MA423 - Fundamentals of Operations Research
Lecture 8: Inventory Models

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Chapter 1

Deterministic inventory models

A crucial decision for a shop's management is to decide how much inventory to keep of each good and to place orders in time. Revenue is lost if the desired item is out of stock, and it can have further negative effects due to loss of customer's goodwill. Shortage can be avoided by keeping excessive amounts of stock. But keeping too much stock will also be disadvantageous, due to storage costs and warehouse space requirement; further, capital is required to purchase stock that could have earned returns instead. Besides retailing, similar problems arise in a wide range of areas, for example, manufacturing: production of a certain product may require a number of components that have to be kept in stock.

Inventory management (also called *inventory control* or *stock control*) aims to build appropriate mathematical models and propose policies for deciding the optimal level of inventory. There are a variety of models, since inventory control problems may occur in many different settings. An important distinguishing feature of the model is the predictability of the demand: in *deterministic* inventory models, the exact amount of demand is known ahead of time, whereas in *stochastic* inventory models, uncertainty is involved, and only the probability distributions of future demands is known. This chapter focuses on deterministic models, whereas the next one will consider the stochastic setting.

1.1 Basic concepts

Deterministic inventory models assume full knowledge of the expected demand. This can be realistic in manufacturing with a steady rate production, or for the retail of goods with very consistent demands. We start with a simple example.

Example 1.1. *An electronics retailer sells a certain laptop model. The demand is fairly consistent at 40 per month, and the selling price to customers is £800 per laptop. Stock can be replenished from the manufacturer at £500 per laptop. The administrative cost of raising an order is £130, and the cost of transporting the order from the manufacturer's warehouse is £520, where both costs do not depend on the size of the order. Further, keeping a laptop in stock incurs a holding cost of £6 per month, reflecting the lost returns on capital and the warehouse's insurance. The company's retail policies do not allow for shortages: all demands* *for laptops must be met. What is a sensible ordering strategy?*

Let us now introduce some important notions.

- The model involves a *sequence of time periods.* In the above example both holding cost and demand are per month so we could define the length of each period of be one month. There is an indefinitely long time to plan for, with the same demand and cost for every month; however, there are models with demands and costs changing over time.

- A crucial part of every model is how the *demand* is given. In the example, we have a *deterministic constant rate demand* d, meaning that the same amount of $d = 40$ will be sold or consumed in every period; we also assume this demand is distributed evenly over the period.

- The *cost of ordering* an amount z can be represented as a function $C(z)$. In the simplest case this is linear: ordering every unit has a *unit cost* of *purchase price* c, and hence the total price is $C(z) = cz$. In the above example, there is also a *fixed cost* K involved. The cost function can be expressed as

$$C(z) = \begin{cases} 0, & \text{if } z = 0, \\ K + cz, & \text{if } z > 0. \end{cases}$$

  In the above example, the fixed cost is $K = 650$: this is the total administration and transportation cost; the *unit cost* is $c = 500$. More complicated cost functions are also possible: for example, ordering in bulk can guarantee lower unit costs for large quantities.

- The *holding cost* or storage cost represents all direct or indirect costs involved in keeping inventory. We will usually assume it is a linear function, assessed on a period-by-period basis. In the above example, $h = 6$ is the holding cost, that has to be paid for each unit in each period of time.

- The *shortage cost* represents the loss incurred from unsatisfied demands. Sometimes items that are not in stock can be backordered, which means that the customer receives the item as soon as the order comes in; however this incurs extra costs as well as customer dissatisfaction with negative implications to future retails and all these are captured in the shortage cost. In the above example, shortages are not allowed at all: hence no shortage cost is incurred.

- The review mode can be *continuous* or *periodic*. In the continuous review model stock levels are monitored continuously through a computerised system and thus orders can be placed at any point of time; whereas in the periodic review model, the stock is checked and new orders are decided at some fixed time periods (say, every half a year). In this course, we will assume the more flexible continuous review mode.

- The time-span between when the order is placed and when the goods arrive is called the *lead time*. This is not specified in the example.

The example also specifies the selling price of £800. If no shortages are allowed, then this is irrelevant for the inventory modelling: the income from selling the good will be always the same, regardless of our inventory policy. The selling price will not directly appear in the model even if shortages are allowed, since the loss will be taken into account in the shortage cost.

Linearity of demands and costs. The demand, holding and shortage costs are given per unit per time period. For fractional periods of time, we simply take the appropriate fraction of these costs. In the example, if we need to store 1 unit for 2.25 months, it will cost $\pounds 6 \cdot 2.25 = \pounds 13.5$. We will use this simple approach to calculate demands and costs unless otherwise specified.

For constant demand rate and continuous review mode as in the example, the choice of time unit, which is the length of the period is arbitrary. We could have chosen a full year instead of a month as the length of the period; then we get an equivalent problem by multiplying both d and h by 12, to become $d = 480$ and $h = 72$. The problem remains equivalent as long as all parameters are with respect to the same time unit.

