Introduction to Game Theory 2

Formalising and Analysing Games

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Version: October 31, 2023

Game Theory Part 1: Outline

1. Game Theory: Science of Strategic Thinking
2. Examples of interesting games
3. **Formalising games**
4. **Solution concept 1: Weak optimality**
5. Solution concept 2: Strategies with (weak) guarantees
6. Solution concept 3: Nash equilibrium

Reading

- **Recommended**
  - Shoham and Leyton-Brown: Chapter 3, sections 3.1-3.3
- **Optional**
  - William Spaniel: Game Theory 101: The Complete Textbook (paperback): *Very accessible and clear, teaching through examples. Accompanying YouTube channel.*
  - N. Nisan, T. Roughgarden, E. Tardos, V. Vazirani: Algorithmic Game Theory. Cambridge UP. *Solid, mathematical. Advanced.*
  - A. Dixit, B. Nalebuff: Thinking Strategically. Norton. *Lots of context and background. Interesting and non-technical.*

Overview

Formalising Games

Solution concept 1: Weak Optimality

- Pareto optimality
- Eliminating dominated strategies
- Best response

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Game theory and strategic agents

- **Game theory** studies **multi-agent decision problems**, that is, problems in which **independent decision-makers interact**.
- What each agent does has an **effect on the other agents** in the group (through utility);
- **Assumptions:**
  - agents have **preferences** encoded in **utility function (pay-off)**
  - **self-interest:** agents strive to maximize their own pay-off;
  - **rational behaviour:** agents reason about the actions of other agents and decide rationally.

A graphical representation: matrix games

In the special case of *two* agents, a strategic game can be graphically represented by a **payoff matrix**, for example:

	left	centre	right
up	$1, 0$	$1, 2$	$0, 1$
down	$0, 3$	$0, 1$	$2, 0$

- Rows correspond to actions of agent 1 and columns to actions of agent 2. Here: $A_1 = \{\text{up}, \text{down}\}$, $A_2 = \{\text{left}, \text{centre}, \text{right}\}$.
- Each entry contains the payoffs (u_1, u_2) of the two agents for each possible joint action. For example, $a = (\text{down}, \text{centre})$ gives $(u_1, u_2) = (0, 1)$.

Normal-form Games (Matrix Games)

- **Players:**
  - make **simultaneous moves** and receive **immediate payoffs**;
  - **payoffs** are specified for the combinations of actions played.

- **Payoff matrix:**
  - Specifies for given action combination $\mathrm{a} = (a_1, a_2, \ldots, a_n)$ the corresponding utility (pay-off) $u_i(\mathrm{a})$ for player $i = 1 \ldots n$

	Player 2 chooses *Left*	Player 2 chooses *Right*
Player 1 chooses *Up*	4, 3	−1, −1
Player 1 chooses *Down*	0, 0	3, 4

Normal form or payoff matrix of a 2-player, 2-strategy game

Formal definition of normal-form game

A n-person **normal-form game** is a tuple (N, A, u):

- N is a set of n **players (agents)**
- **Actions or Strategies** $A = A_1 \times A_2 \times \ldots \times A_n$ where each A_i is the set of actions available to agent i, i.e. set of allowable moves player i can make.
  An A-element $\mathrm{a} = (a_1, a_2, \ldots, a_n)$ is called an **action profile**.
- **Pay-off or utility function:** $\mathrm{u} : A \longrightarrow \mathbb{R}^n$ where $\mathrm{u} = (u_1, u_2, \ldots, u_n)$ and each $u_i : A \longrightarrow \mathbb{R}$ is the corresponding utility function for player i. Notice, payoff $u_i(\mathrm{a})$ for *each* agent depends on the *joint actions* of all agents.

