MA423 - Fundamentals of Operations Research
Lecture 5: Markov Chains

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Chapter 1

Markov chains

1.1 Introduction

Markov chains describe the behaviour of stochastic systems. The systems considered here can be in different states, and they evolve over time in a probabilistic manner. The timeline is described by *discrete time steps* $t = 0, 1, 2, \ldots, T$. A *stochastic process* is a sequence of random variables for every time step. The variable at time t, denoted by X_t, can be in one of M *states*, denoted by $0, 1, 2, \ldots, M-1$. Unless otherwise stated we assume that the set of states in finite.

Example 1.1. The gambler's ruin. *A gambler is repeatedly playing with a fruit machine. He has £1 at the beginning, and in every play, he either wins £1 with probability q, or looses £1 with probability $(1-q)$. He finishes if either he goes broke by losing all his money, or manages to raise his money to £3, in which case he is able to buy a pint. What is the probability of the two outcomes?*

A stochastic process for this example can be defined for $t = 0, 1, 2, \ldots$ by

$$X_t = \text{the budget of the gambler at time } t.$$

The problem has 4 states: $0, 1, 2, 3$. At time 0, $X_0 = 1$. If either state 0 or 3 is reached, the game is over. There is no limit on the time periods: the game could go on forever, if winning and loosing games are exactly alternating (although this has probability 0). With probability 1, the game will terminate in either state 0 or 3; the questions is what is the probability that the game will terminate in each of states 0 and 3. The answer will be given later, after the necessary techniques have been introduced.

An important property of the gambler's process is that the value of X_{t+1} depends only on X_t, but not the prior history of the games. Indeed, for a given budget $X_t = r$ at time t, one will have $X_{t+1} = r + 1$ with probability q, and $X_{t+1} = r - 1$ with probability $1 - q$. The process is memoryless/oblivious: the value of X_{t+1} does not depend on the values of $X_0, X_1, \ldots, X_{t-1}$.

This will be in fact the property defining Markov chains.

A stochastic process is called a *Markov chain*, if the next state X_{t+1} depends only the current state X_t but not on the previous history. Formally, for any time t and for any states $r_0, r_1, \ldots, r_{t+1}$,

$$P(X_{t+1} = r_{t+1} | X_t = r_t) \;=\; P(X_{t+1} = r_{t+1} | X_0 = r_0, X_1 = r_1, \ldots, X_t = r_t)$$

The above property is called the *Markov property*. Recall the notation $P(X_{t+1} = r_{t+1} | X_t = r)$ for conditional probability: this expresses the probability that $X_{t+1} = r_{t+1}$, provided that $X_t = r_t$. In these lectures, we will use one more restriction: these conditional probabilities do not depend on time t, just on the states:

A Markov chain is called *time-homogeneous*, if for every two states i and j,

$$P(X_{t+1} = j | X_t = i) = P(X_1 = j | X_0 = i), \qquad \text{for all } t = 1, 2, \ldots$$

The gambler's example is a time-homogeneous Markov chain: the probability q is the same no matter how many games have been played. A time-homogeneous Markov chain can be fully described by giving the following probabilities for every two states.

The *transition probability* from state i to state j:

$$p_{ij} = P(X_{t+1} = j | X_t = i)$$

All these probabilities can be presented in an $M \times M$ matrix P, where the j'th entry in row i is p_{ij} (with the numbering starting from 0). For the gambler's problem, we obtain the following matrix:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1-q & 0 & q & 0 \\ 0 & 1-q & 0 & q \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In the first and the last row, we have $p_{00} = 1$ and $p_{33} = 1$. This expresses that once the budget runs out or reaches 3, we stay in this state forever. The process was defined to terminate when reaching one of these states; however, it is more convenient to express termination as staying forever in the same state. Formally,

A state i is called an *absorbing state*, if $p_{ii} = 1$.

By the termination of a Markov chain we mean reaching an absorbing state. Note that every row in the matrix sums up to 1. This always holds:

$$\sum_{j=0}^{M-1} p_{ij} = 1, \qquad \text{for all states } i.$$

This simply expresses the fact that from every state, we will move on to some next state (possibly the same). These are disjoint and mutually exclusive events and thus one of them necessarily happens, hence the sum of their probabilities is 1.

