Introduction to Reinforcement Learning

Part 1: MDP and Bellman Equations

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Outline

What is Reinforcement Learning (RL)

- RL is one of **three main learning problems** studied in AI:
- Learning by trial and error and improving over time;
- Natural form of learning, ubiquitous in biology and psychology:


What is Reinforcement Learning?

- **Supervised learning** needs teacher (labeled data!)
- **RL learns on the job:** interact, explore, exploit experience!

What makes RL attractive and interesting?

Natural way to learn complex skills, can we copy this?

RL Example 1: Playing a song by ear

- **Actions and transitions:** Moving from key to key...
- **Immediate reward:** correct key or not;
- **Cumulative reward:** Being able to play whole song
- Relatively easy, one gets immediate valuable feedback...

RL Example 2: Cooking

- **States:**
- **Actions:** e.g. cooking, seasoning, stirring, etc.
- **Transitions:** e.g. from raw to cooked
- **Immediate reward** e.g. smell, look
- **Final reward:** enjoyable dinner
- **Problem:** what's the best sequence of actions (i.e. optimal policy)?

RL Examples: Playing Chess

- **States:** board configurations
- **Actions:** legal moves
- **Value of state/action:** very valuable to determine next move (action),
- **Final reward:** Win(2), lose(0), draw(1)
- **Problem:** what's the optimal action to take in each state? (policy)

What makes Reinforcement Learning challenging?

1. **No Oracle**
    - Supervised learning: compare estimate to correct result (provided by oracle);
    - RL: no information on optimal action, just indicative *reward*
        - i.e. evaluative feedback, not prescriptive!
2. **Sparsity of feedback**
    - RL: only data on states/actions that have been experienced: (might be very small subset of entire state space);
    - For many RL problems, no feedback until target is reached!
    - Initial stages: just random search!
3. **Data generated during training**
    - Online learning: improve policies while interacting w. environment
    - No independent data generation!

policy $\longleftrightarrow$ data generation

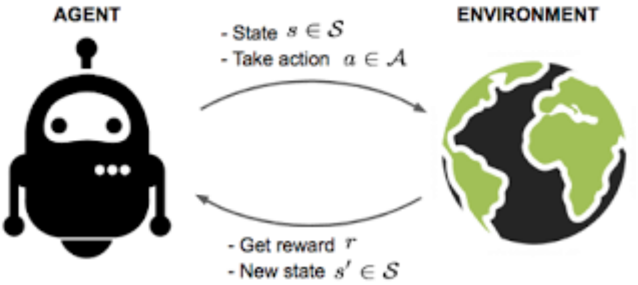
Reinforcement Learning: more abstract view

- **RL:** Learning from (sequential) interaction:
  take actions to move from state to (better!?) state;
- No labeled data, but **feedback** (reward, possibly delayed)
- **Goal:** optimise **end-result** (i.e. long-term/cumulative reward)

RL: Even more abstract version

Agent interacting with Environment:

- states(s), actions (a) and transitions: $s \xrightarrow{a} s'$ $(p(s' \mid s, a))$
- (immediate) reward $r(s, a, s')$

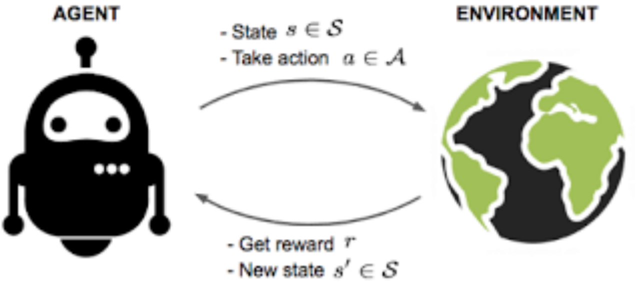


- **Example:** Taking a qualifying test (a), to pass from unqualified (s) to qualified state (s'), or not!

RL GOAL: Find optimal policy!

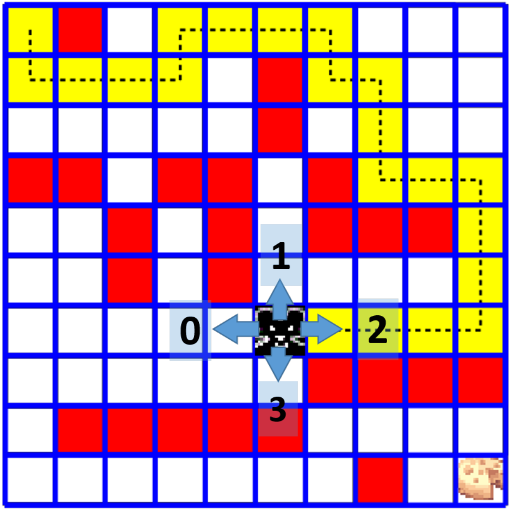
Policy π, tells for each state (s), which action (a) to take:

$$s \xrightarrow{\pi} a$$



RL Goal: Given environment, determine **optimal policy** π^*:
what **action** *to choose in* **each state** *to achieve* **best FINAL reward***?*

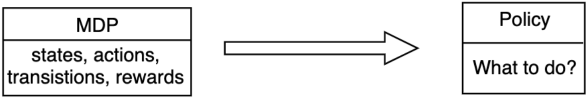
Gridworld Maze as Prototypical RL Example



Overview

MDP
states, actions, transitions, rewards

Overview



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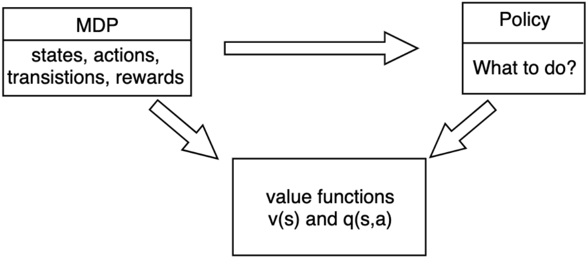
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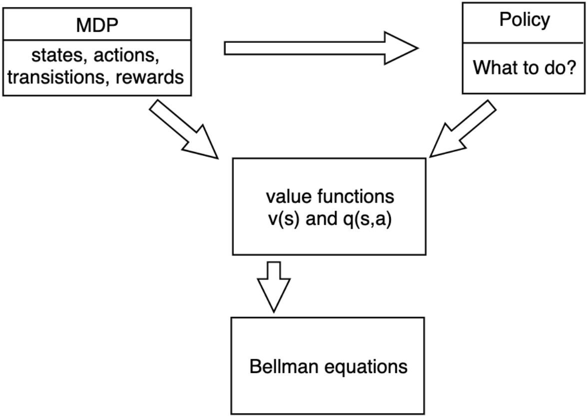
states, actions, transistions, rewards
Policy
What to do?
value functions
v(s) and q(s,a)

Overview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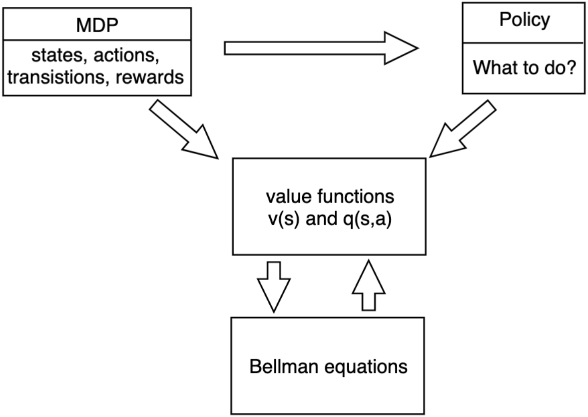
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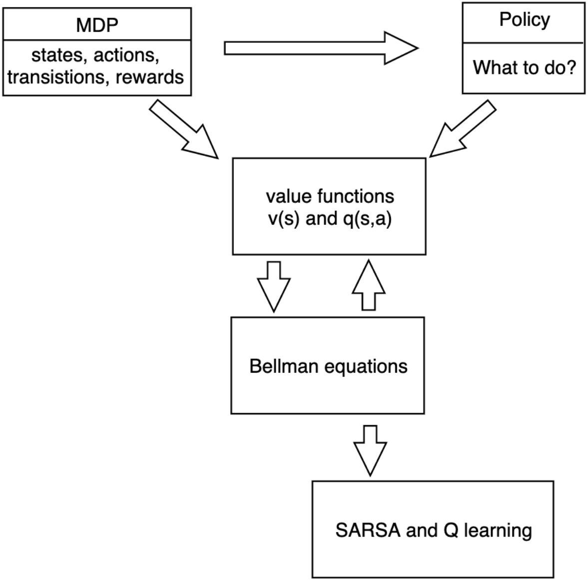
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SARSA and Q learning

