Multi-Agent Systems

Homework Assignment 2

MSc AI, VU

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2 Game Theory: Nash equilibrium

2.1 Dining out

Alice and Bob are going to dinner and plan to split the bill evenly no matter who orders what. There are two meals, a cheap one (C) priced at 10 Euro, which gives each of them 12 Euro's worth of pleasure, and an expensive dinner (E) dinner priced at 20 Euro, which gives them each 18 Euro's worth of pleasure.

1. Write down the pay-off matrix.
2. Assuming that they both order simultaneously and without coordinating, what will they order and why?
3. Alice is quite the romantic type and gets an additional s Euro's worth of pleasure if they happen to pick the same meal (either both cheap or both expensive). Bob, on the other hand, is a bit of a contrarian and gets an additional amount of pleasure (also equivalent to s Euro) when they happen to favour different meal choices. Assume that $0 < s \leq 2$. How does this change the pay-off matrix and the Nash equilibrium (or equilibria) of this game?

2.2 Hawk versus Dove

Two animals are in conflict over some resource worth $v > 0$. Simultaneously, they choose whether to behave like hawks (H) or doves (D). Hawks are willing to fight over the good, whereas doves are not. So if one animal chooses hawk and the other dove, the hawk gets everything leaving nothing for the dove. If both behave like doves, they split the resource equally. If however, both adopt an hawk strategy, they fight and on average get half of the food. The fighting however comes at a cost ($c \leq v$) to both of them.

Questions

- Write down the pay-off matrix for this game.
- Determine the Nash equilibria for this game and discuss how they change as the cost of aggression (c) increases. Do your results make sense?

2.3 Investment in recycling

Two neighbouring countries, $i = 1, 2$, simultaneously choose how many resources r_i (in hours) to spend in recycling activities. The average benefit (for country i) *per hour spent* equals

$$b_i(r_i, r_j) = 10 - r_i + \frac{r_j}{2}$$

Notice that country i's average benefit is increasing in the resources that neighbouring country j spends on his recycling because a clean environment produces positive external effects on other countries. The (opportunity) cost per hour for each country is 4. Hence, the expected utility for country i equals:

$$u_i(r_i, r_j) = b_i(r_i, r_j)r_i - 4r_i.$$

Questions

1. Determine each country's best-response function.
2. Indicate the pure strategy Nash Equilibrium (r_1^*, r_2^*) on the graph;
3. On your previous figure, show how the equilibrium would change if the intercept of one of the countries' average benefit functions b_i fell from 10 to some smaller number. What would this mean for the recycling efforts of both countries?

2.4 Tragedy of the Commons

- n players sharing some common resource (of total size 1)
  - E.g., village green, bandwidth in network, etc.
- Each player i would like to have as big a share $(0 \leq x_i \leq 1)$ as possible!
- However, each player's utility (pay-off) depends on what the others do:

$$u_i(x_i, x_{-i}) = \begin{cases} x_i\,(1 - \sum_{j=1}^{n} x_j) & \text{if } \sum_j x_j < 1 \\ 0 & \text{otherwise} \end{cases}$$

  One way to interpret this utility is that in order to have maximum utility there has to be sufficient (unused) "slack" to accommodate small fluctuations. Think of a highway: initially more cars means more throughput, but at some point the increase in density starts to hamper throughput.

Questions

- Consider the special case where there are only two players (i.e. $n = 2$). Determine the individual shares x_1 and x_2 in the Nash equilibrium for this game.
- Does this Nash equilibrium optimise social welfare which is the aggregated utility of all players (i.e. $\sum_{i=1}^{n} u_i(x)$)?
- Can you generalise this result to arbitrary n?

SOLUTIONS

1 Game Theory: Concepts and Nash Equilibrium

1.1 Dining out

(1) Pay off matrix.

C = Cheap

E = expensive.

	C	E
C	2, 2	−3, 3
E	3, −3	−2, −2

Computation of pay off

$$u_1(C,E) = \underbrace{-\frac{10+20}{2}}_{\text{Cost}} \underbrace{+12}_{\text{value}} = -3$$

etc

(2) Nash eq. 1PNE : (E,E) both take expensive.

