MA423 - Fundamentals of Operations Research
Lecture 10: Game Theory

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Chapter 1

Game Theory: Introduction

1.1 Basic concepts

The theory of games deals with conflict situations in which the outcome depends on the strategies chosen by each of the players. A game is a social situation in which there are several individuals, each pursuing their own interest and in which no single individual can determine the outcome, e.g. many economic situations, warfare. Many conflicts involve an element of chance (e.g. card games, the weather in an agricultural situation). We will not discuss games of pure chance, e.g. roulette, but only games in which the skill and intelligence of the players are useful, that is, games of strategy.

In a game situation there are:

(1) n decision makers (players), where $n \geq 2$.

(2) Rules specifying available courses of action (strategies), known to the players.

(3) A well-defined set of *outcomes*, and rules specifying which outcome will occur.

(4) A *payoff* to each player associated with each outcome, known to all the players.

1.1.1 Classifications of games

Games may be classified according to the following characteristics:

(1) *Number of players.* We distinguish *2-person games* and *more-than-2-person games.* In this course we shall be concerned with 2-person games; that is, the people participating necessarily fall into 2 mutually exclusive groups, each group having identical objectives.

(2) *Nature of payoff.* We distinguish *zero-sum games* from *non-zero-sum games.* A zero-sum 2-person game is one in which for any outcome of the game one players gain is the other players loss. Exactly equivalent is a *constant-sum game*, in which the payoffs to players 1 and 2 sum to a quantity which is constant over the outcomes. In either case the interests of the two players are directly opposed. These two lectures will deal with zero-sum games.

(3) *Communication between players.* We distinguish *cooperative games* and *non-cooperative games.* In non-zero-sum games, there may be mutual benefits to be obtained from the coordination of strategy choices. We shall be concerned with non-cooperative games in which no pre-play communications between players is possible.

1.2 Extensive and Normal Forms

An n-person game in extensive form is a tree representing the moves of the game. We use the following example to demonstrate the extensive form.

Example 1.1. Simplified poker. There are only two cards involved - an Ace and a Two - and only two players, called Alice and Bob. Each puts £1 in the pot and Alice deals Bob one card, which Bob then looks at. If it is the Ace, Bob must say 'Ace', but if it is the Two, Bob can say 'Ace' or 'Two'.

- If Bob says 'Two', he loses the game and his £1 in the pot.
- If Bob says 'Ace' no matter what the card is, he must put another £1 in the pot. In this case Alice can either believe him ('fold'), and so lose her £1 in the pot, or she can demand to see the card ('call'). In this case Alice has to put another

£1 into the pot, and then the card is shown. If Bob had the Ace he wins the pot and so takes £2 from Alice; but if Bob has the Two, Alice wins the pot and so takes £2 from Bob.

This game involves the elements of 'bluffing' and 'calling' involved in real poker, but not surprisingly has not yet taken Las Vegas by storm. The extensive form for the above game is shown in Figure 1.1. It has the following properties.

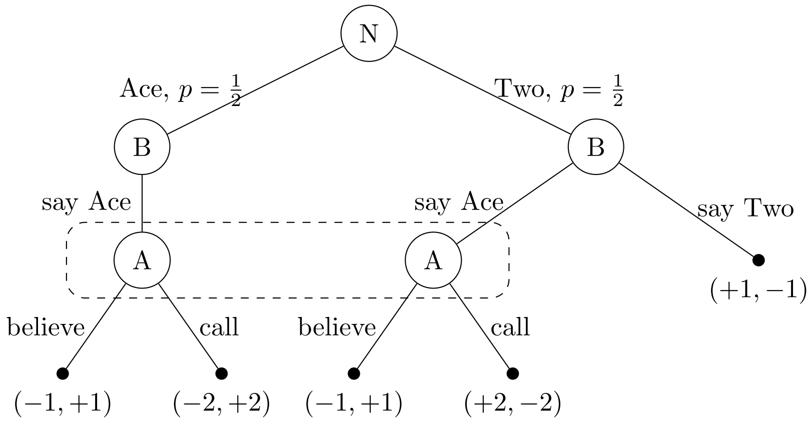


Figure 1.1: The extensive form of Simplified Poker

(i) The tree has an initial node.

(ii) Each node is labelled either with N (for Nature) or with the name of one of the n players.

(iii) A node labelled N represents random variables that are important to the game. The edges that lead out of an N node represent the different outcomes of the random variable. Thus each of these edges is labelled with the outcome and the probability of the outcome.

(iv) The nodes that are labelled with the player names are decision nodes and each edge leading from them represents an action (part of a strategy) of the player.

(v) Each path from the initial node through a leaf describes a scenario of the game: it specifies the outcome of the random variables and the decisions of the players. For each scenario we can calculate the *outcome of the game* and thus calculate the payoff to each player. We label each leaf with the payoff to the players. In a zero sum game we only need to declare the payoff to the first player (since the other player gets the negative of that), but here for emphasis we include both. So $(-1, +1)$ means that Alice loses 1 and Bob gains 1.