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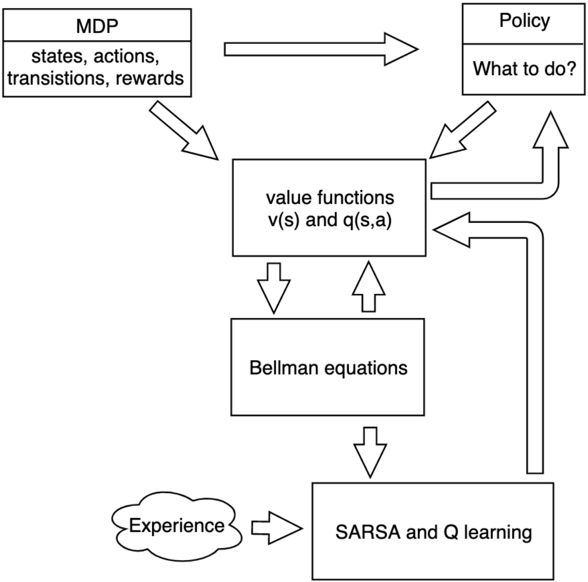
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What to do?
value functions
v(s) and q(s,a)
Bellman equations
Experience
SARSA and Q learning

Outline

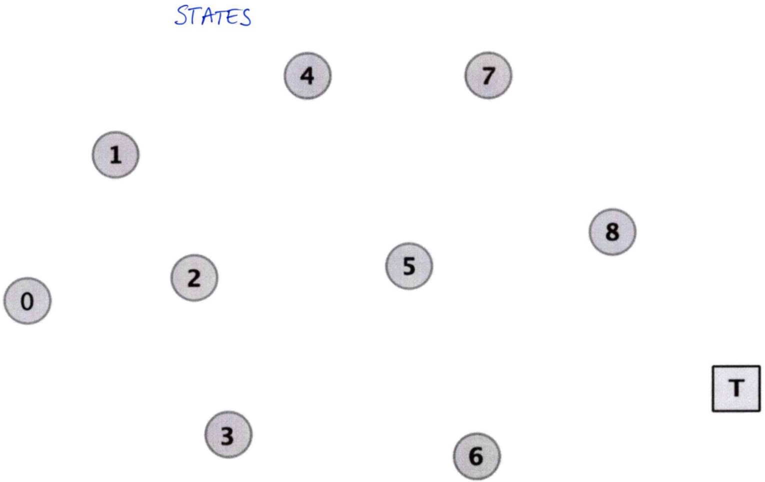
Markov Decision Process (MDP)

Mathematical formalism to capture the **essential characteristics** of the RL problem:

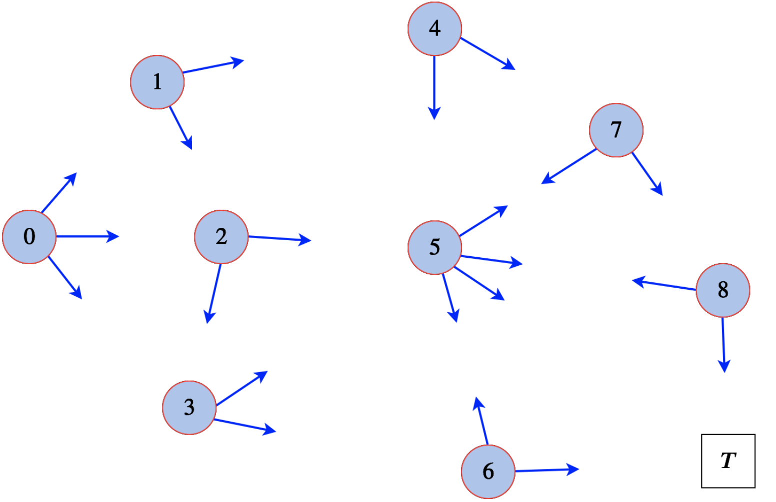
- MDP $=$ states, actions, transitions, rewards, (discount factor)

Markov Decision Process (MDP)
States

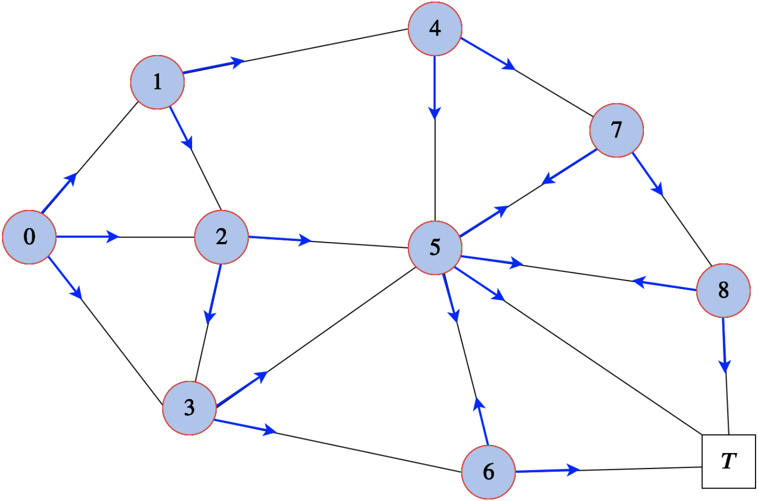
States capture the information that is available to agent



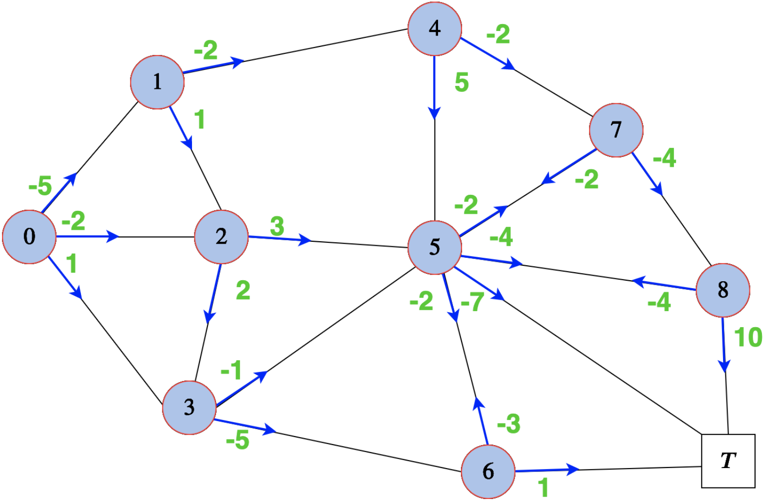
Markov Decision Process (MDP)
States + Actions



Markov Decision Process (MDP)
States + Actions + Transitions



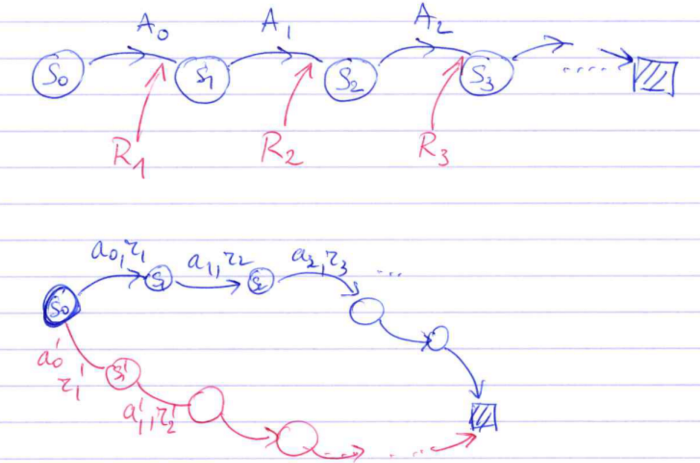
Markov Decision Process (MDP)
States + Actions + Transitions + Rewards



Markov decision processes (MDP)

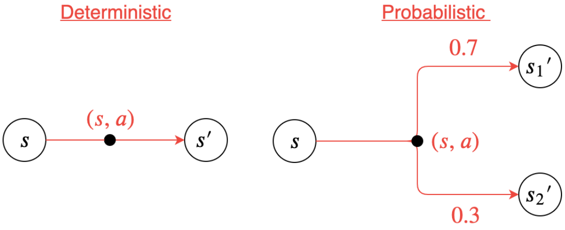
- **Markov decision processes (MDP)** provide a **formal model** for a **sequential decision problem**;
- A **finite MDP** $(\mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma)$ consists of:
  - **Discrete time** $t = 0, 1, 2, \ldots$
  - A discrete set of **states** $s \in S$ (captures relevant information)
  - A discrete set of **actions** $a \in A(s)$ for each s
  - A **transition function** $p(s'|s, a)$: probability of transitioning to state s' when taking action a at state s
    - e.g. prob of passing exam ($s \xrightarrow{a} s'$) when studying (a);
  - A **reward function** $r(s, a, s') = E[R \,|\, s, a, s']$: expected reward when taking action a at state s and transitioning to s'
    - Reward might depend on new state s' (e.g. $s' = pass$ or $s' = fail$)
  - A planning horizon H or **discount factor** γ;
    - How important are future rewards?
    - shortsighted $0 \leftarrow \gamma \rightarrow 1$ farsighted