→ C is strictly dominated.

Figure 1: Dining out: pay-off matrix and corresponding Nash eq.

③ New payoff matrix $(0 < s \leq 2)$

		Bob: C q	E $(1-q)$
Alice p	C	$2+s, 2$	$-3, 3+s$
$1-p$	E	$3, s-3$	$s-2, -2$

Figure 2: New pay-off matrix

Case 1: $0 < s < 1$ C(heap) is still strictly dominated, and therefore the Nash eq. remains unchanged: $NE = (E, E)$

Case 2: $1 < s \leq 2$

Now there are no pure NE, but there is a single mixed NE:

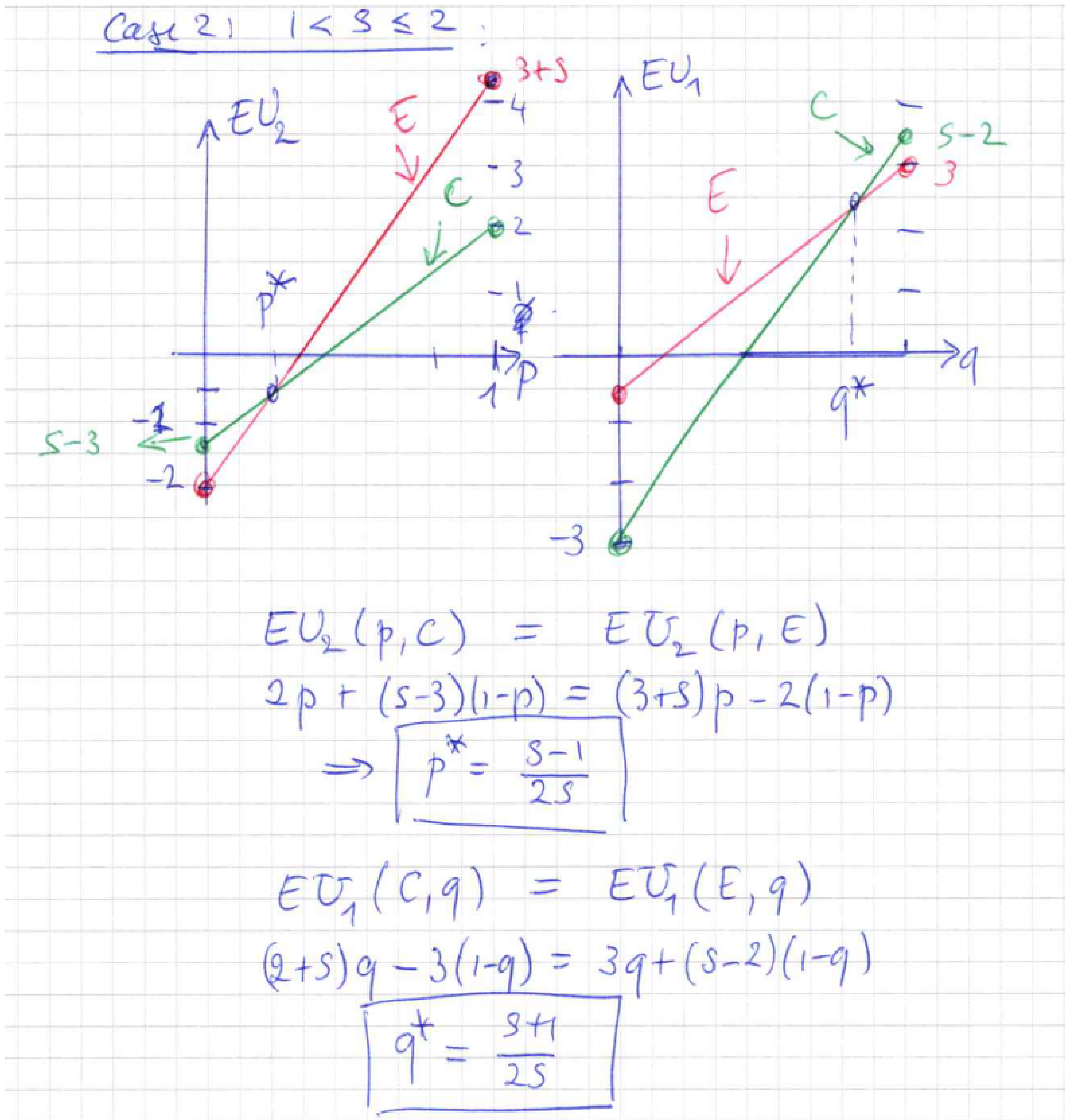


Figure 3: Mixed NE for case 2: $1 < s \leq 2$

Case 3: $s = 1$ In this we can write down the pay-off matrix explicitly:

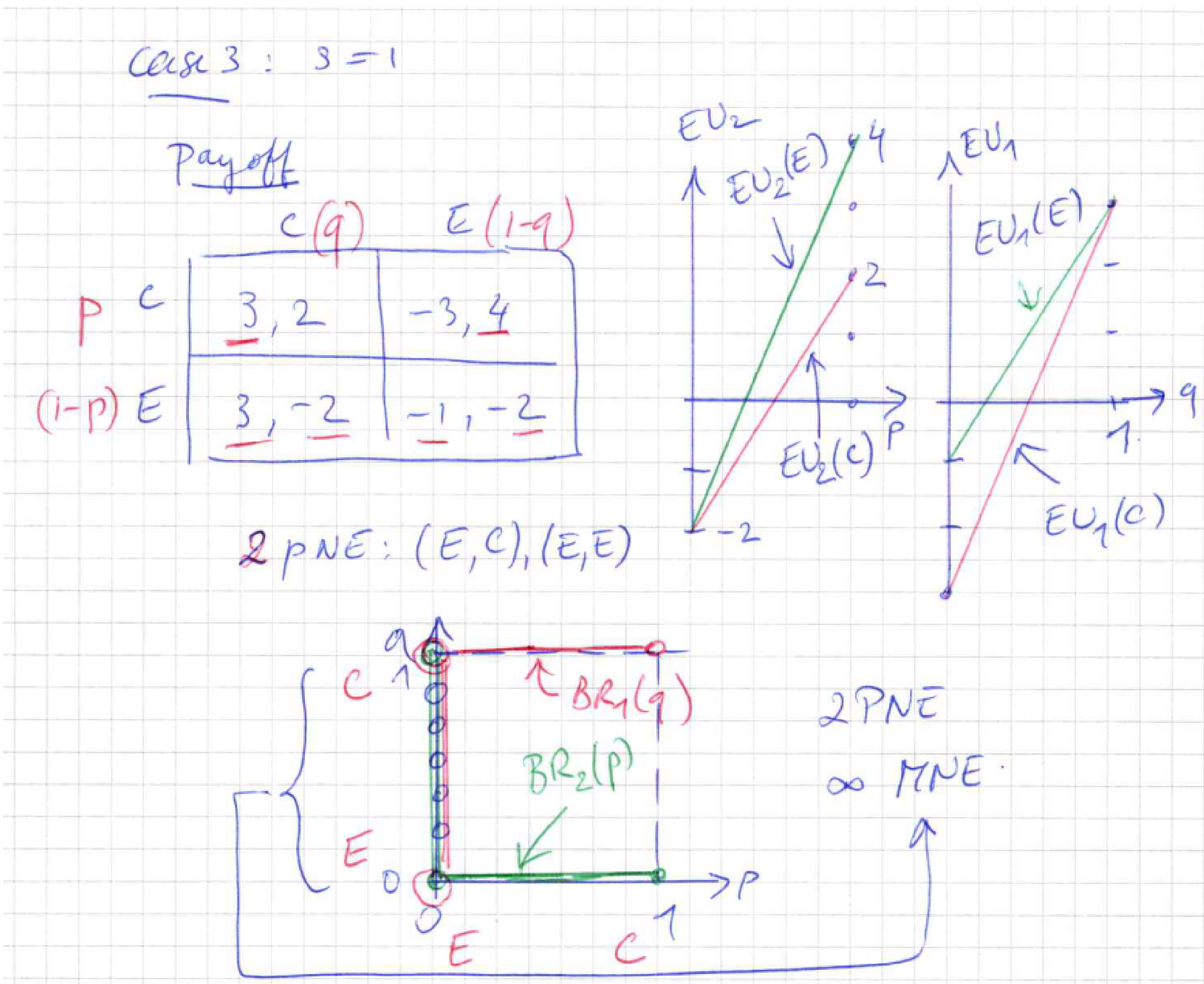


Figure 4: Pay-off matrix, utility function and best-response graphs for $s = 1$. Notice that in this case there are an infinite number of mixed NE in addition to the two PNEs.

Note on the infinite number of NE

- **Value for** p^* To make the column player indifferent, the row player has to pick $p^* = 0$ (see top, middle diagram in Fig, 4). But that means that the row player is playing E, in which case the column player is indifferent between his two options, and could just as well play any mix, hence: $0 \leq q^* \leq 1$. For any other value $p > 0$, the BR_2 is to play E (i.e. $q = 0$). So
$$BR_2(p) = \{(p = 0, 0 \leq q \leq 1\} \bigcup \{(0 < p \leq 1, q = 0)\} .$$

- Similar for q^*