(vi) The decision nodes of each player are partitioned into disjoint subsets called *information sets*. Nodes are in the same information set if the player has the same information on each of these nodes. A player is presumed to know which information set he or she is in, but not which node of the set. Any two nodes in the same set must have the same set of choices (edges with identical labels) leading from them.

On the diagram, the two nodes of Alice belong to the same information set (therefore joined in a dotted rectangle). The two nodes of Bob belong to different information sets: depending on the card he sees, Bob knows which of the two nodes he is currently in.

If all the information sets consist of one node only, the game is one of *perfect information*. A finite game of perfect information can be solved (i.e. optimal actions can be obtained) by "reducing it backward", in a method similar to dynamic programming. We will not cover the latter in this course.

A *strategy* perfectly determines the actions of each player in each situation that the player is in. Specifically, a player's situations are determined by the player's information sets. So a strategy needs to specify an action for each information set. For example for Bob the information sets are $\{\text{Ace}\}$ and $\{\text{Two}\}$. A strategy would be: when you have an Ace say Ace and when you have a Two say Ace. Of course in the case that he has an Ace he only has one action (to say Ace) but we specify the action here for completeness.

1.2.1 Normal form of games

An n-person game in strategic (or normal) form consists of a set of X_i of strategies for each player i, and a set P_i of payoff functions. If player i chooses strategy $s_i \in X_i$ for $i = 1, \ldots, n$, then the payoff to player j is $P_j(s_1, \ldots, s_n)$. We can imagine that the game is played by each player selecting a strategy and handing it to a referee. He then announces all the strategies and the resulting payoffs to each player.

We shall mainly be concerned with two-person games in strategic form. Player 1 can be thought of as choosing a row of rectangular array while player 2 selects a column. In each cell of the array are two numbers giving the resulting payoffs to the two players. A special case arises when one player's gain always equals the other player's loss: the sum of the two entries in each cell is then zero. This is the case in two player zero-sum games.

We can write the Simplified Poker game in a strategic form as follows. Both players have two possible strategies:

Alice	A1	believe Bob when he says 'Ace' *(fold)*;
	A2	do not believe Bob when he says 'Ace' *(call)*;
Bob	B1	say 'Two' when you have a Two;
	B2	say 'Ace' when you have a Two *(bluff)*.

Each of these strategies perfectly determine the decisions to be made by the players (see end of previous section for definition of strategies). We can construct the payoff matrix, containing the expected payoffs for Alice and Bob, with negative values expressing loss. Note that the two numbers always sum up to 0 - this is a zero-sum game.

	B1	B2
A1	$(0, 0)$	$(-1, 1)$
A2	$(-\frac{1}{2}, \frac{1}{2})$	$(0, 0)$

The rules of the game include a probabilistic event, namely the card Bob receives can be Ace or Two, both with probability 1/2. The payoffs here were calculated as the *expected values* with respect to this event, as follows.

A1 vs B1 — If Bob gets an Ace, he says 'Ace' and Alice believes him (according to strategy A1), so Alice has payoff -1. If Bob gets the Two, he says (according to strategy B1) 'Two', and so Alice gets payoff $+1$ straightaway. Both cards appear with a chance 1/2, so the expected payoff to Alice is

$$\frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$$

and thus the expected payoff to Bob is also 0.

A1 vs B2 No matter what Bob gets, he always says ‘Ace’; Alice always believes him and gets payoff -1. So the expected payoff to Alice is

$$\frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot (-1) = 0$$

and the expected payoff to Bob is thus $+1$.

A2 vs B1 If Bob gets an Ace, he says ‘Ace’, but Alice does not believe him, and hence Alice gets -2. If Bob gets a Two, he says so, and Alice gets $+1$. The expected payoff to Alice is

$$\frac{1}{2} \cdot (-2) + \frac{1}{2} \cdot 1 = -\frac{1}{2}.$$

A2 vs B2 Bob always says ‘Ace’, and Alice never believes him. When the card is shown, if it is an Ace, then Alice gets -2; if it is a Two, then she gets $+2$. The expected payoff to Alice is

$$\frac{1}{2} \cdot (-2) + \frac{1}{2} \cdot (+2) = 0.$$

1.3 Two player zero-sum games

Provided a gain to player A is always an equivalent loss to player B (that is, the game is zero-sum) the game can be represented by a payoff matrix where the rows represent strategies of A, and the columns strategies of B. The payoff is conventionally in terms of gain to the row-player, but may be negative, if it represents a loss.

$$
\begin{array}{cc}
 & \text{B} \\
 & \begin{array}{cccc} 1 & 2 & \dots & m \end{array} \\
\text{A} & \begin{array}{c} 1 \\ 2 \\ \vdots \\ n \end{array}
\begin{bmatrix}
r_{11} & r_{12} & \dots & r_{1m} \\
r_{21} & r_{22} & & r_{2m} \\
 & & \ddots & \\
r_{n1} & r_{n2} & & r_{nm}
\end{bmatrix}
\end{array}
$$

In a game, decisions are taken under uncertainty - no probabilities can be ascribed to the occurrence of different opposing strategies. Since one's opponent is assumed rational and has opposing interests, it is reasonable to expect the worst result which your strategy permits. Thus player A can expect player B to try and minimize A's payoff since B's lose is A's payoff. Thus, A can look at the worst case scenario of each strategy (the minimum of each row) and pick the strategy that maximises this.

