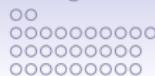


Introduction to Game Theory 3

Solution Concepts: Regret Minimisation, Minimax and Maximin, and Nash Equilibrium

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Version: November 13, 2023



Game Theory Part 1: Outline

1. Game Theory: Science of Strategic Thinking
2. Examples of interesting games
3. Formalising games
4. Solution concept 1: Weak optimality
5. **Solution concept 2: Strategies with (weak) guarantees**
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 - Maximin
 - Minimax
6. **Solution concept 3: Nash equilibrium**



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Regret minimisation

Maximin Value and Safety Strategy

Minimax Value and Punishment Strategy

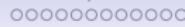
Equilibrium Concept: Nash equilibrium

Nash equilibrium: Definition

Fictitious Play for Nash Equilibrium

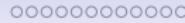
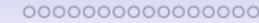
Further examples of Nash equilibria

Nash Equilibria: Additional notes and amplifications



Strategies with (weak) guarantees

- Focus on outcomes that can be **guaranteed by your own actions**
- **Reasons:**
 - Opponent-agnostic: Opponent's utilities might not be known
 - Following a different/hidden agenda:
 - e.g. threatening or punishing opponent;
 - ... others??
- **Strategies**
 1. Regret minimisation
 2. Safety (maximin) and Punishment (minimax) strategies



Regret minimisation

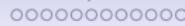
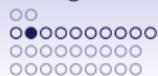
- What is my maximal possible regret if I take this action?
- Regret: diff. btw pay-offs best response and current action:

$$R_1(s_1, s_2) = u_1(BR_1(s_2)) - u_1(s_1, s_2)$$

- Take action to minimise max regret

Regret larger?

		L	R	
		10, ?	1, ?	
		10 0	4 3	
T	B	2, ?	4, ?	max → 3
		10 8	4 0	max → 8
		$BR_1 + \text{Regret}$		



Regret minimisation

- **Regret** (for agent i) is the **difference** between the **actual** and **maximal pay-off** for a **given action profile** (s_i, s_j)

$$R_i(s_i, s_j) = u_i(BR_i(s_j)) - u_i(s_i, s_j) = \max_{s'_i} u_i(s'_i, s_j) - u_i(s_i, s_j)$$

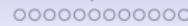
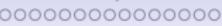
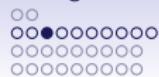
- For each action s_i , there is **maximum regret** depending on s_j :

$$R_i^{\max}(s_i) = \max_{s_j} R_i(s_i, s_j)$$

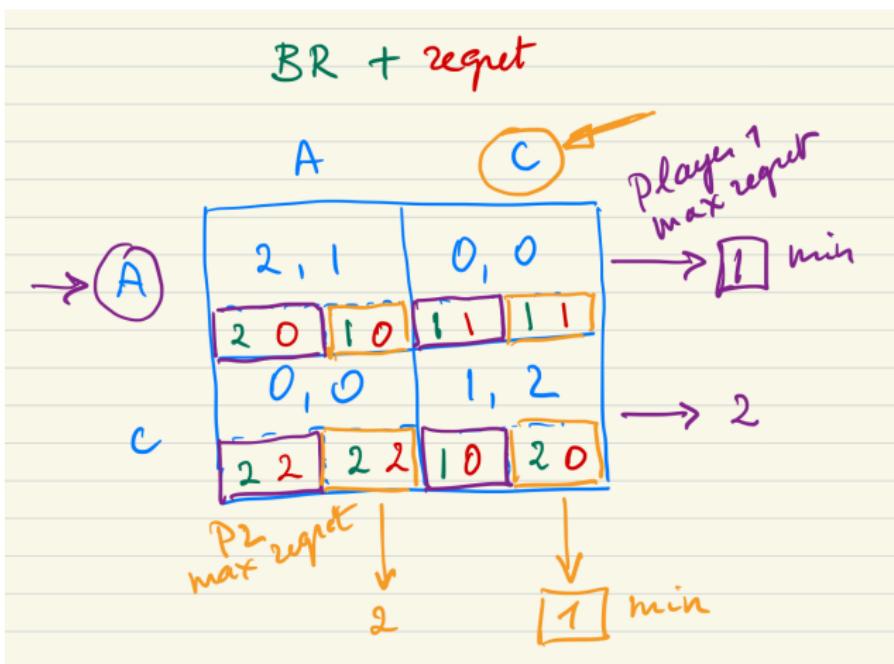
- **Regret minimisation** (**minimax regret**): agent i picks action s_i that **mimimises max regret**:

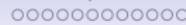
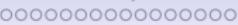
$$s_i^{rm} := \arg \min_{s_i} R_i^{\max}(s_i) = \arg \min_{s_i} \max_{s_j} R_i(s_i, s_j).$$

- **Actual solution concepts:** allows an agent to choose a **strategy with specific guarantees/properties**;



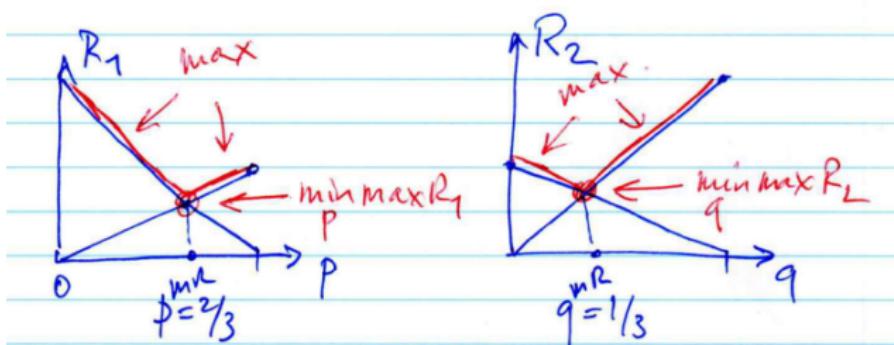
Example of regret minimisation for BoS





BoS: Regret minimisation for mixed strategies

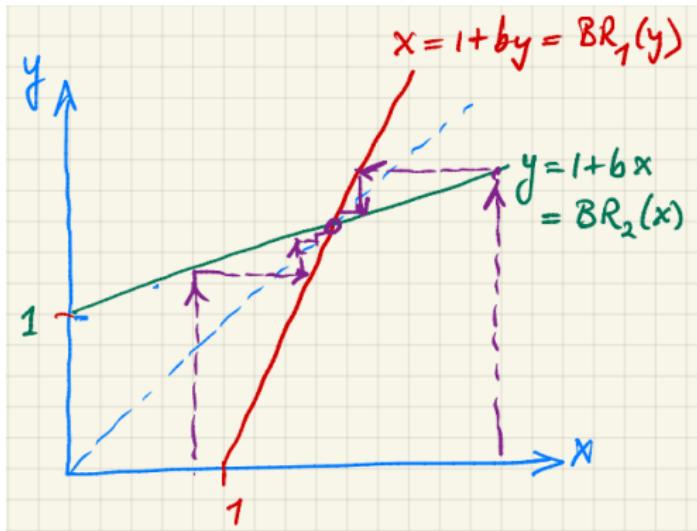
	q	1-q	
p	0, 0	1, 1	1-q
1-p	2, 2	0, 0	2q
	2(1-p)	p	

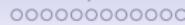
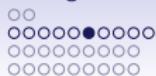




Regret minimisation for partnership game

- Recall:





Regret minimisation for partnership game

Pay-off for player 1:

$$u(x, y) = 2(x + y + bxy) - x^2$$

$$0 \leq b < 1 \quad 0 \leq x, y \leq 4$$

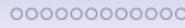
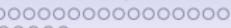
Recall: $\bar{x} = BR_1(y) = 1 + by$

Regret for player 1 if he plays x , opp. y

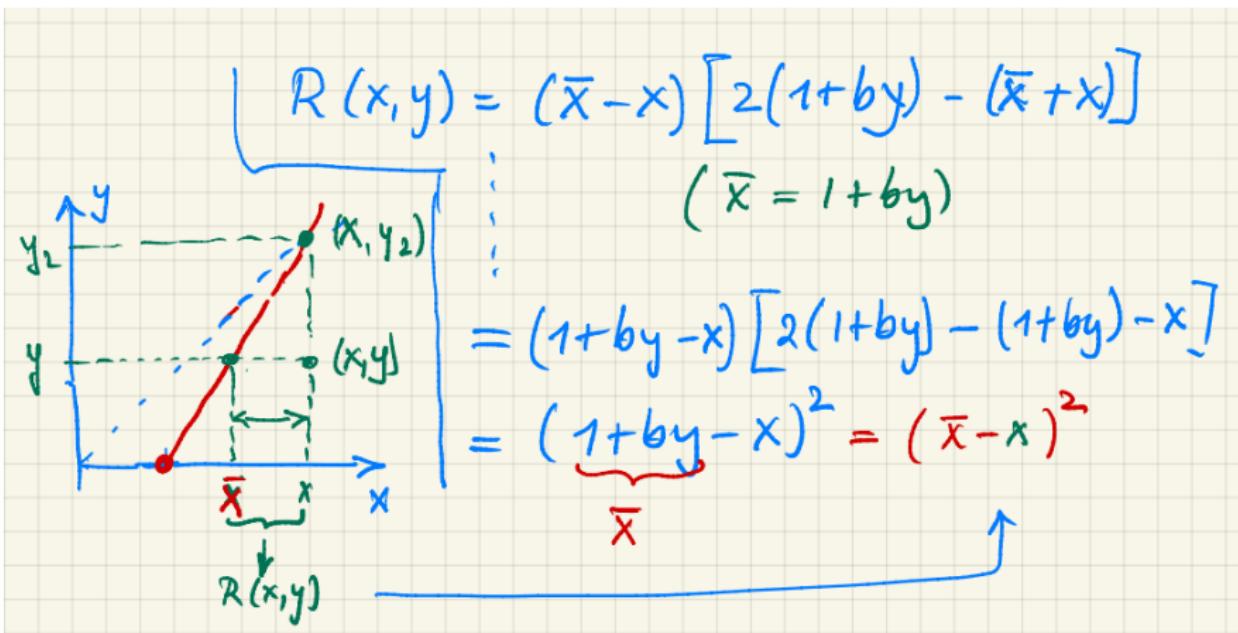
$$R(x, y) = u(\bar{x}, y) - u(x, y)$$

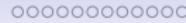
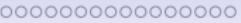
$$= 2(\bar{x} - x + b(\bar{x} - x)y) - (\bar{x}^2 - x^2)$$

$$= (\bar{x} - x)[2(1 + by) - (\bar{x} + x)]$$

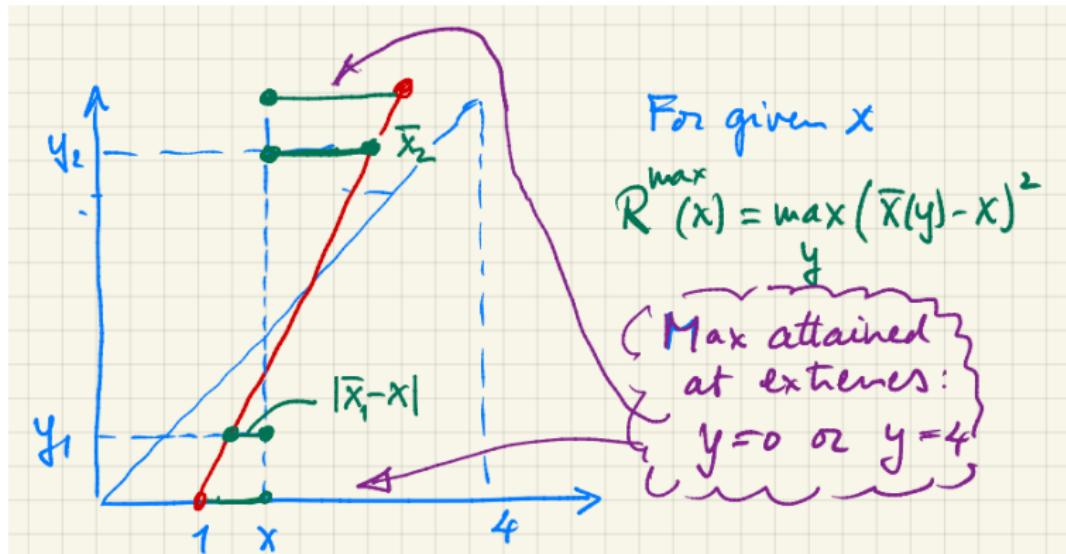


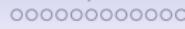
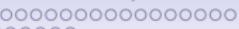
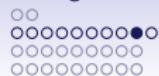
Regret minimisation for partnership game



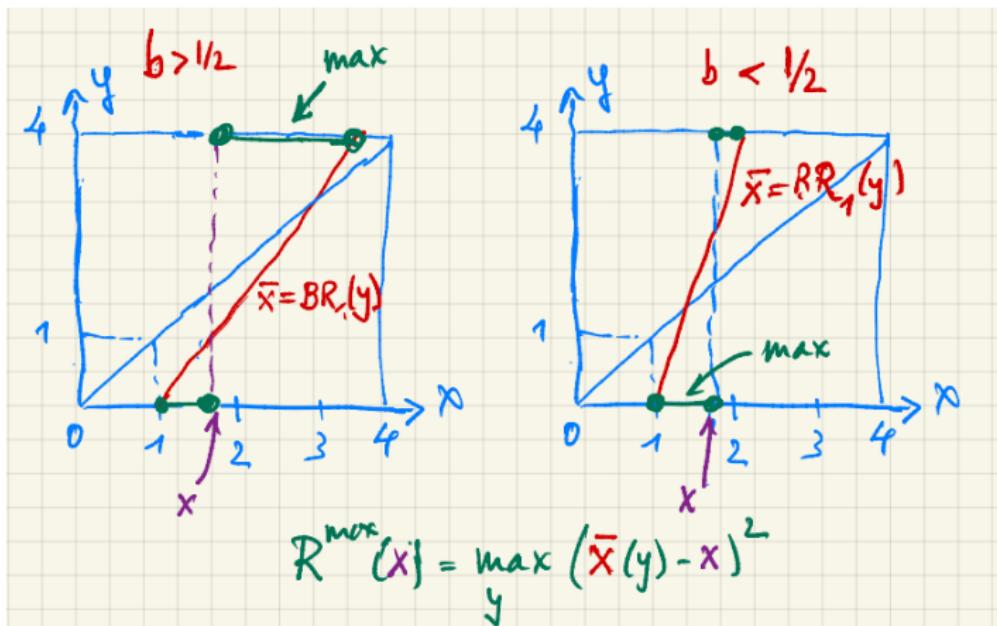


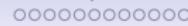
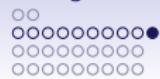
Regret minimisation for partnership game



