MA423 - Fundamentals of Operations Research: Linear Programming Lectures 1 and 2

Katerina Papadaki[1]

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[1]London School of Economics and Political Science. Houghton Street London WC2A 2AE (k.p.papadaki@lse.ac.uk)

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Chapter 1

Lecture 1: Introduction

1.1 Optimization problems

The term Mathematical Programming refers to using **mathematical** tools for optimally allocating and using limited resources when planning (**programming**) activities. Mathematical programming deals therefore with optimization problems, and is also often known with the alternative name of **Mathematical Optimisation**. An optimization problem consists in maximizing or minimizing some function – representing our criterion for how "good" a solution is, such as for example, total profit or cost, or total number of staff, time, or other resources needed to perform certain tasks – subject to a number of constraints, representing limitations on the resources or on the way these can be used. In general, a mathematical programming problem is of the following form:

$$\begin{array}{rc} \max \ \text{(or min)} & f_0(x) \\ \text{s.t.} & f_1(x) \leq b_1 \\ & \vdots \\ & f_m(x) \leq b_m \\ & x \in \mathcal{D} \end{array} \tag{1.1}$$

where

- x is a vector of *decision variables*, $x_1, \ldots, x_n$,
- $\mathcal{D}$ is the domain of the variables,
- $f_0 : \mathcal{D} \to \mathbb{R}$ is the *objective function*,
- $f_i : \mathcal{D} \to \mathbb{R}$, $i = 1, \ldots, m$, are the functions that define the *constraints* $f_i(x) \leq b_i$, where $b_i \in \mathbb{R}$.

If the problem requires to maximize the objective function, then we say that (1.1) is a *maximization problem*, if it requires to minimize, then we say that it is a *minimization problem*.

Definition 1.1 (Feasible and optimal solutions)**.**

- **Feasible solutions** *A point* $\bar{x} \in \mathcal{D}$ *is a* feasible solution *for the optimization problem* (1.1) *if it satisfies all constraints, that is,* $f_i(\bar{x}) \leq b_i$ *for* $i = 1, \ldots, m$.

- **Feasible region** *The* feasible region *for the optimization problem is the set of all feasible solutions*[1].

- **Optimal solutions** *An* optimal solution *for a maximization problem is a feasible solution* x^* *satisfying* $f(x^*) \geq f(x)$ *for every feasible solution* x.

  An optimal solution *for a minimization problem is a feasible solution* x^* *satisfying* $f(x^*) \leq f(x)$ *for every feasible solution* x.

The aim of mathematical programming is to develop theoretical tools and algorithms in order to solve optimization problems, where by "solving" the problem we mean finding an optimal solution, or deciding that an optimal solution does not exist.

Definition 1.2 (Infeasible and unbounded problems)**.**

- **Infeasible problems** *An optimization problem is* feasible *if it has at least one feasible solution. The problem is said to be* infeasible *otherwise.*

- **Unbounded problems***A maximization problem is* unbounded *if, for every* $\alpha \in \mathbb{R}$*, there exists a feasible solution* x *such that* $f(x) \geq \alpha$*. A minimization problem is* unbounded *if, for every* $\alpha \in \mathbb{R}$*, there exists a feasible solution* x *such that* $f(x) \leq \alpha$*. An optimization problem that is not unbounded is said to be* bounded.

Note that, in the situation where the optimization problem is infeasible or unbounded, it does not admit an optimal solution. Indeed, in the former case the problem has no solution at all, whereas in the second case the problem has solutions that are arbitrarily "good". We also observe that an optimization problem may

[1] The feasible region should not be confused with the domain $\mathcal{D}$ of the variables. For example, consider the problem to minimize x^2 subject to the single constraint $-\log(x) \leq 0$; the domain of the problem is $\{x : x > 0\}$ (because the log function is only defined for positive numbers), whereas the feasible region is the set $\{x : x \geq 1\}$

not admit an optimal solution, even if it is feasible and bounded. For example, consider the problem

$$\begin{aligned} \min x_1 \\ x_1 x_2 \geq 1 \\ x_1, x_2 \geq 0. \end{aligned}$$

Obviously the problem is feasible, it is bounded (since $x_1 \geq 0$, hence the objective function cannot take value less than 0), but it does not have an optimal solution. Indeed, it is clear that there are feasible solutions with objective value arbitrarily close to 0, but none can have value 0. (See Figure 1.1.)