A good way to visualise time-homogeneous Markov chains is via the *state transition diagram*. We construct a directed graph by assigning a vertex to every state, and include an arc (i, j) if the state j can follow i with a positive probability: that is, if $p_{ij} > 0$. The graph on Figure 1.1 presents this for the gambler's problem.

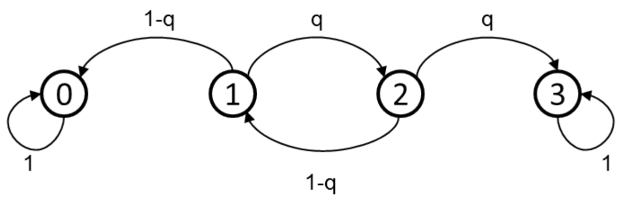

Figure 1.1: State transition diagram for the gambler's problem.

Let us now consider a second example.

Example 1.2. *The weather in London tomorrow only depends on today's weather. There are three types of weather: rainy, cloudy, and sunny. If it is rainy today, then it is equally likely to be cloudy and sunny tomorrow. If it is cloudy today, then it is equally likely to be cloudy, rainy and sunny tomorrow. If it is sunny today, then it will be rainy tomorrow. Given that is it cloudy on Tuesday what is the probability that it will be sunny on Thursday (2 days later)?*

The state transition diagram is given in Figure 1.2.

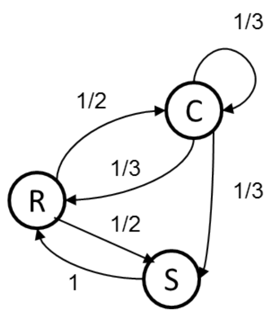

Figure 1.2: State transition diagram for the weather problem.

Let us number the states as 0 =rainy, 1 =cloudy, 2 =sunny. Then the transition matrix is:

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 \end{pmatrix}$$

Let us answer the question. If it was cloudy on Tuesday, then on Wednesday it will be rainy, cloudy and sunny with equal probability $\frac{1}{3}$. Having a sunny day on Thursday has probabilities

$\frac{1}{2}$, $\frac{1}{3}$ and 0 in the three cases (given by the last column of the matrix). Hence the probability of a sunny Thursday is

$$\frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot 0 = \frac{5}{18}.$$

1.2 n-step transition matrix

Motivated by the above example, we now explore how to find the state probabilities multiple steps ahead.

The *n-step transition probability* from state i to state j is:

$$p_{ij}^{(n)} = P(X_{t+n} = j | X_t = i)$$

Clearly, $p_{ij}^{(1)} = p_{ij}$. Let $P^{(n)}$ denote the n-step transition matrix, where the j'th entry of the i'th row is $p_{ij}^{(n)}$. We must also have

$$\sum_{j=0}^{M-1} p_{ij}^{(n)} = 1 \quad \text{for all states } i,$$

since we will end up in one of the states after n steps. Let us first see how to obtain the 2-step transition probabilities $p_{ij}^{(2)}$ - we already did this in the weather example. We first get from state i to one of the states, say k in the next step, with probability p_{ik}. From this state k, we can obtain j with probability p_{kj}. This gives the formula

$$p_{ij}^{(2)} = \sum_{k=0}^{M-1} p_{ik} p_{kj}.$$

The right hand side should be familiar from matrix multiplication: it is the scalar product of the i'th row and the j'th column of the transition matrix P. Consequently,

$$P^{(2)} = P \cdot P = P^2.$$

Consider again the weather example. In fact, we computed $p_{12}^{(2)} = \frac{5}{18}$ as the scalar product of the 1st row and 2nd column of the matrix (recall that numbering starts at 0). The full matrix would be

$$P^{(2)} = \begin{pmatrix} \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{4}{9} & \frac{5}{18} & \frac{5}{18} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

We now generalise the above arguments to n steps.