This leads to the *minimax criterion*:

Minimax criterion. Choose your strategy so as to minimise the maximum loss you can sustain (or equivalently, maximise your minimum gain).

1.3.1 Pure strategy solutions

The minimax criterion may be applied by listing the row minima and column maxima, and finding the largest of the former and the smallest of the latter - in the example below these are starred. The largest row minimum is the payoff A can guarantee himself. The smallest column maximum is a loss B can be sure not to exceed. When they are equal (as in the example) the game has *a solution in pure strategies*, and their joint value is the *value of the game* to A. A game is *fair* if its value is zero.

$$
\begin{array}{cc|cc|c}
 & & \multicolumn{2}{c|}{\text{B}} & \\
 & & 1 & 2 & \\
\hline
\text{A} & 1 & 1 & 5 & \mathbf{1} \\
 & 2 & \mathbf{2} & 3 & \mathbf{2}* \\
\hline
 & & \mathbf{2}* & \mathbf{5} &
\end{array}
$$

We say that a pair of strategies is *a saddle point*, if the payoff for this pair is one of the largest elements in its column, and at the same time, one of the smallest in its row. If either player departs from the saddle-point strategy, he is bound to lose. It is easy to verify that if a game has a solution in pure strategies, such a pair of strategies will be always be a saddle point, and vice versa: every saddle-point corresponds to a solution in pure strategies. A solution in pure strategies was chosen because it was for the row player the minimum in its row and from the column player the maximum in its column, which is the definition of a saddle point. We leave the other direction to the reader to verify.

As we will see later, not every game has a saddle point; if it has, there can be more than one. In this case, all saddle-points must have the same value (see Exercises).

Let us see some further examples. The following game has a solution in pure strategies and is fair because the value is 0.

$$
\begin{array}{cc|cc|c}
 & & \multicolumn{2}{c|}{\text{B}} & \\
 & & 1 & 2 & \\
\hline
\text{A} & 1 & 0 & -2 & -\mathbf{2} \\
 & 2 & \mathbf{0} & 2 & \mathbf{0}* \\
\hline
 & & \mathbf{0}* & \mathbf{2} &
\end{array}
$$

The next one has a solution in pure strategies but is not fair.

$$\begin{array}{cc|c}
 & \text{B} & \\
 & \begin{array}{cc} 1 & 2 \end{array} & \\
\text{A} \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 5 & \mathbf{2} \\ -7 & -4 \end{bmatrix} & \begin{array}{c} \mathbf{2}* \\ -\mathbf{7} \end{array} \\
 & \begin{array}{cc} \mathbf{5} & \mathbf{2}* \end{array} &
\end{array}$$

The next game does not have a solution in pure strategies: the maximum of row minima is 1, whereas the minimum of the row maxima is 2. The pair of strategies (A1,B1) is not a saddle point: A could change his strategy to A2 and get a better payoff.

$$\begin{array}{cc|c}
 & \text{B} & \\
 & \begin{array}{cc} 1 & 2 \end{array} & \\
\text{A} \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 1 & 5 \\ 2 & 0 \end{bmatrix} & \begin{array}{c} \mathbf{1}* \\ \mathbf{0} \end{array} \\
 & \begin{array}{cc} \mathbf{2}* & \mathbf{5} \end{array} &
\end{array}$$

The same method applies to larger games. The following 4×4 game has a solution in pure strategies. The optimal strategies are A3 and B2, and the value of the game is

3.

$$
\begin{array}{cc}
 & B \\
 & \begin{array}{ccccccc}
 & & 1 & 2 & 3 & 4 & \\
 & 1 & \left[\begin{array}{cccc} 7 & 2 & 5 & 1 \\ 2 & 2 & 3 & 4 \\ 5 & \mathbf{3} & 4 & 4 \\ 3 & 2 & 1 & 6 \end{array}\right] & & & & \begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3}* \\ \mathbf{1} \end{array}
\end{array}
\end{array}
$$

$$
A \quad \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} 7 & 2 & 5 & 1 \\ 2 & 2 & 3 & 4 \\ 5 & \mathbf{3} & 4 & 4 \\ 3 & 2 & 1 & 6 \end{array}\right] \\ \begin{array}{cccc} \mathbf{7} & \mathbf{3}* & \mathbf{5} & \mathbf{6} \end{array} \end{array}
\begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3}* \\ \mathbf{1} \end{array}
$$

1.3.2 Dominance

If every value in a row of the payoff matrix, R1, say, is greater than the corresponding values of another row, R2, the former strategy *strictly dominates* the second one. Hence, whatever may be player B's chosen strategy, player A will always be better off playing the strategy corresponding to R1 than if he follows the strategy of R2. This is almost the situation with A3 and A2 above, except that the payoff to A is 4 if B plays B4 in both cases. When every value in a row R1 is either greater than or equal to the corresponding value of another row R2 the former *weakly dominates* the second (but not strictly so). We say that a row dominates that other if there is either strict or weak domination. Thus, A3 dominates A2, so that A should never use A2 and we can remove it from consideration (effectively deleting it from the game).

