta(x) = 0 \quad \rightarrow \quad x = \dfrac{1.5 \pm \sqrt{(1.5)^2 - 4 \times (0.75) \times 1}}{2 \times 0.75} \quad \rightarrow \quad |x| \approx 1.15 > 1 \end{cases}$$

It can be seen that the ARMA(2,2) model is stationary and invertible. based on stationarity, $\mu_t = \mu_{t-k}$, then in the given equation, take Expectation of both sides, we can get

$$\mu_t = 0.5\mu_t + 0.25\mu_t \quad \rightarrow \quad \mu_t = E(Z_t) = 0$$

(b) based on (a), it is stationary and invertible.

(c) For MA representation, replace Z_t with $\mu + \sum_{j=0}^{\infty} \psi_j a_{t-j}$ in the given equation

$$\begin{aligned} \psi_0 a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \ldots = & \ 0.5(\psi_0 a_{t-1} + \psi_1 a_{t-2} + \psi_2 a_{t-3} + \ldots) \\ & + 0.25(\psi_0 a_{t-2} + \psi_1 a_{t-3} + \psi_2 a_{t-4} + \ldots) \\ & + a_t - 1.5a_{t-1} + 0.75a_{t-2} \end{aligned}$$

By comparing the coefficient of a_{t-j}, we can get

$$\begin{cases} \psi_0 = 1 \\ \psi_1 = 0.5\psi_0 - 1.5 \quad \rightarrow \quad \psi_1 = -1 \\ \psi_2 = 0.5\psi_1 + 0.25\psi_0 + 0.75 = 0.5 \\ \ldots \\ \psi_k = 0.5\psi_{k-1} + 0.25\psi_{k-2} \qquad k \geq 3 \end{cases}$$

Thus, the MA representation of the given ARMA(2,2) model is

$$Z_t = \sum_{j=0}^{\infty} \psi_j a_{t-j}$$

where the values of ψ_j follow the above equation

for AR representation, replace a_t with $Z_t - \sum_{j=1}^{\infty} \pi_j Z_{t-j}$ in the given equation, then

$$\begin{aligned} Z_t = 0.5Z_{t-1} + 0.25Z_{t-2} + & (Z_t - \pi_1 Z_{t-1} - \pi_2 Z_{t-2} - \ldots) \\ & - 1.5(Z_{t-1} - \pi_1 Z_{t-2} - \pi_2 Z_{t-3} - \ldots) \\ & + 0.75(Z_{t-2} - \pi_1 Z_{t-3} - \pi_2 Z_{t-4} - \ldots) \end{aligned}$$

by comparing the coefficient of Z_{t-j}, we can get

$$\begin{cases} 0.5 - \pi_1 - 1.5 = 0 \quad \rightarrow \quad \pi_1 = -1 \\ 0.25 - \pi_2 + 1.5\pi_1 + 0.75 = 0 \quad \rightarrow \quad \pi_2 = -0.5 \\ \ldots \\ -\pi_k + 1.5\pi_{k-1} - 0.75\pi_{k-2} = 0 \quad \rightarrow \quad \pi_k = 1.5\pi_{k-1} - 0.75\pi_{k-2} \qquad k \geq 3 \end{cases}$$

Thus, the AR representation of the given ARMA(2,2) model is

$$a_t = Z_t - \sum_{j=1}^{\infty} \pi_j Z_{t-j}$$

where the values of π_j follow the above equation

Problem 6

Consider a time series model

$$Z_t = 5 + 0.3Z_{t-1} + 0.5Z_{t-2} + e^{a_t}$$

where $\{a_t\}$ is an *i.i.d.* sequence with the standard normal distribution

(a) Is it an AR model? Rewrite it in the standard form if your answer is positive.

(b) Is the model stationary? Specify the reasons for your answers.

(c) Find out the general linear process form if it exist.

solution

(a) The model is an AR model. since $\{a_t\} \sim N(0,1)$, then $E[e^{a_t}] = e^{\frac{1}{2}}, Var[e^{a_t}] = e^2 - e$. Thus $\left\{e^{a_t} - e^{\frac{1}{2}}\right\}$ is an i.i.d sequence with mean zero and variance $e^2 - e$, can be assumed as white noise. then the standard formal is:

$$Z_t = 5 + e^{\frac{1}{2}} + 0.3Z_{t-1} + 0.5Z_{t-2} + b_t$$

where $\{b_t\} \sim WN(0, e^2 - e)$.