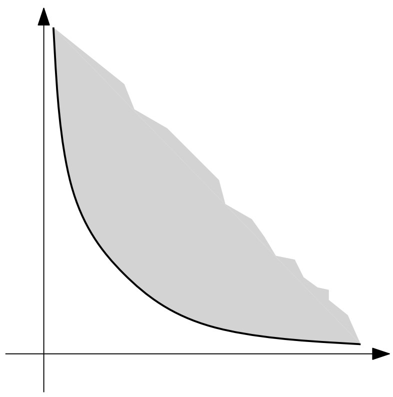

Figure 1.1: The shaded gray area represent the feasible region (which extend indefinitely).

The framework presented so far is however too general. Whether or not an optimization problem can be solved efficiently depends on the specific structure of the problem, namely the domain $\mathcal{D}$ of the decision variables and the form of the functions that define the problem. Here are three classes of optimisation problems:

Linear Programs: in this case $\mathcal{D} = \mathbb{R}^n$ and the functions $f_i : \mathbb{R}^n \to \mathbb{R}$ ($i = 0, \ldots, m$) are *linear functions*, that is, functions of the form $f(x) = \sum_{i=1}^n a_i x_i$ for given numbers $a_1, \ldots, a_n \in \mathbb{R}$.

Convex Programs: This is the case where the functions $f_i : \mathbb{R}^n \to \mathbb{R}$ ($i = 0, \ldots, m$) are convex.

Integer Linear Programs: they are linear programs in which some of the variables are restricted to take integer values.

We remark that, despite the formal similarities between Linear Programs and Integer Linear Programs, Integer Linear Programs are significantly harder to solve than Linear Programs.

Another vast class is that of *combinatorial optimization* problems, which are the subject of the course MA428.

1.2 Examples of Optimization Problems

Example 1.3. A factory produces orange juice (OJ) and orange concentrate (OC), starting from three raw material: electricity, oranges, and water. Every liter of OJ produced gives a profit of £3, while every liter of OC produced gives a profit of £2. Producing a liter of OJ requires 1 unit of electricity, 1 unit of oranges, and 1 unit of water. Producing a liter of OC requires 1 unit of electricity, 2 units of oranges, and it returns 1 unit of water as a byproduct of the concentration process. The factory has at its disposal 6 units of electricity, 10 units of oranges, and 4 units of water. How much OJ and OC should the factory produce in order to maximize profit?

Decision variables. We need to decide how much OJ and how much OC to produce. We introduce two variables:
x_1 = number of liters of OJ produced;
x_2 = number of liters of OC produced.

Objective function. The total profit of producing x_1 liters of OJ and x_2 liters of OC is

$$3x_1 + 2x_2.$$

Constraints. Producing x_1 liters of OJ and x_2 liters of OC we consume

$$x_1 + x_2 \text{ units of electricity,}$$

$$x_1 + 2x_2 \text{ units of oranges,}$$

$$x_1 - x_2 \text{ units of water.}$$

Since we are not allowed to use more resources than the ones at our disposal, we have the following constraints:

$$\begin{aligned} x_1 + x_2 &\leq 6 && (1.2) \\ x_1 + 2x_2 &\leq 10 && (1.3) \\ x_1 - x_2 &\leq 4 && (1.4) \end{aligned}$$

Furthermore, since the variables x_1 and x_2 represent nonnegative quantities, we need also to introduce the constraints $x_1 \geq 0$, $x_2 \geq 0$.

The problem we need to solve is therefore

$$\begin{array}{rrcrcr}
\max & 3x_1 & + & 2x_2 & & \\
s.t. & x_1 & + & x_2 & \leq & 6 \\
 & x_1 & + & 2x_2 & \leq & 10 \\
 & x_1 & - & x_2 & \leq & 4 \\
 & & & x_1, x_2 & \geq & 0
\end{array}$$

The above is a linear program.

Example 1.4. A hospital needs to decide how many nurses to staff. On each day i of the week ($i = 1, \ldots, 7$), the hospital requires d_i nurses (the number of nurses needed might vary from day to day). By contract, each nurse works five consecutive days and then gets two consecutive days of rest each week, but the rest days need not be the same for each nurse. How do we plan the shifts so as to minimize the total number of nurses needed?

Decision variables. In this case the choice of variables is not completely obvious. We choose the following variables: for every day i of the week, $i = 1, \ldots, 7$, we have one variable,

$$x_i = \text{ number of nurses who start working on day } i.$$

Since a nurse works 5 consecutive days and rests the remaining 2, once we have decided how many nurses start on a given day, we have completely determined the nurses' schedule.