Similarly, Player B always wishes to pay out as little as possible, so that if every value of a column C1 is strictly less than the corresponding value of another column C2, the first column strictly dominates the second, and player B should not choose this second strategy. In an analogous way to player A above, weak dominance is defined for the column player. In the above example, the second column B2 weakly dominates the first column B1.

If we delete strategies of the two players that are non-strictly dominated, then the value of the game is still unaltered by deletions, but some saddle-points may be eliminated (where there are multiple saddle points).

From the above matrix, we obtain by domination the following 3×3 matrix.

$$
\begin{array}{cc}
 & \text{B} \\
 & \begin{array}{ccc} 2 & 3 & 4 \end{array} \\
\text{A} \begin{array}{c} 1 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 2 & 5 & 1 \\ 3 & 4 & 4 \\ 2 & 1 & 6 \end{bmatrix} \begin{array}{c} \mathbf{1} \\ \mathbf{3}* \\ \mathbf{1} \end{array} \\
 & \begin{array}{ccc} \mathbf{3}* & \mathbf{5} & \mathbf{6} \end{array}
\end{array}
$$

Note that we cannot reduce it any further using dominance.

Chapter 2

Two-person zero-sum games - mixed strategies

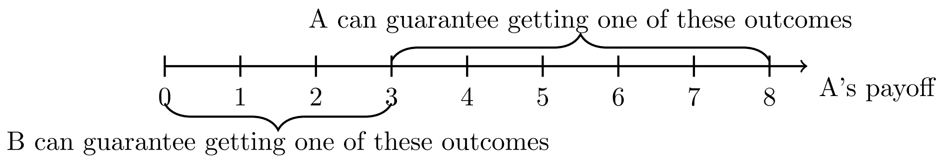
2.1 Mixed strategies

Let us revisit the solution concept from the previous lecture. Consider the following matrix for a two-person zero-sum game.

$$
\begin{array}{cccccc}
 & & & \text{B} & & \\
 & & 1 & 2 & 3 & \\
 & 1 & \left[\begin{array}{c} 7 \\ 5 \\ 0 \end{array}\right. & \begin{array}{c} 2 \\ 3 \\ 1 \end{array} & \left.\begin{array}{c} 8 \\ 4 \\ 6 \end{array}\right] & \begin{array}{c} \mathbf{2} \\ \mathbf{3}* \\ \mathbf{0} \end{array} \\
\text{A} & 2 & & & & \\
 & 3 & & & & \\
 & & \mathbf{7} & \mathbf{3}* & \mathbf{8} &
\end{array}
$$

The row minima have a maximum of 3 and the column maxima have a minimum of 3. Player A can guarantee 3 by playing strategy A2, and Player B can hold A to 3 by playing B2 (i.e. can guarantee losing 3 or less). The central '3' is a saddle point of the matrix. There are two good reasons for claiming that the strategy pair $(2, 2)$ is a solution to this game:

(a) What A can guarantee equals what B can hold him to. Since the game is zero-sum this outcome must be the solution: A will not accept less than he can guarantee, not will B let him get more than she can hold him to. Diagrammatically the possible outcomes (payoffs to A) are the following:

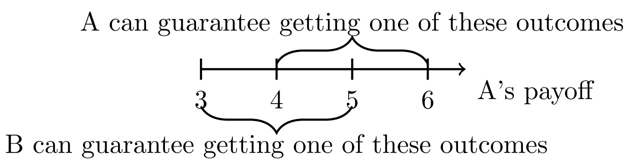


(b) The strategy pair $(2, 2)$ is a saddle point, and therefore an *equilibrium*. This means that neither player has any incentive to change his strategy since this point is the maximum payoff in its column (thus A doesn't want to move) and it is the minimum in its row (thus B doesn't want to move).

Now consider a game matrix with no saddle point.

$$
\begin{array}{ccc}
 & & \text{B} \\
 & & \begin{array}{cc} 1 & 2 \end{array} \\
\text{A} & \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 3 & 6 \\ 5 & 4 \end{bmatrix} \begin{array}{c} \mathbf{3} \\ \mathbf{4}* \end{array} \\
 & & \begin{array}{cc} \mathbf{5}* & \mathbf{6} \end{array}
\end{array}
$$

In this game the maximum of the row minima is 4, whilst the minimum of the column maxima is 5. We get the following diagram:



No pair of *pure* strategies can be considered to be a solution. Is there any way that A can guarantee more than 4 and B hold him to less than 5? There is, if they randomise their strategy choices. A *mixed strategy* is a combination of pure strategies in a certain proportion, the choice of which strategy to play being made by a random device with the appropriate probabilities.