Stochastic actions and returns: interpretation



Long-term return: $G_0(s_0) = R_1 + R_2 + \ldots + R_T = \sum_{t=1}^{T} R_t$

Detailed Definition and Notation (1)



$p(s' \mid s, a) = 1$

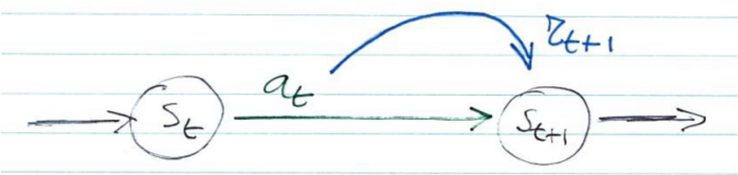
$p(s_1' \mid s, a) = 0.7$

$p(s_2' \mid s, a) = 0.3$

Transition probability to state s' when taking action a in state s :

$$p(s' \mid s, a) := P(S_{t+1} = s' \mid S_t = s, A_t = a)$$

Deterministic versus probabilistic transitions



- **Expected immediate reward** when transitioning from state s to state s' under action a:

$$r(s, a, s') := E(R_{t+1} = r \mid S_t = s, A_t = a, S_{t+1} = s')$$

The Markov property

Markov property: informally

- The present (state) has all the information necessary to predict the future: no need to keep track of the past.
- *"The future is independent of the past, given the present"*

$$P(S_{t+1}, R_{t+1} | s_t, a_t, r_t, s_{t-1}, a_{t-1}, \dots, r_1, s_0, a_0) = P(S_{t+1}, R_{t+1} | s_t, a_t)$$

If reward R_{t+1} does NOT depend on successor state S_{t+1}:

$$P(S_{t+1} | s_t, a_t, r_t, s_{t-1}, a_{t-1}, \dots, r_1, s_0, a_0) = P(S_{t+1} | s_t, a_t)$$

$$P(R_{t+1} | s_t, a_t, r_t, s_{t-1}, a_{t-1}, \dots, r_1, s_0, a_0) = P(R_{t+1} | s_t, a_t)$$

Joint versus marginal distribution

$$P(R_{t+1}, S_{t+1} \mid \ldots) = P(R_{t+1} \mid S_{t+1} \ldots) P(S_{t+1} \mid \ldots)$$

Dependence versus independence

- Action = hard work
- S_{t+1} = success or fail
- Reward R_{t+1}, could be:
  - dependent on next state: e.g. bonus if you succeed;
  - independent of next state, e.g. energy expenditure for action

Markov Property

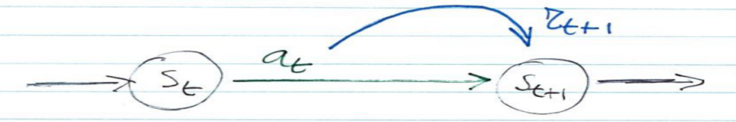
- MP simplifies the mathematics and makes it tractable;
- MP is sufficiently powerful as states can be expanded to include all the information necessary to predict future;
  - E.g.: to estimate the speed of a vehicle, we need at least two positions!

Example of Markov game

- **Game of chess**
- The current state of the board provides all information needed to determine next (optimal) move;
- No need to know the history (i.e. the path leading up to the current state)

Outline

Policies for a MDP

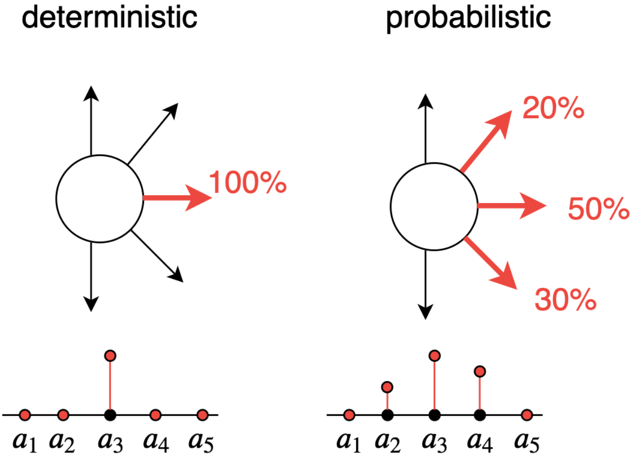


- **Policy** specifies what actions should be taken in a given state
- Hence, a **policy** π **maps states into actions;**
  - **Deterministic policy:** $a = \pi(s)$
  - **Stochastic policy:**

$$\pi(a \mid s) := P(A_{t+1} = a \mid S_t = s)$$

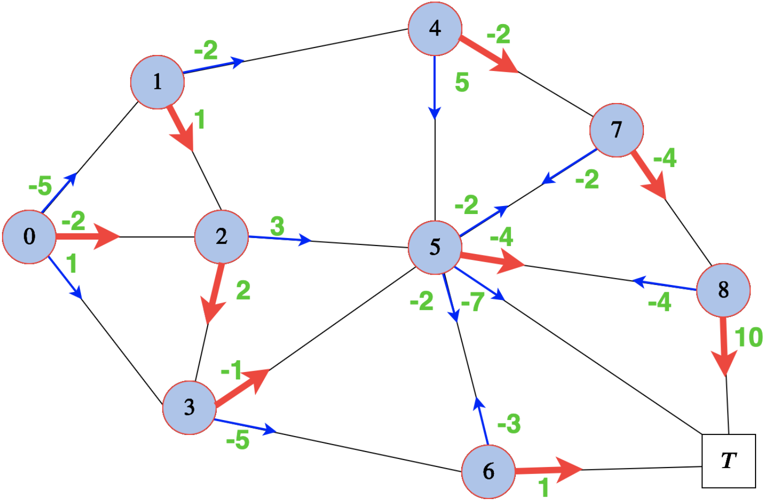
- **Important: policies** are **not** part of the MDP!

Deterministic vs. Probabilistic Policy

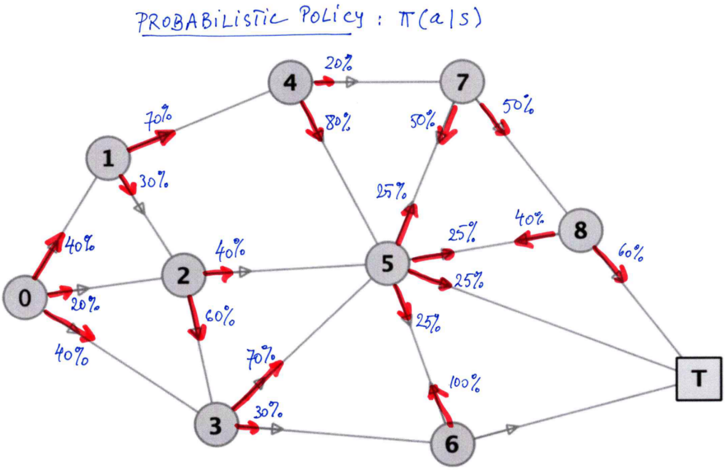


$$\boxed{s \to \pi(a \mid s)}$$

MDP + Policy (deterministic)



MDP + Policy (Probabilistic/Stochastic)



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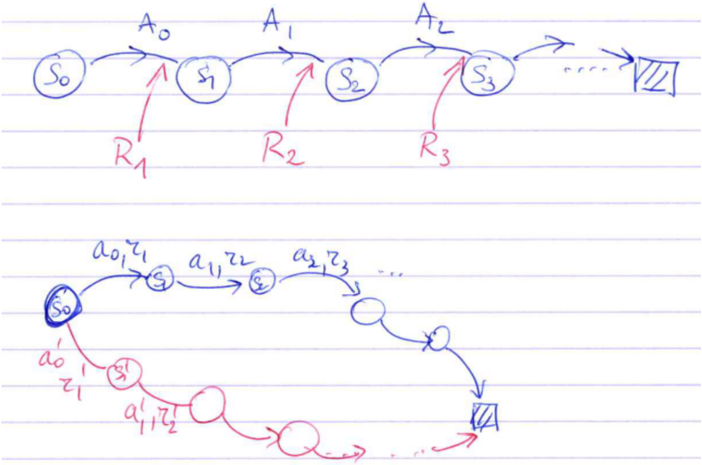


Long-term return: $G_0(s_0) = R_1 + R_2 + \ldots + R_T = \sum_{t=1}^{T} R_t$

Return and Rewards

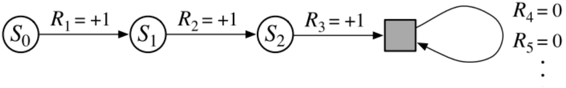
- The goal of the agent is to maximize the expected **(long-term) return**, a (discounted) sum over the immediate rewards received:

$$G_0 = R_1 + \gamma R_2 + \gamma^2 R_3 + \ldots = \sum_{k=1}^{\infty} \gamma^{k-1} R_k$$

or more generally:

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots = \sum_{k=1}^{\infty} \gamma^{k-1} R_{t+k}$$

- **Episodic tasks** can be interpreted as infinite-horizon, if we represent episode termination as transition to an **absorbing state** with self-transitions and zero reward.