1.2 Investment in recycling

Let $i, j \in \{1, 2\}$ but $i \neq j$. For a given investment of resources r_i, r_j the unit benefit for country i equals:

$$b_i(r_i, r_j) = 10 - r_i + r_j/2$$

with corresponding utility:

$$u_i(r_i, r_j) = b_i(r_i, r_j)r_i - 4r_i.$$

Best response Given r_2, the best response for country 1 ($r_1^* = BR_1(r_2)$) is obtained by setting the first derivative to zero:

$$\begin{aligned} 0 = \frac{\partial u_1}{\partial r_1} &= 10 - r_1 + r_2/2 + (-1)r_1 - 4 \\ \implies & \quad r_1^* := BR_1(r2) = 3 + r_2/4 \end{aligned}$$

Similarly:

$$r_2^* = 3 + r_1/4.$$

The NE is obtained by combining both best responses:

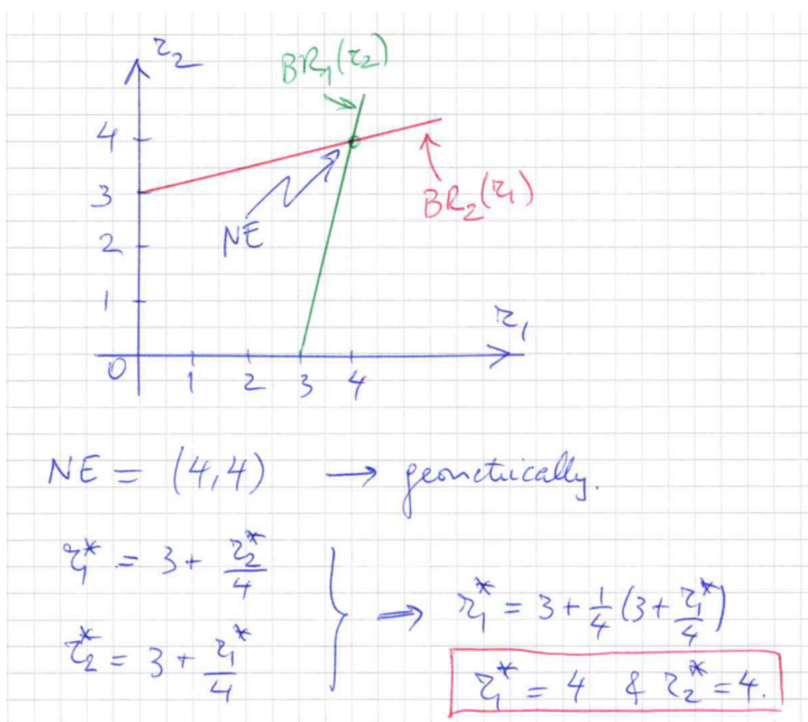


Figure 5: Best response functions and Nash equilibrium for recycling problem.