1.2 Economic Order Quantity model

The *Economic Order Quantity model (EOQ)* is the simplest inventory model, with the following assumptions:

- The unit of time is one period.
- Demand is at a deterministic constant rate d per period.
- The cost of ordering $z > 0$ units is $K + cz$.
- The holding cost is h per unit per time period.
- No backlogging is allowed.
- Inventory can be replenished anytime (continuous review mode).
- The lead time is 0: the ordered quantity arrives all at once, just when desired.

In this model, we only need to replenish inventory when it falls to zero. Since the model is deterministic, the lead time does not make much difference. We assume the lead time is 0 to simplify the model, but the same model can easily incorporate a lead time of l periods $\ell > 0$: if we know exactly when the new batch will be needed, we can place the order ℓ time before that.

We will always order a new batch of the same (optimal) size Q. The time between two consecutive replenishments will be called a *cycle*. Figure 1.1 shows the inventory level over

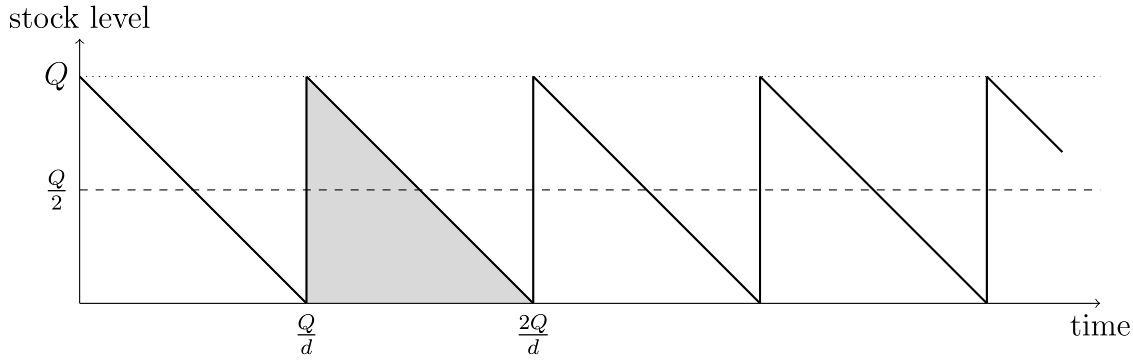

Figure 1.1: The inventory level over time in the EOQ model.

time in the EOQ model. The main decision is to determine the optimal *order quantity* Q. For a too small value of Q, the fix ordering cost K must be paid too often, increasing the ordering cost per unit. On the other hand, for a value too large, the holding cost can be excessive. We need to find the optimal tradeoff somewhere in-between. Given the continuous demand rate d, we get:

$$\textit{Length of cycle in periods: } t = \frac{Q}{d}.$$

The cost of each cycle comprises the following two terms:

$$\begin{aligned} \textit{Ordering cost per cycle} &= K + cQ, \\ \textit{Holding cost per cycle} &= \frac{hQ^2}{2d}. \end{aligned}$$

To understand the holding cost, note that the average level of stock over a cycle is $\frac{Q}{2}$. We have to pay h per unit per period and since on average we have $\frac{Q}{2}$ units and in each cycle we have $t = \frac{Q}{d}$ periods, we obtain the above cost as the product of these: $h\frac{Q}{2}\frac{Q}{d} = \frac{hQ^2}{2d}$. Note that $\frac{Q^2}{2d}$ is precisely the integral of the stock level over a cycle, that is, the area of the shaded region in Figure 1.1).

The total cost of a cycle is the sum of these too terms: $K + cQ + \frac{hQ^2}{2d}$. What we are really interested in is the average cost to be paid per period, that is the above quantity divided by the cycle length of t periods.