Objective function. Each nurse we hire starts on a specific day of the week. Thus the total number of nurses hired is

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7.$$

Constraints. We need to guarantee that on any given day there is a sufficient number of nurses working. For example, the nurses working on Monday are the ones who started either on Monday, or within four days before. Thus the total number of nurses working on Monday is $x_1 + x_4 + x_5 + x_6 + x_7$. This gives the constraint

$$x_1 + x_4 + x_5 + x_6 + x_7 \geq d_1.$$

The other constraints can be derived similarly. Note also that we have also the conditions $x_i \geq 0$, $i = 1, \ldots, 7$. Another condition that we need to impose is that the number of nurses starting the shift on a given day is an integer number, we have therefore the integrality constraints:

$$x_i \text{ integer}, \ i = 1, \ldots, 7.$$

The problem we need to solve is therefore

$$\begin{array}{rccccccccc}
\min & x_1 & +x_2 & +x_3 & +x_4 & +x_5 & +x_6 & +x_7 & & \\
s.t. & x_1 & & & +x_4 & +x_5 & +x_6 & +x_7 & \geq & d_1 \\
& x_1 & +x_2 & & & +x_5 & +x_6 & +x_7 & \geq & d_2 \\
& x_1 & +x_2 & +x_3 & & & +x_6 & +x_7 & \geq & d_3 \\
& x_1 & +x_2 & +x_3 & +x_4 & & & +x_7 & \geq & d_4 \\
& x_1 & +x_2 & +x_3 & +x_4 & +x_5 & & & \geq & d_5 \\
& & +x_2 & +x_3 & +x_4 & +x_5 & +x_6 & & \geq & d_6 \\
& & & +x_3 & +x_4 & +x_5 & +x_6 & +x_7 & \geq & d_7 \\
& & \multicolumn{5}{l}{x_1, x_2, x_3, x_4, x_5, x_6, x_7} & \geq 0 & & \\
& & \multicolumn{5}{l}{x_1, x_2, x_3, x_4, x_5, x_6, x_7} & \text{integer.} & &
\end{array}$$

The above is an Integer Linear Program.

Example 1.5. (Markowitz portfolio optimization) We are given n assets or stocks, to be held over a given period of time. We want to decide how to allocate our budget B among the different assets during the time period. Let w_i denote the proportion of budget B invested in asset i during the time period. We assume that $\sum_{i=1}^n w_i = 1$ (i.e. we invest all our budget). Here we consider the case in which we cannot hold short positions, that is, $w_i \geq 0$, $i = 1, \ldots, n$.

The return r of the portfolio at the end of the time period is a random variable, depending on our choice of $w \in \mathbb{R}^n$. We denote by $\bar{r}$ the expected value of r. In Markowitz's model, the *risk* of the portfolio is measured as the variance of r (i.e. $\mathbb{E}[(r - \bar{r})^2]$). Given a *target return* $r_{\min}$, we want to determine the portfolio allocation vector w that minimizes risk, subject to the constraint that the expected return $\bar{r}$ is at least the target $r_{\min}$. That is, we want to determine the optimal solution for the following optimization problem.

$$\begin{gathered}
\min \ \operatorname{Var}(r) \\
\text{s.t.} \\
\bar{r} \geq r_{\min} \\
\textstyle\sum_{i=1}^n w_i = 1 \\
w \geq 0
\end{gathered}$$

To express $\bar{r}$ and $\operatorname{Var}(r)$ in terms of w, consider the random vector $p \in \mathbb{R}^n$, where p_i is the return of asset i at the end of the period. We denote by $\bar{p}$ the vector of means of p (i.e. $\bar{p}_i = \mathbb{E}[p_i]$), and by Σ the covariance matrix of p (i.e. $\Sigma_{ij} = \mathbb{E}[(p_i - \bar{p}_i)(p_j - \bar{p}_j)]$). (We are not concerned here on how the vector $\bar{p}$ and matrix Σ are determined, typically they are estimated based on historical data.)

Since the return of asset i is $p_i w_i$, the return of the portfolio is given by

$$r = p^\top w.$$

and therefore

$$\bar{r} = \bar{p}^\top w, \qquad \mathrm{Var}(r) = w^\top \Sigma w,$$

where you get the second one by plugging $r = p^\top w$ into $\mathrm{Var}(r) = \mathbb{E}[(r - \bar{r})^2]$ and using the formula for Σ_{ij} given above.

Thus Markowitz portfolio optimization problem is

$$\begin{gathered} \min\ w^\top \Sigma w \\ \text{s.t.} \\ \bar{p}^\top w \geq r_{\min} \\ \textstyle\sum_{i=1}^n w_i = 1 \\ w \geq 0 \end{gathered}$$

Note that all constraints are linear (and therefore convex), and that the objective function is a quadratic convex function (this can be shown because covariance matrices are always positive semidefinite). Therefore the above is an example of a convex optimization problem (indeed, it is a so called *quadratic program*, meaning that the objective function is convex quadratic and the constraints are linear).