Next, assume that the benefit $b_2^{new} = 8 - r_1 + r_2/2$, then it follows that

$$BR_2(r_1) = r_1^* = 2 + r_1/4$$

i.e. same slope but lower intercept. Hence, effort of country 1 reduces slightly, but effort of country 2 reduces substantially.

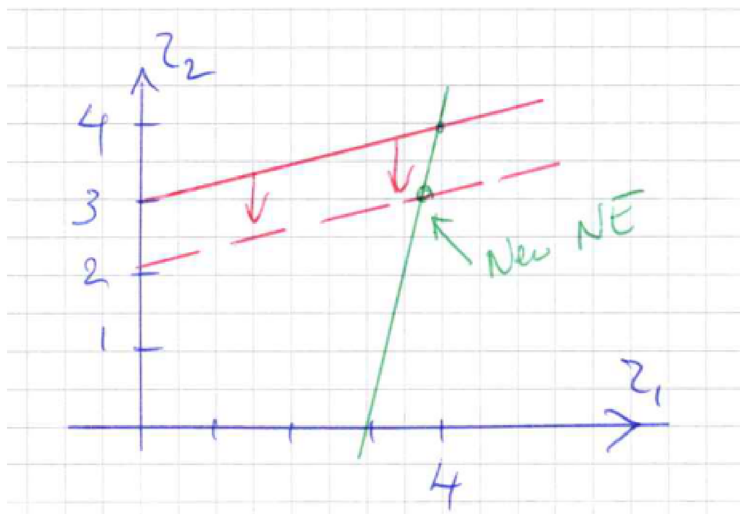


Figure 6: Effect of new benefit for country 2 on Nash equilibrium.

1.3 Tragedy of the Commons

1. shared resource (eg: shared bandwith, common green) $\longrightarrow$ total capacity $= 1$

2. $\#$Agents $= n$

3. Each agent can choose a strategy, determining the amount/fraction x_i of the resource he wants to consume:

$$0 \leq x_i \leq 1$$

   We denote the corresponding strategy profile by $\mathbf{x} = (x_1 \dots x_n)$.

4. Pay off: (for agent i)

$$u_i(\mathbf{x}) = \begin{cases} 0 & \text{if} \quad \sum_{j=1}^{n} x_i \geq 1 \\ \underbrace{x_i}_{\substack{\text{proportional} \\ \text{to own} \\ \text{share}}} \quad \underbrace{(1 - \overbrace{\sum_{j=1}^{n} x_j}^{\text{includes i}})}_{\substack{\text{proportional to} \\ \text{slack capacity}}} & \text{otherwise} \end{cases} \tag{1}$$

1.4 Solution for two players ($n = 2$)

$$u_1(x_1, x_2) = \begin{cases} 0 & \text{if} \quad x_1 + x_2 \geq 1 \\ x_1(1 - (x_1 + x_2)) & \text{if} \quad x_1 + x_2 < 1 \end{cases} \tag{2}$$

$$u_1(x_1,x_2) = \begin{cases} 0 & \text{if} \quad x_1+x_2 \geq 1 \\ x_1((\underbrace{1-x_2}_{a})-x_1) & \text{if} \quad x_1+x_2 < 1 \end{cases} \tag{3}$$

$$u_1(x_1,x_2) = \begin{cases} 0 & \text{if} \quad x_1+x_2 \geq 1 \\ x_1(a-x_1) & \text{if} \quad x_1+x_2 < 1 \end{cases} \tag{4}$$

Best response: Given x_2, what's the best response for agent 1 ($x_1^* = BR_1(x_2)$)? To this end we compute the derivative with respect to x_1 and equate it to zero:

$$\begin{aligned} \frac{\partial}{\partial x_1} u_1(x_1,x_2) &= \frac{\partial}{\partial x_1}(x_1(1-x_2-x_1)) && (5) \\ &= 1-(x_1+x_2)+x_1(-1) && (6) \\ &= 1-2x_1-x_2 = (1-x_2)-2x_1 && (7) \end{aligned}$$