$$\textit{Cost per period: } \frac{Kd}{Q} + dc + \frac{hQ}{2}.$$

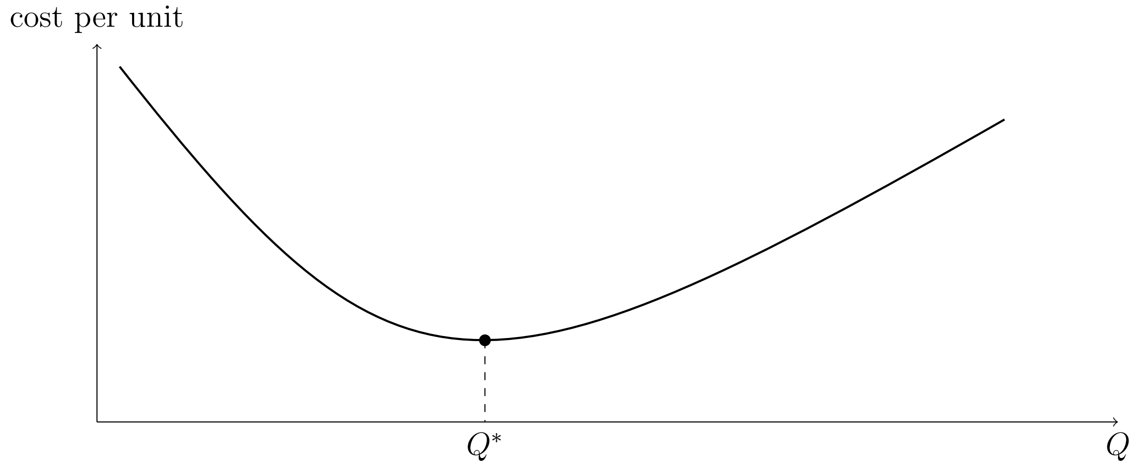

Figure 1.2: The cost per unit as a function of Q.

Our objective is hence finding the value of Q that minimises the above expression. This is a convex function (see Figure 1.2) and hence the minimum is taken where the derivative is 0:

$$-\frac{Kd}{Q^2} + \frac{h}{2} = 0.$$

This gives the optimum value

$$Q^* = \sqrt{\frac{2Kd}{h}}.$$

The optimum length of the time cycle is then given as

$$t^* = \frac{Q^*}{d} = \sqrt{\frac{2K}{dh}}.$$

Observe that these values **do not** depend on the unit cost c. Indeed, c appears only in the term dc of the cost per unit time, and hence it does not influence the choice of Q.

Let us now return to Example 1.1. The parameter values were

$$K = 650, \quad c = 500, \quad h = 6, \quad d = 40.$$

giving

$$Q^* = 93.09, \quad t^* = 2.33.$$

The total cost per period will be £21,117.14. Hence the optimal policy suggests ordering 93.09 laptops every 2.33 months. Of course, fractions of laptops cannot be ordered, and we have to round it to 93 laptops.

1.3 The EOQ model with planned shortages

No shortages were allowed in the basic EOQ model. We now relax this assumption (but leave the model otherwise unchanged). In many cases, customers may accept some delay in receiving the orders; this is called backlogging orders. We will allow a shortage cost p per unit per time period. If this is not very high compared to other costs, it might be a reasonable decision to order with shortages planned. In the next chapter, we shall also see an inventory model with uncertain demands that can be approximated by this deterministic model.

The optimal replenishing policy will be defined by two parameters:

$Q =$ the order quantity;

$S =$ the inventory level just after a new batch arrives and delayed demands are satisfied.

That is, $Q - S$ units of demand will be satisfied with delay. The inventory levels will be as in Figure 1.3.

As in the basic EOQ model, the ordering cost for each cycle is $K + cQ$. The holding cost per cycle is

$$\frac{hS^2}{2d},$$

since the average level of inventory between the arrival of the batch and the full consumption of the stock after S/d units of time is $\frac{S}{2}$.

The shortage cost per cycle will be a new term. Its value is

$$\frac{p(Q-S)^2}{2d},$$

since there will be shortage for a duration of $(Q - S)/d$ at the end of every cycle, with an average shortage of $\frac{Q-S}{2}$. Note that the quantities $\frac{S^2}{2d}$ and $\frac{(Q-S)^2}{2d}$ equal the areas of the shaded triangles in Figure 1.3 above and below the horizontal axes, respectively. We get the total cost per period as the cycle cost divided by the cycle length Q/d:

$$\textit{Cost per period: } C(Q,S) = \frac{Kd}{Q} + dc + \frac{hS^2}{2Q} + \frac{p(Q-S)^2}{2Q}.$$

We need to find the values of Q and S minimising this expression (with $S \leq Q$). For that, we need to set the partial derivatives $\partial C(Q,S)/\partial Q$ and $\partial C(Q,S)/\partial S$ to 0. This leads to the optimum values:

$$Q^* = \sqrt{\frac{2dK}{h} \cdot \frac{p+h}{p}}, \quad S^* = \sqrt{\frac{2dK}{h} \cdot \frac{p}{p+h}}, \quad t^* = \frac{Q^*}{d} = \sqrt{\frac{2K}{dh} \cdot \frac{p+h}{p}}.$$

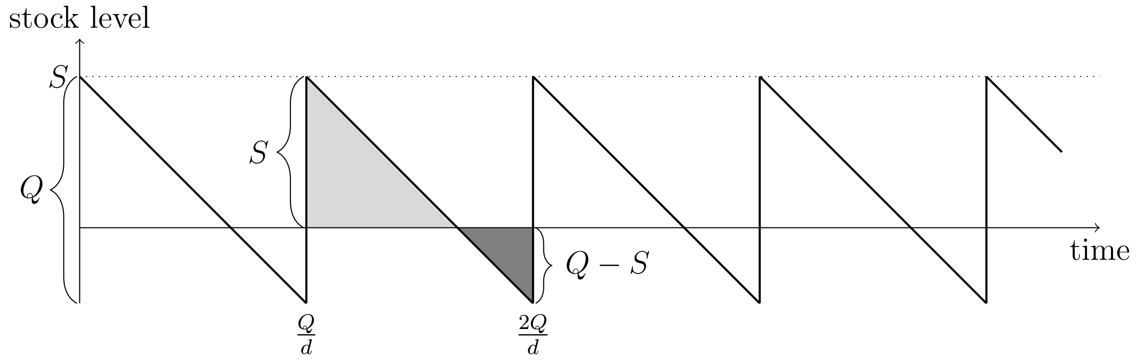

Figure 1.3: The inventory level over time in the EOQ model with shortages.