1.3 Notations and definitions

In these notes, every vector $x \in \mathbb{R}^n$ is considered as the column vector,

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

and treated as an $n \times 1$ matrix. For any matrix M, we denote by $M^\top$ the transpose of M.

Hence, given $x, y \in \mathbb{R}^n$,

$$x^\top y = y^\top x = \sum_{j=1}^n x_j y_j = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

is the *scalar product* of x and y.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is *linear* if there exists a vector $a \in \mathbb{R}^n$ such that $f(x) = a^\top x$ for all $x \in \mathbb{R}^n$. The function f is said to be *affine* if there exists a vector $a \in \mathbb{R}^n$ and a number $b \in \mathbb{R}$ such that $f(x) = a^\top x + b$ for all $x \in \mathbb{R}^n$.

Given $x, y \in \mathbb{R}^n$, we will write

$$x \geq y$$

to indicate that $x_j \geq y_j$ for every $j \in \{1, \dots, n\}$; we will write

$$x > y$$

if $x_j > y_j$ for every j, $j \in \{1, \dots, n\}$; and we will write

$$x \neq y$$

if there exists $j \in \{1, \dots, n\}$ such that $x_j \neq y_j$.

We denote by 0 the zero vector, where the dimension of such vector will be clear from the context. Thus, given $x \in \mathbb{R}^n$, when we write $x \geq 0$ it will be understood that the dimensions of the vectors x and 0 are compatible, and thus 0 is the zero vector in $\mathbb{R}^n$.

1.4 References

- D. Bertsimas J.N. Tsitsiklis, *Introduction to Linear Optimization*, Athena Scientific, (1997).

  This is an excellent modern treatment of linear programming, very clearly exposed. Also nice chapter with problem formulation examples.

- V. Chvátal, *Linear Programming*, Freeman, (1983).

  The classic textbook in linear programming. Probably the most widely read and influential text on linear programming.

- S. Boyd, L. Vandenberghe, *Convex Optimization*, Cambridge University Press, (2004).

  Excellent reference for the part on Convex Optimization. Freely available online at `www.stanford.edu/~boyd/cvxbook/bv_cvxbook.pdf`.

- L.H. Wolsey, *Integer Programming*, Wiley, (1998).

  A concise first book in integer programming by one of the leading figures in the field. Several examples of problem formulations.

  A book on problem formulation.

- L. Schrage, *Optimization modeling with LINGO*, LINDO Systems Inc, 1998

  Freely available at `http://www.lindo.com/`. It is the manual of LINGO, a modeling language for linear and integer programming. The book contains many example of how to formulate problems.

Chapter 2

Lecture 1: Linear Programming Problems

2.1 Definitions and Fundamental Theorem

A *Linear Programming* (LP) problem consists of finding the maximum (resp. minimum) of a linear function subject to linear equations or inequality constraints. Thus, a linear programming problem can be written in the form

$$\begin{array}{llcll} \max(\text{resp.}\min) & c^\top x & & & \\ & a_i^\top x & = & b_i & i = 1, \dots, k \\ & a_i^\top x & \leq & b_i & i = k+1, \dots, r \\ & a_i^\top x & \geq & b_i & i = r+1, \dots, m \end{array} \tag{2.1}$$

where $b_i \in \mathbb{R}$, $i = 1, \dots, m$, $c, a_i \in \mathbb{R}^n$, $i = 1, \dots, m$, and x is a vector of variables in $\mathbb{R}^n$.

Linear programming problems are the most basic type of optimization problems, they have a rich theory and several nice properties that allow to solve them efficiently. A first such property is that, unlike general optimization problems, Linear Programs always admit an optimal solution whenever they are feasible and bounded, as we express formally in the following theorem.

Theorem 2.1 (Fundamental Theorem of Linear Programming)**.** *For any linear programming problem, exactly one of the following holds.*

1. *The problem has a finite optimum;*
2. *The Problem is infeasible;*
3. *The problem is unbounded.*

We do not prove the theorem at this stage. The validity of the statement will actually follow from the *simplex algorithm* which is probably the most used and widely known method to solve LP problems, and will be presented later in this course. As we will see, the algorithm terminates when it determines that one of the three outcomes of Theorem 2.1 holds.

We illustrate the three possible outcomes outlined in Theorem 2.1 in the next three examples.