Utility functions capture preferences

von Neumann and Morgenstern, 1944

If there exists a preference relation $\succcurlyeq$ on the outcomes of a game that satisfies a number of "natural conditions" (completeness, transitivity, substituability, decomposability, monotonicity and continuity), then there exists a function $u : \mathcal{O} \longrightarrow \mathbb{R}$ such that:

- $u(o_1) \geq u(o_2) \quad \text{iff} \quad o_1 \succcurlyeq o_2$
- $u(\{(o_1 : p_1), (o_2 : p_2), \ldots, (o_n : p_n)\}) = \sum_{i=1}^{n} p_i u(o_i)$

Examples of competitive and cooperative (matrix) games

A strategic game can model a variety of situations where agents interact. These are two well-known cases:

Matching Pennies

	head	tail
head	$1, -1$	$-1, \ 1$
tail	$-1, \ 1$	$1, -1$

In a **strictly competitive** or **zero-sum** game, $\sum_i u_i(\mathrm{a}) = 0$ for all a (anti-coordination game).

Going to the Movies

	action	comedy
action	$1, 1$	$0, 0$
comedy	$0, 0$	$1, 1$

In a **strictly cooperative** game, a type of **coordination** game, $u_i(\mathrm{a}) = u_j(\mathrm{a})$ for all i, j, a.

More examples

Chicken

	swerve	straight
swerve	$0, 0$	$-1, 1$
straight	$1, -1$	$-5, -5$

Stag Hunt

	stag	hare
stag	$2, 2$	$0, 1$
hare	$1, 0$	$1, 1$

Battles of the Sexes 1

	action	comedy
action	$3, 2$	$0, 0$
comedy	$0, 0$	$2, 3$

Battle of the Sexes 2

	action	comedy
action	$3, 2$	$2, 1$
comedy	$0, 0$	$2, 3$

Continuous action space

Hotelling's Game (ice-cream time):

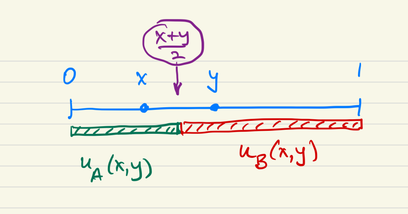
- **Two players**
- **Continuous (infinite) action space:**
  - each player can choose any position between 0 and 1.
  - Assume first player choses x while second player chooses y where for simplicity: $0 \leq x < y \leq 1$;
- **Utility**

$$
\begin{aligned}
u_1(x, y) &= x + \frac{y - x}{2} = \frac{x + y}{2} \\
u_2(x, y) &= 1 - y + \frac{y - x}{2} = 1 - \frac{x + y}{2}
\end{aligned}
$$

Hotelling's game

- **Utility**

$$
\begin{aligned}
u_A(x,y) &= x + \frac{y-x}{2} = \frac{x+y}{2} \\
u_B(x,y) &= 1 - y + \frac{y-x}{2} = 1 - \frac{x+y}{2}
\end{aligned}
$$



- Both simultaneous and sequential version (same outcome);

Strategies

- A player's **strategy** is the **algorithm** that determines the action the player will take at **any stage of the game.**
- **Pure strategy:** Select single action and play it.
- **Mixed strategy:** Select single action according to probability distribution and play it. :

Rationale? *matching pennies*

	Heads	Tails
Heads	1, -1	-1 , 1
Tails	-1, 1	1, -1

Mixed strategy: using randomness NOT to be *outsmart-ed* by opponent.

- **Strategy profile:** s $= (s_1, s_2, \ldots, s_n)$, i.e. one specified strategy for each agent.