Long-term (cumulative) return: some remarks

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots = \sum_{k=1}^{\infty} \gamma^{k-1} R_{t+k}$$

Remarks

- Notice that G depends on both the starting state (s) and the policy applied (π):

$$G \equiv G_\pi(s)$$

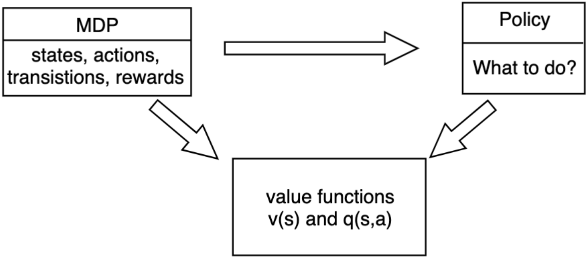
- Recursion relation:

$$\begin{aligned} G_t &= R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots \\ &= R_{t+1} + \gamma(R_{t+2} + \gamma R_{t+3} + \ldots) \\ &= R_{t+1} + \gamma G_{t+1} \end{aligned}$$

Long-term reward

- Pole balancing

Overview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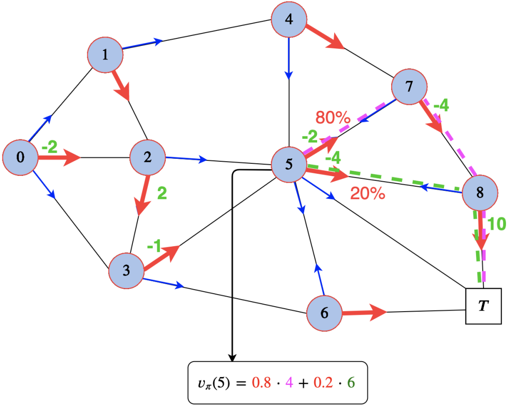
MDP + Policy $\longrightarrow$ state value function $v_\pi(s)$

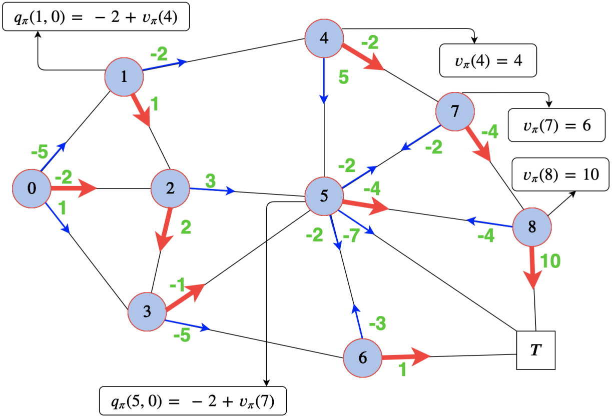
$v_\pi(s)$: expected value of state s when actions are specified by π:

$$v_\pi(s) = E_\pi\left(\sum_{t=1}^{T} R_t \mid s_0 = s\right)$$



MDP + Policy $\longrightarrow$ state-action value function $q_\pi(s, a)$

$$q_\pi(s, a) = E_\pi\left(\sum_{t=1}^{T} R_t \mid s_0 = s,\ a_0 = a\right)$$



Value functions: Tools for reasoning about future reward

- The **state value function** $v_\pi(s)$ assigns to each state s the **expected total return** when **starting** in that state s and **applying policy** π;

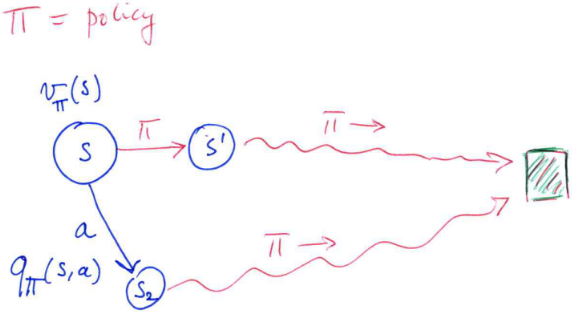
$$v_\pi(s) := E_\pi\Big[G_0 \,|\, S_0 = s\Big] = E_\pi\Big[\sum_{k=0}^{\infty} \gamma^k R_{k+1} | S_0 = s\Big]$$

- The **state-action value function** $q_\pi(s, a)$ of a policy π is:

$$q_\pi(s, a) := E_\pi\Big[G_0 \,|\, S_0 = s, A_0 = a\Big] = E_\pi\Big[\sum_{k=0}^{\infty} \gamma^k R_{k+1} \,|\, S_0 = s, A_0 = a\Big]$$

- $q(s, a)$ state space typically much larger than $v(s)$-state space! Hence, more difficult to learn.

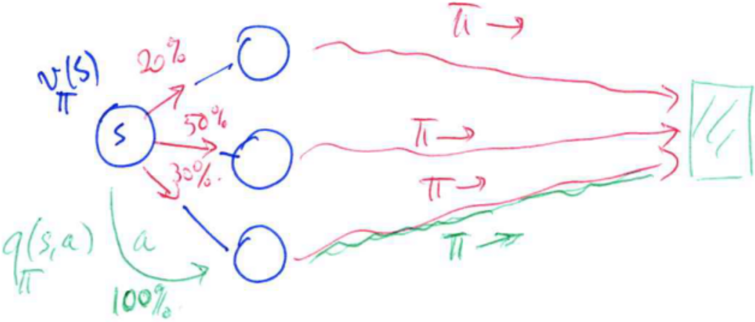
Difference between value functions $v_\pi(s)$ and $q_\pi(s, a)$



- $v_\pi(s)$: policy π dictates every action along the path;
- $q_\pi(s, a)$: first action a is taken independently of the policy π, from then onward, π dictates the remaining actions taken along the rest of the path.

Difference between value functions $v_\pi(s)$ and $q_\pi(s, a)$

Similar interpretation for stochastic policy.



Relationship between value and value-action function

Notice that $v(s)$ is a weighted mean of $q(s, a)$, where the weights are determined by the policy π:

$$v_\pi(s) = \sum_a \pi(a \mid s) q_\pi(s, a)$$

Indeed:

$$\begin{aligned} v_\pi(s) &= E_\pi\Big[G_0 \mid S_0 = s\Big] \\ &= \sum_a E_\pi\Big[G_0 \mid S_0 = s, A_0 = a\Big]\pi(a \mid s) \\ &= \sum_a q_\pi(s, a)\pi(a \mid s) \end{aligned}$$

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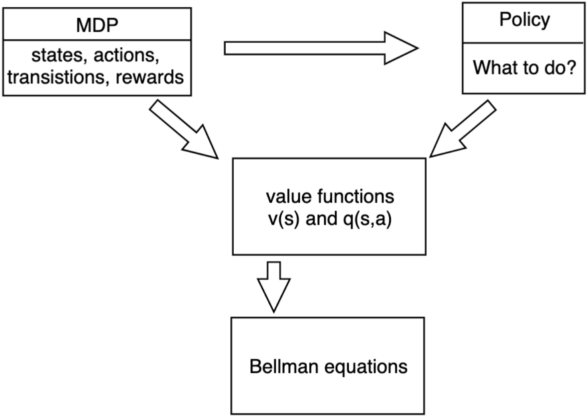
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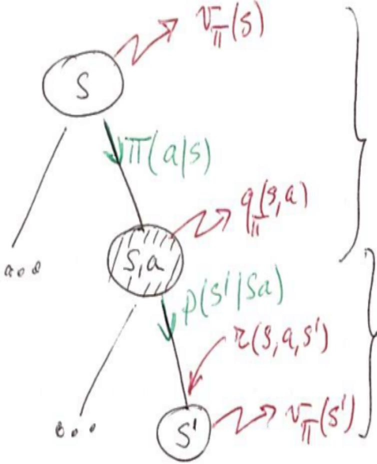
Bellman equation for value functions