Example (An LP with optimal solution). Consider the LP problem from the previous chapter, on deciding how much Orange Juice and Orange Concentrate to manufacture.

$$\begin{array}{rrcrcr}
\max & 3x_1 & + & 2x_2 & & \\
s.t. & x_1 & + & x_2 & \leq & 6 \\
 & x_1 & + & 2x_2 & \leq & 10 \\
 & x_1 & - & x_2 & \leq & 4 \\
 & & & x_1, x_2 & \geq & 0
\end{array}$$

The feasible region and the objective function are depicted in Figure 2.1. It can be seen that the optimum is the point $x^* = (5, 1)$, with objective value 17.

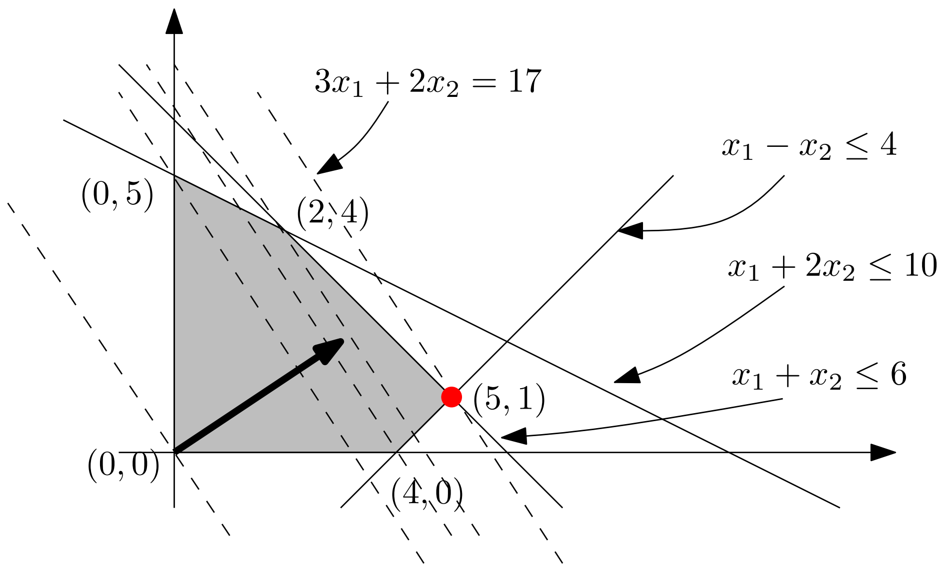

Figure 2.1: The shaded gray region is the feasible region of the problem. The boldfaced arrow represent the direction of improvement of the objective function. All points in the same dashed line have the same objective value.

Example (An infeasible LP). Consider the following LP problem.

$$\begin{array}{rrcrcrcr} \max & x_1 & + & x_2 & & & & \\ s.t. & x_1 & - & 2x_2 & + & 2x_3 & \leq & 2 \\ & -2x_1 & + & 6x_2 & - & 2x_3 & \leq & -6 \\ & & & x_1, x_2, x_3 & & & \geq & 0 \end{array}$$

The problem has no feasible solution. To see this, note that if we multiply the second constraint by $\frac{1}{2}$ and sum it to the first constraint, we obtain the inequality

$$x_2 + x_3 \leq -1.$$

Because the above inequality is derived by multiplying inequalities of the LP by nonnegative numbers and then summing them up, it follows that the derived inequality should be satisfied by every feasible solution. However, since $x_2, x_3 \geq 0$ for every feasible solution, it must be the case that

$$x_2 + x_3 \geq 0$$

holds for every feasible solution. Clearly the two inequalities can never be satisfied both at the same time. We thus conclude that there cannot be any feasible solution.

Example (An unbounded LP). Consider the following LP problem.

$$\begin{array}{rrcrcr} \max & 2x_1 & - & 3x_2 & & \\ s.t. & x_1 & + & x_2 & \geq & 1 \\ & -x_1 & + & x_2 & \leq & 2 \\ & x_1 & - & 2x_2 & \leq & 2 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

Consider the following family of solutions, defined by the parameter t.

$$\begin{array}{rcr} x_1(t) & = & 1 + 2t \\ x_2(t) & = & t. \end{array}$$

Note that, for any $t \geq 0$, the point $x(t)$ is always feasible. The objective value of $x(t)$ is

$$2(1 + 2t) - 3t = 2 + t$$

which goes to infinity as t goes to infinity. Therefore the problem is unbounded, since there are feasible solutions with arbitrarily large values.

The example is illustrated in Figure 2.2. Note that the family of solutions $x(t)$, $t \geq 0$, describes a half-line starting from the point $(1, 0)$ with direction $(2, 1)$. The half line is entirely contained in the feasible region.