We include a derivation of the above formulas in Appendix 3.

Let us now modify Example 1.1 by allowing shortages at a cost of $p = 18$ per unit. In the basic EOQ model, we obtained $Q^* = 93.09$ and $t^* = 2.33$, at a cost of £21,117.14 per time period. This now has to be multiplied by $\sqrt{\frac{p+h}{p}} = \sqrt{\frac{6+18}{18}}$, giving $Q^* = 107.49$, $S^* = 80.62$ and $t^* = 2.68$, at a cost of £20,483.74 per time period. We can round Q^* to either 107 or 108.

1.4 Supply chain management

The models above addressed the inventory problem of a single good in a single warehouse. The problems modern businesses face are much more complex. A distributor has to coordinate the flow of goods between several regional and national warehouses and field distribution centres; a manufacturer has to coordinate between a number of different production places and warehouses. Every stage in this procedure is referred to as an *echelon*, and the entire system as a *multiechelon inventory system*. An even more general problem is *supply chain management*, involving not only the inventory, but also procurement and manufacturing phases. Supply chain management can comprise a large number of echelons, and successfully designed supply chains are instrumental for leading companies. See Section 19.8 in [1] 7th ed. (Sec. 18.8 in 9th ed.) for the example of Hewlett-Packard.

Serial two echelon systems are the simplest possible multiechelon system. There are two echelons, E1 and E2. The inventory at E2 has to be periodically replenished from E1. For example, E1 could be a factory and E2 a distribution centre. In a different application, E2 could be a factory and E1 another facility where the components needed for production are received. Section 18.5 in the 9th ed. of [1] (not contained in the 7th ed.) provides the analysis of such models, already substantially more complicated than the ones described above.

Chapter 2

Stochastic inventory models

In stochastic inventory models, our knowledge and expectations about future demand are expressed by a probability distribution.

2.1 Stochastic continuous review model

Our first stochastic model will be based on a simple order policy. Reordering of products will be according to the following rule for some values of R and Q:

> **Order-quantity policy:** Whenever the inventory level of the product drops to R units, place an order for Q units to replenish the inventory. The parameters to be determined are the *reorder point* R and the *order quantity* Q.

A traditional implementation of such a policy is the *two-bin system*, used for storing e.g. nuts & bolts. The items would be held in two bins, the first one of size R. Units would be withdrawn from the second bin. A new order is placed once the second bin runs empty; until it arrives, the first bin will be used. Upon arrival of the order, the first bin is replenished to the level R, and the rest is put in the second bin. Today, such manual replenishment policies have been replaced by computerized inventory systems, that monitor inventory levels and place orders whenever necessary.

The model studied in this chapter will share the following assumptions with the EOQ model with planned shortages:

- The cost of ordering $z > 0$ units is $K + cz$.
- The holding cost is h per unit per time period.
- Backorders are possible, with a shortage cost p incurred for each backordered unit.
- Inventory can be replenished at any time (continuous review mode).

The following features will be different due to the stochastic nature of the problem:

- The demand is uncertain, with the *expected number* of units demanded per time unit known to be d.
- There is a lead time $\ell > 0$ between the placement and the arrival of an order.
- The demand during the lead time period is a random variable X, given by the probability density function $f(x)$.

We use continuous random variables to represent the demand, although in reality, the demand should be integer. However, continuous variables typically give a good approximation for the integer demands, and are easier to deal with. Given the above information, the goal is to devise an optimal order-quantity policy, that is, determine the optimal reorder point R and order quantity Q.

2.1.1 Choosing the order quantity Q

Computing the precise optimal value of Q is not easy. Instead, we use the EOQ model as a fairly good approximation. We simply consider the EOQ model with planned shortages, and a constant demand rate d. The formula from the previous chapter gives

$$Q = \sqrt{\frac{2dK}{h} \cdot \frac{p+h}{p}}.$$

2.1.2 Choosing the reorder point R

Whereas the lead time was not important in the deterministic models, it plays a key role in stochastic models. For lead time 0, we could set $R = 0$, as the new order immediately arrives when the stock runs out. We now have to take into account the demand distribution over the lead time. We have not yet established the criteria to judge how good a certain choice of R can be. Several different measures could be used; the one we choose here is the following:

> *Service level requirement:* the probability of a stockout between the time an order is placed and received should not be more than q. The value q will be called the *tolerated stockout probability.*

To keep the holding cost low, we shall aim for the smallest possible value of R under this condition. Let us start with an example.