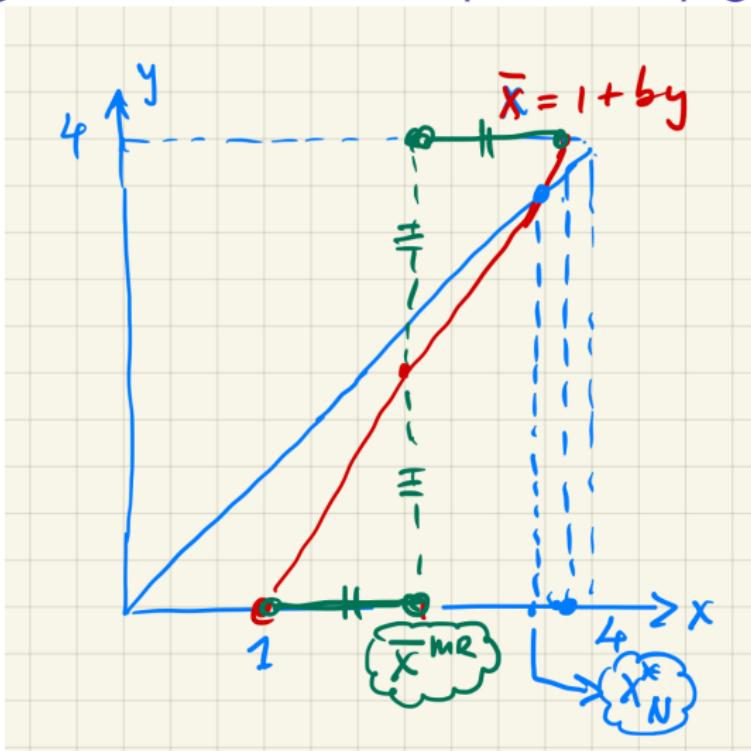


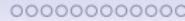
Regret minimisation for partnership game





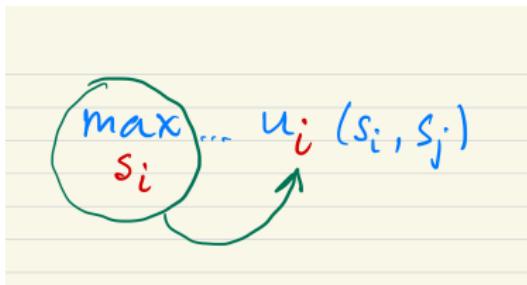
Regret minimisation for partnership game

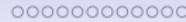




Maximin Value and Safety Strategy

- **Safety strategy:** What is the **best outcome** i can secure for myself, **no matter** what the opponent does?
- **No need to know** the corresponding opponent's pay-offs!
- We consider player i 's point of view: (i.e. maximise over s_i)





Maximin strategy (safety strategy): Algorithm

1. Ag_i computes for all his actions the worst possible outcome:

$$s_i \longrightarrow \min_{s_j} u_i(s_i, s_j)$$

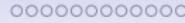
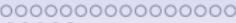
2. Next, ag_i chooses action s_i to maximise his minimal pay-off:

$$v_i^{\text{mami}} := \max_{s_i} \min_{s_j} u_i(s_i, s_j)$$

Agent tries to maximise pay-off of worst possible outcome

		L	C	R	
		2, 1	5, 6	7, 1	min
		3, 0	2, 4	1, 2	max
(1)	U	2, 1	5, 6	7, 1	→ 2
	D	3, 0	2, 4	1, 2	→ 1

$v_i^{\text{mami}} = \max_{s_i} \min_{s_j} u_i(s_i, s_j) = 2$



Maximin Value and Strategy

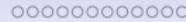
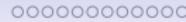
Context: 2-player, general sum game

- **Maximin value** (or **safety level** for WORST CASE) is the guaranteed minimal pay-off for agent i playing strategies in S_i :

$$v_i^{\text{mami}} := \max_{s_i \in S_i} \min_{s_j \in S_j} u_i(s_i, s_j)$$

- **Maximin strategy** (or **safety strategy**) for agent i maximizes his worst case pay-off (and therefore yields **at least** v^{mami} as utility):

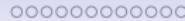
$$s_i^{\text{mami}} = \underbrace{\arg \max_{s_i} \min_{s_j} u_i(s_i, s_j)}_{\text{security level}}$$



Maximin (safety) value and strategy

- Why play maximin strategy?

- Pay-off guarantee! Highest pay-off agent i can guarantee for himself irrespective of the actions taken by other agent(s).
- Worst case analysis: assume that opponent is malicious!



Example: both agents play maximin-strategy

Both players play maximin.

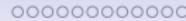
②

	L	CL	CR	R
U	3, 1	4, 6	8, 7	5, 2
M	4, 2	7, 4	1, 5	4, 5
D	6, 2	1, 3	7, 0	0, 4

min ↓ ↓ max

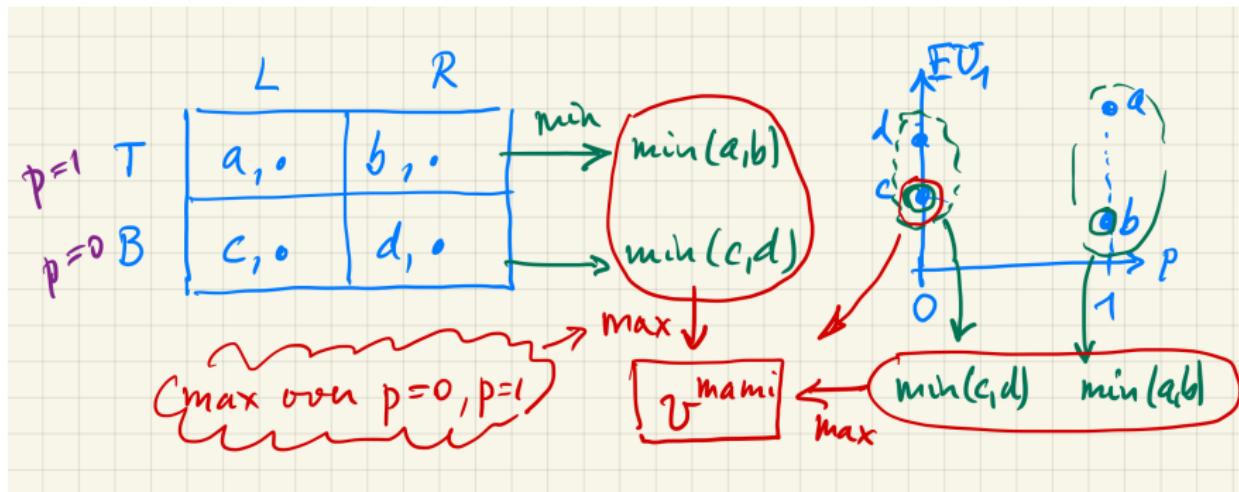
1 3 0 2

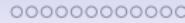
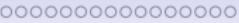
$$\left. \begin{array}{l} u_1^{\text{max}} = 3, s_1^{\text{max}} = U \\ u_2^{\text{max}} = 3, s_2^{\text{max}} = CL \end{array} \right\} \rightarrow \boxed{u_1(U, CL) = 4} \\ \boxed{u_2(U, CL) = 6}$$



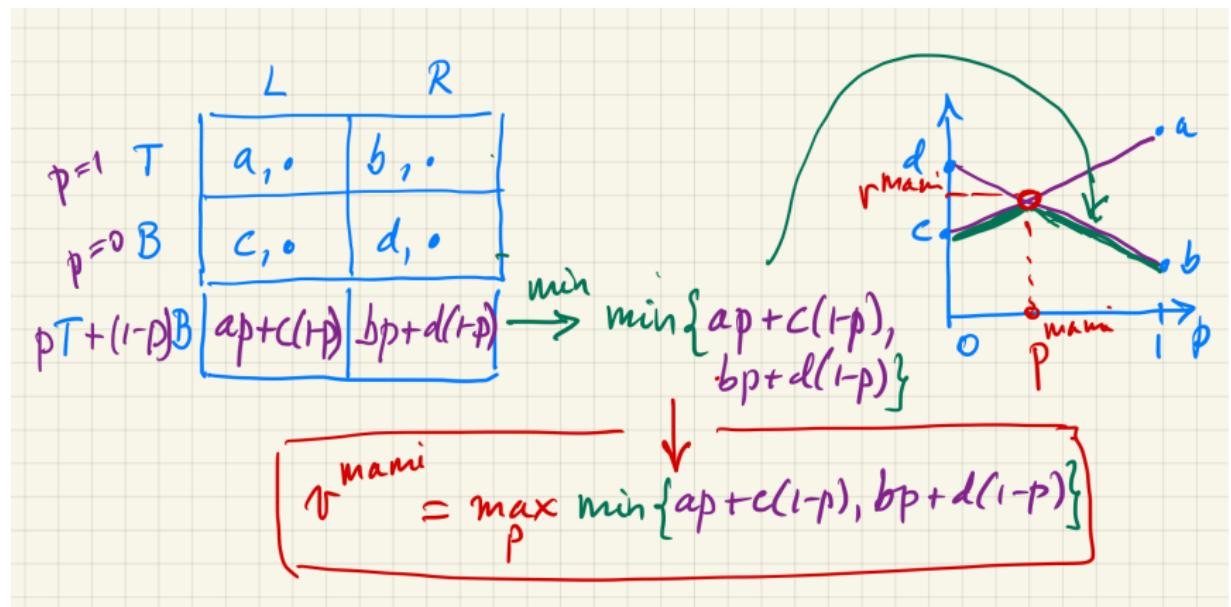
From Pure to Mixed safety strategies

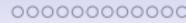
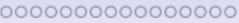
- safety value for pure safety strategy





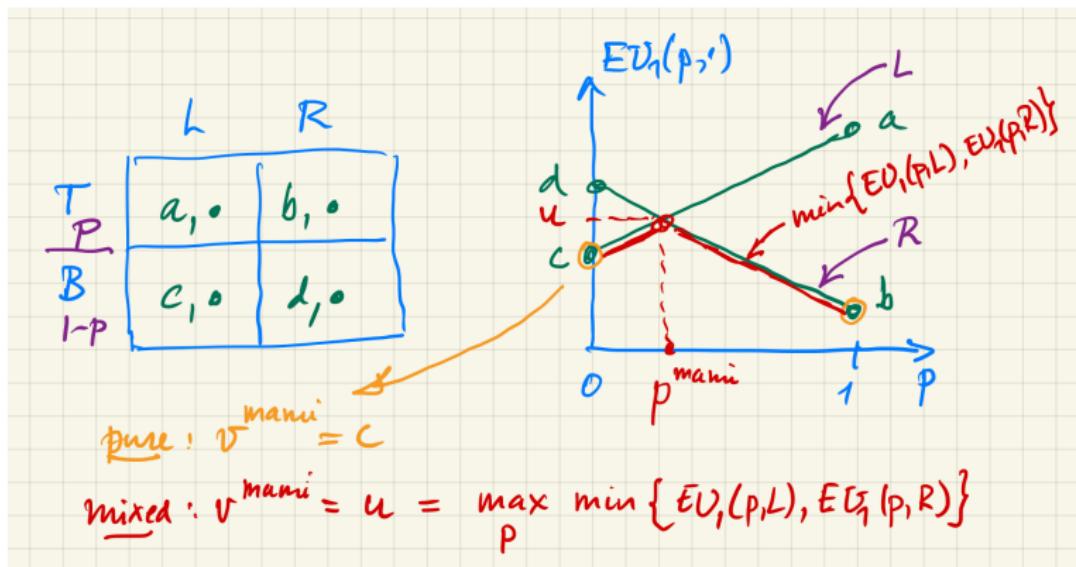
From Pure to Mixed safety strategies

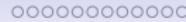
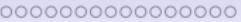




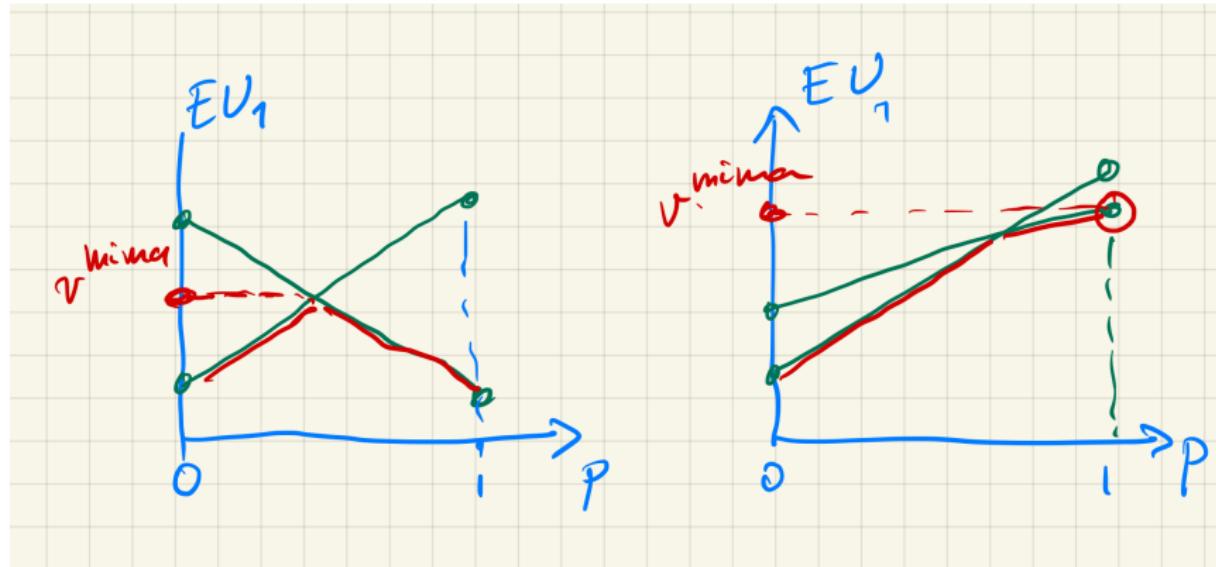
Mixed safety strategies

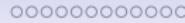
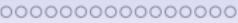
- If $EU_1(p, \cdot)$ graphs intersect, it **might** be better to look at mixed strategies





Sometimes pure safety strategy is best





Punishment (or minimax) Strategy (of player j) yields Minimax Value (for player i)

1. Player i computes best response utility for each of j's strategies s_j : $BR_i(s_j) = \max_{s_i} u_i(s_i, s_j)$;
2. Then player j (who wants to punish i) picks action to minimise player i best pay-off. This punishment strategy of player j results in the minimax value for player i:

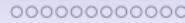
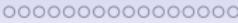
$$v_i^{\text{minimax}} := \min_{s_j} \max_{s_i} u_i(s_i, s_j) = \min_{s_j} u_i(BR_i(s_j))$$

(i) j

	2, 1	5, 6	7, 1
3, 0	3, 0	2, 4	1, 2
	3	5	7

$v_i^{\text{minimax}} = 3$

$\max(82)$



Algo: Minimax Value and Punishment Strategy

- **Minimax value** for agent i :

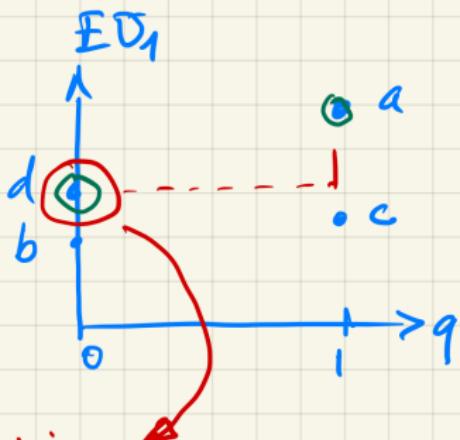
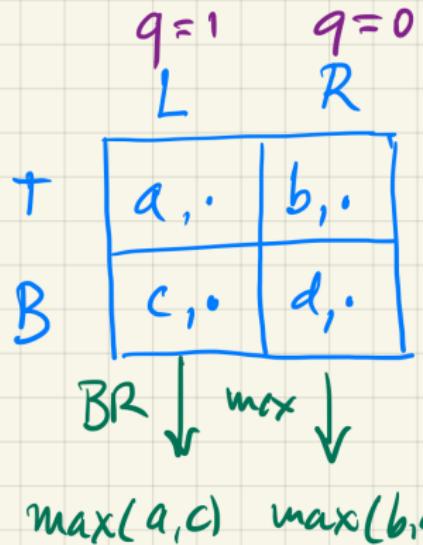
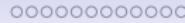
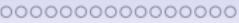
- Given the strategy s_j of his opponent, agent i will play its **best response**, resulting in a pay-off:

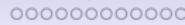
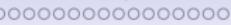
$$\max_{s_i} u_i(s_i, s_j)$$

- The opponent is aware of this and wants to "punish" i by *minimizing* this pay-off, yielding the **minimax value** for player i :

$$v_i^{\text{mima}} := \min_{s_j} \max_{s_i} u_i(s_i, s_j)$$

- The corresponding **minimising strategy** s_j^{mima} is called the **minimax strategy** for player j .
- If j plays his minimax strategy s_j^{mima} , then i cannot do better than v_i^{mima} (even if i plays best response $BR_i(s_j^{\text{mima}})$).

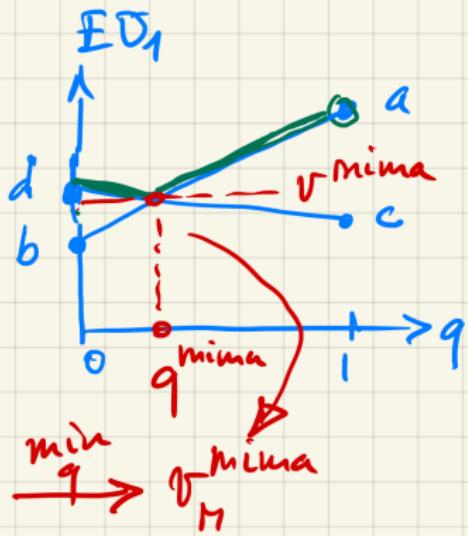


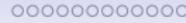


Punishment strategy: Mixed

	L	R	$qL + (1-q)R$
T	a, \cdot	b, \cdot	$aq + b(1-q)$
B	c, \cdot	d, \cdot	$cq + d(1-q)$

$$\max \left\{ \begin{array}{l} cq + b(1-q), \\ cq + d(1-q) \end{array} \right\}$$





Comparing minimax and maximin value for player i

- $\forall s_i, s_j :$

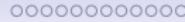
$$\underbrace{\min_{s_j} u_i(s_i, s_j)}_{\phi(s_i)} \leq u_i(s_i, s_j) \leq \underbrace{\max_{s_i} u_i(s_i, s_j)}_{\psi(s_j)}$$

- Since the above inequality holds for all s_i and s_j , it follows:

$$\max_{s_i} \phi(s_i) \leq \min_{s_j} \psi(s_j)$$

- Hence:

$$v_i^{mami} = \max_{s_i} \min_{s_j} u_i(s_i, s_j) \leq \min_{s_j} \max_{s_i} u_i(s_i, s_j) = v_i^{mima}$$



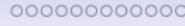
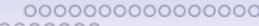
Minimax and Maximin Value for Player i

- In general:

$$v_i^{\text{mami}} = \max_{s_i} \min_{s_j} u_i(s_i, s_j) \leq \min_{s_j} \max_{s_i} u_i(s_i, s_j) = v_i^{\text{mima}}$$

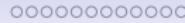
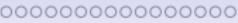
- Your guaranteed pay-off is a **lower bound** for the worst your opponent can force onto you!
- The worst your opponent can force onto you is an **upper bound** on your guaranteed pay-off.

	0,.	3,.	$\xrightarrow{\min}$ 0
	2,.	1,.	$\xrightarrow{\max}$ $1 = v_i^{\text{mami}} \leq v_i^{\text{mima}} = 2$
$\xrightarrow{\max}$	2	1	"safety" "forced"



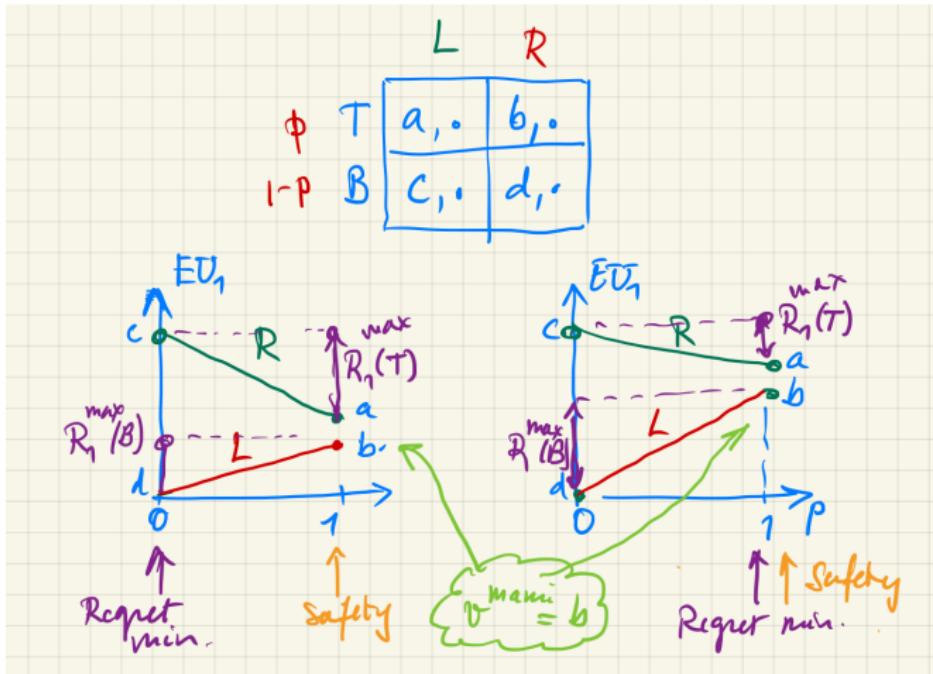
Recap Safety and Punishment Strategy

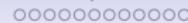
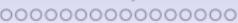
- Player i 's **maximin strategy** is **safety strategy**
 - player i concerned about his own safety
 - Malicious or adversarial opponent
 - Multi-agent setting
 - strategy yields highest guaranteed outcome for player i
 - Viable **solution algorithm**.
- Player i 's **minimax strategy** is **punishment strategy**
 - i 's strategy is directed **against** player j
 - player i tries to minimize best (i.e. maximum) pay-off for j
 - Useful as **threat** (e.g. in repeated games);
 - i 's **maximin strategy** gives rise to j 's **maximin value**



Which is better? Regret Minimisation or Safety?

- Neither, not comparable!





Which is better? Regret Minimisation or Safety?

- Comparison for $(0 \leq T \leq 2, -1 \leq S \leq 1)$ parametrisation for social dilemmas.

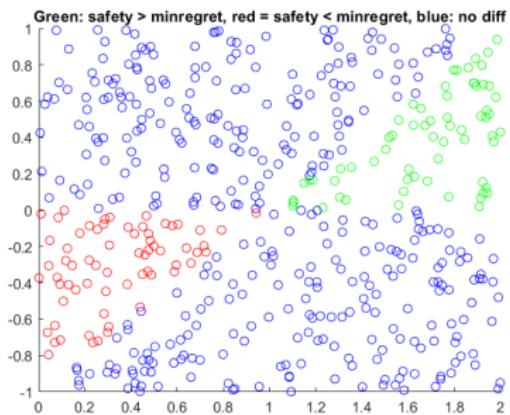


Figure: Utilities (assuming both agents play same strategy): : green: safety > min regret, red = min regret > safety, blue: no difference

Table of Contents

Strategies with (weak) Guarantees

Regret minimisation

Maximin Value and Safety Strategy

Minimax Value and Punishment Strategy

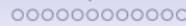
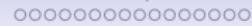
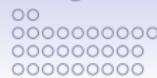
Equilibrium Concept: Nash equilibrium

Nash equilibrium: Definition

Fictitious Play for Nash Equilibrium

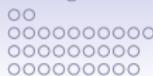
Further examples of Nash equilibria

Nash Equilibria: Additional notes and amplifications



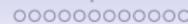
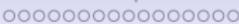
Nash equilibrium: Context

- Von Neumann's minimax theorem established game theory as a discipline;
- Nash equilibrium: Extension of von Neumann's minimax theorem
 1. From two person to n person game;
 2. from zero sum to general utilities
- For more info on the minimax theorem: See addendum;

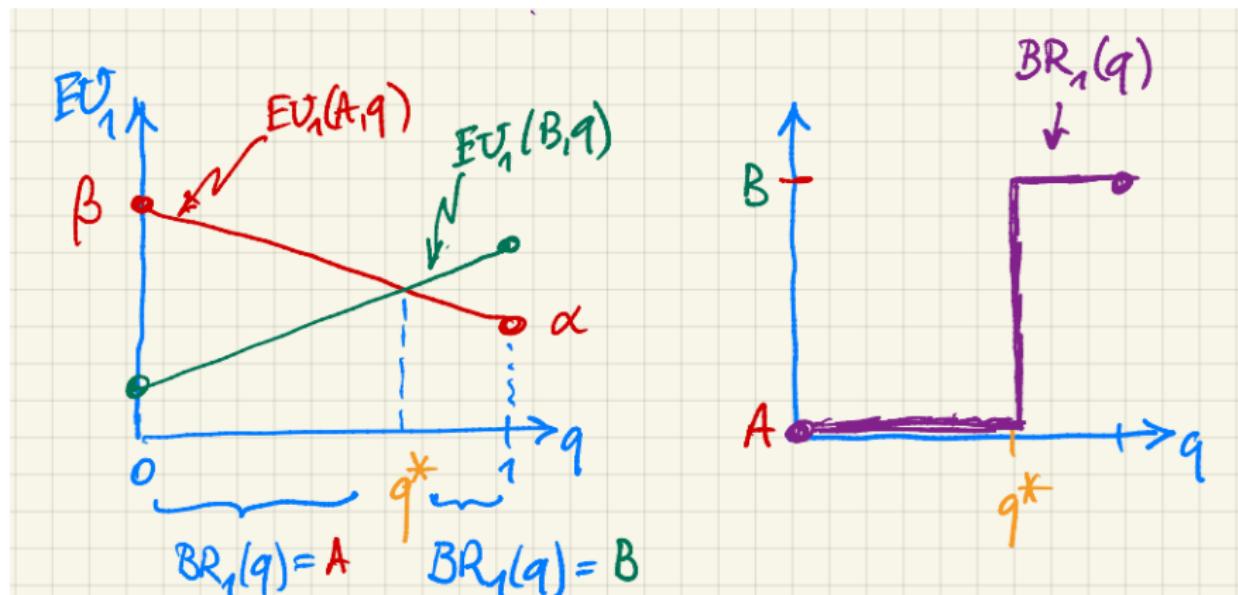


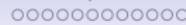
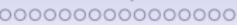
Nash Equilibrium: Intro

- Optimal choice no longer applies, as your outcome depends on opponent's actions.
 - Opposing "forces" at play;
- From optimum to equilibrium:
 - Opposing "forces" balance out;
- Recall: Best response (BR) dynamics ...
 - sometimes converged to equilibrium position;
 - sometimes gave rise to cycles ... Nash to the rescue!



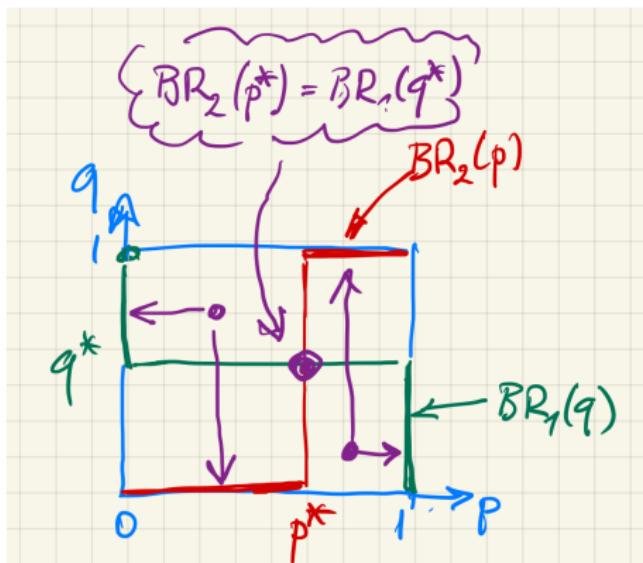
Recap: Best response to mixed strategy

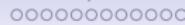
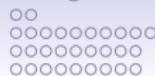




Nash equilibrium: intro

- Consider general mixed strategies in 2p-game, with two actions each;
- Imagine one player is allowed to change his mind!



