Exam Advanced Machine Learning
7 January 2020, 18.30–21.15

This exam consists of 6 problems, each consisting of several questions. All answers should be motivated, including calculations, formulas used, etc. The use of a calculator is not allowed.

Question 1: Short questions
Please provide an argument for your answer on the following questions.

(a) Is the following statement true or false? A convolutional neural network (CNN) for image analysis can be trained for unsupervised learning tasks, whereas an ordinary neural network cannot.

False

(b) Is the following statement true or false? A perceptron model can achieve zero training error on *any* linearly separable dataset.

True

(c) How does the bias-variance decomposition of a ridge regression estimator compare with that of ordinary least squares regression?

Ridge has larger bias, smaller variance

(d) Is the following statement true or false? Logistic regression is equivalent to a neural network without hidden units and using the cross-entropy loss.

True

(e) What are some practical problems with the sigmoidal activation functions in neural networks?

Vanishing gradients, slow learning

Question 2: Neural networks

(a) Suppose that you build a neural network for a specific purpose. You train your network with cost function $J = \frac{1}{2}|\mathbf{y} - \mathbf{z}|^2$.

- When the input $\mathbf{x}$ is given to the network, first a weight matrix V mapping the input layer to the hidden layer is applied to yield $\mathbf{g} = V\mathbf{x}$.
- The vector of hidden unit values $\mathbf{g}$ are then activated using a ReLU activation function r, yielding $\mathbf{h} = r(\mathbf{g})$.
- Finally, the activated values are transformed to the output $\mathbf{z}$ by a weight matrix W given by $\mathbf{z} = W\mathbf{h}$.

Derive $\partial J/\partial W_{ij}$ and $\partial J/\partial V_{ij}$ for this network.

$\frac{\partial J}{\partial W_{ij}} = (\mathbf{z} - \mathbf{y})^\top \frac{\partial \mathbf{z}}{\partial W_{ij}} = (z_i - y_i)h_j$
$\frac{\partial J}{\partial V_{ij}} = (\mathbf{z} - \mathbf{y})^\top \frac{\partial \mathbf{z}}{\partial V_{ij}} = (\mathbf{z} - \mathbf{y})^\top W \frac{\partial \mathbf{h}}{\partial V_{ij}} = ((\mathbf{z} - \mathbf{y})^\top W)_i r'(g_i) x_j$

(b) Suppose that you have a 3-dimensional input $x = (x_1, x_2, x_3) = (2, 2, 1)$ fully connected to 1 neuron with activation function g_i. The forward propagation can be written as

$$a_i = g_i\left(w_1 x_1 + w_2 x_2 + w_3 x_3 + b\right).$$

After training this network, the values of the weights and bias are $w = (w_1, w_2, w_3) = (0.5, -0.2, 0)$ and $b = 0.1$. You try 4 different activation functions $g_1, \ldots, g_4$, which respectively output the values $a_1 = 0.67$, $a_2 = 0.70$, $a_3 = 1.0$, and $a_4 = 0.70$. What is a valid guess for the activation functions $g_1, \ldots, g_4$? You can choose for each activation function from the set of sigmoid, tanh, linear, ReLU, leaky ReLU, and indicator functions.

g_1 = sigmoid
g_2 = linear / ReLU / leaky ReLU
g_3 = indicator function
g_4 = g_2

(c) Explain what effect the following operations generally have on the bias and variance of your model.

(1) Regularizing the weights;
bias increases, variance decreases

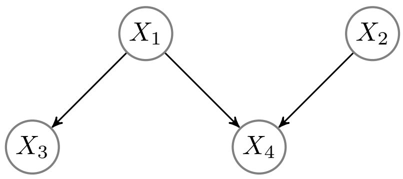
(2) Increasing the size of the layers (more hidden units per layer);
bias decreases, variance increases

(3) Using dropout to train a deep neural network;
bias increases, variance decreases

(4) Getting more training data (from the same distribution as before).
no change in bias, variance decreases

Question 3: Graphical models

The following figure shows a graphical model over four binary valued variables $X_1, \ldots, X_4$. We do not know the parameters of the probability distribution associated with the graph.



(a) Write the expression for the joint probability $\mathbb{P}(X_1, X_2, X_3, X_4)$ of the network in its *reduced* factored form.

$\mathbb{P}(X_1)\mathbb{P}(X_2)\mathbb{P}(X_3|X_1)\mathbb{P}(X_4|X_1, X_2)$

(b) Would it typically help to know the value of X_3 so as to gain more information about X_2?

No

(c) Assume we already know the value of X_4. Woud it help in this case to know the value of X_3 to gain more information about X_2?

Yes

(d) List three different conditional independence statements between the four variables that can be inferred from the graph. You can include marginal independence by saying "given nothing".