Value function $v_\pi(s)$**:** $v_\pi(s) = \sum_a \pi(a \mid s) q_\pi(s,a)$
Value function $q_\pi(s,a)$

$$
\begin{aligned}
q_\pi(s,a) &= E_\pi\Big[G_0 \mid S_0 = s, A_0 = a\Big] \\
&= \sum_{s'} E_\pi\Big[G_0 \mid S_0 = s, A_0 = a, S_1 = s'\Big] p(s' \mid s,a) \\
&= \sum_{s'} E_\pi\Big[R_1 + \gamma G_1 \mid S_0 = s, A_0 = a, S_1 = s'\Big] p(s' \mid s,a) \\
&= \sum_{s'} \Big[r(s,a,s') + \gamma E_\pi(G_1 \mid S_1 = s')\Big] p(s' \mid s,a) \\
&= \sum_{s'} \Big[r(s,a,s') + \gamma E_\pi G_0(s')\Big] p(s' \mid s,a) \\
&= \sum_{s'} p(s' \mid s,a) \Big[r(s,a,s') + \gamma v_{\pi(}s')\Big]
\end{aligned}
$$

Bellman equations (schematically): Backup Diagram



$$\to v_\pi(s) = \sum_a \pi(a|s)\, q(s,a)$$

$$\to q(s,a) = \sum_{s'} p(s'|sa)\left[r(s,a,s') + \gamma V(s')\right]$$

Bellman equation: Summary

- **Back-up**

$$v_\pi(s) = \sum_a \pi(a \mid s) q(s,a)$$

$$q(s,a) = \sum_{s'} p(s' \mid s,a) \left[r(s,a,s') + \gamma v(s') \right]$$

- **Combined into recursion equations**

$$v_\pi(s) = \sum_a \pi(a \mid s) \sum_{s'} p(s' \mid s,a) \Big[r(s,a,s') + \gamma v_\pi(s') \Big]$$

$$q_\pi(s,a) = \sum_{s'} p(s' \mid s,a) \Big[r(s,a,s') + \gamma \sum_{a'} \pi(a' \mid s') q_\pi(s',a') \Big]$$

Bellman equation: Summary

- The definition of v_π can be rewritten recursively by making use of the transition model, yielding the **Bellman equation**:

$$v_\pi(s) = \sum_a \pi(a \mid s) \sum_{s'} p(s' \mid s, a)\Big[r(s, a, s') + \gamma v_\pi(s')\Big]$$

- This is a set of **linear equations,** one for each state, the solution of which defines the value of π
- A similar recursive relation holds for Q-values:

$$q_\pi(s, a) = \sum_{s'} p(s' \mid s, a)\Big[r(s, a, s') + \gamma \sum_{a'} \pi(a' \mid s') q_\pi(s', a')\Big]$$

Matrix form of Bellman equation

$$v_\pi(s) = \sum_a \pi(a \mid s) \sum_{s'} p(s' \mid s, a) \Big[r(s, a, s') + \gamma v_\pi(s') \Big]$$

We can rewrite this as:

$$\begin{aligned} v_\pi(s) \quad &= \quad \gamma \sum_{s'} \underbrace{\left(\sum_a \pi(a \mid s) p(s' \mid s, a) \right)}_{P_\pi(s,s')} v_\pi(s') \\ &\quad + \sum_a \underbrace{\left(\sum_{s'} p(s' \mid s, a) r(s, a, s') \right)}_{R(s,a)} \pi(a \mid s) \end{aligned}$$

Matrix form of Bellman equation

Under policy π:

- square matrix $P_\pi(s, s')$ is **transition probability** $s \to s'$:

$$P_\pi(s, s') := \sum_a \pi(a \mid s) p(s' \mid s, a)$$

- $R(s, a)$ is the **expected (immediate) reward** when taking action a in state s:

$$R(s, a) = \sum_{s'} p(s' \mid s, a) r(s, a, s')$$

- $\mathbf{r}_\pi(s)$ is the **expected (immediate) reward** in state s:

$$\mathbf{r}_\pi(s) = \sum_a \pi(a \mid s) R(s, a)$$

Matrix formulation of Bellman equation

- Bellman in matrix form:

$$\mathbf{v}_\pi = \gamma P_\pi \mathbf{v}_\pi + \mathbf{r}_\pi \qquad \text{or again} \qquad (I - \gamma P_\pi)\mathbf{v}_\pi = \mathbf{r}_\pi$$

- Solving for the value function v_π:

$$\mathbf{v}_\pi = (I - \gamma P_\pi)^{-1}\mathbf{r}_\pi$$

- Notice that:

$$\mathbf{v} = (I - \gamma P)^{-1}\mathbf{r} = (I + \gamma P + \gamma^2 P^2 + \ldots)\mathbf{r}$$

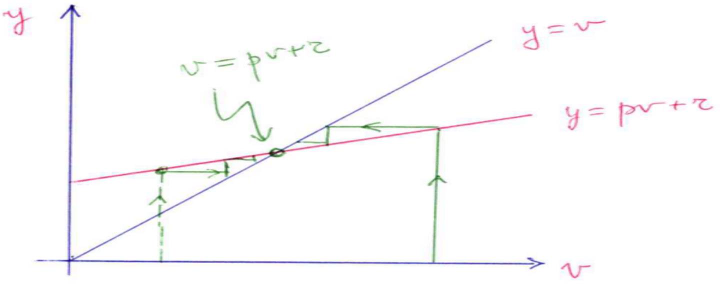
Solving Bellman eqs: Iteration to Fix-Point

Matrix form of **Bellman equation** corresponds to fix-point:

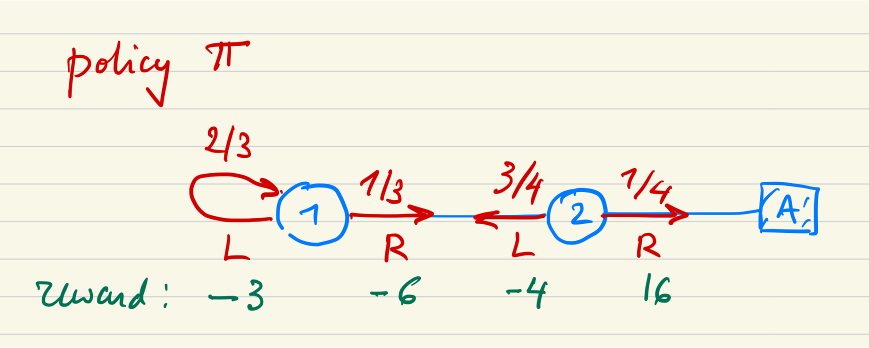
$$\mathbf{v}_\pi = \gamma P_\pi \mathbf{v}_\pi + \mathbf{r}_\pi \qquad \Longrightarrow \qquad \mathbf{v}_\pi = (I - \gamma P_\pi)^{-1} \mathbf{r}_\pi$$

Iterative solution (fix-point solution): update rule:

$$\mathbf{v}^{k+1} = \gamma P \mathbf{v}^k + \mathbf{r}$$

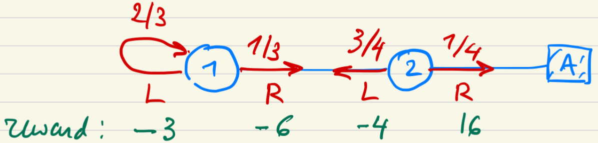


Solving Bellman eqs: Worked example



Solving Bellman eqs: Worked example

policy π



$$P_\pi = \begin{array}{c|ccc} s \backslash s' & A & 1 & 2 \\ \hline A & 1 & 0 & 0 \\ 1 & 0 & 2/3 & 1/3 \\ 2 & 1/4 & 3/4 & 0 \end{array}$$

$$R(s,a) = \begin{array}{c|cc} & L & R \\ \hline A & 0 & 0 \\ 1 & -3 & -6 \\ 2 & -4 & 16 \end{array}$$

Solving Bellman eqs: Worked example

$$R(s,a) = \begin{array}{c|cc} & L & R \\ \hline A & 0 & 0 \\ 1 & -3 & -6 \\ 2 & -4 & 16 \end{array}$$

$$\boxed{r_\pi(s) = \sum_a R(s,a)\,\pi(a|s)}$$

$$r(1) = -3 \cdot \frac{2}{3} + (-6)\frac{1}{3} = -4$$

$$r(2) = -4 \cdot \frac{3}{4} + 16 \cdot \frac{1}{4} = 1$$

$$r(A) = 0$$

$$r_\pi = (0, -4, 1)^T$$

Solving Bellman eqs: Worked example

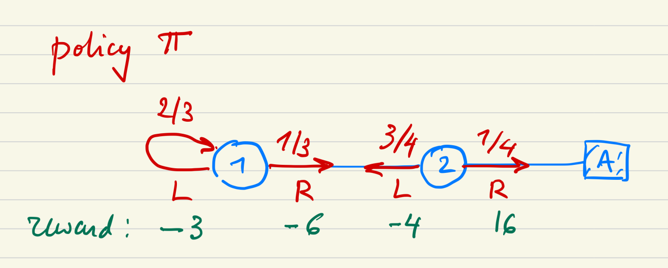
$$\boxed{v_\pi = \gamma P_\pi v_\pi + r_\pi} \qquad \gamma = 1$$

$$\underbrace{\begin{pmatrix} v(A) \\ v(1) \\ v(2) \end{pmatrix}}_{v} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2/3 & 1/3 \\ 1/4 & 3/4 & 0 \end{pmatrix} v + \begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix}$$

$$v_0 = r \;, \quad v_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2/3 & 1/3 \\ 1/4 & 3/4 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -7/3 \\ -3 \end{pmatrix} + \begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -19/3 \\ -2 \end{pmatrix}$$