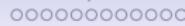
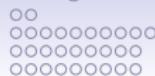


Nash equilibrium (1)

- A **Nash equilibrium** (NE, 1950) is a solution concept based on *conditions* instead of an *algorithm*.
- **Mutual best response:** NE is joint strategy profile s^* such that **for each agent i** the strategy s_i^* is a **best response** to s_{-i}^* ;
- **Formally:** A strategy profile $s^* = (s_1^*, s_2^*, \dots, s_n^*)$ is a **strict NE** if:

$$\forall \text{agents } i, \forall s'_i \neq s_i^* : u(s_i^*, s_{-i}^*) > u(s'_i, s_{-i}^*).$$

- **Strict ($>$) versus weak (\geq) NE**
- **No Regret/Self-enforcing:** a (strict) NE is a stable strategy profile for which no agent has an incentive to **unilaterally deviate**;



Nash equilibrium: Computation of NE (pure strategy)

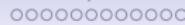
Find mutual best responses:

- **Battle of the Sexes** (two pure strategy NEs)

		action	comedy
		action	<u>2, 1</u>
		comedy	0, 0
action	comedy	0, 0	<u>1, 2</u>

- **Prisoner's dilemma** (single NE, not Pareto-optimal!)

		hush	confess
		hush	<u>-1, -1</u>
		confess	0, -12
hush	confess	<u>-12, 0</u>	<u>-8, -8</u>

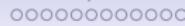
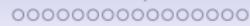
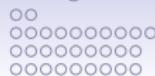


Nash's Theorem

Existence of Nash Equilibrium (Nash, 1950)

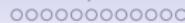
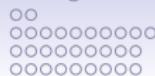
A **finite strategic game** (i.e. finite number of players and actions) always has **at least one Nash equilibrium** (allowing mixed strategies).

- A **pure** Nash equilibrium can be **strict** or **weak**;
- A **mixed** Nash equilibrium is necessarily **weak**;



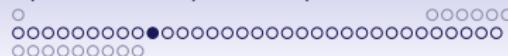
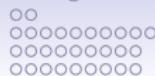
Nash equilibrium: Nash's Theorem

- A **finite strategic game** is a game with a **finite number of agents** and a **finite number of actions**;
- A game may have **zero, one, or more pure-strategy NE**.
- If there's a **single NE**: **natural solution concept**, but might be (Pareto) sub-optimal!
- If there are **multiple NEs**: there might be no compelling reasons to pick a particular one; but ...
 - Utility dominant NE, Schelling's focal points
- Since **humans are not always rational**, Nash equilibria **might not agree** with experiments or observations..



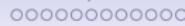
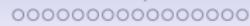
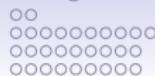
Note on **multiple** Nash Equilibria

- A unique best alternative is exceptional.
- In general, non-cooperative game theory is plagued by a multiplicity of equilibria.
- Hence, prescriptions of how to act without any coordination or cooperation are in general impossible.
- Non-cooperative game theory tells us what to exclude from choice.
- Since in non-coop GT, there are no binding contracts, a player's announcements and promises in a pre-play phase are credible only, if they are totally in line with his best interests.



Computation of NE

- **Pure NE** for each agent i the strategy s_i^* is a **best response** to s_{-i}^* ; (mutual best response);
 - Matrix games (discrete state/action) space;
 - Continuous action space
- **Mixed NE**: make opponent indifferent (matrix games only);



Nash equilibrium: Computation of **mixed** NE

Matching pennies (zero-sum game)

- No pure strategy NE ...

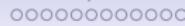
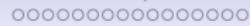
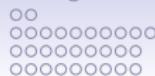
	heads	tails
heads	<u>1</u> , -1	-1, <u>1</u>
tails	-1, <u>1</u>	<u>1</u> , -1

- ... hence, at least one **mixed strategy NE!**
- Intuitively** this is obvious; play each action with prob = 1/2:

$$s_1 = s_2 = \{(H, 1/2), (T, 1/2)\}$$

- Expected utility (pay-off):**

$$u_1(s_1, s_2) = \frac{1}{4}u_1(H, H) + \frac{1}{4}u_1(T, H) + \frac{1}{4}u_1(H, T) + \frac{1}{4}u_1(T, T) = 0.$$



Matching pennies: Computation of mixed NE

		q	$1-q$
	H	H	T
p	H	$1, -1$	$-1, 1$

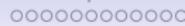
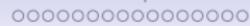
		$1-p$	T
	T	$-1, 1$	$1, -1$

$$\begin{aligned} u_2(p, H) &= (-1)p + 1 \cdot (1-p) \\ &= 1 - 2p. \end{aligned}$$

$$\begin{aligned} u_2(p, T) &= p + (-1)(1-p) \\ &= 2p - 1 \end{aligned}$$

$$u_1(H, q) = q - (1-q) = 2q - 1$$

$$u_1(T, q) = -1 \cdot q + (1-q) = 1 - 2q$$



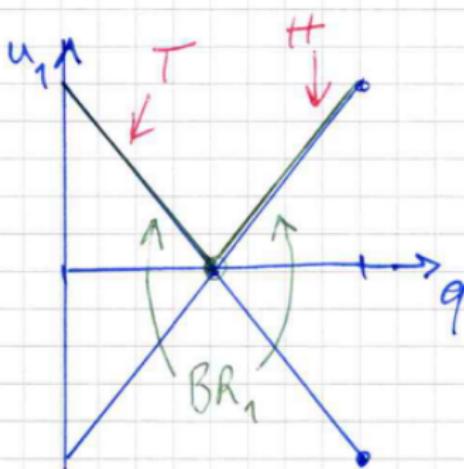
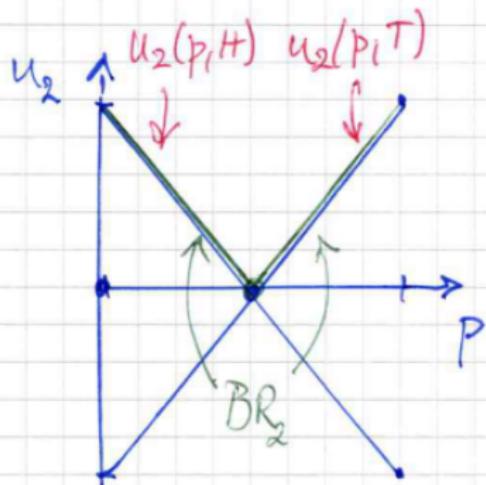
Matching pennies: Computation of best response

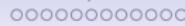
$$u_2(p, H) = 1 - 2p$$

$$u_2(p, T) = 2p - 1$$

$$u_1(H, q) = 2q - 1$$

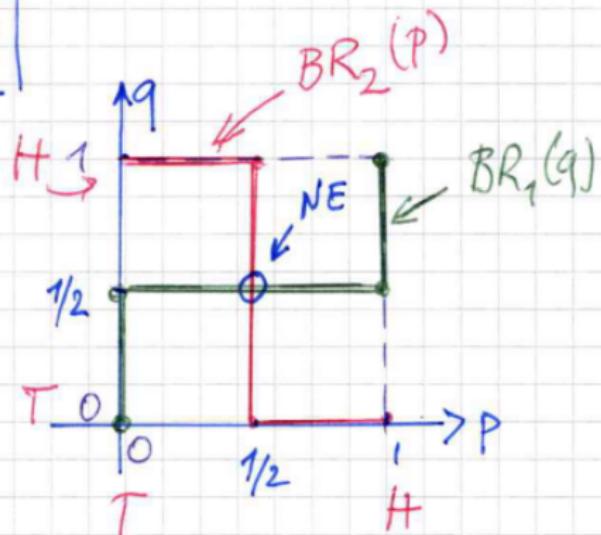
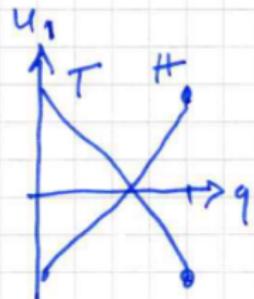
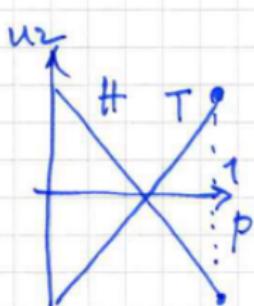
$$u_1(T, q) = 1 - 2q$$

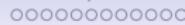




Matching pennies: Best response graph

	q	$1-q$				
	H	T				
p	H	<table border="1"> <tr> <td>1, -1</td> <td>-1, 1</td> </tr> <tr> <td>-1, 1</td> <td>1, -1</td> </tr> </table>	1, -1	-1, 1	-1, 1	1, -1
1, -1	-1, 1					
-1, 1	1, -1					
$1-p$	T					





Matching pennies: Mixed Nash Equilibrium

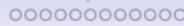
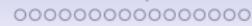
Nash equilibrium characteristics:

- At the intersection point ($p = 1/2, q = 1/2$), players are simultaneously playing best response to each other;
- No player can do strictly better by unilaterally deviating:
 - If player 2 keeps playing $q = 1/2$ then player 1's utility $u_1(p, q = 1/2)$ can be computed as follows:

$$\begin{aligned} u_1(p, 1/2) &= (1 \cdot p + (-1) \cdot (1 - p) + (-1) \cdot p + 1 \cdot (1 - p)) \cdot \frac{1}{2} \\ &= 0 \quad (\text{independent of } p) \end{aligned}$$

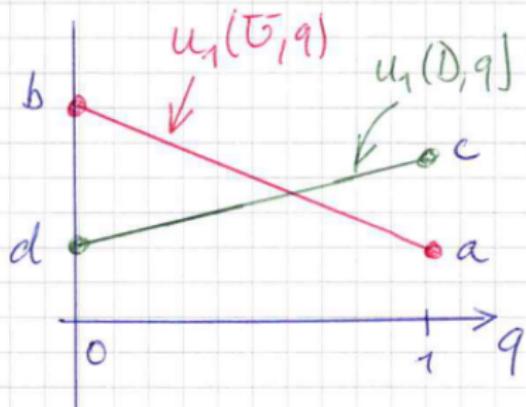
Player 1: no incentive to change his strategy (i.e. change p)

- Same consideration for player 2.



Computation of mixed Nash equilibrium

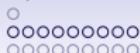
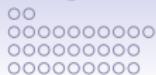
	q	$1-q$
	L	R
$p \bar{v}$	a, A	b, B
$1-p \bar{D}$	c, C	d, D



(Expected) utility for player 1:

$$u_1(p, q) = apq + bp(1-q) + c(1-p)q + d(1-p)(1-q)$$

Linear in p (for fixed q) and vice versa.



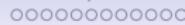
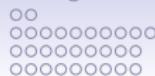
Computation of mixed Nash equilibrium

$$u_1(p, q) = apq + bp(1-q) + c(1-p)q + d(1-p)(1-q)$$

Player 1 : no incentive to change

$$0 = \frac{\partial u_1}{\partial p} = \underbrace{(aq + b(1-q))}_{\text{if } u_1(U, q)} - \underbrace{(cq + d(1-q))}_{\text{if } u_1(D, q)} = 0$$

Player 2 needs to pick q such that
player 1 is indifferent btw U and D.



Computing mixed Nash equilibrium

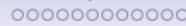
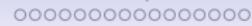
Player 1: T or B, player 2: L or R

- **Strategies:** Player 1: T or B, Player 2: L or R
- Player 1: fix $p = p^*$, such that player 2 is indifferent (between L and R):

$$EU_2(p^*, L) = EU_2(p^*, R)$$

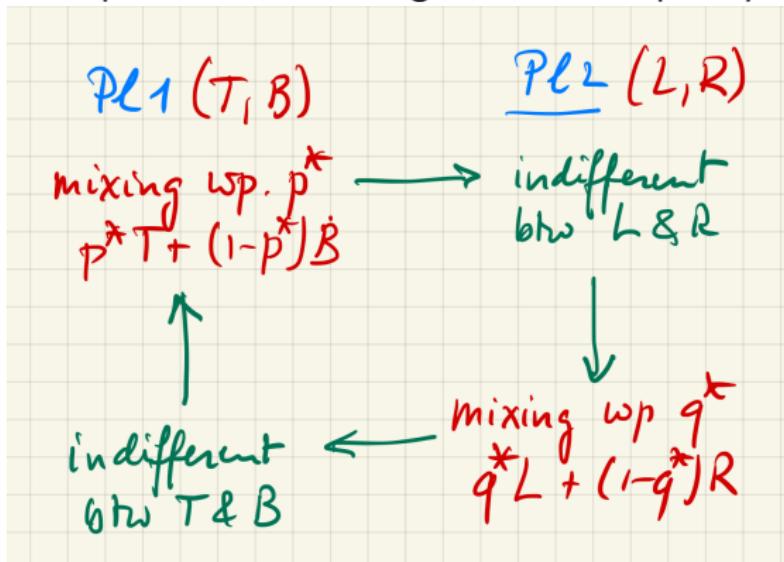
- Player 2: fix $q = q^*$, such that player 1 is indifferent (between T and B):

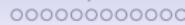
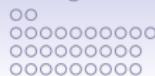
$$EU_1(T, q^*) = EU_1(B, q^*)$$



Mixed Nash equilibrium: “mutual stranglehold”

- P1 mixes (with prob p^*) to make P2 indifferent;
- Since P2 is indifferent, he's OK to mix too!
- This allows P2 to mix with prob q^* to make P1 indifferent;
- As a consequence P1 is willing to mix with prob p^* .

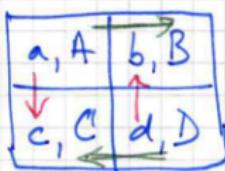




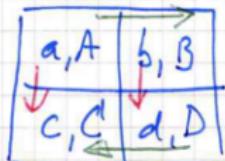
Some useful graphical representations

GRADIENT
PLOT

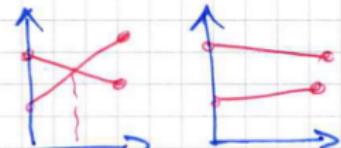
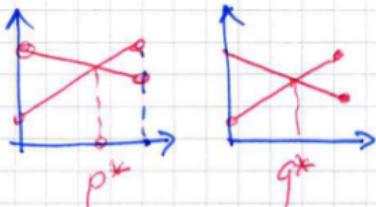
MNE

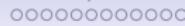
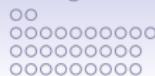


no MNE



UTILITY
PLOT





Nash equilibrium: Computation of mixed equilibrium

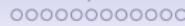
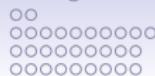
- **Battle of the Sexes** (two pure strategy NEs)
- Additional NE by **mixing pure strategies?**

$$s_1 = \{(A, p), (C, 1-p)\} \quad \text{and} \quad s_2 = \{(A, q), (C, 1-q)\}.$$

		Action (q)	Comedy (1 - q)
		Action (p)	2, 1
		Comedy (1 - p)	0, 0
Action (p)	Comedy (1 - p)	0, 0	1, 2

- Determining the mixture parameters p and q
 - Ag 1 chooses p such that Ag 2 is **indifferent** btw actions A and C; if not Ag 2 would focus on the most lucrative option.