Expected Utility for Mixed Strategies

- **Pure strategy:** (Expected) utility u_i for agent i selecting action a_i equals $u_i(a_i, a_{-i})$.
- **Mixed strategy:** Agent i plays strategy s_i which is a probability distribution over k possible actions:

$$s_i = \{(a_{i1}, p_{i1}), (a_{i2}, p_{i2}), \ldots, (a_{ik}, p_{ik})\} \qquad \text{(where } p_k = P(a_k))$$

- **Expected utility** for mixed strategies:
  - agent i playing mixed strategy $s_i = \{(a_{i1}, p_{i1}) \ldots (a_{in}, p_{in})\}$
  - agent j playing mixed strategy $s_j = \{(a_{j1}, p_{j1}) \ldots (a_{jm}, p_{jm})\}$

$$EU_i(s_i, s_j) = \sum_{k=1}^{n} \sum_{\ell=1}^{m} u_i(a_{ik}, a_{j\ell}) p_{ik} p_{j\ell}$$

Expected Utility for Mixed Strategies

B

A		b_1 (q)	b_2 $(1-q)$
	a_1 (p)	α, α' pq	β, β' $p(1-q)$
	a_2 $(1-p)$	γ, γ' $(1-p)q$	δ, δ' $(1-p)(1-q)$

Strategies

$$S_A = \{(a_1, p), (a_2, 1-p)\}$$

$$S_B = \{(b_1, q), (b_2, 1-q)\}$$

$$EU_A = \alpha pq + \beta p(1-q) + \gamma(1-p)q + \delta(1-p)(1-q)$$

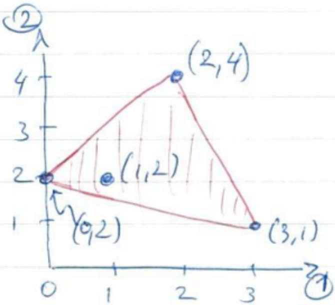
$$EU_B = \alpha' pq + \beta' p(1-q) + \gamma'(1-p)q + \delta'(1-p)(1-q)$$

Expected utility for mixed strategies

- Utility of mixed strategy lies within convex hull of utilities for (pure) support strategies

Expected utility

1 \ 2	L	R
U	1,2	0,2
D	2,4	3,1



Analysing games: Solution concepts for games

Consider point of view of a **single (self-interested) agent:**

- Given all game information: **what strategy** should he adopt?
- Complicated: depends on **actions of other agents**!
- Solving a game means trying to predict its outcome.
- **From (weak) optimality ...**
  - Pareto Optimality
  - Best Response (BR) given the actions of the other agents;
  - Iterated elimination of strictly dominated strategies (IESDS)
- ... **over strategies with weak guarantees ...**
  - Regret minimisation, Maximin and Minimax
- **... to Equilibrium,** i.e. no incentive to deviate:
  - Nash equilibrium (John Nash, 1950)

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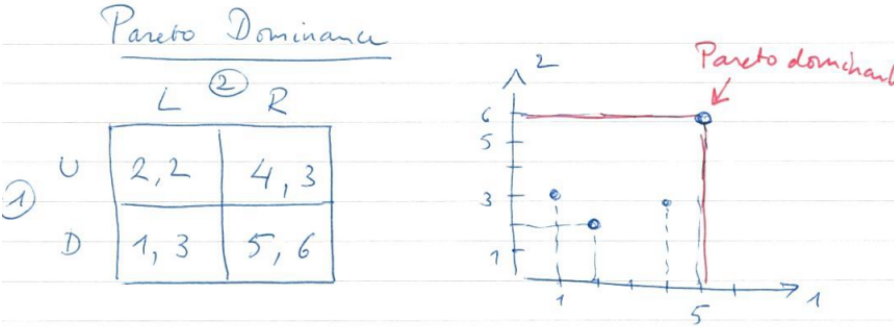
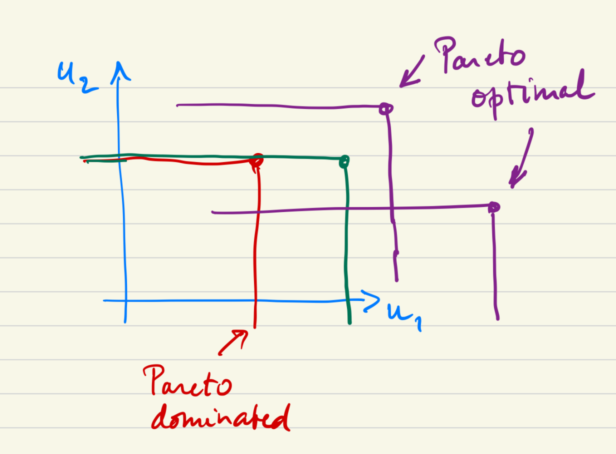
Pareto optimality