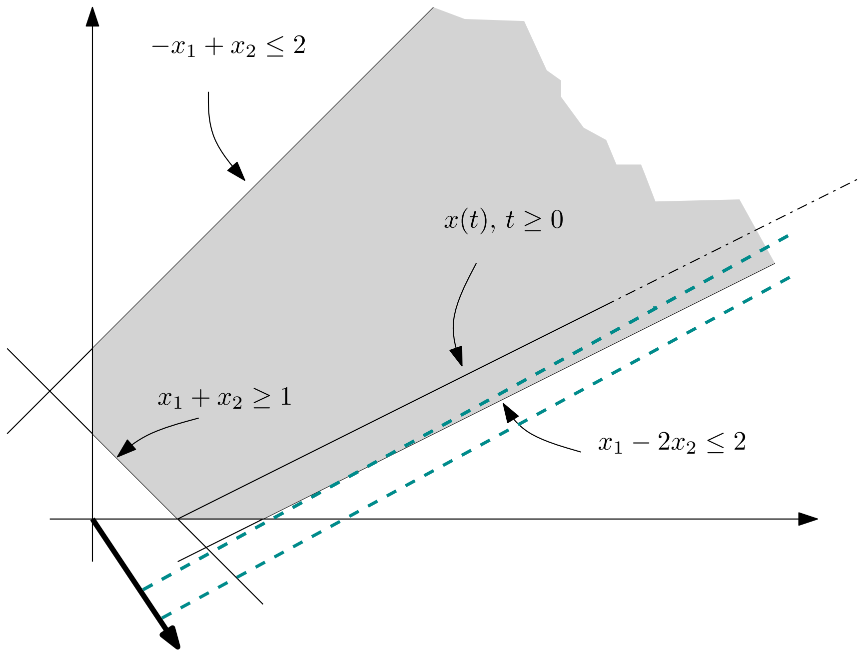

Figure 2.2: An unbounded problem.

2.2 Standard forms

When dealing with Linear Programming problems, it often convenient to assume that they are of some specific form. We will often consider problems in one of the following forms.

An LP problem is in *standard form* if it is of the form

$$\begin{aligned} z^* = \quad & \max\, c^\top x \\ Ax \leq \quad & b \\ x \geq \quad & 0 \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and x is a vector of indeterminates in $\mathbb{R}^n$.

An LP problem is in *standard equality form* if it is of the form

$$\begin{aligned} z^* = \quad & \max\, c^\top x \\ Ax = \quad & b \\ x \geq \quad & 0 \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and x is a vector of indeterminates in $\mathbb{R}^n$.

We want to observe here that any general LP problem (2.1) "can be brought" in one of the above standard forms, in the sense that we can always write an "equivalent " LP problem in standard form or standard equality form.

Standard form. First, observe that, if an LP problem (2.1) is a minimization problem, such problem is the same as solving $\max -c^\top x$ subject to the same constraints. We may therefore assume without loss of generality, whenever convenient, that (2.1) is a maximization problem.

"$\geq$" and "$=$" type constraints: Any "greater than or equal" constraint

$$a_i^\top x \geq b_i$$

is equivalent to the constraint

$$-a_i^\top x \leq -b_i$$

(they have the same set of feasible solutions). Similarly, any equality constraint

$$a_i^\top x = b_i$$

is equivalent to the two constraints

$$a_i^\top x \leq b_i, \quad -a_i^\top x \leq -b_i.$$

Hence we can assume that all constraints of (2.1) are of the "less than or equal" type, $a_i^\top x \leq b_i$,.

Nonpositive and free variables: If the problem has a constraint of the type $x_i \geq 0$, the variable x_i is said a *nonnegative variable.* If the problem has a constraint of the type $x_i \leq 0$, the variable x_i is said a *nonpositive variable.* Otherwise x_i is said to be a *free variable.*

- Suppose the problem has a nonpositive variable x_i. We introduce a new nonnegative variable, x_i' and we set $x_i' = -x_i$. Substituting x_i with $-x_i'$ in the constraints and in the objective function of (2.1), we get a new problem where variable x_i does not appear, but we have one new nonnegative variable. Clearly the problem we get is equivalent.