So if A plays 1 with probability $\frac{1}{3}$ and 2 with probability $\frac{2}{3}$ then

- if B plays 1, the expected gain to A is $\frac{1}{3} \cdot 3 + \frac{2}{3} \cdot 5 = 4\frac{1}{3}$; and
- if B plays 2, the expected gain to A is $\frac{1}{3} \cdot 6 + \frac{2}{3} \cdot 4 = 4\frac{1}{3}$.

Hence the mixed strategy $(\frac{1}{3}, \frac{2}{3})$ guarantees A at least $4\frac{1}{3}$ in expectation.

But how can we find the optimal mixed strategies? For 2x2 games this is quite straightforward. Before we proceed let us look at a graphical representation of this game as shown in Figure 2.1.

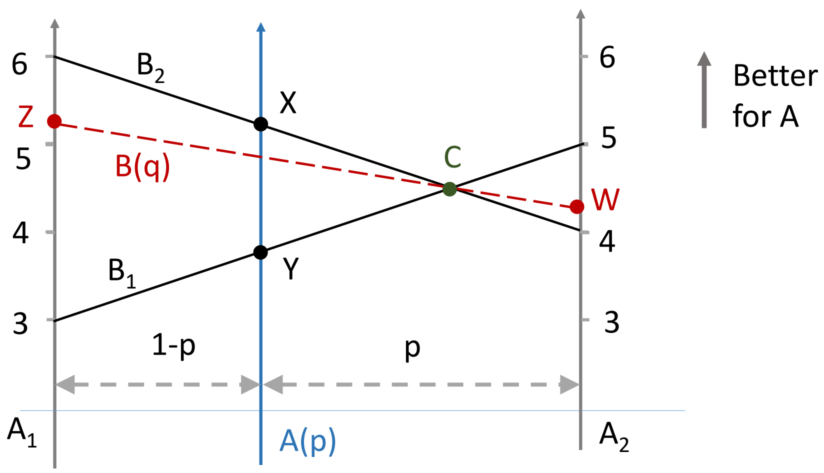


Figure 2.1: Graphical representation of 2x2 game. The equilibrium is at point C.

Each of A's strategies is represented by an axis A_1 on the left hand side and A_2 on the right hand side. The vertical coordinate is the value of the game and A wants it as high as possible and B as low as possible. We let $V(P)$ denote the vertical coordinate of some point P. The horizontal coordinate depends on p, which is the probability that A plays A_1 whereas $1 - p$ is the probability that A plays A_2. A's decision would be to choose the value of p and this limits the game to the line $A(p)$ shown on the diagram. For example, if A plays $A(p)$ and B plays B_2 then the solution would be point X and the payoff to A (loss to B) would be $V(X)$. Given that A plays $A(p)$, B can mix between B_1 and B_2 and arrive at any point between points X and Y. If p and $A(p)$ are as shown on the diagram then the best B can do is play B_1 and end up at point Y. Whatever A plays B will try to lower his payoff as much as possible, thus A needs to pick a point where the lowest point B can take him is maximized. This is point C, the intersection of B_1 and B_2 and the payoff of A will be $V(C)$.

This is the point where $V(X) = V(Y)$, that is this is the point where the expected payoff from B_1 is equal to the expected payoff from B_2 given that A plays $A(p)$. Equating these to the value v we get:

$$3p + 5(1-p) = v$$
$$6p + 4(1-p) = v,$$

which gives $p = \frac{1}{4}$ and $v = 4\frac{1}{2}$.

Now let's look at B's strategies. B has to pick a q to play strategy B_1 and will play B_2 with $1-q$. Mixing those for some q will create a line $B(q)$ as shown on the diagram, which is a weighted combination of the lines of B_1, B_2 that passes through their intersection point C. Now B knows that A will try and settle at a point that has the highest vertical coordinate and thus would like to pick q in such a way that the worst case scenario given by $B(q)$, that is its highest point, is minimized: the way to do that is to make $B(q)$ horizontal (flat) which will be at the height of point C and thus have payoff $V(C)$. To find the value of q that gives this we note that the expected payoff from A_1 and $B(q)$, which is $3q + 6(1-q)$, should equal to the expected payoff from A_2 and $B(q)$, which is $5q + 4(1-q)$, that should also equal to v (since $B(q)$ is horizontal):

$$3q + 6(1-q) = v$$
$$5q + 4(1-q) = v.$$

This gives $q = \frac{1}{2}$ and $v = 4\frac{1}{2}$ as we got earlier.

Thus the best mixture for A is $(\frac{1}{4}, \frac{3}{4})$, which guarantees him $4\frac{1}{2}$. The mixed strategy $(\frac{1}{2}, \frac{1}{2})$ guarantees B a loss of no more that $4\frac{1}{2}$. If A plays $(\frac{1}{4}, \frac{3}{4})$ and B plays $(\frac{1}{2}, \frac{1}{2})$, then neither player has any incentive to change his strategy. Hence by allowing mixed strategies we regain the two properties (a) and (b) that we noted for the pure strategy solution to the first game.