Pareto dominance and Partial Ordering



Pareto optimality

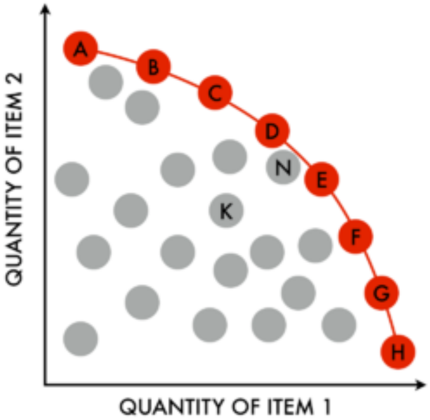
Pareto optimality is a **solution property** (not solution concept itself)

A joint action/strategy profile a is **Pareto dominated** by another joint action a′ if $u_i(\mathrm{a}') \geq u_i(\mathrm{a})$ for all agents i and $u_j(\mathrm{a}') > u_j(\mathrm{a})$ for some j.

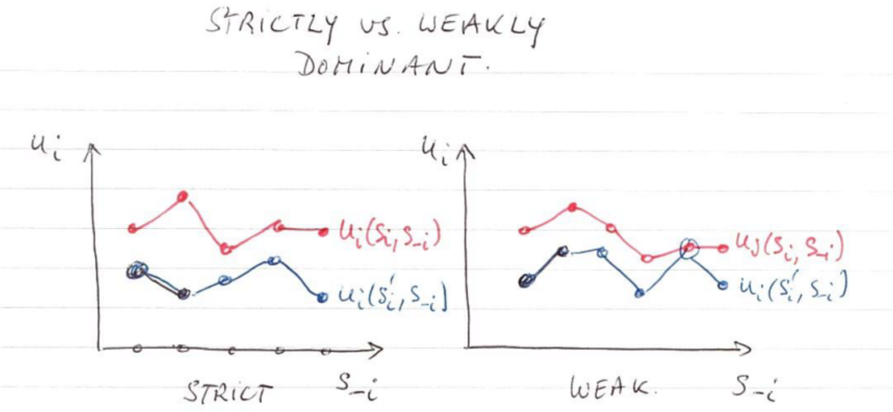
A joint action/strategy profile a is **Pareto optimal** if there is no other joint action a′ that Pareto dominates it.

Pareto dominance defines a **partial ordering** over strategy profiles.