Example 2.1. The lead time for orders in a bicycle shop is 10 days for a certain road bike. According to historical data, the demand during these 10 days is uniform between 21 and 40. That is, all values 21, 22, ..., 39, 40 occur with equal probability $\frac{1}{20}$, and no other values are possible (such a distribution is not very realistic, but it is just for the sake of illustration). The company expects that the probability of a stockout should be no more than 15%.

Let X be the demand for bicycles during the lead time of 10 days, $X \in \{21, \ldots, 40\}$. We have $q = 0.15$, and want to set a level R such that the probability $\mathbb{P}(X > R) \leq 0.15$. We have:

$$\begin{aligned}
\mathbb{P}(X > 40) &= 0 \\
\mathbb{P}(X > 39) &= \frac{1}{20} = 0.05 \\
\mathbb{P}(X > 38) &= \frac{2}{20} = 0.10 \\
\mathbb{P}(X > 37) &= \frac{3}{20} = 0.15 \\
\mathbb{P}(X > 36) &= \frac{4}{20} = 0.20 \\
&\ldots
\end{aligned}$$

This gives $R = 37$ since it gives the exact probability: $\mathbb{P}(X > 37) = \frac{3}{20} = 0.15$, whereas note that for $R = 36$ the probability of stockout would be already 0.20.

Another important quantity is the expected amount of stock at the time the new batch arrives. Note that the stock value may be 0 in case of stockout; however, a certain amount may be left in case of lower demand.

> The *safety stock* is the expected inventory level just before the new stock arrives. It can be computed as $R - \mathbb{E}(X)$, where $\mathbb{E}(X)$ is the expected demand during the lead time.

In the above example, $\mathbb{E}(X) = 30.5$, and therefore if we let $R = 37$, the safety stock is $37 - 30.5 = 6.5$. In general, we can compute the reorder point R and the safety stock using the probability density function $f(x)$ of the demand, as follows.

> For a given service level requirement q, choose the reorder point R such that
>
> $$P(X > R) = q,$$
>
> which can be written as
>
> $$\int_R^{\infty} f(x)dx = q.$$

Let us recall the notion of the cumulative distribution function (CDF)

$$F(d) = \int_0^d f(x)dx,$$

that equals the probability $P(X \leq d)$ for some positive value d. In other words, $F(d)$ is the probability that the demand will be at most d. Using that $\int_0^{\infty} f(x)dx = 1$, the condition can be rewritten as

$$F(d) = 1 - q.$$

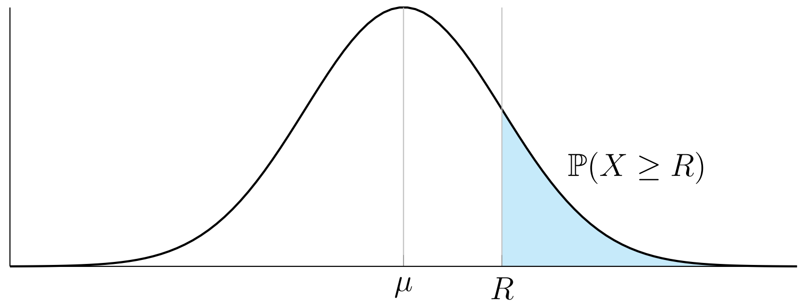

Figure 2.1: A normal distribution with the tail probability for R

See Figure 2.1 as an illustration of this value for a Normal distribution $N(\mu, \sigma^2)$.[1] You do not have to compute these integrals in most cases - for common distributions, the CDF are given in tables in the statistics handbooks, or can be computed with spreadsheet software such as Excel. For example, assume X has a normal distribution with mean $\mu = 30$ and standard deviation $\sigma = 3$, and the requirement is $q = 0.05$.

We want:

$$\begin{aligned}
\mathbb{P}(X > R) &= q \\
\mathbb{P}(X \leq R) &= 1 - q \\
\mathbb{P}\left(\frac{X-\mu}{\sigma} \leq \frac{R-\mu}{\sigma}\right) &= 1 - q \\
\mathbb{P}\left(Z \leq \frac{R-\mu}{\sigma}\right) &= 1 - q \\
\Phi(\frac{R-\mu}{\sigma}) &= 1 - q,
\end{aligned}$$

where $Z \sim N(0,1)$ is the standard normal distribution, and Φ is its cumulative density function. We can get a value for r for which $\Phi(r) = 1 - q$ from normal probability tables. Then we have: $\frac{R-\mu}{\sigma} = r$ or:

$$R = \mu + r\sigma$$

Setting $q = 0.05$ we get from the normal tables: $\Phi(r) \approx 0.95$ imples $r = 1.65$ which gives:

$$R = \mu + r\sigma = 30 + 3 \cdot 1.65 = 34.95$$

Since R also needs to be integer, we shall set the reorder point at 35.