1) X_1 is conditionally independent of X_2 given nothing
2) X_3 is conditionally independent of X_2 given nothing
3) X_3 is conditionally independent of X_4 given X_1

Question 4: Hidden Markov Models (HMMs)
Consider a two-state hidden Markov model specified by (π, A, φ) that can output 4 possible values. Thus, the hidden states $z_i \in \{1, 2\}$, and the output values $x_i \in \{1, 2, 3, 4\}$. The further specification of the hidden Markov model is given as follows:

$$\pi = (0.5, 0.5), \qquad A = \begin{pmatrix} 0.99 & 0.01 \\ 0 & 1 \end{pmatrix}, \qquad \varphi = \begin{pmatrix} 0 & 0.199 & 0.8 & 0.001 \\ 0.1 & 0 & 0.7 & 0.2 \end{pmatrix}.$$

(a) Give an example of an output sequence of length 2 that cannot be generated by the hidden Markov model specified above.

1, 2

(b) We generated a sequence of 2020^{2020} observations from the hidden Markov model, and found that the last observation in the sequence was 3. What is the most likely hidden state corresponding to that last observation?

2

(c) Consider an output sequence with a 3 followed by another 3. What is the most likely sequence of hidden states corresponding to these observations?

1, 1

(d) Consider an output sequence with a 3 followed by another 3 followed by a 4. What are the *first two states* of the most likely sequence of hidden states corresponding to these observations?

2, 2

Question 5: Reinforcement learning
We are using Q-learning to learn a policy in a system with two states s_1 and s_2. State s_1 has two actions (a and b), and state s_2 has only one action (c). Suppose that the discount factor is γ, and the learning rate is α. We initialize the Q-table with all zero values.

(a) On the first transition, you start in state s_1, apply action a, receive an immediate reward of 1, and then land in state s_2. What is the resulting Q-table? Give your answer as an algebraic expression that may include one or both of the symbols γ and α.

$Q(s_1, a) = \alpha$
$Q(s_1, b) = 0$
$Q(s_2, c) = 0$

(b) On the second transition, you apply action c, receive an immediate reward of 0, and then land in state s_1. What is the resulting Q-table? Give your answer as an algebraic expression that may include one or both of the symbols γ and α.

$Q(s_1, a) = \alpha$
$Q(s_1, b) = 0$
$Q(s_2, c) = \alpha^2\gamma$

(c) On the third transition, you apply action b, receive an immediate reward of 1, and then land in state s_2. What is the resulting Q-table? Give your answer as an algebraic expression that may include one or both of the symbols γ and α.

$Q(s_1, a) = \alpha$
$Q(s_1, b) = \alpha + \alpha^3\gamma^2$
$Q(s_2, c) = \alpha^2\gamma$

(d) What is the optimal policy that Q-learning has learned so far?

action b

Question 6: Regularized linear regression
Recall that the objective function for the L_2 regularized linear regression is given by

$$J(\mathbf{w}) = ||X\mathbf{w} - \mathbf{y}||_2^2 + \lambda||\mathbf{w}||_2^2,$$

where X is the design matrix (the rows of X are the data points). During the lecture, we derived that the global minimizer of J is given by

$$\mathbf{w}^* = (X^TX + \lambda I)^{-1}X^T\mathbf{y}.$$

Now, let us consider running Newton's method to minimize J. Thus, let $\mathbf{w}_0$ be an arbitrary initial guess for Newton's method. One update step is then given by

$$\mathbf{w}_1 = \mathbf{w}_0 - [H(J(\mathbf{w}))]^{-1}\,\nabla_{\mathbf{w}}J(\mathbf{w}),$$

where H is the Hessian, and ∇ is the gradient.

(a) Show that $\mathbf{w}_1$, the value of the weights after one Newton step, is equal to $\mathbf{w}^*$.

$\nabla_{\mathbf{w}}J(\mathbf{w}) = 2X^TX\mathbf{w} - 2X^T\mathbf{y} + 2\lambda\mathbf{w} = 2(X^TX + \lambda I)\mathbf{w} - 2X^T\mathbf{y}$
$H(J(\mathbf{w})) = 2(X^TX + \lambda I)$
$\mathbf{w}_0 - [H(J(\mathbf{w_0}))]^{-1}\,\nabla_{\mathbf{w_0}}J(\mathbf{w_0}) = (X^TX + \lambda I)^{-1}X^T\mathbf{y}$

partial grade	1	2	3	4	5	6
(a)	1	3	1	1	1	3
(b)	1	2	1	1	1	
(c)	1	2	1	1	1	
(d)	1		1	1	1	
(e)	1					

Final grade is: (sum of partial grades) / 9.0 + 1.0