Pareto Front



Dominating and Dominated Strategies: strict vs. weak



Domination for strategies

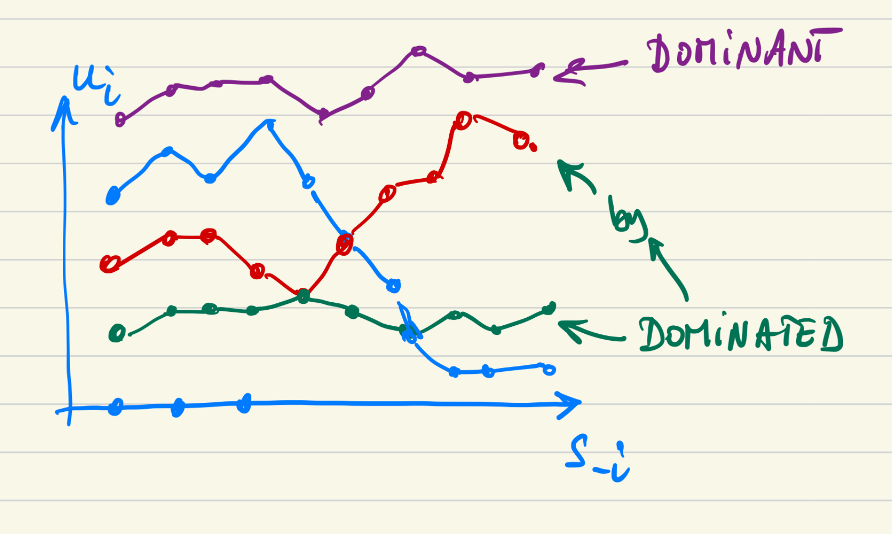
Let s_i and s_i' be two strategies for player i, and S_{-i} set of all strategy profiles for the other players:

- s_i **strictly dominates** s_i' if

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \qquad \forall s_{-i} \in S_{-i}$$

- s_i **weakly dominates** s_i' if
    1. $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \qquad \forall s_{-i} \in S_{-i}$, and
    2. $u_i(s_i, s_j) > u_i(s_i', s_j)$ for at least one $s_j \in S_{-i}$

Dominant and Dominated Strategies



Dominant and dominated strategies

- **(Strictly/Weakly) Dominant Strategy:** (strictly/weakly) dominates every other strategy of the agent;
- **Strictly/Weakly) Dominanted Strategy:** is (strictly/weakly) dominated by at least one of this agent;
- A strictly dominated strategy will never be the best response to anything!
- For a dominating strategy, we don't have to worry what the opponents are going to do!
- Dominance plays important role in **mechanism design.**

What NOT to do? IESDS: Iterated elimination of strictly dominated strategies

IESDS (a.k.a. What NOT to do?) is based on the following assumptions:

- It is **common knowledge** that all agents are rational.
- Rational agents **never** play strictly dominated actions.
- Hence, strictly dominated actions can be **eliminated.**

	left	centre	right
up	$13, 3$	$1, 4$	$7, 3$
middle	$4, 1$	$3, 3$	$6, 2$
down	$-1, 9$	$2, 8$	$8, -1$

What would IESDS predict in this game?

Iterated elimination of strictly dominated strategies (2)

Centre **strictly dominates** *right*. Row player knows that column player will never play the dominated action *right*. Hence he can eliminate that action and only needs to consider the simpler game:

	left	centre
up	13, 3	1, 4
middle	4, 1	3, 3
down	-1, 9	2, 8

For the row player, action *middle* strictly dominates *down*; hence eliminate! We are left with the simpler game where *centre* dominates *left*:

	left	centre
up	13, 3	1, 4
middle	4, 1	**3, 3**

Application Dominated Strategies: Prisoner's Dilemma

	Quiet	Confess
Quiet	$-1, -1$	$-12, 0$
Confess	$0, -12$	$-8, -8$

Prisoner's Dilemma

- *Quiet* is a strictly dominated strategy for both players, hence can be **eliminated**.
- Players will therefore both play confess, yielding pay-off $(-8, -8)$.
- Notice that this action profile is Pareto dominated!

Cournot Duopoly (discrete version)

Unit production cost: $c = 1$;
$Q_A(Q_B) =$ quantity produced by A (B)
P= Market price (per unit) : $P = 12 - 2(Q_A + Q_B)$
Pay-off:

$$u_A(A2, B3) = Q_A(P(Q_A, Q_B) - c) = 2(P(2,3) - c) = 2(12 - 2{\cdot}5 - 1) = 2$$