Let us mention that instead of the service level requirement used above, various other measures could also be used, e.g. the average number of stockouts per year. For example,

[1]The demand will be an integer number, whereas a normal distribution allows for fractional outcomes. Nevertheless, it is usually a good approximation to model discrete quantities with continuous distributions.

suppose that the order quantity Q has been set at 10% of the annual demand, so that an average of 10 orders are placed per year. If the probability is 0.2 that a stockout will occur during the lead time until an order is received, then the average number of stockouts per year would be $10 \cdot (0.2) = 2$. There are other measures that we could consider but this is not covered here.

2.1.3 Example

Let us revisit Example 1.1 from the previous chapter, allowing a shortage cost $p = 18$ per unit per period. The demand was assumed to be deterministic at $d = 40$; we now assume it follows a Poisson distribution at a monthly rate of 40. We will also assume that the lead time is 0.5 months, and the tolerated stockout probability is $q = 10\%$.

According to the Poisson distribution, demand occurs at a given rate, independently from prior demand. Therefore the demand X during the lead period of 0.5 months will also follow a Poisson distribution with parameter $\lambda = 20$ (and therefore $E(X) = Var(X) = 20$).

We already computed the optimal order quantity $Q^* = 107$ using the EOQ model with planned shortages in the previous chapter. We need to find the value of R with $P(X > R) \leq 0.1$. This can be easily computed by Excel: the function

$$\text{POISSON.DIST}(R,20,\text{TRUE})$$

returns the cumulative distribution $P(X \leq R)$. The smallest value of R with cumulative distribution > 0.9 will be $R = 26$. Consequently, the optimal reorder policy will be to order 107 new laptops once the stock falls below 26.

Alternatively, the Poisson distribution can be approximated by the normal distribution: let $Y \sim Poisson(\lambda)$ where λ is greater than 10. Then we can use the normal distribution to approximate Y: $Y \sim N(\lambda, \lambda)$. For $\lambda > 10$ the approximation is good and for $\lambda > 100$ the approximation is very good.

2.2 The newsvendor problem

The previous models addressed inventory problems over a longer period of time. We will now look at a different kind of inventory model, suitable for problems involving *perishable goods*. Planning is for a single time period only, but units left unsold at the end expire or loose value. The name derives from the standard example: a newspaper is a perishable product as it becomes outdated in a day. The newsvendor needs to order an appropriate amount of stock so that it can meet all or most demand; at the same time, only a small amount should be left unsold. The problem would be trivial for a deterministic demand - however, the number of customers is uncertain, and will be modelled by a random variable.

Besides newspapers, perishable products can include flowers, produce, restaurant food, fashion goods, or phones and laptops when a new model is introduced. Another type of example is airplane tickets: seats left empty get wasted, whereas overbooking can occur in case of excess demand. Hotel bookings is another similar case. *Revenue management* is the technique developed for such problems.

In this section, we will focus on the simplest case of inventory management for perishable goods. Our model is based on the following assumptions.

- There is a single good and a single time period involved.
- The good must be sold in this time period, and excess amounts loose all or part of their value.
- The demand is uncertain, given by a random variable X, with probability density function $f(x)$.
- Unit purchase cost (or purchase price) is c.
- For every unit of unmet demand, there is a shortage cost p, representing lost revenue and loss of customer goodwill.
- For every unsold unit, there is a holding cost h that represents storage and disposal costs, minus the salvage value, that is, the amount that can be recovered from selling these items. (h can be negative if the salvage value is higher than the disposal cost).
- There is no initial inventory.

The only decision to be made is the amount S of the good to be purchased.

Selling price is taken into account through the shortage cost p: p represents the lost revenue for unmet demand, which is calculated based on selling price.

The amount sold is given by

$$\min\{X, S\} = \begin{cases} X & \text{if } X < S, \\ S & \text{if } X \geq S. \end{cases}$$

Hence the total cost is given by the random variable:

$$C(X, S) = cS + p\max\{0, X - S\} + h\max\{0, S - X\}.$$

We will now express the expected value of the cost of ordering S units, denoted by $C(S)$, in terms of the probability density function $f(x)$.

$$\begin{aligned} C(S) = \mathbb{E}[C(X, S)] = \int_0^\infty C(x, S) f(x) dx = \\ = \int_0^\infty (cS + p\max\{0, x - S\} + h\max\{0, S - x\}) f(x) dx = \\ = cS + p\int_S^\infty (x - S) f(x) dx + h\int_0^S (S - x) f(x) dx. \end{aligned} \tag{2.1}$$

It can be shown that $C(S)$ is a convex function similar to the one on Figure 1.2, and we can find the minimum by setting its derivative to 0. Computing the derivative is not immediate and will not be covered here but it is:

$$C'(S) = c - p\int_S^\infty f(x)dx + h\int_0^S f(x)dx = 0.$$

Substituting the definition of the cumulative density function $F(S) := \int_0^S f(x)dx$ and using that $\int_0^\infty f(x)dx = 1$, we obtain

$$c - p(1 - F(S)) + hF(S) = 0,$$

therefore the optimal value S^* is given for

$$F(S^*) = \frac{p-c}{p+h}.$$

Since $F(S)$ is a monotone increasing continuous function from 0 to 1, there will be a unique value S^* satisfying this property. The numerator $p - c$ is the unit cost of underordering: this is the loss incurred for every unit of unmet demand. Note that $p > c$ can always be assumed, because p accounts for the lost revenue, hence it must be at least the selling price of the good, which must be larger than the purchase price c.