Chapman-Kolmogorov equations. For any two states i and j, and any two integers $m < n$,

$$p_{ij}^{(n)} = \sum_{k=0}^{M-1} p_{ik}^{(m)} p_{kj}^{(n-m)}. \tag{1.1}$$

The formula on $p_{ij}^{(2)}$ above was the special case for $n = 2$, $m = 1$. The argument is exactly the same: we decompose the n step process into two parts, corresponding to the first m steps and the last $(n-m)$ steps. Since $p_{ik}^{(m)}$ for $k = 0, \ldots, M-1$ is the i^{th} row of $P^{(m)}$ and $p_{kj}^{(n-m)}$ for $k = 0, \ldots, M-1$ is the j^{th} column of $P^{(n-m)}$, equation (1.1) can be written in matrix form as follows:

$$P^{(n)} = P^{(m)} \cdot P^{(n-m)} \quad \text{for all integers } m < n.$$

Iterating the argument for the special case of $m = 1$, we get

$$P^{(n)} = P \cdot P^{(n-1)} = P \cdot P \cdot P^{(n-2)} = \ldots = P^n.$$

That is, the n-step transition matrix can be obtained as the n'th power of the one-step transition matrix.

1.3 Long-run properties of Markov chains

The weather example

Let us again consider the weather example, and compute $P^{(n)}$ for some larger values of n.

$$P^{(2)} = \begin{pmatrix} 0.6667 & 0.1667 & 0.1667 \\ 0.4444 & 0.2778 & 0.2778 \\ 0 & 0.5000 & 0.5000 \end{pmatrix} \qquad P^{(5)} = \begin{pmatrix} 0.3210 & 0.3395 & 0.3395 \\ 0.3868 & 0.3066 & 0.3066 \\ 0.5185 & 0.2407 & 0.2407 \end{pmatrix}$$

$$P^{(10)} = \begin{pmatrix} 0.4104 & 0.2948 & 0.2948 \\ 0.4017 & 0.2991 & 0.2991 \\ 0.3844 & 0.3078 & 0.3078 \end{pmatrix} \qquad P^{(20)} = \begin{pmatrix} 0.4002 & 0.2999 & 0.2999 \\ 0.4000 & 0.3000 & 0.3000 \\ 0.3997 & 0.3001 & 0.3001 \end{pmatrix}$$

It appears that as n increases, all rows of the matrix will converge to the vector $(0.4, 0.3, 0.3)$. This suggests that if we wait long enough, the weather on the first day will not matter anymore: we will have 40% probability of a rainy day on the long run.

These will be called the *steady state probabilities*. Later on we shall see that under some general conditions, the probability of every state will stabilise after a large number of steps.

The gambler's example

Consider now the gambler's example with probability $q = \frac{1}{3}$ for winning. The long term behaviour in this case is quite different.

$$P^{(2)} = \begin{pmatrix} 1.0000 & 0 & 0 & 0 \\ 0.6667 & 0.2222 & 0 & 0.1111 \\ 0.4444 & 0 & 0.2222 & 0.3333 \\ 0 & 0 & 0 & 1.0000 \end{pmatrix} \qquad P^{(5)} = \begin{pmatrix} 1.0000 & 0 & 0 & 0 \\ 0.8477 & 0 & 0.0165 & 0.1358 \\ 0.5432 & 0.0329 & 0 & 0.4239 \\ 0 & 0 & 0 & 1.0000 \end{pmatrix}$$

$$P^{(10)} = \begin{pmatrix} 1.0000 & 0 & 0 & 0 \\ 0.8567 & 0.0005 & 0 & 0.1428 \\ 0.5711 & 0 & 0.0005 & 0.4283 \\ 0 & 0 & 0 & 1.0000 \end{pmatrix} \qquad P^{(20)} = \begin{pmatrix} 1.0000 & 0 & 0 & 0 \\ 0.8571 & 0.0000 & 0 & 0.1429 \\ 0.5714 & 0 & 0.0000 & 0.4286 \\ 0 & 0 & 0 & 1.0000 \end{pmatrix}$$

As n increases, the probabilities in the two columns in the middle will converge to 0. This is no surprise, given that the game will terminate with a very high probability within the first 10 iterations. As n goes to infinity, the matrix $P^{(n)}$ will converge to

$$P^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{6}{7} & 0 & 0 & \frac{1}{7} \\ \frac{4}{7} & 0 & 0 & \frac{3}{7} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This is substantially different from the weather example, where the matrix converged to one with all rows identical. The matrix P^* can be interpreted as follows: if we start the game with £1, we will eventually go broke with probability $\frac{6}{7}$ and get the pint with probability $\frac{1}{7}$. If we start with £2, the chances are better: $\frac{4}{7}$ for loosing and $\frac{3}{7}$ for the pint.