Hence:

$$\frac{\partial}{\partial x_1} u_1(x_1,x_2) = 0 \quad \Longrightarrow \quad x_1^* := BR_1(x_2) = \frac{1-x_2}{2} \tag{8}$$

Similarly:

$$x_2^* := BR_2(x_1) = \frac{1-x_1}{2}. \tag{9}$$

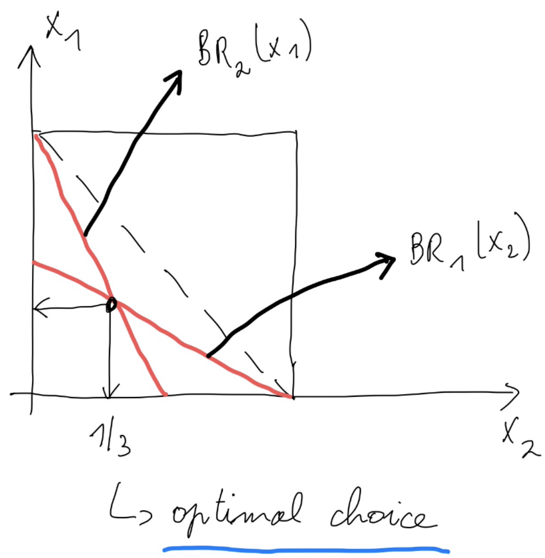


Figure 7: BR graphs for $n = 2$

Nash equilibrium In the Nash equilibrium, both agents play best response to one another. Hence, we can substitute eq. (9) into eq.(8) to obtain:

$$x_1^* = \frac{1-(1-x_1^*)/2}{2} \quad \Longrightarrow \quad x_1^* = 1/3 \quad (= x_2^* \quad \text{due to symmetry}).$$

1.5 General solution

- n agents $\longrightarrow$ division $(x_1 \ldots x_n) = \mathbf{x}$
- pay off for agent i:

$$u_1(x_1 \ldots x_n) = u_i(\mathbf{x}) = \begin{cases} 0 & \text{if } \sum x_i \geq 1 \\ x_i(1 - \sum_{j=1}^{n} x_j) & \text{otherwise} \end{cases} \tag{10}$$

Best response:

- Denote strategy profile: $\mathbf{x} = (x_1 \ldots x_n) \equiv (x_i, x_{-i})$
- Best response of agent i to choices for all other agents (i.e. x_{-i})
- **Optimum:**

$$\begin{aligned}
\frac{\partial}{\partial x_i} u_i(x_i, x_{-i}) &= \frac{\partial}{\partial x_i}\left(x_i(1 - \sum_{j=1}^{n} x_j)\right) \\
&\text{denote, } S_{-i} = \sum_{j \neq i} x_j \\
&= \frac{\partial}{\partial x_i}\left[x_i(1 - (x_i + \underbrace{S_{-i}}_{\text{independent of } x_i}))\right] \\
&= (1 - (x_i + S_{-i})) + x_i(-1) \\
&= 1 - 2x_i - S_{-i}
\end{aligned}$$

- **Best response:**

$$\begin{aligned}
0 &= 1 - 2x_i^* - S_{-i} \\
&\Downarrow \\
x_i^* &= \frac{1 - S_{-i}}{2}
\end{aligned}$$

Nash equilibrium:

$$\begin{aligned}
\text{Because of symmetry:} \quad & x_1^* = x_2^* = \ldots = x_n^* \\
\text{Hence} \quad S_{-i}^* & = (n-1)x_i^* \\
\text{from which:} \quad x_i^* & = \frac{1 - S_{-i}^*}{2} \qquad \Downarrow
\end{aligned}$$

$$2x_i^* = 1 - (n-1)x_i^*$$

$$\boxed{x_i^* = \frac{1}{n+1}}$$

Maximal social welfare:

$$\text{Social welfare} = \sum_{i=1}^{n} u_i(\mathbf{x}) = \sum_{i=1}^{n} x_i(1 - \underbrace{\sum_{j=1}^{n} x_j}_{=S}) = S(1-S)$$

which is maximal for $S = 1/2$.