This is always true for 2 person zero-sum games. What A can guarantee (in expectation) by choice of the best mixed strategy will always equal what B can hold him to. This is called the *value* of the game. Mixed strategies that guarantee the value of the game in expectation are called *optimal*. Any pair of optimal mixed strategies for A and B will be in equilibrium, and will give both players their guarantees (in expectation).

In a 2×2 zero-sum game, it is possible to write formulas to calculate the optimal mixed strategies explicitly. The existence of such strategies is true for an arbitrary (finite) number of strategies of the player. Von Neumann's famous theorem shows that this holds in arbitrary two-person zero-sum games.

Theorem 2.1 (John von Neumann, 1944)**.** *In every two-person zero-sum game, the maximum expected value the row player A can guarantee equals the minimum expected value the column player B can guarantee.*

In the next section, we will derive this theorem using Linear Programming. Before that, let us remark for the interested reader that the theorem is true under much more general circumstances. In a finite n person game, we consider a mixed strategy for every player. We say that these mixed strategies form a *Nash equilibrium*, if no player can achieve a better expected payoff by unilaterally changing his strategy, that is, if all other players' strategies remain unchanged.

Theorem 2.2 (John Nash, 1951)**.** *In every game with a finite number of players where every player has a finite number of strategies, there exists a Nash equilibrium.*

2.2 Games and Linear Programming

We derive von Neumann's theorem, by showing that optimal mixed strategies of both players can be obtained using Linear Programming in every two-person zero-sum game. We demonstrate the method on a particular example.

Consider the 3×3 game below; there is no pure strategy solution, and the game is irreducible by dominance. Let (x_1, x_2, x_3) express the mixed strategy of row player A; that is, x_i is the probability of playing strategy i, for $i = 1, 2, 3$. We require $x_1 + x_2 + x_3 = 1$ in order to have a probability distribution. Similarly, let (y_1, y_2, y_3) express the mixed strategy of the column player B.

$$
\begin{array}{cc}
 & \text{B} \\
 & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\text{A} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 2 & -1 \\ -2 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \\
 & \begin{array}{ccc} y_1 & y_2 & y_3 \end{array}
\end{array}
$$

For each of the strategies of A, we can express the expected payoff against the B's strategy (y_1, y_2, y_3):

$$
\begin{aligned}
A1 :&\quad y_1 + 2y_2 - y_3, \\
A2 :&\quad -2y_1 + y_2 + y_3, \\
A3 :&\quad 2y_1 + y_3.
\end{aligned}
$$

Since player B wants to minimise the expected payoff A can achieve using any of his strategies, his objective is to keep the maximum among these three values as small as possible. This is captured by the following linear program.

$$
\begin{aligned}
&\min v \\
y_1 + 2y_2 - y_3 &\leq v \\
-2y_1 + y_2 + y_3 &\leq v \\
2y_1 + y_3 &\leq v \\
y_1 + y_2 + y_3 &= 1 \\
y_1, y_2, y_3 &\geq 0.
\end{aligned}
$$

We introduce free variable v to be greater than all the above payoffs and when we minimize v its value will be equal to the maximum of these payoffs.

Using a linear programming solver, we obtain the optimal solution

$$y_1 = \frac{2}{17}, \quad y_2 = \frac{8}{17}, \quad y_3 = \frac{7}{17}, \quad v = \frac{11}{17}.$$

A similar reasoning can be used to find the optimal mixed strategy (x_1, x_2, x_3) of A. We calculate the expected payoff against every strategy of B. Since he wants to maximise the payoff, his objective is to keep the minimum of these values as high as possible:

$$\begin{aligned} &\max u \\ x_1 - 2x_2 + 2x_3 &\geq u \\ 2x_1 + x_2 &\geq u \\ -x_1 + x_2 + x_3 &\geq u \\ x_1 + x_2 + x_3 &= 1 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

Solving this LP we obtain the optimal solution $u = \frac{11}{17}$, equal to the value of u in the previous program.

Using the duality theorem in linear programming, one can show that the two LPs are duals to each other. The same method can be applied for arbitrary two-person zero-sum games: we obtain two LPs describing the optimal mixed strategies of the two players. These will have the same optimum value, and the two programs will be duals of each other.

2.3 Best response strategies

We define best response strategies as follows:

Definition 2.3. Suppose row player A plays mixed strategy $x = (x_1, x_2, \ldots, x_n)$ then a mixed strategy for column player B, $y = (y_1, y_2, \ldots, y_m)$, is a *best response strategy* if it guarantees B the minimum expected payoff against A's x strategy.

It turns out that best response strategies have a specific structure. Assume that A plays strategy $x = (x_1, x_2, \ldots, x_n)$ which guarantees A a payoff of u. Let d_i be the expected payoff to A if B plays pure strategy B_i. If the payoff matrix is R and r_{ij} is the ith row jth column entry of R then we have:

$$d_i = x_1 r_{1i} + x_2 r_{2i} + \ldots + x_n r_{ni}.$$

The values $(d_1, d_2, \ldots, d_m)$ are the expected payoff to A when B plays one of the pure strategies $B_1, B_2, \ldots, B_m$. Thus, the value that A can guarantee is the minimum of these values:

$$u = \min\{d_1, d_2, \ldots, d_m\}$$

This is because even if B plays a mixture of his pure strategies the weighted average will be at least u. We have the following result.