- Suppose the problem has a free variable x_i. We perform a change of variable based on the simple observation that any number can be written as the difference of two nonnegative numbers (for example, $-3 = 0 - 3$, or also

$-3 = 2 - 5$). We introduce two nonnegative variables, x_i^+ and x_i^-, and we set

$$x_i = x_i^+ - x_i^-, \quad x_i^+, x_i^- \geq 0.$$

Substituting x_i with $x_i^+ - x_i^-$ in the constraints and in the objective function of (2.1), we get a new problem where variable x_i does not appear, but we have two new nonnegative variables. Clearly the problem we get is equivalent: for any feasible solution of the new problem, we can construct a feasible solution of (2.1) with the same objective value by setting $x_i = x_i^+ - x_i^-$. Viceversa, given a feasible solution of (2.1), we get a feasible solution of the new problem with the same objective value by setting $x_i^+ = \max\{0, x_i\}$ and $x_i^- = \max\{0, -x_i\}$. For example, if $x_i = 3$, then $x_i^+ = 3$ and $x_i^- = 0$, and if $x_i = -2$ then $x_i^+ = 0$ and $x_i^- = 2$.

Standard equality form. We have seen above that any problem can be written in standard form. To transform a problem in standard equality form we have to "get rid" of all the "less than or equal" constraints, and substitute them with equality constraints. If the problem has a constraint of the form

$$a_i^\top x \leq b_i,$$

we can introduce a new nonnegative variable s_i and replace the previous constraint with the two constraints

$$a_i^\top x + s_i = b_i, \quad s_i \geq 0.$$

The variable s_i is said a *slack variable*, since $s_i = b_i - a_i^\top x$ represent the slack of the constraint $a_i^\top x \leq b_i$. Obviously, x satisfies $a_i^\top x \leq b_i$, if and only if $s_i = b_i - a_i^\top x$ satisfies $s_i \geq 0$. Since the objective function remains the same, the two problems are equivalent.

Example. Consider the following LP.

$$\begin{array}{llllll}
\min & 3x_1 & -2x_2 - & x_3 & & \\
 & -x_1 & + \ x_2 + & 2x_3 & = & 4 \\
 & 2x_1 & \quad\ - & x_3 & \geq & -2 \\
 & -2x_1 - & x_2 + & x_3 & \leq & 1 \\
 & \multicolumn{5}{l}{\quad x_1 \leq 0, \ x_3 \geq 0}
\end{array}$$

Using the procedure described above, the problem can be brought into standard from as follows:

$$
\begin{array}{lllllll}
\max & 3x_1' & +2x_2^+ & -2x_2^- & + \ x_3 & & \\
 & x_1' & + \ x_2^+ & - \ x_2^- & +2x_3 & \leq & 4 \\
 & -x_1' & - \ x_2^+ & + \ x_2^- & -2x_3 & \leq & -4 \\
 & 2x_1' & & & + \ x_3 & \leq & 2 \\
 & 2x_1' & - \ x_2^+ & + \ x_2^- & + \ x_3 & \leq & 1 \\
 & & \multicolumn{3}{l}{x_1',\ x_2^+,\ x_2^-,\ x_3 \geq 0} & &
\end{array}
$$

The optimal solution of the above problem (computed using an LP solver) is $x_1' = 1$, $x_2^+ = 3$, $x_2^- = 0$, $x_3 = 0$, with value 9. This means that the original problem has optimal value -9, and the optimal solution is $x_1 = -x_1' = -1$, $x_2 = x_2^+ - x_2^- = 3$, $x_3 = 0$.

The problem can be then brought into standard equality form by introducing slack variables:

$$
\begin{array}{lllllllllll}
\max & 3x_1' & +2x_2^+ & -2x_2^- & + \ x_3 & & & & & & \\
 & x_1' & + \ x_2^+ & - \ x_2^- & +2x_3 & +s_1 & & & & = & 4 \\
 & -x_1' & - \ x_2^+ & + \ x_2^- & -2x_3 & & +s_2 & & & = & -4 \\
 & 2x_1' & & & + \ x_3 & & & +s_3 & & = & 2 \\
 & 2x_1' & - \ x_2^+ & + \ x_2^- & + \ x_3 & & & & +s_4 & = & 1 \\
 & & \multicolumn{7}{l}{x_1',\ x_2^+,\ x_2^-,\ x_3,\ s_1,\ s_2,\ s_3,\ s_4 \geq 0} & &
\end{array}
\tag{2.2}
$$

Incidentally, observe that, if the purpose was to bring the problem into standard equality form, we could have avoided the step to turn the equality constraint in the original problem into two "$\leq$" constraints, and simply write the problem in the form

$$
\begin{array}{lllllllll}
\max & 3x_1' & +2x_2^+ & -2x_2^- & + \ x_3 & & & & \\
 & x_1' & + \ x_2^+ & - \ x_2^- & +2x_3 & & & = & 4 \\
 & 2x_1' & & & + \ x_3 & +s_1 & & = & 2 \\
 & 2x_1' & - \ x_2^+ & + \ x_2^- & + \ x_3 & & +s_2 & = & 1 \\
 & & \multicolumn{5}{l}{x_1',\ x_2^+,\ x_2^-,\ x_3,\ s_1,\ s_2 \geq 0} & &
\end{array}
\tag{2.3}
$$

Note that from the first two constraints of (2.2) we get that $s_1 = -s_2$ and since $s_1, s_2 \geq 0$, we must have $s_1 = s_2 = 0$. Substituting this into the first two constraints of (2.2) we get the first constraint of (2.3). Thus the formulations (2.2) and (2.3) are identical.