Hence maximal social welfare if:

$$x_i = \frac{1}{2n} \qquad \forall i$$

Conclusion Whereas optimal social welfare is achieved when agents claim quantity $x_i = 1/(2n)$, the problem is that the Nash equilibrium results in strictly larger individual shares $x_i^* = 1/(n+1)$. This individual overuse results in reduced social welfare (hence *tragedy of the commons*).

2 Hawk versus Dove

2.1 Pay-off matrix for this game

	H	D
H	$\frac{v}{2} - c, \frac{v}{2} - c$	$v, 0$
D	$0, v$	$\frac{v}{2}, \frac{v}{2}$

2.2 Nash equilibria and how they change as cost of aggression (c) increases

Case 1: Low cost of aggression

$$\frac{v}{2} - c > 0 \qquad \Longrightarrow \qquad c < \frac{v}{2}$$

H is strictly dominant $\Rightarrow$ Single PNE:(H,H)

Case 2: High cost of aggression

$$\frac{v}{2} - c < 0 \qquad \Longrightarrow \qquad c > \frac{v}{2}$$

Denote $w := c - v/2, \quad (w > 0)$:

	H	D
H	$-w,\ -w$	$\underline{v},\ \underline{0}$
D	$\underline{0},\ \underline{v}$	$\frac{v}{2},\ \frac{v}{2}$

$\Rightarrow$ 2 PNE (H, D) and (D, H)

Mixed Nash Equilibrium

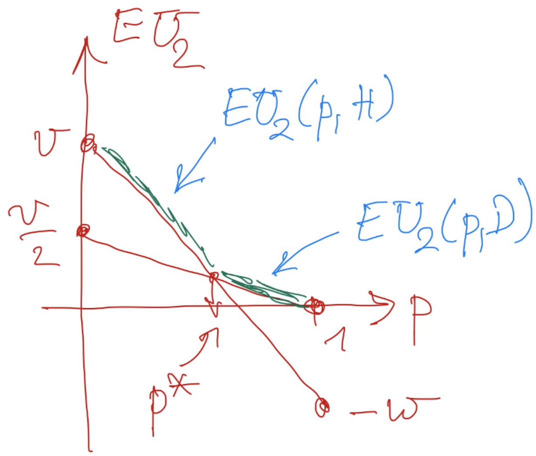
		q	1- q
		H	D
p	H	$-w,\ -w$	v,0
1-p	D	0,v	$\frac{v}{2},\ \frac{v}{2}$

$$
\begin{aligned}
\mathbb{E}\,U_2(p, H) &= -wp + v(1-p) \\
&= -(v+w)p + v \\
\mathbb{E}\,U_2(p, D) &= 0\,p + \frac{v}{2}(1-p)
\end{aligned}
$$

Indifference:

$$
\begin{aligned}
\frac{v}{2}(1-p) &= -(v+w)p + v \\
(v+2w)p &= v
\end{aligned}
$$

$$
\begin{aligned}
p^* &= \frac{v}{v+2w} = \frac{v}{v+2(c-\frac{v}{2})} = \frac{v/2}{c} \\
q^* &= p^*(\text{symmetry})
\end{aligned}
$$

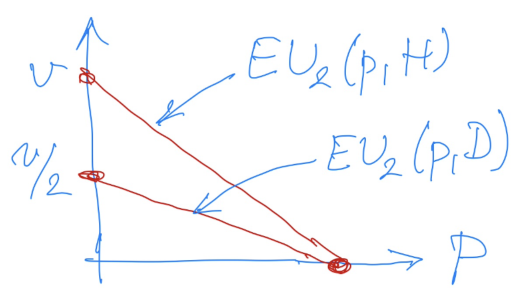


Case 3

$$\frac{v}{2} - c = 0 \qquad \Longrightarrow \qquad c = \frac{v}{2}$$

	H	D
H	$\underline{0}, \underline{0}$	$\underline{v}, \underline{0}$
D	$\underline{0}, \underline{v}$	$\frac{v}{2}, \frac{v}{2}$

H weakly dominates D.