Solving Bellman eqs: Worked example



$$
\begin{aligned}
v_2 &= (0, -6.3333, -2.0000) \\
v_3 &= (0, -8.8889, -3.7500) \\
\ldots. &= \ldots. \\
v_\infty &= (0, -44, -32)
\end{aligned}
$$

Example: MDP + value functions

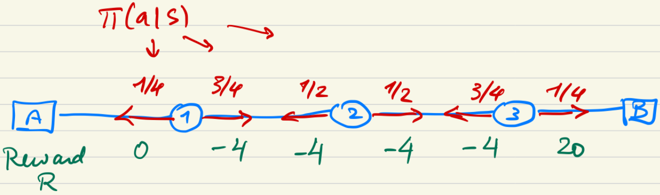
Example: - MDP + policy : see fig.

Assumptions:

1. No discounting: $\gamma = 1$
2. Deterministic transitions: $s \xrightarrow{a} s'$

   i.e.: $p(s' \mid s, a)$ = degenerate.

   ↳ either 1 or 0.



Example: MDP + value functions

$$P_\pi(s,s') = \sum_a \pi(a|s)\, p(s'|s,a)$$

$$= P(\text{transition } s \to s' \text{ in 1 time step})$$

$$P = \begin{array}{c|ccccc} s \backslash s' & A & 1 & 2 & 3 & B \\ \hline A & 1 & 0 & 0 & 0 & 0 \\ 1 & 1/4 & 0 & 3/4 & 0 & 0 \\ 2 & 0 & 1/2 & 0 & 1/2 & 0 \\ 3 & 0 & 0 & 3/4 & 0 & 1/4 \\ B & 0 & 0 & 0 & 0 & 1 \end{array}$$

Example: MDP + value functions

$$R(s,a) = \sum_{s'} p(s'|s,a)\, r(s,a,s')$$

= expected immediate reward when taking action a in state s.

$$R = \begin{array}{c} \\ A \\ 1 \\ 2 \\ 3 \\ B \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \boxed{\begin{array}{cc} 0 & 0 \\ 0 & -4 \\ -4 & -4 \\ -4 & 20 \\ 0 & 0 \end{array}} \end{array}$$

Example: MDP + value functions

$$r(s) = \sum_a R(s,a)\,\pi(a|s)$$

$$r = \boxed{\begin{matrix} r(A) \\ r(1) \\ r(2) \\ r(3) \\ r(B) \end{matrix}} = \underline{\text{diag}} \left(\underbrace{\begin{array}{c} \begin{matrix} L & R \end{matrix} \\ \boxed{\begin{matrix} 0 & 0 \\ 0 & -4 \\ -4 & -4 \\ -4 & 20 \\ 0 & 0 \end{matrix}} \end{array}}_{R(s,a)} \; \underbrace{\begin{array}{cc} & \begin{matrix} A & 1 & 2 & 3 & B \end{matrix} \\ \begin{matrix} L \\ R \end{matrix} & \boxed{\begin{matrix} * & 1/4 & 1/2 & 3/4 & * \\ * & 3/4 & 1/2 & 1/4 & * \end{matrix}} \end{array}}_{\pi(a|s)} \right)$$

Example: MDP + value functions

$$r(s) = \sum_a R(s,a)\,\pi(a|s)$$

= expected immediate return in s

$$r = \begin{bmatrix} 0 & \left(\frac{1}{4}\cdot 0 + \frac{3}{4}(-4)\right) = -3 & -4 & \frac{3}{4}(-4) + \frac{1}{4}20 = 2 & 0 \end{bmatrix}$$

A 1 2 3 B

Example: MDP + value functions

Bellman eqs in matrix form:

$$v = \begin{bmatrix} v(A) \\ v(1) \\ v(2) \\ v(3) \\ v(B) \end{bmatrix} = \begin{array}{c} \begin{array}{ccccc} A & 1 & 2 & 3 & B \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} v + \underbrace{\begin{bmatrix} 0 \\ -3 \\ -4 \\ 2 \\ 0 \end{bmatrix}}_{=r}$$

$$v = P \cdot v + r$$

Example:

$$v(2) = \tfrac{1}{2} v(1) + \tfrac{1}{2} v(3) + (-4)$$

$$v(3) = \tfrac{3}{4} v(2) + \tfrac{1}{4} \underbrace{v(B)}_{=0} + 2$$

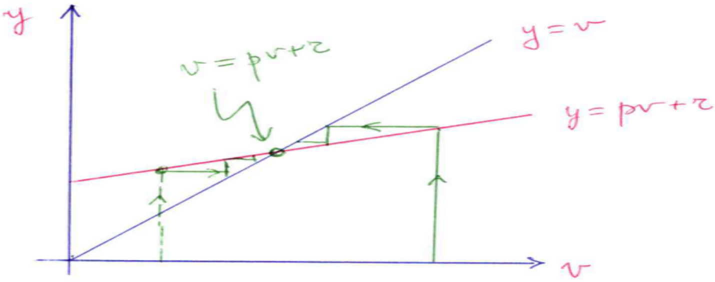
Solving Bellman eqs: Iteration to Fix-Point

Matrix form of **Bellman equation** corresponds to fix-point:

$$\mathbf{v}_\pi = \gamma P_\pi \mathbf{v}_\pi + \mathbf{r}_\pi \qquad \Longrightarrow \qquad \mathbf{v}_\pi = (I - \gamma P_\pi)^{-1} \mathbf{r}_\pi$$

Iterative solution (fix-point solution): update rule:

$$\mathbf{v}^{k+1} = \gamma P \mathbf{v}^k + \mathbf{r}$$

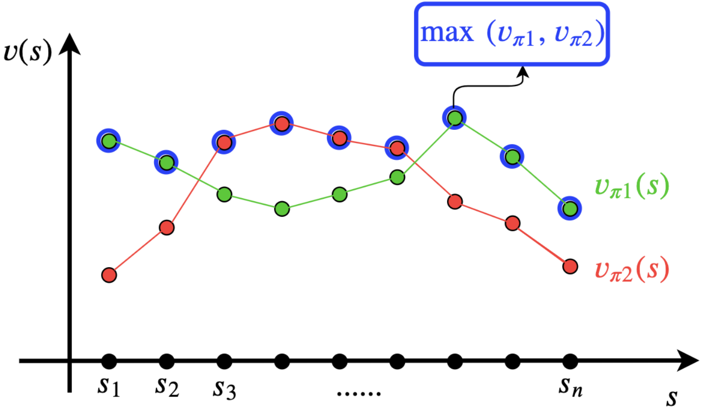


Outline

Optimal value functions

The optimal value functions are defined by the **pointwise maximum over policies**:

$$\forall s, a : \quad v^*(s) := \max_\pi v_\pi(s) \quad \text{and} \quad q^*(s,a) := \max_\pi q(s,a)$$

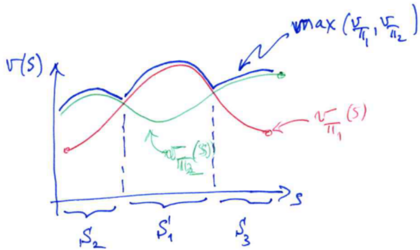


Optimal policy π^*

There exists an **optimal policy** π^* such that its value functions corresponds to the optimal value functions:

$$v_{\pi^*}(s) = v^*(s) \qquad \text{and} \qquad q_{\pi^*}(s,a) = q^*(s,a)$$

Intuition:



$$\pi^*(s) = \begin{cases} \pi_1(s) & \text{if } s \in S_2 \\ \pi_2(s) & \text{if } s \in S_1 \cup S_3 \end{cases}$$

Optimal value functions

- Value functions define a partial ordering over policies:

$$\pi \succ \pi' \Rightarrow v_{\pi(s)} \geq v_{\pi'}(s), \forall s \in S$$

- There can be multiple optimal policies but they all share the same **optimal state-value function**:

$$v^*(s) = \max_\pi v_\pi(s), \quad \forall s \in S$$

- They also share the same **optimal action-value function**:

$$q^*(s,a) = \max_\pi q_\pi(s,a), \quad \forall s \in S, a \in A$$

Optimal value functions: from weighted mean to max

- State-value function
  - General:

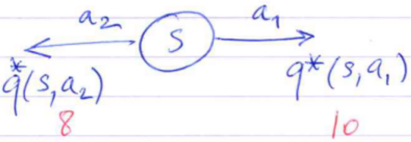
$$v_\pi(s) = \sum_a \pi(a \mid s) q_\pi(s, a)$$

  - Optimal

$$v^*(s) = v_{\pi^*}(s) = \max_a q_{\pi^*}(s, a) = \max_a q^*(s, a)$$

$$v^*(s) = \max_a q^*(s, a)$$

Bellman Optimality Equations



$q^*(s, \underline{a_1})$ = best possible expected return if taking action $\underline{\underline{a_1}}$ in s. $= 10$

$q^*(s, \underline{\underline{a_2}})$ = best possible expected return if taking action $\underline{\underline{a_2}}$ in s $= 8$

$v^*(s)$ = best possible expected return in s.

$$\boxed{v^*(s) = \max_a q^*(s,a)} \quad (= 10).$$

Bellman Optimality Equations

Deterministic transition $s \xrightarrow{a} s'$

$$\underbrace{q^*(s,a)}_{\downarrow} = r(s,a,s') + \gamma \underbrace{v^*(s')}_{\downarrow}$$

Best possible expected Return when Committed to action a

Best possible expected return in this state s'

$$q^*(s,a) = \sum_{s'} p(s'|sa)\left[r(s,a,s') + \gamma v^*(s')\right]$$

General

Optimal value functions: from weighted mean to max

- State-action value function
  - General:

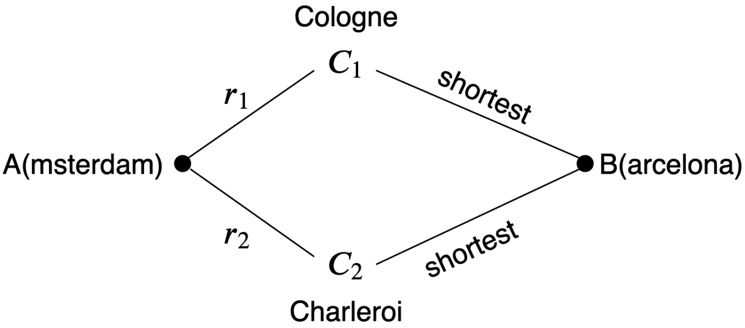
$$q_\pi(s,a) = \sum_{s'} p(s' \mid s,a)(r(s,a,s') + \gamma v_\pi(s))$$

  - Optimal:

$$q_{\pi^*}(s,a) = \sum_{s'} p(s' \mid s,a)(r(s,a,s') + \gamma v_{\pi^*}(s))$$

$$q^*(s,a) = \sum_{s'} p(s' \mid s,a)(r(s,a,s') + \gamma v^*(s))$$

Bellman Optimality equation: Travel Distance Analogy



$$d^*(A, B) = \min_{C_i} \{r_i + d^*(C_i, B)\}$$

Bellman optimality conditions (for deterministic transitions)

- Travel Analogy: Shortest distance paths:

$$d^*(A, B) = \min_{C_i} \{r_i + d^*(C_i, B)\}$$

- Deterministic transitions: $s \xrightarrow{a} s_a$

$$v^*(s) = \max_a \{r(s, a, s_a) + v^*(s_a)\}$$

$$q^*(s, a) = r(s, a, s_a) + v^*(s) = r(s, a, s_a) + \max_{a'} q^*(s_a, a')$$

Bellman optimality equations (General form)

- **Optimal state value function**

$$v^*(s) = \max_a q^*(s, a)$$

- **Optimal state-action value function**

$$q^*(s, a) = \sum_{s'} p(s' \mid s, a) \left[r(s, a, s') + \gamma v^*(s') \right]$$

- **Combined**

$$v^*(s) = \max_{a \in A} \sum_{s'} p(s' \mid s, a) \Big[r(s, a, s') + \gamma v^*(s') \Big]$$

$$q^*(s, a) = \sum_{s'} p(s' \mid s, a) \Big[r(s, a, s') + \gamma \max_{a' \in A} q^*(s', a') \Big]$$

Backup Diagram for Bellman Optimality Equations

Optimize over actions!

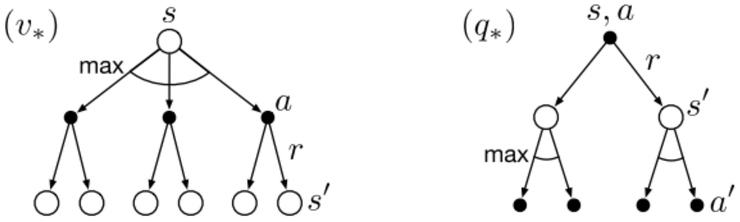


Figure 3.5: Backup diagrams for v_* and q_*

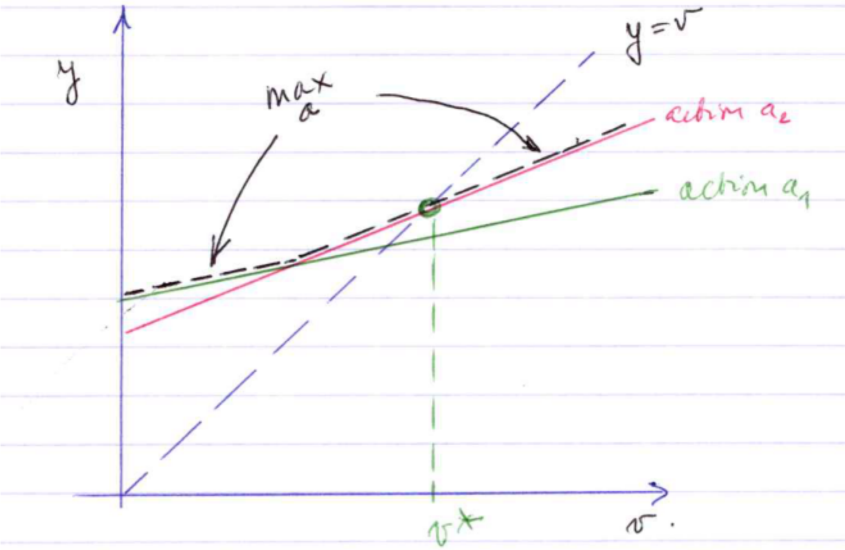
Bellman optimality equation in matrix form

$$
\begin{aligned}
v^*(s) &= \max_{a \in A} \sum_{s'} p(s' \mid s, a)\Big[r(s, a, s') + \gamma v^*(s')\Big] \\
&= \max_{a} \left(R(s, a) + \gamma \sum_{s'} \underbrace{p(s' \mid s, a)}_{T_a(s,s')} v^*(s') \right) \\
&= \max_{a} \left(R(s, a) + \gamma \sum_{s'} T_a(s, s') v^*(s') \right)
\end{aligned}
$$

or in matrix notation:

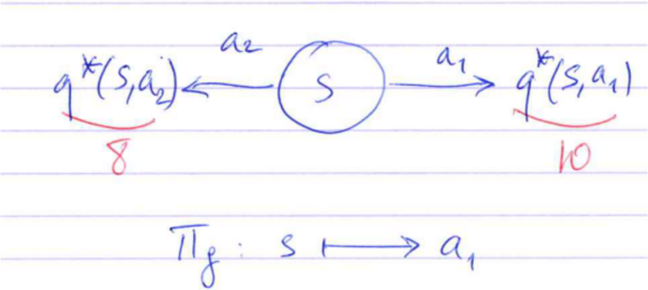
$$
\mathbf{v}^* = \max_{a} \left(R_a + \gamma T_a \mathbf{v}^* \right)
$$

Bellman optimality equation



Importance of v^* and q^*

- Markov property: Optimal decision only depends on current state:
- **Greedification:** In state s pick action a with highest $q^*(s, a)$



Bellman Optimality Conditions

- v^* and q^* satisfy a set of (non-linear) equations analogous to Bellman equations for v_π and q_π;
- these non-linear equations do not refer explicitly to the optimal policy;
- As a consequence we can find the optimal policy π^* by
  1. first solving these equations to find $v^*(s)$ (and $q^*(s,a)$) and
  2. then use **greedification** to determine the corresponding optimal policy π^*:

$$\forall s : \quad a_{opt} = \arg\max_a q^*(s, a)$$

Example: Bellman optimality conditions

MDP

Example : - MDP + policy : see fig.