Theorem 2.4. *B's strategy $y = (y_1, y_2, \ldots, y_m)$ is a best response to A's strategy $x = (x_1, x_2, \ldots, x_n)$ if and only if a positive weight $y_i > 0$ is assigned to B_i only if $d_i = u = \min\{d_1, d_2, \ldots, d_m\}$.*

Proof. If B plays $y = (y_1, y_2, \ldots, y_m)$ and A plays $x = (x_1, x_2, \ldots, x_n)$ the expected payoff to A is:

$$e = y_1 d_1 + y_2 d_2 + \ldots + y_m d_m$$

since d_i is the expected payoff to A when A plays x and B plays pure strategy B_i (as defined above). We know that $d_i \geq u$ for all i, then we have:

$$e = \sum_{i=1}^{m} y_i d_i \geq \sum_{i=1}^{m} y_i u = u. \tag{2.1}$$

If y is a best response strategy then by definition it will use values for y_i that minimize e. If B uses $y_i > 0$ for a strategy B_i that has $d_i > u$ then the above inequality will be strict: $e > u$. The inequality $e \geq u$ will become an equality and thus B a best response if and only if B uses weights: $y_i > 0$ for $d_i = u = \min\{d_1, \ldots, d_m\}$ and $y_i = 0$ for $d_i > u$. Therefore, B is a best response if and only if positive weights on B_i are used only if $d_i = u$. □

This leads to the following:

Theorem 2.5. *Let $x = (x_1, x_2, \ldots, x_n)$ be an optimal strategy for A. Then B's strategy $y = (y_1, y_2, \ldots, y_m)$ is a best response to A's strategy x if and only if y is optimal.*

Proof. Since x is optimal for A and $u = \min\{d_1, \ldots, d_m\}$ is what A can guarantee then the value of the game is u.

If y is the best response to x then the expected payoff of the game to A will be $u = \min\{d_1, \ldots, d_m\}$ as we have seen in the proof of the previous theorem. We showed that a best response strategy can reach the value of u by picking the appropriate weights and thus B can also guarantee of not losing more than u when he plays y and thus y is optimal for B.

If y is optimal then it will have to guarantee that B does not lose more than u (since u is the value of the game). Looking at equation (2.1) we would need to have $e = u$ and the only way to do that is if B picks positive weights $y_i > 0$ only if $d_i = u$, which means that y would be a best response strategy according to Theorem 2.4. $\square$

The above is important because if we know the optimal strategy of one player then we can find the optimal strategy of the other player by finding the best response to the optimal strategy. We will use this in the next section.

2.4 $2 \times n$ games

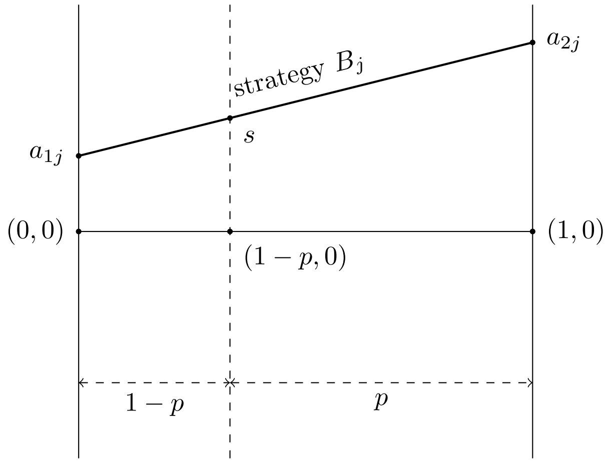
In the linear programming lectures we have used a geometric method to solve linear programs with two variables. This provides a particularly intuitive method to solve $2 \times n$ games. Consider a game, where the row player has only two strategies, while the column player has n strategies:

$$
\begin{array}{cc}
 & \text{B} \\
 & \begin{array}{cccc} 1 & 2 & \ldots & n \end{array} \\
\text{A} \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \end{bmatrix}
\end{array}
$$

Let us denote the mixed strategy of A by $(p, 1-p)$. That is, he plays strategy A_1 with probability p, and A_2 with the remaining probability $(1-p)$. Then A's LP can be written as:

$$\begin{aligned} \max u \\ a_{11}p + a_{21}(1-p) &\geq u \\ a_{12}p + a_{22}(1-p) &\geq u \\ \dots \\ a_{1n}p + a_{2n}(1-p) &\geq u \\ 1 \geq p &\geq 0. \end{aligned}$$

We use the following graphical representation. The two strategies of A correspond to two vertical lines, namely the y axis and another parallel line through the point $(1,0)$. The j'th mixed strategy B_j of the column player is represented by the line connecting the points $(0, a_{1j})$ and $(1, a_{2j})$.