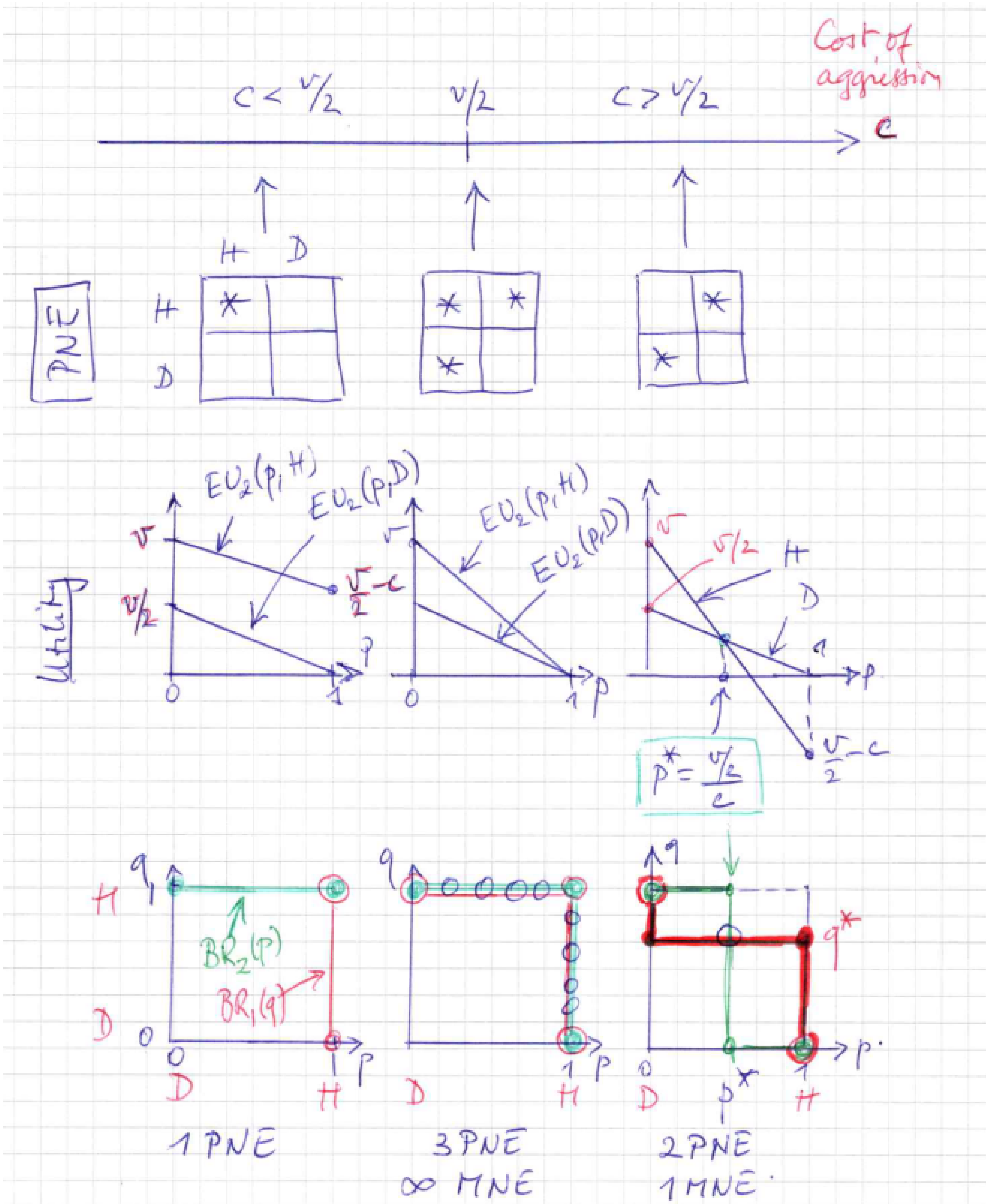


Figure 8: Evolution of NE and utilities as function of cost of aggression (c).