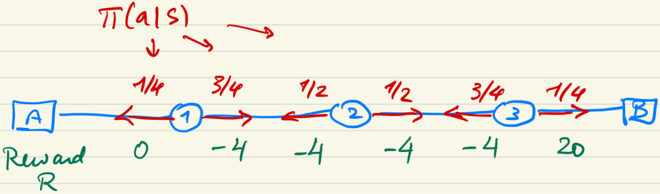
Assumptions :

① No discounting, $\gamma = 1$

② Deterministic transitions, $s \xrightarrow{a} s'$

i.e. : $\cancel{p}(s'|s,a)$ = degenerate.

↳ either 1 or 0.



Example: Bellman optimality conditions

Bellman Optimality Conditions

$$v^*(s) = \max_a \sum_{s'} p(s'|s a)\left[r(s,a,s') + \gamma v^*(s')\right]$$

$$= \max_a \left\{ R(s,a) + \gamma \sum_{s'} p(s'|s a)\, v^*(s') \right\}$$

Figure: General expression for v^*

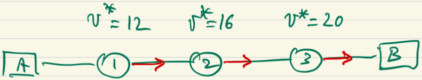
Bellman optimality conditions: Deterministic transition

Deterministic transition : $s \xrightarrow{a} s_a$

$$v^*(s) = \max_a \left[R(s,a) + \gamma v^*(s_a) \right]$$

In this example, optimal policy is simple

$\boxed{\pi^* : \text{always move right}}$



Eg, $S = 2$

$a = L$: $S_a = 1$, $v^*(S_a) = 12$, $R(s,a) = -4$

$R(s,a) + v^*(S_a) = -4 + 12 = 8$

$a = R$: $S_a = 3$, $v^*(S_a) = 20$, $R(s,a) = -4$

$R(s,a) + v^*(S_a) = -4 + 20 = \boxed{16}$

Bellman optimality conditions: Deterministic transition

Computing v^*, Worked example.

$$R = \begin{bmatrix} 0 & 0 \\ 0 & -4 \\ -4 & -4 \\ -4 & 20 \\ 0 & 0 \end{bmatrix}$$

Deterministic: $s \xrightarrow{a} s_a$

$$v^*(s) = \max_a \left[R(s,a) + \gamma v^*(s_a) \right]$$

$\gamma = 1$

Figure: $R(s, a)$ and $v^*(s)$ for deterministic transitions

Bellman optimality condition: Computing v^* using iteration

Initialise

$t=0 \longrightarrow t=1$

$$v^* = 0 \qquad v^* = \max_a \left(\begin{array}{c} \overbrace{\begin{array}{cc} L & R \end{array}}^{a} \\ \begin{bmatrix} 0 & 0 \\ 0 & -4 \\ -4 & -4 \\ -4 & 20 \\ 0 & 0 \end{bmatrix} \end{array} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right) = \begin{pmatrix} 0 \\ 0 \\ -4 \\ 20 \\ 0 \end{pmatrix}$$

$$t=2: \quad v^* = \max_a \left\{ \begin{bmatrix} 0 & 0 \\ 0 & -4 \\ -4 & -4 \\ -4 & 20 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -4 \\ 0 & 20 \\ -4 & 0 \\ 0 & 0 \end{bmatrix} \right\} = \begin{bmatrix} 0 \\ 0 \\ 16 \\ 20 \\ 0 \end{bmatrix}$$

Example: Bellman optimality conditions

$$t=3: \quad v^{*} = \max_{a} \left\{ \begin{bmatrix} 0 & 0 \\ 0 & -4 \\ -4 & -4 \\ -4 & 20 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 16 \\ 0 & 20 \\ 16 & 0 \\ 0 & 0 \end{bmatrix} \right\} = \begin{bmatrix} 0 \\ 12 \\ 16 \\ 20 \\ 0 \end{bmatrix}$$

Figure: $t = 3$

Example: Bellman optimality conditions

$\underline{t=4}$

$$v^* = \max_a \left\{ \begin{bmatrix} 0 & 0 \\ 0 & -4 \\ -4 & -4 \\ -4 & 20 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 16 \\ 12 & 20 \\ 16 & 0 \\ 0 & 0 \end{bmatrix} \right\} = \begin{bmatrix} 0 \\ 12 \\ 16 \\ 20 \\ 0 \end{bmatrix}$$

Converged!

Figure: Convergence for $t = 4$

For known MDP: compute $q^*(s, a)$ based on $v^*(s)$

$$s \xrightarrow{a} s_a$$

$$q^*(s,a) = r(s,a,s_a) + \gamma v^*(s_a)$$

$$v^* = \begin{array}{cc} \begin{pmatrix} 0 \\ 12 \\ 16 \\ 20 \\ 0 \end{pmatrix} & \begin{array}{c} A \\ 1 \\ 2 \\ 3 \\ B \end{array} \end{array} \qquad q^* = \begin{array}{cc} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} A \\ 1 \\ 2 \\ 3 \\ B \end{array} & \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & -4 \\ -4 & -4 \\ -4 & 20 \\ 0 & 0 \end{bmatrix}}_{R(s,a)} \end{array} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 16 \\ 12 & 20 \\ 16 & 0 \\ 0 & 0 \end{bmatrix}}_{v^*(s_a)} = \begin{bmatrix} 0 & 0 \\ 0 & 12 \\ 8 & 16 \\ 12 & 20 \\ 0 & 0 \end{bmatrix}$$

$$a^*(s) = \arg\max\ q^*(s,a)$$

Hence $a^*(s) = R$ for $s = 1, 2, 3$

Figure: Computing $q^*(s, a)$ from $v^*(s)$

Bellman optimality equations

If MDP is fully known, it suffices to compute v^*, as q^* can then be derived:

- **Optimal state-action value function**

$$q^*(s,a) = \sum_{s'} p(s' \mid s,a) \left[r(s,a,s') + \gamma v^*(s') \right]$$

- **Recall: Optimal state value function**

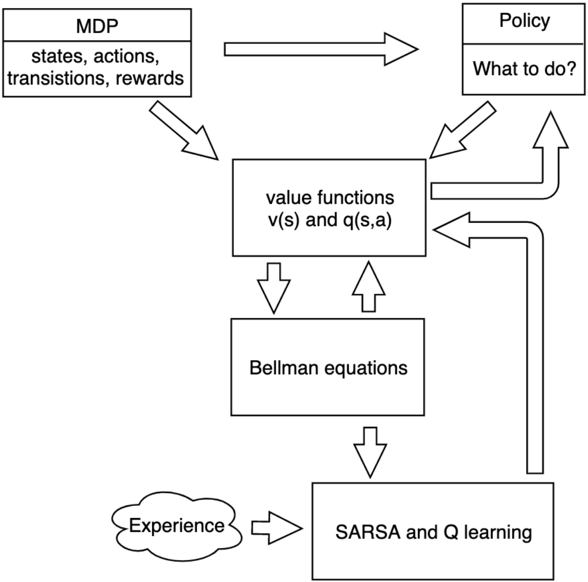
$$v^*(s) = \max_a q^*(s,a)$$

Model-based vs model-free

- **Model-based (PLANNING):** the MDP $= (\mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma)$ is completely specified;
  - Solve the Bellman equations (DP)!
- **Model-free (LEARNING):** only **direct experience**, i.e. only sample paths (states, actions and rewards) are given. Put differently, only experience-based information is given!
  - Random search but Bellman equations allow to propagate values!

Outline

Overview


MDP
states, actions, transistions, rewards
Policy
What to do?
value functions
v(s) and q(s,a)
Bellman equations
Experience
SARSA and Q learning

Bellman equations: General versus Optimal

- **State value functions**:
  - General:

$$v_\pi(s) = \sum_a \pi(a \mid s) \sum_{s'} p(s' \mid s, a) \Big[r(s, a, s') + \gamma v_\pi(s') \Big]$$

  - Optimal:

$$v^*(s) = \max_a \sum_{s'} p(s' \mid s, a) \Big[r(s, a, s') + \gamma v^*(s') \Big]$$

- **State-action value function**
  - General:

$$q_\pi(s, a) = \sum_{s'} p(s' \mid s, a) \Big[r(s, a, s') + \gamma \sum_{a'} \pi(a' \mid s') q_\pi(s', a') \Big]$$

  - Optimal:

$$q^*(s, a) = \sum_{s'} p(s' \mid s, a) \Big[r(s, a, s') + \gamma \max_{a'} q^*(s', a') \Big]$$

Taxonomy of RL problems

	Prediction *Estimation:* *Given π, what is v?*	**(Optimal) Control** *Optimisation:* *What is optimal π?*
model-based (MDP given)	Policy evaluation using Dyn. Programming (DP)	Policy improvement (+ Policy evaluation) = Policy iteration
model-free (MDP unknown)	Monte Carlo (MC) Temporal Diff^{ing} (TD) = "impatient MC" *bootstrapping!*	Generalized Policy Iteration *"simultaneous"*