Condition for Ag 1: $u_2(s_1(p), A) = u_2(s_1(p), C)$



Nash equilibrium: Computation of mixed equilibrium

- **Battle of the Sexes** (mixed strategy NEs)

	Action (q)	Comedy ($1 - q$)
Action (p)	2, 1	0, 0
Comedy ($1 - p$)	0, 0	1, 2

- Determining the mixture parameters p and q

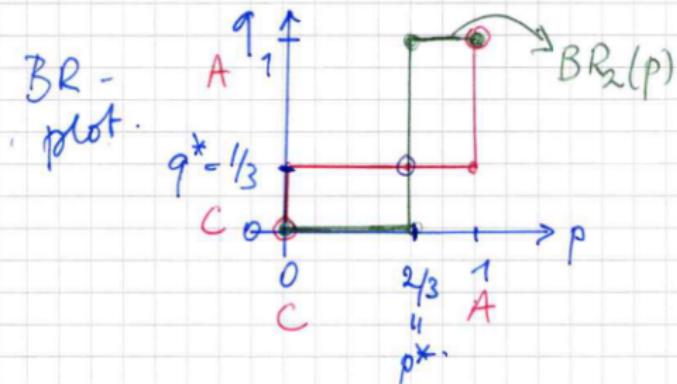
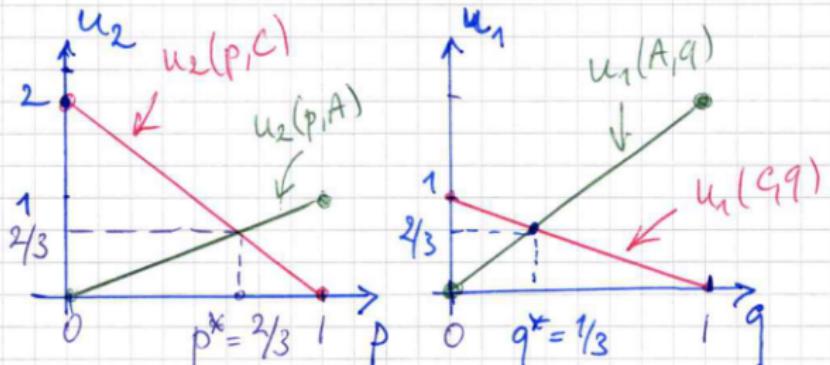
Ag 1: $EU_2(s_1(p), A) = EU_2(s_1(p), C) \implies p = 2(1 - p)$

Ag 2: $EU_1(A, s_2(q)) = EU_1(C, s_2(q)) \implies 2q = (1 - q)$

Conclusion: $p = \frac{2}{3}$, $q = \frac{1}{3}$ $EU_1(s_1, s_2) = EU_2(s_1, s_2) = 2/3.$



Battle of the sexes: Mixed Nash Equilibrium



Nash equilibrium: Computation of mixed equilibrium

- **Prisoner's Dilemma** (does mixed strategy NE exist?)

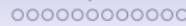
	Quiet (q)	Confess ($1 - q$)
Quiet (p)	-1, -1	-12, 0
Confess ($1 - p$)	0, -12	-8, -8

- Determining the mixture parameters p and q

$$\text{Ag 1: } EU_2(s_1, Q) = EU_2(s_1, C) \implies -p - 12(1-p) = -8(1-p)$$

$$\text{Ag 2: } EU_1(Q, s_2) = EU_1(C, s_2) \implies -q - 12(1-q) = -8(1-q)$$

Conclusion: $p = q = \frac{4}{3}$ **impossible!**



Mixed NE for game with three actions

Rock-Paper-Scissors: no PURE NE!

		v	w	$1 - (v + w)$
		R	P	S
p	R	0, 0	-1, 1	1, -1
q	P	1, -1	0, 0	-1, 1
$1 - (p + q)$	S	-1, 1	1, -1	0, 0

Determine p, q by insisting that player 2 is indifferent btw actions:

$$0 \cdot p + (-1) \cdot q + 1 \cdot (1 - p - q) = 1 \cdot p + 0 \cdot q + (-1) \cdot (1 - p - q)$$

$$0 \cdot p + (-1) \cdot q + 1 \cdot (1 - p - q) = -1 \cdot p + 1 \cdot q + 0 \cdot (1 - p - q)$$

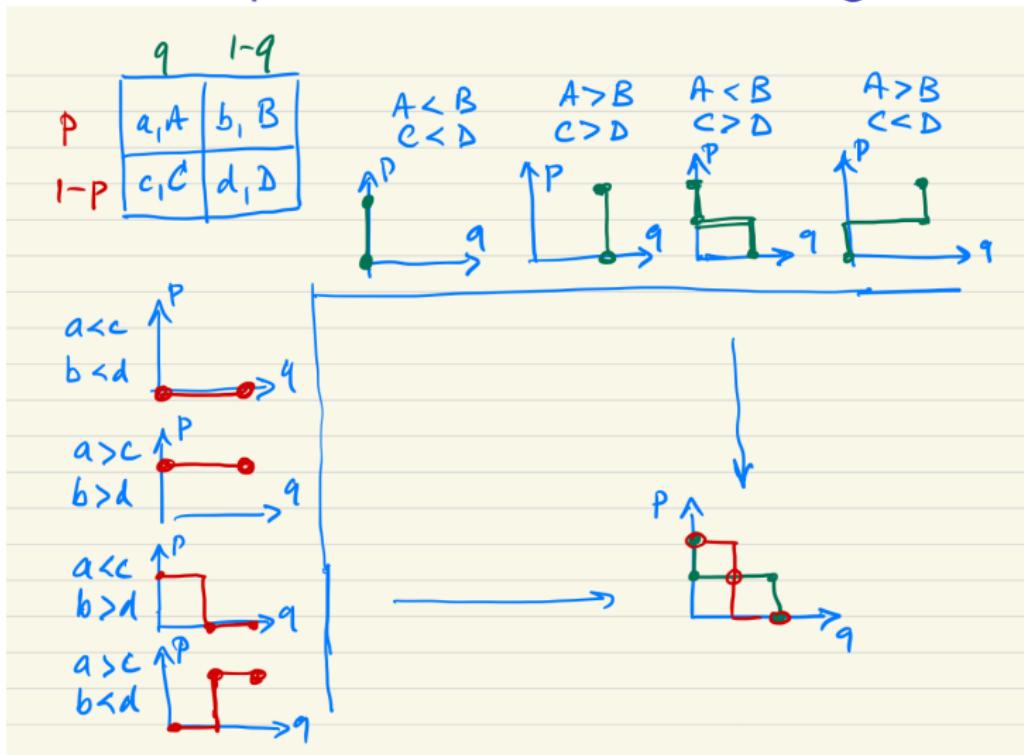
Result: $p = q = 1/3$. Determine mixing parameters u, v similarly.

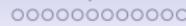
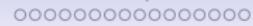
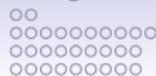


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Nash equilibrium for 2×2 matrix game





Nash eq. for coordination and anti-coordination games

Coordination

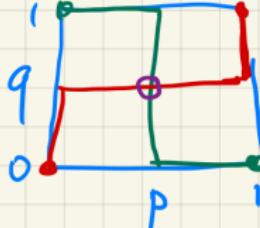
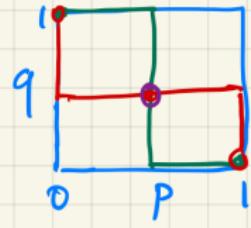
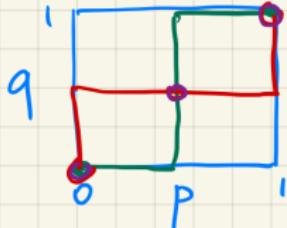
1,1	0,0
0,0	1,1

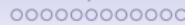
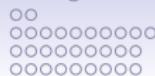
anti-coord

0,0	1,1
1,1	0,0

Coord +
anticoord

1,-1	-1,1
-1,1	1,-1



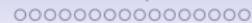


NE 2×3 : Example of NE for 2×3 game

- Pay-off matrix and best response

		u	v	$1 - u - v$	
		L	C	R	
p	T	3, 5	1, 1	2, 4	
	B	4, 1	0, 3	6, 2	

- No pure Nash eq.
- There must be at least one mixed Nash eq.
 - Player 1: mixing: $\pi_1 = (p, 1 - p)$
 - Player 2: mixing: $\pi_2 = (u, v, 1 - (u + v))$



NE 2x3: Player 2 mixing to make Player 1 indifferent

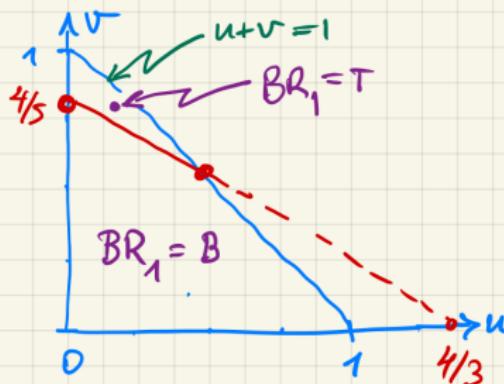
Pl 2 mixing to make Pl.1 indifferent:

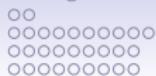
$$EU_1(T, \pi_2) = 3u + v + 2(1-u-v) = u - v + 2$$

$$EU_1(B, \pi_2) = 4u + 0.v + 6(1-u-v) = -2u - 6v + 6$$

indifference
↓

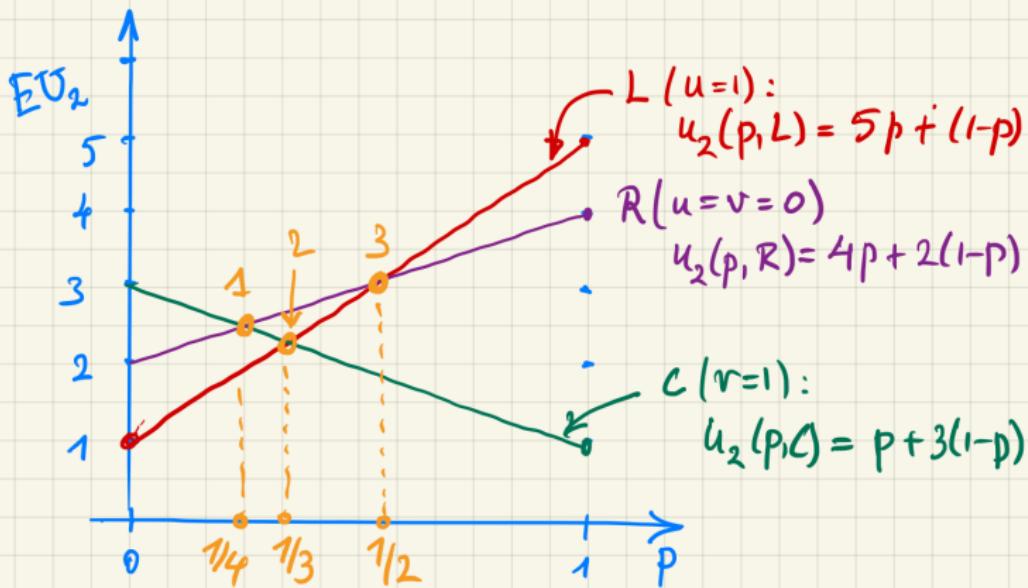
$$\begin{aligned} u - v + 2 &= -2u - 6v + 6 \\ \Rightarrow \boxed{3u + 5v = 4} \end{aligned}$$





NE 2x3: Player 1 mixing to make Player 2 indifferent

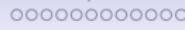
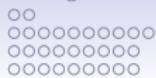
Pt 1 mixing: (L, L) , (L, R) , (R, L) , (R, R)





NE 2x3: Inspection of the Player 1 mixing options

- **M1: mixing R and C at $p = 1/4$**
 - Since L is not included in mix: $u = 0$
 - There is a unique point on indifference line for Player 1, corresponding to $u = 0$ and $v = 4/5$
 - Hence **mixed NE at $p = 1/4$ and $u = 0, v = 4/5$**
 - **M2: mixing L and C at $p = 1/3$**
 - Not NE as deviating to R is better!
 - **M3: Mixing L and R at $p = 1/2$**
 - Since C is not included: $v = 0$
 - **No solution** (point on indifference line outside feasible triangle!)
 - **Mixing all three strategies** dilutes pay-off and therefore is **not optimal**



NE 2x3: Expected utilities in Nash Eq.

3	5	1	1	2	4
4	1	0	3	6	2

π_1 { $\frac{1}{4}$
 $\frac{3}{4}$ } π_2 { 0
 $\frac{4}{5}$
 $\frac{1}{5}$ }

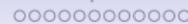
0	$\frac{4}{20}$	$\frac{1}{20}$
0	$\frac{12}{20}$	$\frac{3}{20}$

$$u_1(\pi_1, \pi_2) = \frac{1}{20} [1 \cdot 4 + 0 \cdot 12 + 2 \cdot 1 + 6 \cdot 3]$$

$$= \frac{24}{20} = \frac{6}{5} = \underline{\underline{1.2}}$$

$$u_2(\pi_1, \pi_2) = \frac{1}{20} [1 \cdot 4 + 3 \cdot 12 + 4 \cdot 1 + 2 \cdot 3]$$

$$= \frac{50}{20} = \frac{5}{2} = \underline{\underline{2.5}}$$



NE 2x3: Compare safety strategy to Nash Eq.