	$B0$	$B1$	$B2$	$B3$	$B4$	$B5$
$A0$	$0, 0$	$0, 9$	$0, 14$	$0, 15$	$0, 12$	$0, 5$
$A1$	$9, 0$	$7, 7$	$5, 10$	$3, 9$	$1, 4$	$-1, -5$
$A2$	$14, 0$	$10, 5$	$6, 6$	$2, 3$	$-2, -4$	$-2, -5$
$A3$	$15, 0$	$9, 3$	$3, 2$	$-3, -3$	$-3, -4$	$-3, -5$
$A4$	$12, 0$	$4, 1$	$-4, -2$	$-4, -3$	$-4, -4$	$-4, -5$
$A5$	$5, 0$	$-5, -1$	$-5, -2$	$-5, -3$	$-5, -4$	$-5, -5$

Cournot Duopoly (discrete version)

1. A3 (B3) strictly dominates A5 (B5), eliminate A5/B5
2. A3 (B3) strictly dominates A4 (B4), eliminate A4/B4
3. A1 (B1) strictly dominates A0 (B0), eliminate A0/B0

	$B1$	$B2$	$B3$
$A1$	$7,7$	$5,10$	$3,9$
$A2$	$10,5$	$6,6$	$2,3$
$A3$	$9,3$	$3,2$	$-3,-3$

4. A2(B2) strictly dominates A3 (B3), eliminate A3/B3
5. A2(B2) strictly dominates A1 (B1), resulting in strategy profile (A2,B2) with utility $(6,6)$;
6. Notice: **not Pareto-optimal!** (dominated by (A1,B1), with utilitity $(7,7)$

Iterated elimination of strictly dominated actions (3)

More challenging example:

- Rules of the game:
  - Game played in large group (e.g. auditorium)
  - Each player picks number between 1 and 100.
  - Collect all numbers and compute the mean.
  - Winner is player whose number was closest to 1/2 of mean.
- What strategy should you use when picking your number?
- **Bounded rationality** vs. "homo economicus"! Rationality is bounded by limits to our resources (Simon, 1982):
  - cognitive capacity, available information, time, emotion, etc.

Domination by mixed strategy

It is possible that the dominant strategy is mixed!

	L	R
U	3, 1	0, 1
M	1, 1	1, 1
D	0, 1	4, 1

$$\frac{1}{2}U + \frac{1}{2}D > M$$

Best response (BR): example

- Simultaneous games are difficult because we don't know what the opponent will do!
- If we would know what he will do, then we can play our best response to his action:
- Example:

	L	C	R
T	3, 3	2 4	5 1
→ M	0 0	0 6	6 0
B	4 2	1 1	5 3

$BR_1(L) = B$

$BR_2(M) = C$

Best response

- **Best response** from agent i's point of view:
- Let's assume that we know the strategies of all the other agents, i.e. s_{-i} is known;
- Agent i's best response s_i^* to strategy profile s_{-i}, is (a possibly mixed) strategy $s_i^* \in S_i$ such that

$$s_i^* = BR_i(s_{-i}) \quad \Leftrightarrow \quad \forall s_i \in S_i : \; u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}).$$

- Of course, . . . in a realistic setting we don't know the strategies of the other agents!
- **Not** solution concept, but **essential ingredient** for Nash eq.!
- BR dynamics yields equilibrium in some cases.

Best response

Chicken

	swerve	straight
swerve	$0, 0$	$-1, 1$
straight	$1, -1$	$-5, -5$

Stag Hunt

	stag	hare
stag	$2, 2$	$0, 1$
hare	$1, 0$	$1, 1$

- Chicken:

$$BR_1(a_2 = \text{swerve}) = \text{straight} \qquad BR_1(a_2 = \text{straight}) = \text{swerve}$$

- Stag hunt:

$$BR_1(a_2 = \text{stag}) = \text{stag} \qquad BR_1(a_2 = \text{hare}) = \text{hare}$$