Consider the probability that the gambler will eventually go broke (endup with £0) at some long term time n given that currently he has £1 at some time t. Also assume that t is very large. Then we have:

$$\begin{aligned} Pr(\text{broke}|X_t = 1) &= P_{10} \cdot Pr(\text{broke}|X_{t+1} = 0) + P_{12} \cdot Pr(\text{broke}|X_{t+1} = 2) \\ &= \frac{2}{3} \cdot 1 + \frac{1}{3} \cdot \frac{4}{7} = \frac{6}{7}, \end{aligned}$$

where we got $Pr(\text{broke}|X_{t+1} = 0)$ and $Pr(\text{broke}|X_{t+1} = 2)$ from P^* since n is very large. The final sum product above, $\frac{6}{7}$, is the entry P^*_{01} and thus this argument supports these values. Similarly,

$$\begin{aligned} Pr(\text{pint}|X_t = 1) &= P_{10} \cdot Pr(\text{pint}|X_{t+1} = 0) + P_{12} \cdot Pr(\text{pint}|X_{t+1} = 2) \\ &= \frac{2}{3} \cdot 0 + \frac{1}{3} \cdot \frac{3}{7} = \frac{1}{7}, \end{aligned}$$

which also holds. We can do the same when the gambler starts with $X_t = 2$:

$$\begin{aligned}
Pr(\text{broke}|X_t = 2) &= P_{21} \cdot Pr(\text{broke}|X_{t+1} = 1) + P_{23} \cdot Pr(\text{broke}|X_{t+1} = 3) \\
&= \frac{2}{3} \cdot \frac{6}{7} + \frac{1}{3} \cdot 0 = \frac{4}{7}. \\
Pr(\text{pint}|X_t = 2) &= P_{21} \cdot Pr(\text{pint}|X_{t+1} = 1) + P_{23} \cdot Pr(\text{pint}|X_{t+1} = 3) \\
&= \frac{2}{3} \cdot \frac{1}{7} + \frac{1}{3} \cdot 1 = \frac{3}{7}.
\end{aligned}$$

We found the answer for the gambler's game for $q = \frac{1}{3}$: the gambler can eventually reach £3 with probability $\frac{1}{7}$, given that he starts in state 1 (starts with £1). However, this involved numerical experimentation to guess the right values - and it does not provide the answer for arbitrary q. The general technique to answer the question will be presented later.

Chapter 2

Markov chains: steady state probabilities

2.1 Classification of states

Consider a time-homogeneous Markov chain with transition matrix P. We say that state j *is accessible from* state i, if $p_{ij}^{(n)} > 0$ for some $n \geq 0$, that is, if we can eventually get to state j from state i with positive probability. (For convenience, we assume $p_{ii}^{(0)} = 1$, that is, every state is accessible from itself.)

> Two states i and j *communicate*, if j is accessible from i, and i is accessible from j. A Markov chain is *irreducible* if any two states can communicate.

The state transition diagram is helpful to decide which states are accessible from which other states. State j is accessible from state i if and only if there is a directed path from i to j in the state transition diagram. A Markov chain is irreducible if and only if every state can be reached from every other state on a directed path.

In the gambler's example (Fig 1.1) state 3 is accessible from state 2 but not vice versa; hence 2 and 3 do not communicate. Consequently, the gambler's problem is *not* irreducible. On the other hand, states 1 and 2 communicate. In the weather problem (Fig 1.2) any state is accessible from any other state and the Markov chain is therefore irreducible.

Communication between states has the following important properties.

(i) *Reflexivity:* Any state communicates with itself.

(ii) *Symmetry:* If state i communicates with state j, then state j can also communicates with state i.

(iii) *Transitivity:* If state i communicates with state j, and state j communicates with state g, then state i also communicates with state g.

The first two properties directly follow from the definition of communication between states. For the third property, we show that g is accessible from i; the proof that i is accessible from g is the same. These two claims together yield that i and g communicate. If j is accessible from i, then there exists a value n such that $p_{ij}^{(n)} > 0$, that is, we can reach j from i in finite number of steps. Similarly, there exists a value n' such that $p_{jg}^{(n')} > 0$. But then it follows (e.g. by the Chapman-Kolmogorov equations), that $p_{ig}^{(n+n')} \geq p_{ij}^{(n)} p_{jg}^{(n')} > 0$.