3	5	1	1	2	4
4	1	0	3	6	2

$\min \rightarrow 1$

$$v_1^{\text{max}} = 1$$

$\rightarrow 0$

$\min \downarrow$

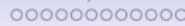
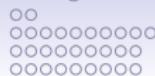
1	1	2
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$\downarrow \quad \downarrow \quad \downarrow$

$1 \quad 1 \quad 2$

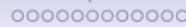
\max

$$v_2^{\text{max}} = 2$$



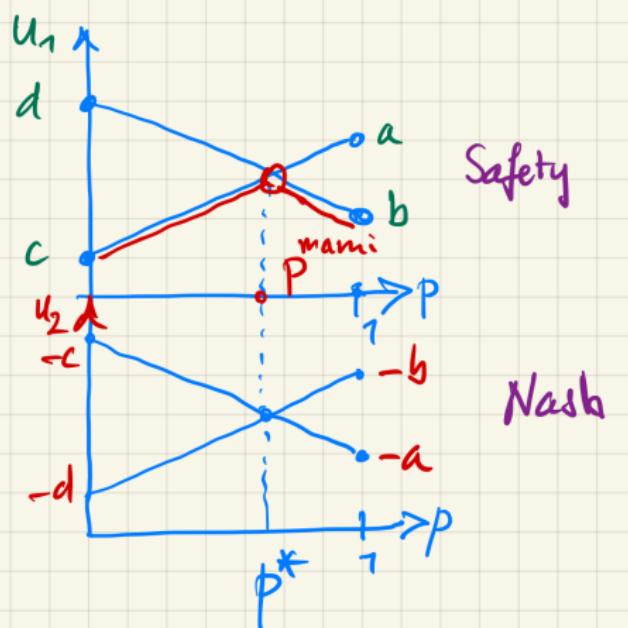
Fictitious Play (FP)

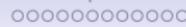
- FP uses simulation of many game iterations to learn about equilibria;
- FP refers to a dynamic process where at each stage, players play a (pure) best response to the empirical distribution of their opponent's play (which they interpret as a mixed strategy);
- If FP converges (in distribution), the limit distribution coincides with mixed Nash strategies.
- Reminiscent of best response dynamics, but more general (includes not just last action, but whole history)
- Form of learning, works even opponent's utilities are unknown;



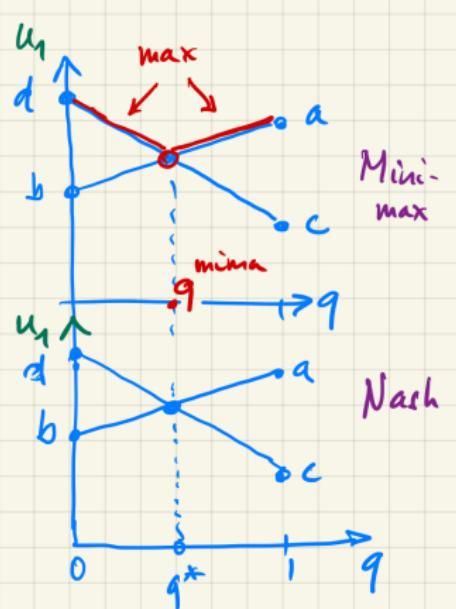
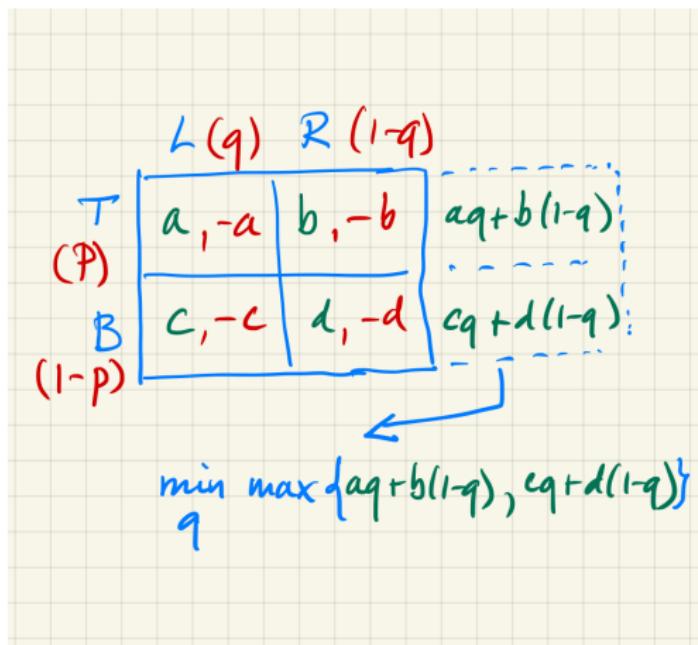
2 player zero-sum games: Nash = Maximin = Minimax

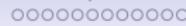
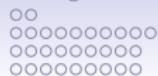
	$L(q)$	$R(1-q)$
$T(p)$	$a, -a$	$b, -b$
$B(1-p)$	$c, -c$	$d, -d$





2 player zero-sum games: Nash = Maximin = Minimax





Fictitious Play: NE 2×3 example

	L	C	R
T	3, 5	1, 1	2, 4
B	4, 1	0, 3	6, 2

Suppose after $n=40$ iterations (simulations)

$$N_1 = \begin{bmatrix} T & B \\ 30 & 10 \end{bmatrix}$$

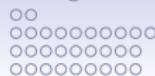
$$\hat{\pi}_1 = \begin{bmatrix} 3/4 & 1/4 \end{bmatrix}$$

$$N_2 = \begin{bmatrix} L & C & B \\ 15 & 5 & 20 \end{bmatrix}$$

$$\hat{\pi}_2 = \begin{bmatrix} 3/8 & 1/8 & 4/8 \end{bmatrix}$$

$$u_1(T, \hat{\pi}_2) = \frac{3}{8} \cdot 3 + \frac{1}{8} \cdot 1 + \frac{4}{8} \cdot 2 = \frac{9}{4}$$

$$u_1(B, \hat{\pi}_2) = \frac{3}{8} \cdot 4 + \frac{1}{8} \cdot 0 + \frac{4}{8} \cdot 6 = \frac{18}{4} \quad \leftarrow \text{OK}$$



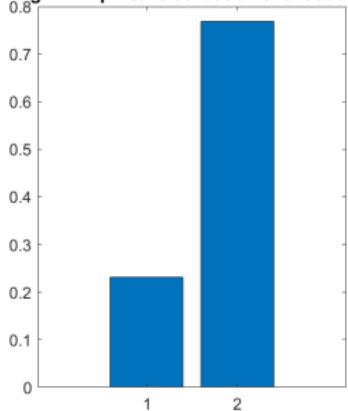
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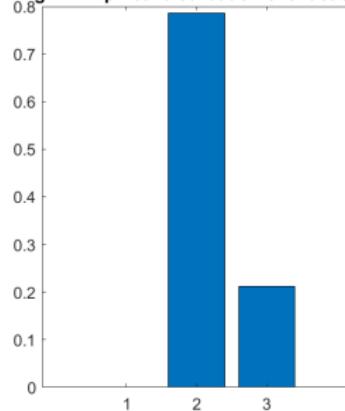
Fictitious Play: NE 2×3 example

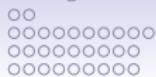
- Recall: MNE at P1 ($1/4, 3/4$) and P2 ($0, 4/5, 1/5$)
- Results for FP (300 iterations)

Ag 1: Empirical distribution over actions

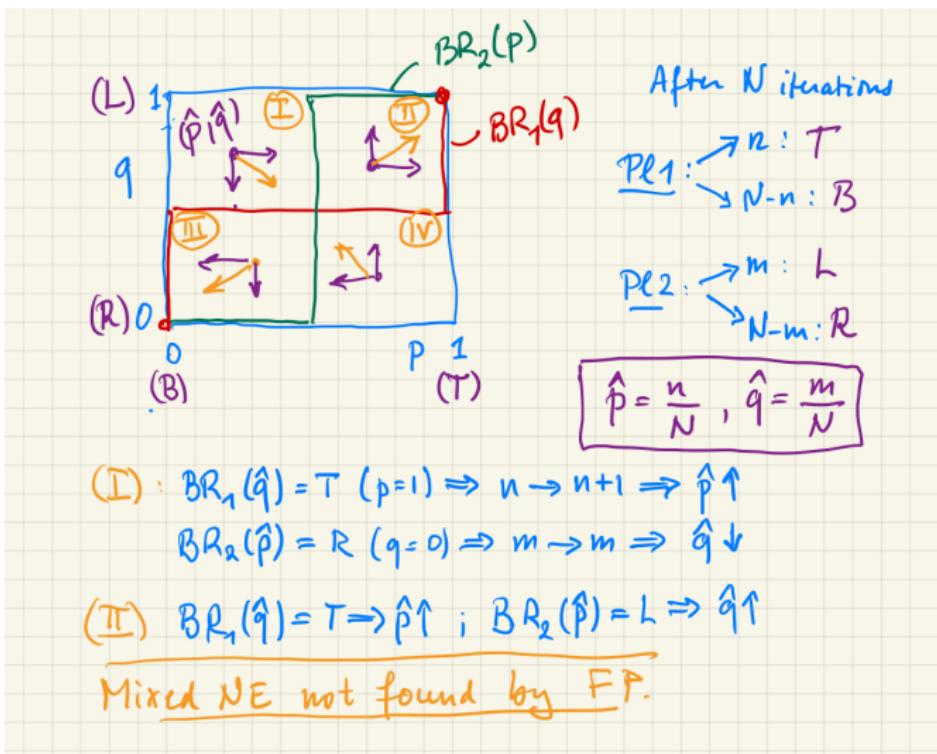


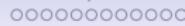
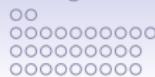
Ag 2: Empirical distribution over actions





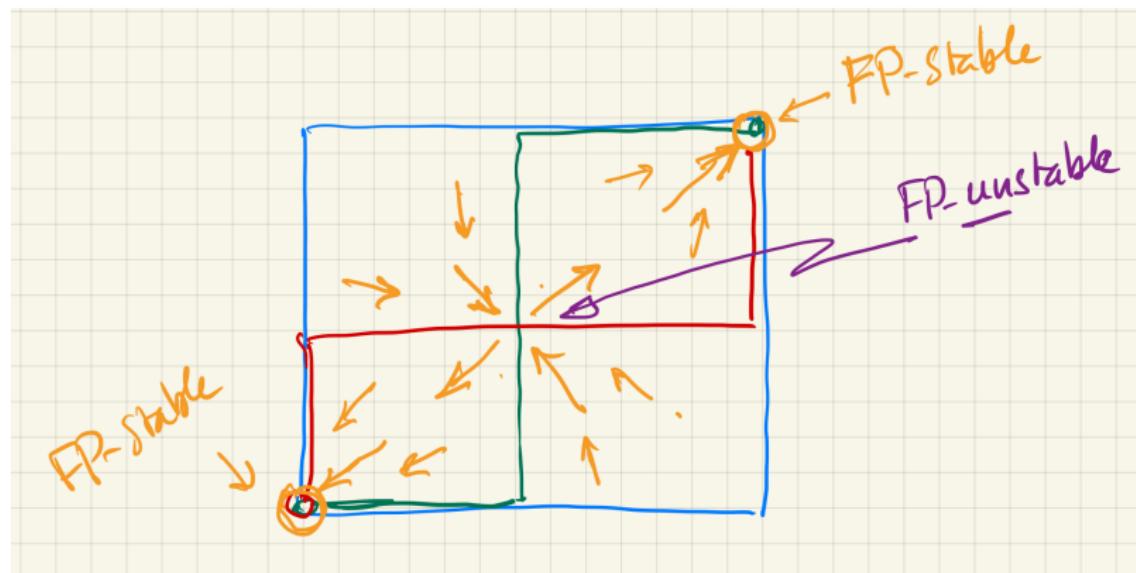
Caution: FP for 2 PNE and 1 MNE

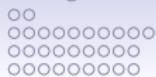




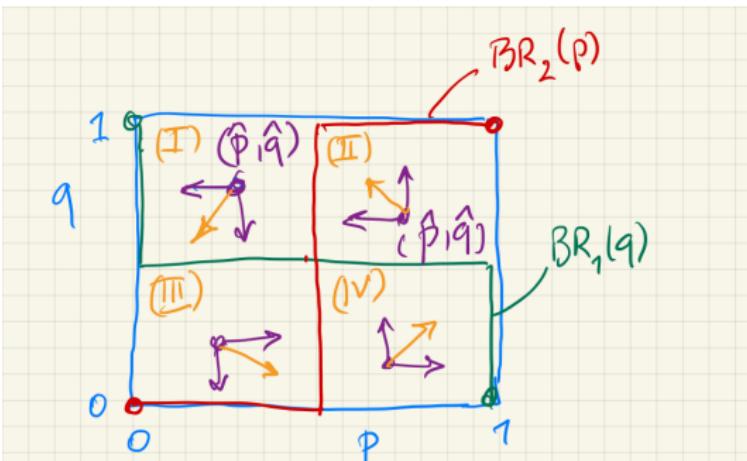
FP No PNE, 1 MNE

PNE are stable but MNE is **unstable** under FP-dynamics





FP No PNE, 1 MNE



$$\text{(I)} \quad BR_1(\hat{q}) = (p=0) \Rightarrow (n \rightarrow n) \Rightarrow \hat{p} \downarrow$$

$$BR_2(\hat{p}) = (q=0) \Rightarrow (m \rightarrow m) \Rightarrow \hat{q} \downarrow$$

$$\text{(II)} \quad BR_1(\hat{q}) = (p=0) \Rightarrow (n \rightarrow n) \Rightarrow \hat{p} \downarrow \\ BR_2(\hat{p}) = (q=1) \Rightarrow m \rightarrow m+1 \Rightarrow \hat{q} \uparrow$$



FP No PNE, 1 MNE

Cycle yields approximate empirical distribution

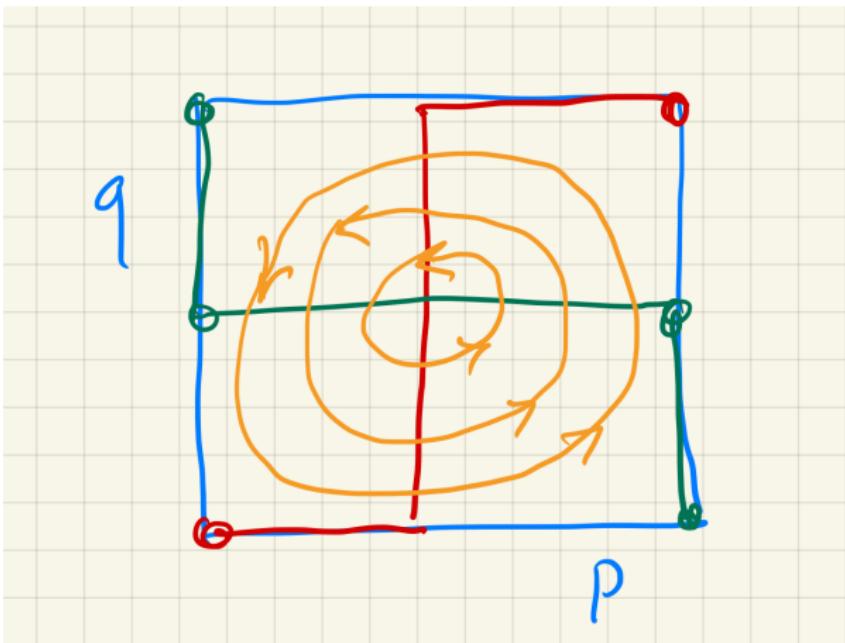


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Strategies with (weak) Guarantees

Regret minimisation

Maximin Value and Safety Strategy

Minimax Value and Punishment Strategy

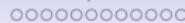
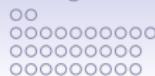
Equilibrium Concept: Nash equilibrium