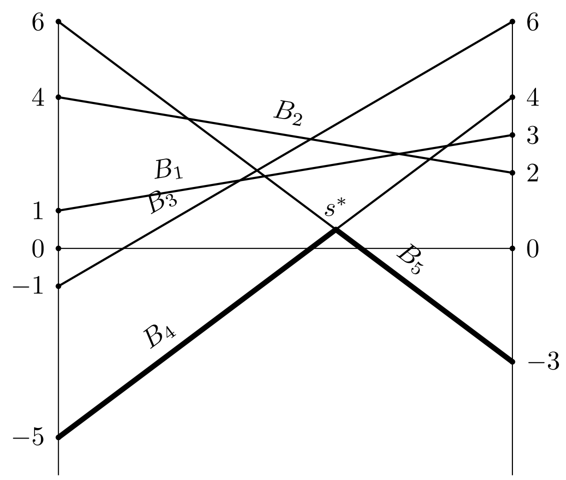
Consider the mixed strategy $(p, 1-p)$ of A, that is, playing A_1 with probability p and A_2 with probability $(1-p)$. This corresponds to the dashed vertical line shown in the figure that we will denote $A(p)$. Note that when $p = 1$ this line $A(1)$ becomes the x-axis, which corresponds to strategy A_1, and when $p = 0$ this line $A(0)$ becomes the vertical line through point (1.0) which corresponds to strategy A_2.

The payoff of B_1 against strategy $A(p)$ is $a_{1j}p + a_{2j}(1-p)$. This corresponds to the intersection of the line representing B_j, and the vertical line through the point $(1-p, 0)$, denoted by s on the picture.

We represent every strategy B_j for $j = 1, \dots, n$ this way. Consider now a certain mixed strategy $(p, 1-p)$ of A, represented by line $A(p)$. The *best response* strategy of B corresponds to the line that intersects $A(p)$ at the lowest point. (This lowest point is $(1-p, u)$ for the variable u in the LP above.) For all values of p between 0 and 1, we can identify the best responses of B with the *lower envelope* of these lines. To find the optimal mixed strategy $(p, 1-p)$ of A, we need to find the value of p where the best response of B is the largest possible, which corresponds to the highest point of this lower envelope. (The optimal value of the variable u.) We demonstrate this method on the following example.

$$
\begin{array}{cc}
 & \text{B} \\
 & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\text{A} \begin{array}{c} 1 \\ \\ 2 \end{array} & \begin{bmatrix} 1 & 4 & -1 & -5 & 6 \\ \\ 3 & 2 & 6 & 4 & -3 \end{bmatrix}
\end{array}
$$

We obtain the graph below. The lower envelope is denoted by the thick lines. Its highest point is denoted by s^*.



We now wish to obtain the exact optimal mixed strategies, which correspond to the intersections of the lines B_4 and B_5. That is, we have

$$\begin{aligned} -5p + 4(1-p) &= u \\ 6p - 3(1-p) &= u. \end{aligned}$$

Solving this system, we obtain $p = \frac{7}{18}$, and $u = \frac{1}{2}$. Hence the value of the game is $\frac{1}{2}$, and the optimal mixed strategy of A is $(\frac{7}{18}, \frac{11}{18})$.

To obtain B's optimal strategies, we use theorem 2.5. Given that we have the optimal strategy for A we only need to find B's best response to this strategy and from the theorem we know that this is optimal. From theorem 2.4 we know that B's the best response strategies will only use pure strategies that give minimum expected payoff to the optimal strategy of A. A's optimal strategy is played on $A\left(\frac{7}{18}\right)$ and on this line the pure B strategies that give the lowest expected payoff are B_4 and B_5. Thus B can only play B_4 and B_5 with positive (non-zero) probabilities. Thus, the optimal solution for B will be a weighted combination of B_4 and B_5, which is a line that passes through their intersection and lies between the lines for B_4 and B_5. This line has to be horizontal because according to the minimax criterion B expects worst play from A and thus he has to minimize the maximum value that this line takes. This is achieved by a horizontal line through point s^*. Thus, if we assign probability q to B_4 and $(1-q)$ to B_5, and let v be the height of this line (equal at both ends since horizontal) we get

$$\begin{aligned} -5q + 6(1-q) &= v \\ 4q - 3(1-q) &= v, \end{aligned}$$

This gives $q = \frac{1}{2}$, and $v = \frac{1}{2}$. Hence the optimal mixed strategy of B is $(0, 0, 0, \frac{1}{2}, \frac{1}{2})$.

Another way to recover B's optimal strategy from the diagram is to realise that this is a horizontal line that can be derived by a weighted sum of the all the other pure strategy lines. As argued above, B will chose a horizontal line at height v to make sure that he can guarantee that value for all $A(p)$ strategies of A for $0 \le p \le 1$. The lowest horizontal line that he can get as a weighted combination of the lines of pure strategies will pass from s^* which is the intersection of strategies B_4 and B_5. Continuing as above we get B's optimal strategy.