Best response

- Best response **not necessarily unique**!
- When the best response includes two (or more) actions, then the agent must be **indifferent** among them!
- In fact: **any mixture** of these actions would also be a best response (mixed) strategy.
- Indeed,
  - If a_{i1} and a_{i2} are best both best response actions to s_{-i}, then $u_i(a_{i1}, \mathrm{s}_{-i}) = u_i(a_{i2}, \mathrm{s}_{-i}) =: u_i^*$.
  - Then, for any mixed strategy $s_i = \{(a_{i1}, p_1), (a_{i2}, p_2)\}$:

$$u_i(s_i, \mathrm{s}_{-i}) = p_1 u_i(a_{i1}, \mathrm{s}_{-i}) + p_2 u_i(a_{i2}, \mathrm{s}_{-i}) = (p_1 + p_2) u_i^* \equiv u_i^*,$$

since $p_1 + p_2 = 1$.

Best response: Continuous state space

Partnership game (two players)

- **Actions:** choice of individual contributions to joint project

$$0 \leq x, y \leq 4$$

- **Utilities:** *utility* = *profit* − *cost*

$$\begin{cases} u_1(x,y) & = & 2(x+y+bxy) - x^2 \qquad (0 \leq b < 1) \\ u_2(x,y) & = & 2(x+y+bxy) - y^2 \end{cases}$$

- Utility is quadratic (high input is very costly);

Best response: Continuous state space

Partnership game (two players): continued

- **Best response:** For a given input x of player 1, what input y of player 2 maximizes the latter's utility $(u_2(x, y))$?
- **Finding the maximum utility $u_2(x, y)$ for given x:**

$$\frac{\partial u_2}{\partial y} = 2(1 + bx) - 2y = 0$$

- **Best response solution:**

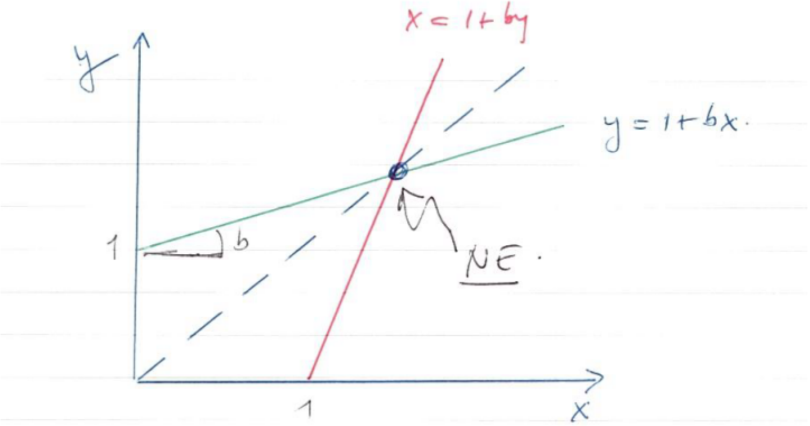
$$y^* \equiv BR_2(x) = 1 + bx$$

- **Similarly:**

$$x^* \equiv BR_1(y) = 1 + by$$

Best response: Continuous state space

- Spoiler Alert! Intersection of BR curves yields (Nash) equilibrium!



Best response to mixed strategy

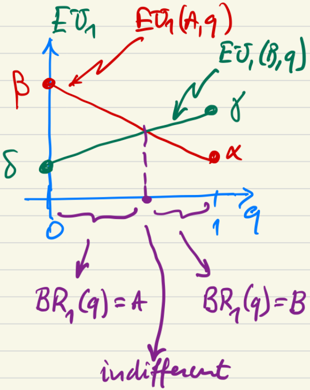
Best response of pl 1, to mixed strateg of pl. 2