Relations having the three properties reflexivity, symmetry and transitivity are known as *equivalence relations*. Such a relation enables us to partition the set of all states into disjoint classes:

> A *class* of states in a Markov chain is a maximal set of states that can mutually communicate with each other. The classes form a partition of the states: every state belongs to exactly one class.

The Markov chain is irreducible if every state belongs to the same class. Classes can have just a single element. In the gambler's problem there are three classes: $\{0\}, \{1, 2\}$ and $\{3\}$. The Markov chain on Figure 2.1 has 3 classes: $\{0\}, \{1, 2\}, \{3, 4, 5\}$.

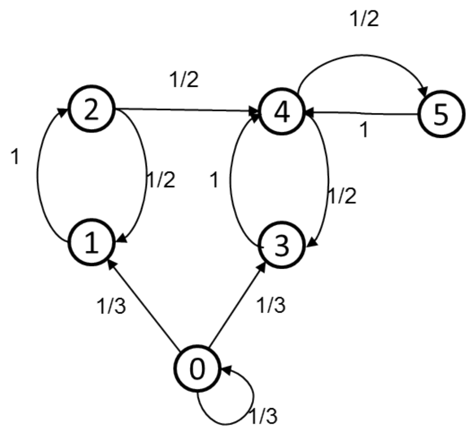

Figure 2.1: A Markov chain with three classes

Recall the notion of *absorbing states*: once such a state is reached, the process remains in the same state forever. Clearly, every absorbing state forms a single element class (as states 0 and 3 in the gambler's problem); on the other hand, not every single element class is an absorbing state (e.g. state 0 in the previous example).

There is a notable difference between classes $\{1, 2\}$ and $\{3, 4, 5\}$ in the example. After the process enters the class $\{1, 2\}$, it will eventually leave: in state 2, there is a chance 1/2 to move to state 4, and once this happens, the process never visits states 1 or 2 again. Once the process enters the class $\{3, 4, 5\}$, it will stay inside this class forever. This motivates the following distinction between states:

- *Transient state:* Upon leaving this state, the process may never return again. Equivalently, the state i is transient if there is another state j that is accessible from i, but i is not accessible from j.

- *Recurrent state:* Upon leaving such a state, the process will certainly return to the same state.

Hence every state is either transient or recurrent. Absorbing states are a specific type of recurrent states. States in the same class are either all transient or all recurrent. Every state in an irreducible Markov chain is recurrent.

Transient states may belong to different classes as we have seen in the example in Figure 2.1. Note that recurrent states may also belong to different classes. In the gambler's problem, both states 0 and 3 are absorbing and thus recurrent, but they are in different (single element) classes.

Periodicity and ergodicity

Another important property of Markov chains will be *periodicity.* We define periodicity for states that the system can return to: for states i such that $p_{ii}^(n) > 0$ for some $n > 0$.

The *period* of state i is the largest integer k such that $p_{ii}^{(n)} > 0$ implies that $n > 0$ is an integer multiple of k. A state is called *periodic* if it has a period $k > 1$, and *aperiodic* if its period is $k = 1$.

That is, if the system is in state i at time zero, then we can only observe state i in periods that are multiples of k. Thus k will be the Greatest Common Divisor (GCD) of these periods. We can also write:

$$period(i) = gcd\{n \in \mathbb{N}^+ : p_{ii}^n > 0\}. \tag{2.1}$$

In Figure 2.1, if we start from state 3, we can only get back here in even time steps - hence state 3 has a period 2.

On the other hand, all states in the weather example are aperiodic. For example, if it is rainy on day 0, it can be rainy again on day 2, but also possibly on days 3 and 4. If we can find two consecutive time periods t and $t + 1$ such that the system can reach state i in both these periods, then i must be aperiodic.

We now give the definition describing a class of Markov chains that behaves nicely.

A Markov chain is *ergodic*, if all states are recurrent and aperiodic.