The optimal value $F(S^*)$ is called the *optimal service level*: if it is e.g. 0.7, the demand will be met in 70% of the cases.

Example 2.2. *A haulage company has a fleet of 35 lorries to transport goods around the country. It costs £200 per day to run a lorry, whether it is needed or not (this might include insurance, depreciation, road fund licence, and maintenance). If the demand is higher then they hire other lorries from suppliers for £350 per day each. The demand per day is for X lorries, with an exponential distribution with mean 45. The company is planning to downsize its fleet - how many lorries would you advise them to sell?*

Recall that the exponential distribution with mean 45 has probability density function

$$f(x) = \frac{1}{\mu}e^{-\frac{x}{\mu}} = \frac{1}{45}e^{-\frac{x}{45}}$$

on the interval $[0, \infty)$. The CDF is

$$F(d) = \int_0^d \frac{1}{45}e^{-\frac{x}{45}}dx = 1 - e^{-\frac{d}{45}}.$$

The cost of keeping a lorry is £200 and thus this is the purchase price of an item: $c = 200$. If the demand is high and we need another lorry it will cost us £350 per lorry and thus the shortage cost $p = 350$. For any unused lorry there is no cost or benefit so we set $h = 0$. Thus the parameter values are $c = 200$, $p = 350$, and $h = 0$.

Hence we need to solve

$$F(S^*) = \frac{p-c}{p+h}$$
$$1 - e^{-\frac{S^*}{45}} = \frac{350-200}{350+0}$$

reordering and taking the logarithm we get

$$-\frac{S^*}{45} = \ln\frac{4}{7},$$

therefore,

$$S^* = 25.18$$

Hence for the given demand, the optimal amount of lorries to maintain is 25; the company is advised to sell 10 of their lorries to maximise profit.

2.2.1 Initial inventory and fixed costs

In the above model, we assumed that the ordering cost is linear, at unit cost c. We now extend it to the more general case where we consider fixed costs: ordering $z > 0$ units will be $K + cz$. As a further extension of the model, we also assume that some *initial inventory* $I \geq 0$ is already present. The newsvendor therefore has to choose between two options:

1. Do not order at all, but only use the initial inventory.
2. Order some quantity $T > 0$ at price $K + cT$.

Let S denote the total stock after the order arrives. That is, $S = I$ if no order is placed and $S = I + T$ if an order is placed. Let $C(S)$ denote the cost associated with a stock level S as in (2.1) for the basic newsvendor problem in the previous section. This formula included a purchase cost cS for ordering S units. Since we receive the initial I units for free, we have to subtract cI for our model. Also, we add fixed cost K in the case that we do place an order. Therefore the total cost can be obtained in the two cases as

$$\bar{C}(S) = \begin{cases} C(I) - cI, & \text{if no order is made,} \\ C(S) - cI + K, & \text{if a quantity } S - I > 0 \text{ is ordered.} \end{cases}$$

Using the formula for the basic model, let S^* denote the value minimising $C(S)$, given by

$$F(S^*) = \frac{p-c}{p+h}.$$

We know that $C(S)$ is a convex function of the same shape as the one in Figure 1.2. $C(S)$ is depicted in Figure 2.2 where the optimal point S^* is shown.

Let us see what happens when $S^* \leq I$. In this case, from Figure 2.2 we can see that $C(I) \geq C(S^*)$, since $C(S)$ is increasing for $S \geq S^*$. Now suppose we order to a level $S > I$ then $C(S) \geq C(I)$ and the cost to bring it to a level S will be:

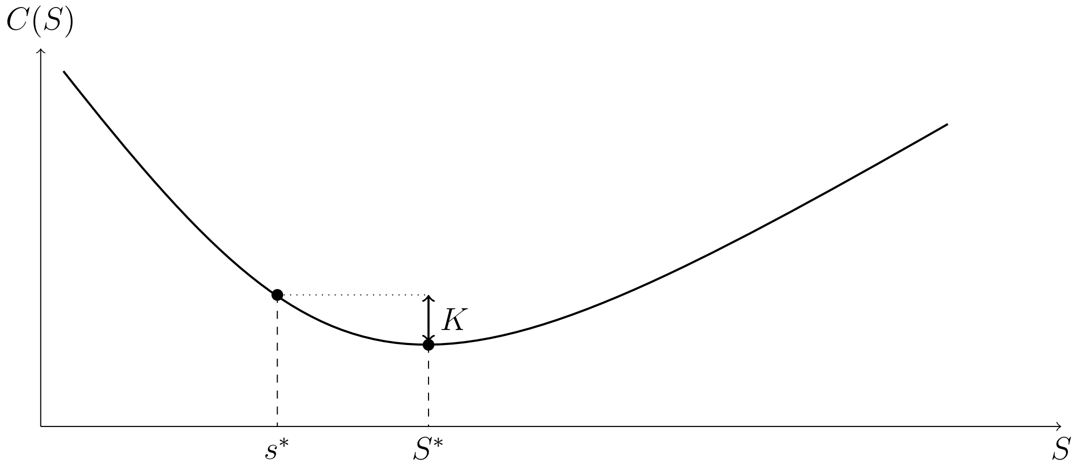

Figure 2.2: $C(S)$ versus S, where S is the total inventory; S^* minimises $C(S)$; s^* is such that $c(s^*) = K + c(S^*)$;

$$\begin{aligned}\bar{C}(S) &= C(S) + K - cI \\ &\geq C(I) + K - cI \\ &> C(I) - cI \\ &= \text{cost of not ordering}\end{aligned}$$

Thus, when $S^* \leq I$ we should never order.

Now we assume that $I < S^*$. For this case we define stock level s^* to be the stock level such that $s^* \leq S^*$ and:

$$C(s^*) = C(S^*) + K,$$

as shown in Figure 2.2. We now have two cases:

The first case is when $I < s^*$, where we have $C(I) \geq C(s^*)$ since $C(S)$ is non-increasing for $S \leq S^*$. This gives $C(I) - cI \geq C(S^*) + K - cI$ which means it is better to bring stock to level S^* than to not order. Further, given that we have I in stock the best level $S > I$ to bring the stock level to is S^* since

$$\bar{C}(S^*) = C(S^*) + K - cI \leq C(S) + K - cI = \bar{C}(S).$$

The second case is when $s^* \leq I \leq S^*$. In this case we have $C(s^*) \geq C(I)$ which implies $C(S^*) + K \geq C(I)$. Thus, for any $S > I$ we have:

$$\begin{aligned}&C(S) + K - cI \\ &\geq C(S^*) + K - cI \\ &> C(I) - cI \\ &= \text{cost of not ordering}\end{aligned}$$

Thus in this case we should not order.

To summarize, the optimal order policy is the following:

- If $s^* \leq I$ then *do not order*;
- If $I < s^*$ then *order* $S^* - I$ *units.*

Chapter 3

Appendix: derivation of batch-size and stock level for EOQ model with shortages

Recall that we needed to derive the quantities Q and S in order to minimise the function:

$$\textit{Cost per period: } C(Q,S) = \frac{Kd}{Q} + dc + \frac{hS^2}{2Q} + \frac{p(Q-S)^2}{2Q}.$$

To do this, we need to set the partial derivatives $\partial C(Q,S)/\partial Q$ and $\partial C(Q,S)/\partial S$ to 0. We compute the partial derivatives:

$$\begin{aligned} \frac{\partial C(Q,S)}{\partial Q} &= -\frac{Kd}{Q^2} - \frac{hS^2}{2Q^2} + \frac{2p(Q-S)Q - p(Q-S)^2}{2Q^2} \\ \frac{\partial C(Q,S)}{\partial S} &= \frac{hS}{Q} - \frac{p(Q-S)}{Q}. \end{aligned}$$

Setting them to zero yields the equations:

$$-2Kd - hS^2 + 2p(Q-S)Q - p(Q-S)^2 = 0 \tag{3.1}$$

$$hS - p(Q-S) = 0. \tag{3.2}$$

From (3.2), we have $p(Q-S) = hS$, hence, if we replace $p(Q-S)$ with hS in (3.1), we obtain

$$-2Kd - hS^2 + 2hSQ - hS(Q-S) = 0.$$

After two cancellations, the previous equation reduces to

$$-2Kd + hSQ = 0. \tag{3.3}$$

From (3.2), we get

$$Q = \frac{(h+p)}{p} S,$$

hence substituting for Q in (3.3), we obtain

$$-2Kd + \frac{h(h+p)}{p} S^2 = 0,$$

which gives the optimal value of S

$$S^* = \sqrt{\frac{2dK}{h} \cdot \frac{p}{p+h}}.$$

Finally

$$Q^* = \frac{(h+p)}{p} S^* = \sqrt{\frac{2dK}{h} \cdot \frac{p+h}{p}}.$$

Bibliography

[1] Hillier & Liebermann, *Introduction to Operations Research*, McGraw-Hill Series in Industrial Engineering and Management Science, 7th Edition or 9th Edition