Nash equilibrium: Definition

Fictitious Play for Nash Equilibrium

Further examples of Nash equilibria

Nash Equilibria: Additional notes and amplifications

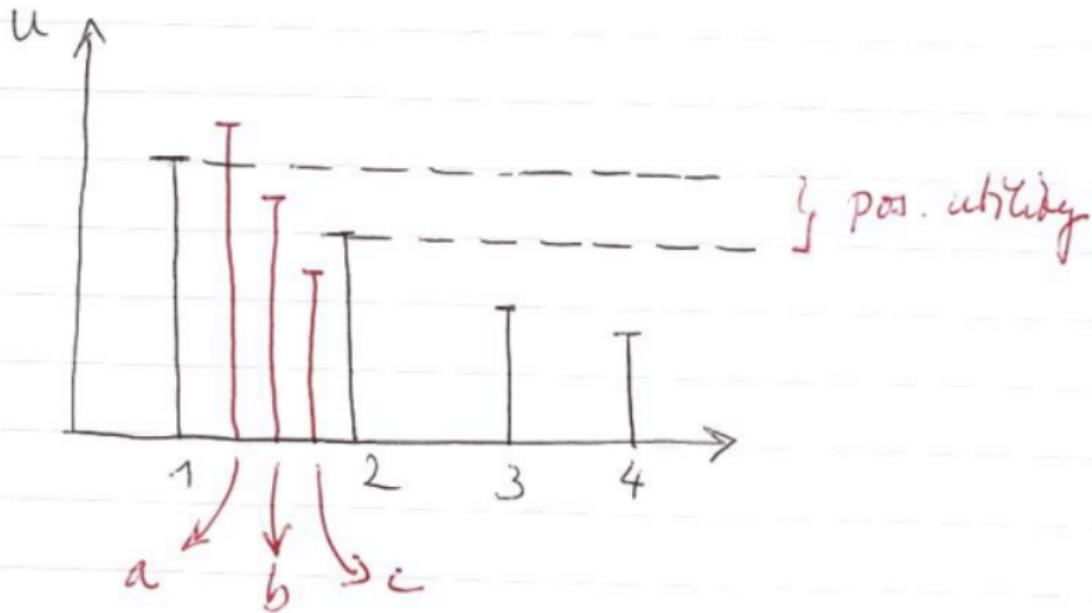


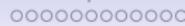
Vickrey auction: Second Price Auction

- **Vickrey Auction:**
 - n sealed bid auction for single item;
 - highest bid wins, but pays 2nd highest price;
- **Truth-telling is (weakly) dominant strategy;**
- NE: Neither winner nor loser(s) have incentive to deviate:
- **Winner:**
 - Higher: still winner, same price;
 - Lower: might lose, but if still winner, still paying 2nd price;
- **Loser:**
 - Higher: possibly winner, but at higher price;
 - Lower: still loser;
- Example of **mechanism design**.

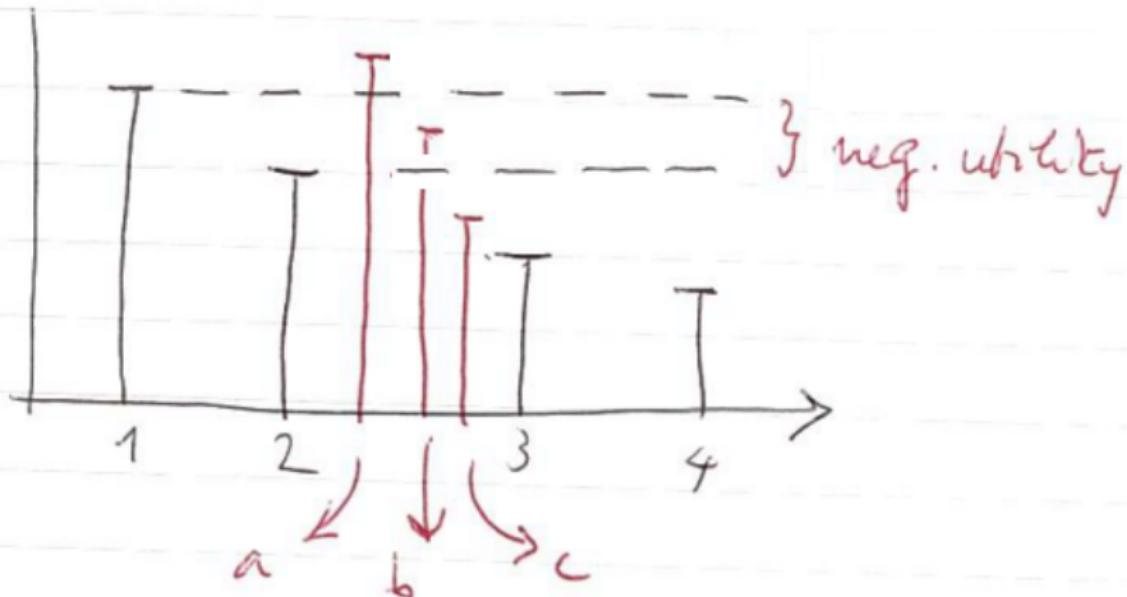


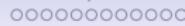
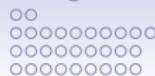
Vickrey Auction: Alternatives for winner





Vickrey Auction: Alternatives for runner-up



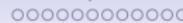


Hawk or Dove

- Equilibria as a function of **exogenous pay-off variables**;
- **Exogenous variables** are imposed on the game (not by players);
- **Strategy: Hawk or dove:**
 - Two parties are in conflict over some good of value $2v > 0$;
 - Two doves share, each gets v ;
 - Hawks fight, on average each gets half, but at a cost c (e.g. due to injury)
 - A dove is no match for a hawk and yields;
 - **Pay-off matrix:**

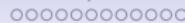
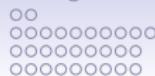
		<i>Hawk</i>	<i>Dove</i>
<i>Hawk</i>	$v - c, v - c$	$2v, 0$	
<i>Dove</i>	$0, 2v$	v, v	

- Different outcomes depending on cost of aggression c !



Hawk or Dove

- **Aggression is cheap ($c < v$)** H is dominant, hence: NE = (H,H) with utility $(v - c, v - c)$;
- **Aggression is risky ($c > v$)**
 - Two PNE = (H,D) and (D,H)
 - One MNE at $P(H) = p^* = v/c$.
 - If **risk/cost increases** $c \uparrow$, then $p^* \downarrow$.

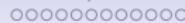
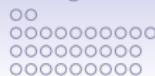


Investment game

- Many agents (n) but only two strategies;
- **Strategies:** Each agent can either
 - I: invest 10 euro
 - N: not invest
- **Pay-offs:**
 - N: No investment: pay-off = 0;
 - I: Pay-off = 15:

$$\text{net pay-off} = \begin{cases} 5 & (= 15 - 10) \quad \text{if at least 90\% of ag. invest} \\ -10 & (= 0 - 10) \quad \text{otherwise} \end{cases}$$

- **Coordination game!**



Strategic Effects

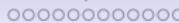
- **Symmetric penalty kick game:**

		goal keeper	
		left	right
kicker	left	0, 0	1, -1
	right	1, -1	0, 0

- Suppose kicker has weak left kick: **direct and indirect effect!**

		goal keeper	
		left	right
kicker	left	0, 0	$a, -a$
	right	1, -1	0, 0

$0 < a < 1$



Strategic Effects

- Kicker has weak left side, so is inclined to kick less with left;
- Goalkeeper knows this, so anticipates less kicks to the left, so will tend to jump less to the left ...
- Since kicker knows this, he might reconsider and kick more with left, since goalie has tendency to jump to the right ... ,
- but goalie knows this too, so might reconsider ...
- and so on ...
- Is there a way out of infinite regress???

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Strategic Effects

goalie

	(q)	$(1-q)$
L	$0, 0$	$a, -a$
R	$1, -1$	$0, 0$

$0 < a < 1$.

striker

$$EU_2(p, L) = 0 \cdot p + (-1)(1-p) = p - 1 \quad \} \text{ goalie.}$$

$$EU_2(p, R) = -ap + 0(1-p) = -ap. \quad \}$$

$$EU_1(L, q) = 0 \cdot q + a(1-q) = a(1-q) \quad \} \text{ striker}$$

$$EU_1(R, q) = q + 0 = q$$

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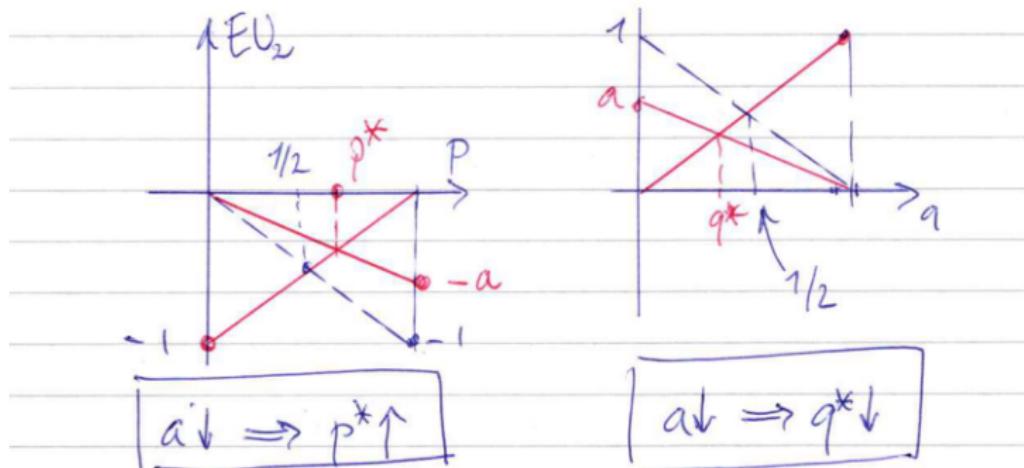
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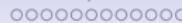
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Strategic Effects

	L	R
(p)	$0, 0$	$a, -a$
$(1-p)$	$1, -1$	$0, 0$

Striker.





Strategic Effects

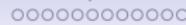
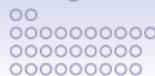
- Mixed Nash equilibrium at:

$$p^* = \frac{1}{1+a} = 1 - q^* \quad \Rightarrow \quad p^* > 1/2, \quad q^* < 1/2$$

- Zero-sum game: Utilities for kicker and goalie:

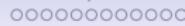
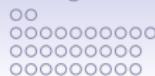
$$EU_1(p^*, q^*) = \frac{a}{1+a} = -EU_2(p^*, q^*)$$

- Notice that $p^* \rightarrow 1$ as $a \rightarrow 0$, i.e. if left kick is powerless ($a \approx 0$), make sure to kick left ($p^* \approx 1$). Is NE the right tool to think about this??



Homework dilemma: TA versus Wikipedia

- **Student:** either is honest (H), or cheats, copying Wikipedia (W), which means that the homework is solved with much less effort. However, there is a severe penalty if caught.
- **TA:** either trusts the students (no check: N) or conducts a thorough Wikipedia check (C). The latter requires considerably more effort from the TA!



Homework dilemma: TA versus Wikipedia

		Student			
		$H(q)$		$W(1-q)$	
TA	N (P)	2 —	0 —	-1 —	5 —
	C (1-P)	1 —	1 —	5 —	-10 —

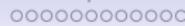
Mixed NE:

$$p^*: 0p + 1(1-p) = 5p + (-10)(1-p)$$

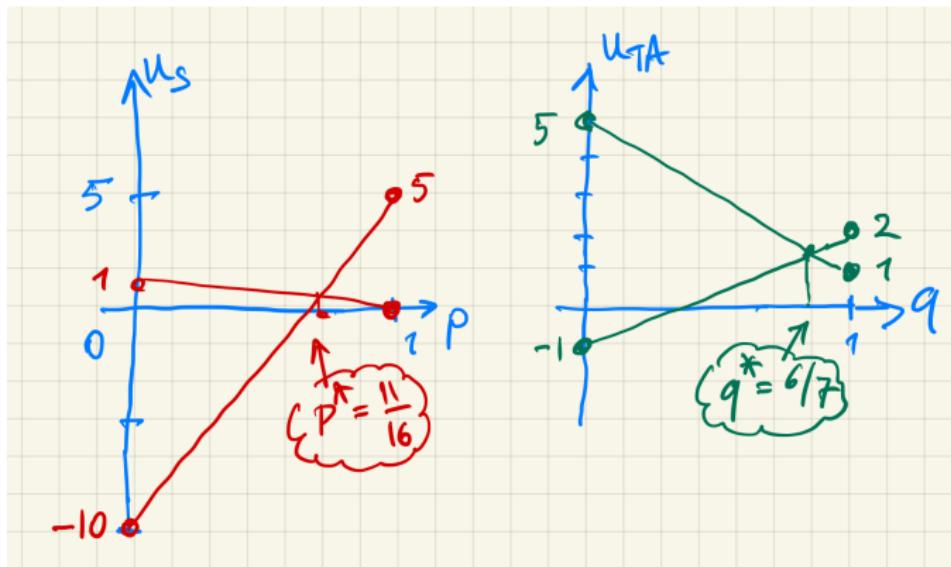
$$1-p = 5p + 10p - 10 \Rightarrow p^* = 11/16$$

$$q^*: 2q + (-1)(1-q) = 1q + 5(1-q) \Rightarrow q^* = 6/7$$

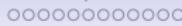
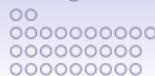
- Interpretation p^* : TA's actual randomisation probability.
- Interpretation q^* : Fraction of student population that cheats



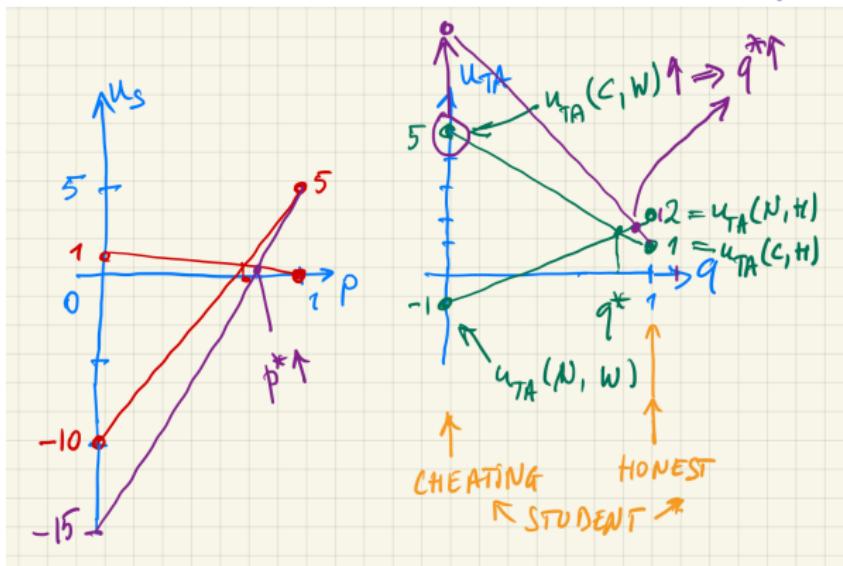
Homework dilemma: TA versus Wikipedia



- How to increase the fraction q^* of honest students?
 - Increase penalty for cheating?
 - Increase pay-off for TA for catching cheater?