The gambler's example and the one on Figure 2.1 are not ergodic since they contain transient states. Restricting Figure 2.1 to states 3, 4 and 5 would make all the states recurrent but the Markov chain would not be ergodic, since all states have period 2. However, the weather example is ergodic (all states are both recurrent and aperiodic).

2.2 Steady-state probabilities

Computing the n-step transition matrices indicated that the weather problem converges to a phase with fixed probabilities for every state, however, this property does not hold for the gambler's problem. The main reason turns out to be that the weather example is *irreducible and ergodic*, while the gambling example is not. A fundamental theorem on Markov chains claims that such a convergence is observed for every irreducible ergodic Markov chain.

> For every irreducible ergodic Markov chain, there exists positive probabilities $\pi_j > 0$ for each state, called the *steady-state probabilities*, such that starting from an arbitrary state i,
>
> $$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j$$
>
> holds. The π_j's are the unique solutions of the *steady state equations*:
>
> $$\begin{aligned} \pi_j &= \sum_{i=0}^{M-1} \pi_i p_{ij}, \qquad \text{for all } j = 0, 1, \ldots, M-1, \\ \sum_{j=0}^{M-1} \pi_j &= 1. \end{aligned}$$

This means that if we examine the system after a large number of steps, we shall find it in state j with probability π_j, no matter where we started from. The equalities capture precisely this: we can reach state j from state i with probability p_{ij}, and the system is in state i with probability π_i. In a more concise matrix form, the equalities can be written as

$$\begin{aligned} P^T\pi &= \pi \,(\text{ or } \pi P = \pi)\,, \\ \mathbf{1}\pi &= 1, \end{aligned}$$

where $\mathbf{1}$ denotes the M-dimensional row vector of all ones.

Consider yet again the weather example. The equations give

$$\begin{aligned} \pi_0 &= \frac{1}{3}\pi_1 + \pi_2, \\ \pi_1 &= \frac{1}{2}\pi_0 + \frac{1}{3}\pi_1, \\ \pi_2 &= \frac{1}{2}\pi_0 + \frac{1}{3}\pi_1, \\ 1 &= \pi_0 + \pi_1 + \pi_2. \end{aligned}$$

The vector $(0.4, 0.3, 0.3)$ identified on the previous lecture is the unique solution to this system.

Note that there are $M+1$ equations for M variables; at least one of the equations must be therefore redundant. The last one cannot be redundant, since the zero vector satisfies the first M equations but it does not satisfy the last one. Thus, one of the M equations $P^T\pi = \pi$ is redundant; the reason is that the P matrix has the property that each row sums to 1. Thus, taking any $M-1$ of the $P^T\pi = \pi$ equations plus the last one, $\mathbf{1}\pi = 1$, would give a unique solution as long as the Markov chain is irreducible and ergodic.

The irreducibility and ergodicity guarantees the uniqueness of the solutions. This is not true in general (see exercises).

2.2.1 Necessity of conditions

Why assume that the Markov chain is irreducible and ergodic? Let us examine these properties individually. Recall that ergodic means that every state is recurrent and aperiodic. For transient states, there is a positive probability that the process will not return anymore after a certain time period. After waiting long enough, the probability of observing the Markov chain in a transient state tends to 0.

Consider now periodicity, illustrated in the simple example of the following transition matrix.

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Every state in this Markov chain has a period 3, as it keeps traversing a length 3 cycle. Solving the steady state equations yields $\pi_0 = \pi_1 = \pi_2 = \frac{1}{3}$, which makes sense: the Markov chain will be on average in all three states with equal probability. However, the main requirement

$$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j$$

does not hold. We will always have $p_{00}^{(n)} = 1$ if n is divisible by 3 and $p_{00}^{(n)} = 0$ otherwise. The system will oscillate between different states rather than converging to a steady state.

In the exercises you will be asked to study an ergodic but not irreducible Markov chain.

2.3 The PageRank algorithm

Google's PageRank algorithm for website ranking can be seen as an application of Markov chains. It is named after Google co-founder Larry Page, and was developed in the late 1990s. The purpose of this algorithm is to determine an importance ranking of the websites. The basic variant we study here provides an absolute ranking; for practical implementation, it needs to be further refined to match the topics and queries.