(b) Rewrite the equation as:

$$Z_t - 0.3Z_{t-1} - 0.5Z_{t-2} = 5 + e^{\frac{1}{2}} + b_t$$

Then $\phi(x) = 1 - 0.3x - 0.5x^2$ and let is equals to 0, we can get

$$x = \frac{0.3 \pm \sqrt{(-0.3)^2 - 4 \times (-0.5) \times 1}}{2 \times (-0.5)} \qquad \to \qquad |x| = 1.75 \ or \ 1.15 > 1$$

Thus, the model is stationary

(c) replace Z_t with $\mu + \sum_{j=0}^{\infty} \psi_j b_{t-j}$, we can get

$$\begin{aligned} \mu + \psi_0 b_t + \psi_1 b_{t-1} + \psi_2 b_{t-2} + \ldots = 5 + e^{\frac{1}{2}} + b_t + 0.3(\mu + \psi_0 b_{t-1} + \psi_1 b_{t-2} + \psi_2 b_{t-3} + \ldots) \\ + 0.5(\mu + \psi_0 b_{t-2} + \psi_1 b_{t-3} + \psi_2 b_{t-4} + \ldots) \end{aligned}$$

by comparing the coefficient of b_{t-j}, we get

$$\begin{cases} \mu = 5 + e^{\frac{1}{2}} + 0.3\mu + 0.5\mu & \to \qquad \mu = 5(5 + e^{\frac{1}{2}}) \\ \psi_0 = 1 \\ \psi_1 = 0.3\psi_0 = 0.3 \\ \psi_2 = 0.3\psi_1 + 0.5\psi_0 = 0.59 \\ \ldots \\ \psi_k = 0.3\psi_{k-1} + 0.5\psi_{k-2} & \qquad k \geq 3 \end{cases}$$

Thus, the GLP form of the given model is

$$Z_t = \mu + \sum_{j=0}^{\infty} \psi_j b_{t-j}$$

where the values of μ and ψ_j follow the above equation

Some conclusions used in the assignment

if $X \sim \mathcal{N}(\mu, \sigma^2)$, then

$$
\begin{aligned}
E(e^{nX}) &= \int_{-\infty}^{+\infty} e^{nx} \frac{1}{\sqrt{2\pi}\sigma} exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} exp\left\{-\frac{(x-(\mu+n\sigma^2))^2 - 2n\mu\sigma^2 - n^2\sigma^4}{2\sigma^2}\right\} \\
&= exp\left\{n\mu + \frac{n^2}{2}\sigma^2\right\} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} exp\left\{-\frac{(x-(\mu+n\sigma^2))^2}{2\sigma^2}\right\} \rightarrow\ cdf\ of\ \mathcal{N}(\mu+n\sigma^2, \sigma^2) \\
&= exp\left\{n\mu + \frac{n^2}{2}\sigma^2\right\}
\end{aligned}
$$

$$
\begin{aligned}
E(Xe^{nX}) &= \int_{-\infty}^{+\infty} xe^{nx} \frac{1}{\sqrt{2\pi}\sigma} exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} \\
&= \int_{-\infty}^{+\infty} x \frac{1}{\sqrt{2\pi}\sigma} exp\left\{-\frac{(x-(\mu+n\sigma^2))^2 - 2n\mu\sigma^2 - n^2\sigma^4}{2\sigma^2}\right\} \\
&= exp\left\{n\mu + \frac{n^2}{2}\sigma^2\right\} \int_{-\infty}^{+\infty} x \frac{1}{\sqrt{2\pi}\sigma} exp\left\{-\frac{(x-(\mu+n\sigma^2))^2}{2\sigma^2}\right\} \rightarrow\ expectation\ of\ \mathcal{N}(\mu+n\sigma^2, \sigma^2) \\
&= (\mu+n\sigma^2) exp\left\{n\mu + \frac{n^2}{2}\sigma^2\right\}
\end{aligned}
$$