②

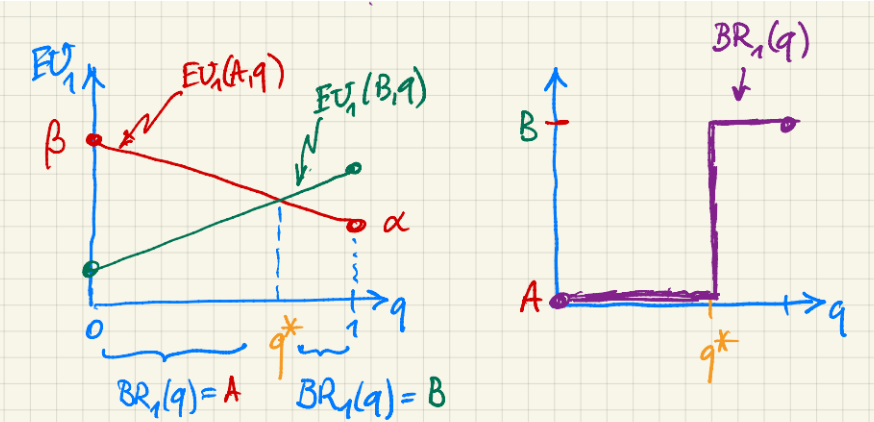
player 1	L(q)	R(1-q)
A	α, •	β, •
B	γ, •	δ, •

$$EU_1(A,q) = \alpha q + \beta(1-q)$$

$$EU_1(B,q) = \gamma q + \delta(1-q)$$



Best response to mixed strategy



Best Response Dynamics

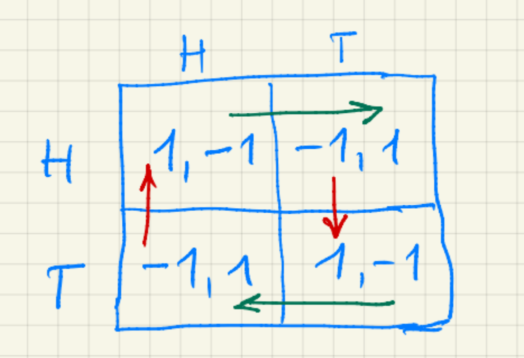
- **Best Response Dynamics:**
  - Imagine the simultaneous game to be sequential;
  - Players take turns to play best response (BR) to opponent;
  - An equilibrium might be reached
- Example: in action profile (M, C) equilibrium is reached!

	L	C	R
U	$4, 3$	$2, 0$	$8, 2$
M	$8, 2$	$4, 6$	$-1, 1$
D	$6, -3$	$0, 0$	$1, -1$

- Counter-example: matching pennies produces a cycle!
- In finite game, converges to either equilibria or cycles!
  - This presages the concept of **Nash equilibrium**

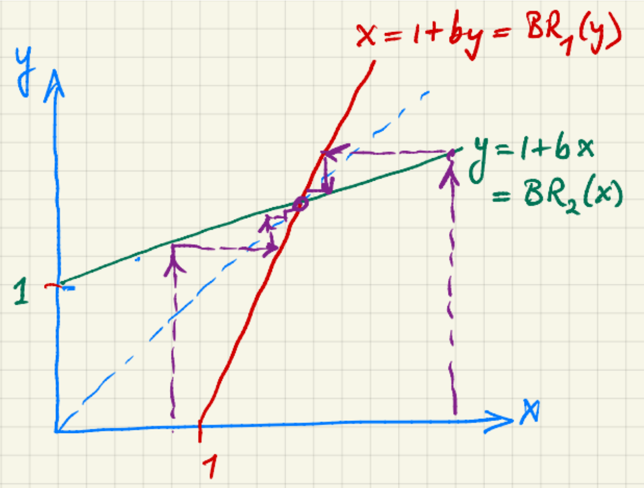
BR dynamics might yield a cycle!

- Matching pennies:



BR dynamics for continuous state space

- Partnership game:



Best response and IESDS

- Strictly dominated strategies are **never a best response;**
- **Church-Rosser property:** Order of elimination does not matter for IESDS (strict dominance)!
- Eliminating **weakly dominated** strategies might be too drastic!

Game Theory Part 1: Outline