Homework dilemma: TA versus Wikipedia



- Increase penalty for cheating $p^* \uparrow$ i.e. **less checking** by TA.
- Increase TA's pay-off for catching cheater: $q^* \uparrow$, i.e. **fewer students cheat!**

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Strategies with (weak) Guarantees

Regret minimisation

Maximin Value and Safety Strategy

Minimax Value and Punishment Strategy

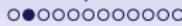
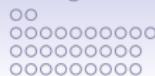
Equilibrium Concept: Nash equilibrium

Nash equilibrium: Definition

Fictitious Play for Nash Equilibrium

Further examples of Nash equilibria

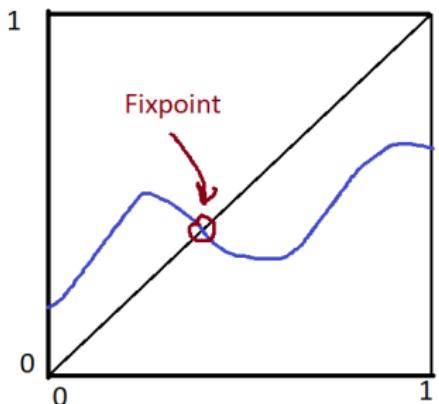
Nash Equilibria: Additional notes and amplifications



Aside: Sperner's Lemma (1928) and fix-point theorems

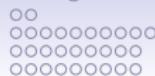
Defintion: Fix-point

Point $a \in A$ is a **fix-point** for function $f : A \rightarrow A$, iff $f(a) = a$.



$f: [0,1] \rightarrow [0,1]$ continuous

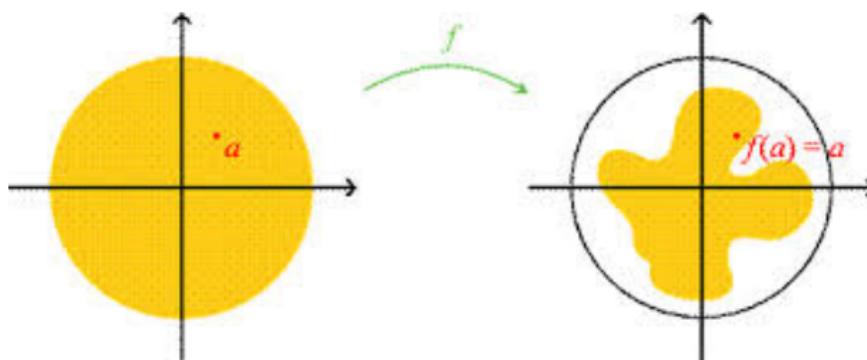
Sometimes(!), fixpoints can be computed using **function iteration**.



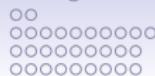
Nash Theorem is based on Fixed-Point Theorem

Brouwer's Fixed Point Thm

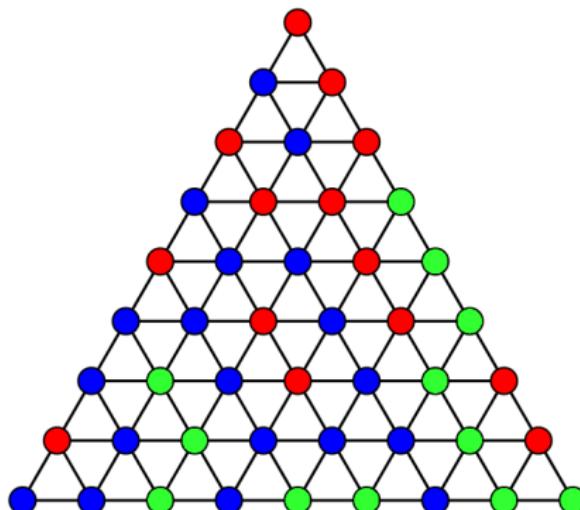
Let $K \subset \mathbb{R}^n$ be a compact and convex, and $f : K \rightarrow K$ continuous. Then f has a fix-point in K , i.e. there exists a $x_0 \in K : f(x_0) = x_0$.



: Non-constructive existence proof!



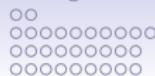
Aside: Sperner's Lemma (1928) and fix-point theorems



If in triangle:

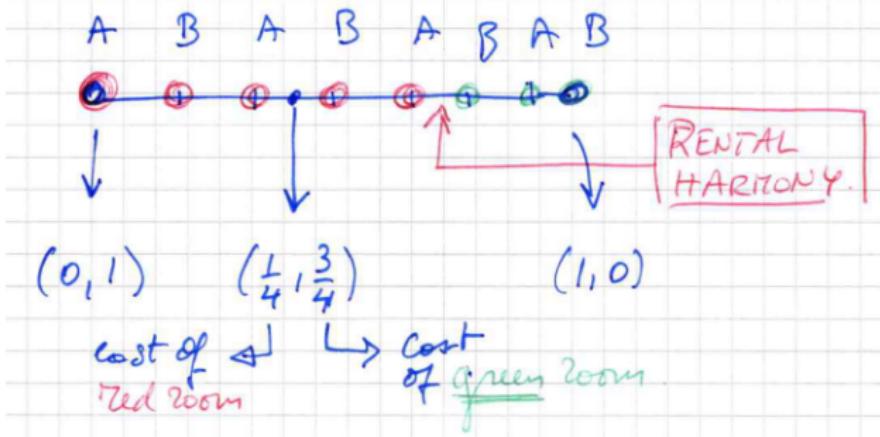
- corner nodes have different colors
- nodes on outer edges have two possible colors (determined by corners)

Then there is a subtriangle with 3 differently colored corners.



Sperner's lemma: Rental Harmony: Fair division of rent

Room 1 = red Room 2 = green

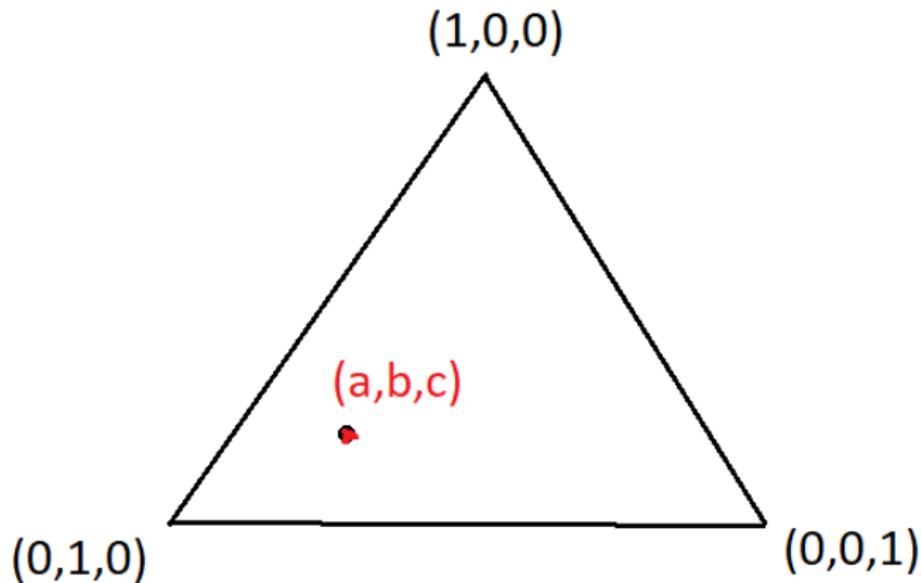


- Divide interval in segments by adding points
- Assign alternating decision makers (A and B)
- Decision maker decides which room he picks (for given rental division)



Sperner's lemma – Application 1: Rental Harmony

Flat sharing: three bedroom flat shared by three friends but rooms are not of equal quality: how much should each contribute to rent?



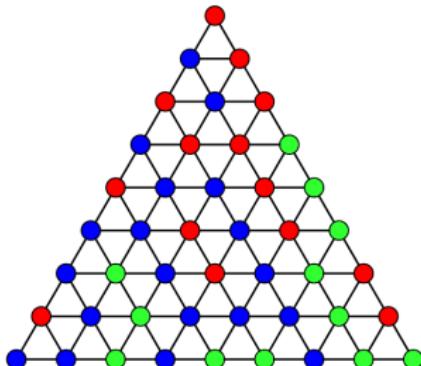
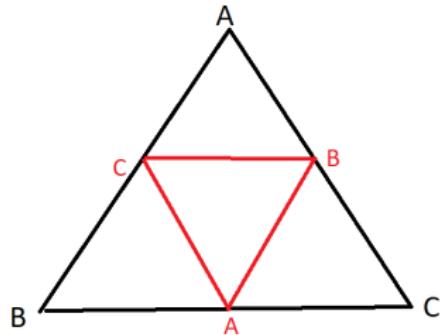
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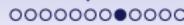
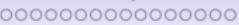
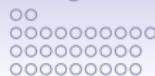
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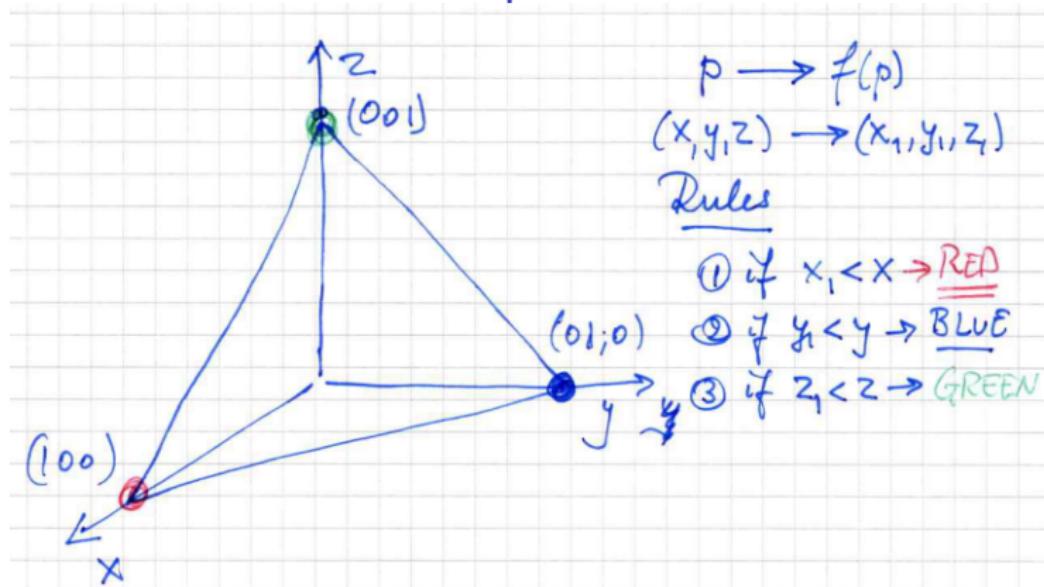
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Sperner's lemma – Application 1: Rental Harmony

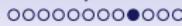
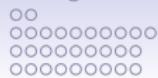




Sperner's lemma – Application 2: Proof of Brouwer's fixpoint thm



Youtube: Trefor Bazett: A beautiful combinatorial proof of the Brouwer Fixed Point Theorem - Via Sperner's Lemma

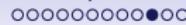


Nash equilibrium: NE and IEWDS

Eliminating **weakly dominated** strategies might erase NEs!

	<i>L</i>	<i>R</i>
<i>U</i>	2, 3	4, 3
<i>D</i>	3, 3	1, 1

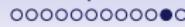
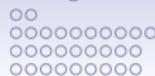
- Two NEs: (4,3) and (3,3)
- L **weakly** dominates R;
- Eliminating R would result in single solution (3,3);
- Notice that the **Pareto-optimal NE would be eliminated.**



Games with NO Nash Equilibrium

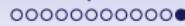
- **Dollar auction:** Sealed bid auction: highest bid gets item, 2nd highest pays this amount!
- More generally: One can construct games without NE by making sure that either
 - state space is **not compact**
 - utility function is **not continuous**
- **Ex. for 2-player games with continuous state spaces:**
 - Non-compact: $u_i(x, y) = xy \quad \text{for } 0 \leq x, y < 1$
 - Non-continuous: $0 \leq x, y \leq 1$ and

$$u_i(x, y) = \begin{cases} xy & \text{if } x, y < 1 \\ 0 & \text{if } x, y = 1 \end{cases}$$



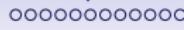
Nash equilibrium: Computational aspects

- Finding a **Nash equilibrium** for **2-player zero-sum** games can be done efficiently by formulating a linear program. Notice that in this case: NE = minimax = maximin.
- Finding a Nash equilibrium is not known to be NP-complete because it is not a decision problem
- PPAD (polynomial parity argument, directed version) is a class describing problems for which a solution always exists
- Daskalakis, Goldberg, and Papadimitriou showed that finding a sample **Nash equilibrium** of a **general-sum finite game** with two or more players is **PPAD-complete** (i.e. “difficult!”).



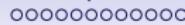
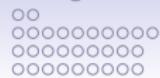
Other solution concepts

- **Minimax equilibrium:** zero-sum special case of Nash
- **Trembling-hand perfect equilibrium:** each player's action is a best-response even if other players make small mistakes
- **ϵ -Nash equilibrium:** deviating benefits no agent more than ϵ
- **Correlated equilibrium:** agents can condition strategy on external signal: "If there is intelligent life on other planets, in a majority of them, they would have discovered correlated equilibrium before Nash equilibrium." -Myerson
- **Evolutionary stable state:** "A population is said to be in an evolutionarily stable state if its genetic composition is restored by selection after a disturbance, provided the disturbance is not too large." -Maynard Smith



Summary

- Game theory studies utility-based multiagent decision making.
- Solving a game means trying to predict its outcome.
- Rational agents never play strictly dominated actions.
- No agent has an incentive to **unilaterally deviate** from a **Nash equilibrium**.
- In finite games, there's always a NE (possibly in mixed strategies).
- Nash equilibria need not be Pareto optimal.



Some additional literature

- Presh Talwalkar: The Joy of Game Theory: An introduction to strategic thinking.
- Avinash K. Dixit, Barry J. Nalebuff: The Art of Strategy: A Game Theorist's Guide to Success in Business and Life
- William Poundstone: Prisoner's Dilemma: John von Neumann, Game Theory, and the Puzzle of the Bomb