The Internet can be modelled by a directed graph, with the vertices representing individual websites, and a directed arc (i, j) denoting a hyperlink from site i to site j. As a first idea, we could associate importance based on the number of hyperlinks pointing to the site. Such a system could be gamed very easily: to promote your own website, you could simply create numerous phantom websites linked to it i.e. pointing to your website.

Obviously, a distinction has to be made between a link from a humble LSE lecturer's personal website and a link from HM the Queen's website (`https://twitter.com/BritishMonarchy`). The intuition would be to find a ranking with the property that

a page is important if it is linked by important pages.

This is a circular definition; but as it turns out, Markov chains are very appropriate to capture such a definition.

To formalise the intuition on importance, assume that if a website i has importance x, then it "gives" an equal share of this importance to all sites it links. That is, if website i has ℓ outgoing links, all its neighbours receive importance x/ℓ. The importance of every website would be equal to the importance given by those linking to it.

This is very similar to studying the steady-state distribution of a time-homogeneous Markov chain with states corresponding to websites, and transition probabilities

$$p_{ij} = \frac{1}{\ell} \text{ if } i \text{ has a hyperlink to } j \text{ and } i \text{ has } \ell \text{ hyperlinks in total.}$$

The above constraints on importance are precisely given by the steady-state equalities $P^T\pi = \pi$.

The condition $\mathbf{1}\pi = 1$ was not imposed. In fact, 0 is a solution to $P\pi = \pi$, and if π_0 is a solution, then so is $\pi = \alpha\pi_0$ for every real number α. We can impose $\mathbf{1}\pi = 1$ for the purpose of normalisation, in order to pick a single importance measure. This is a natural choice as it gives a probability distribution. Another possible choice would be $\mathbf{1}\pi = n$, where n is the total number of websites; this gives an average page rank of 1.

The model captures the browsing of a random surfer, who starts from an arbitrary website and moves following the hyperlinks. Let us define a time-homogeneous Markov chain with the states being the individual website. It is assumed that from each site, the random surfer picks a new hyperlink uniformly at random.

2.3.1 Dealing with non-ergodicity

The steady-state probabilities in this process seem good candidates to rank the pages. Nevertheless, these are only guaranteed to exists in irreducible ergodic Markov chains. The link graph of the web is *not ergodic*. A simple reason is that there are websites not linked by any other; thus once you leave them you can never come back: all these are transient states.

This can occur in less obvious ways as well: there can be a set of pages linked to each other, with no links to the outside world. In this context, such sets are called *"spider traps"*.

An easy way to get around this is a simple trick called *"random teleportation"*. Fix some constant $0 < d < 1$, and modify the random surfer model so that with probability d, he restarts the browsing by jumping to an arbitrary page on the web uniformly at random; with probability $1 - d$, he follows one of the outgoing links uniformly. This way we can get from an arbitrary website to an arbitrary other one, and thus the Markov chain becomes ergodic and irreducible.

There is one more issue to cope with: pages with no outgoing links, (called *"dead ends"*). The original process is undefined for such pages. Even the modified process does not work, since no further moves can be made with probability $1 - d$.

The solution is rather simple: once a dead end is reached, we jump to an arbitrary website uniformly at random with probability 1, instead of probability $d < 1$.

This way we really get a well-defined irreducible and ergodic Markov chain. The PageRank algorithm aims to determine the steady-state distribution of this Markov chain for a certain value of d.

2.3.2 The Power Iteration Method

With the above modifications, the steady state equations are guaranteed to have a unique solution. Computing these raises further difficult problems: the number of websites is in the order of billions, and therefore it is infeasible to solve via Gaussian elimination. Instead, the following *Power Iteration Method* is used. Let P be the transition matrix of the Markov chain, and start with an arbitrary positive vector $r^{(0)}$ on the states such that the components sum up to 1. For example, if there are M states, we can start from all components being $r_i^{(0)} = 1/M$. Let us repeatedly compute

$$r^{(t+1)} = P^T r^{(t)}$$

As $t \to \infty$, $r^{(t)}$ will converge to the steady state probability vector π. In practice, it will converge at a fairly rapid rate. However, even computing this matrix multiplication is very challenging for matrices of astronomical size. It requires lots of computational cleverness, exploiting e.g. the fact that the link structure on the web is sparse - typical pages are linked to at most dozens of others.