Chapter 3

Lecture 2: Linear Programming Duality

Ideally, given an LP problem one would like to be able to find an optimal solution. Here we address a more basic question: given an LP problem and a feasible solution, how can decide that the solution is optimal? The answer to such a question is of paramount importance when designing a method to solve LP problems, since in order to find an optimal solution we should first be able to recognize one.

Let us consider the usual example (in standard form):

$$\begin{array}{rrcr} \max & 3x_1 + 2x_2 & & \\ s.t. & x_1 + \ \ x_2 & \leq & 6 \\ & x_1 + 2x_2 & \leq & 10 \\ & x_1 - \ \ x_2 & \leq & 4 \\ & x_1, x_2 & \geq & 0 \end{array}$$

We have seen graphically that the optimal solution is $(5, 1)$, with value 17. How can we convince ourself that $(5, 1)$ is indeed the optimum? We would need to show that any feasible solution cannot have value greater than 17.

For example, if we multiply the first constraint by 3, we obtain the inequality $3x_1 + 3x_2 \leq 18$. This inequality must be satisfied by every feasible solution. Furthermore, since $x_1, x_2 \geq 0$ for every feasible solution, we have that

$$3x_1 + 2x_2 \leq 3x_1 + 3x_2 \leq 18$$

for every feasible solution. This shows that the objective value of any feasible solution x, that is $3x_1 + 2x_2$, cannot exceed 18, therefore the optimum value cannot be more than 18.

This does not yet prove optimality. Let us now, instead, multiply the first constraint by $\frac{5}{2}$ and the third by $\frac{1}{2}$, and let us sum them. We obtain the inequality

$$\frac{5}{2}(x_1 + x_2) \leq 15$$

and

$$\frac{1}{2}(x_1 - x_2) \leq 2.$$

Both inequalities are satisfied by any feasible solution. If we sum them, we obtain the inequality

$$3x_1 + 2x_2 \leq 17,$$

which is satisfied by every feasible solution. Note that $3x_1 + 2x_2$ is precisely the objective function, thus no feasible solution can have value more than 17. But the solution $(5, 1)$ has value exactly 17, therefore it is an optimal solution.

We will see that, according to the theory of duality, it is always possible to prove optimality with this kind of arguments.

Let us first slightly abstract the method we used in the example. Choose nonnegative multipliers for the three first constraints, say y_1, y_2, and y_3. Because $y_1, y_2, y_3 \geq 0$, the inequalities

$$\begin{array}{rcr} y_1(x_1 + x_2) & \leq & 6y_1 \\ y_2(x_1 + 2x_2) & \leq & 10y_2 \\ y_3(x_1 - x_2) & \leq & 4y_3 \end{array}$$

are satisfied by all feasible solutions x. If we sum them up, we obtain the inequality

$$(y_1 + y_2 + y_3)x_1 + (y_1 + 2y_2 - y_3)x_2 \leq 6y_1 + 10y_2 + 4y_3,$$

which is also satisfied by every feasible solution x.

Suppose now that the multipliers y_1, y_2, y_3 have been chosen so that the coefficients of the variables in the above inequality are greater than or equal to the coefficients of the variables in the objective function, which is $3x_1 + 2x_2$. That is, suppose that

$$\begin{array}{rcl} y_1 + y_2 + y_3 & \geq & 3 \\ y_1 + 2y_2 - y_3 & \geq & 2 \end{array} \tag{3.1}$$

Since $x_1, x_2 \geq 0$ for every feasible solution, we have that

$$3x_1 + 2x_2 \leq (y_1 + y_2 + y_3)x_1 + (y_1 + 2y_2 - y_3)x_2 \leq 6y_1 + 10y_2 + 4y_3$$

for every feasible solution. In this way, we would have shown that no feasible solution has value greater than $6y_1 + 10y_2 + 4y_3$. That is, any choice of nonnegative values y_1, y_2, y_3 satisfying (3.1) provides us with an upper bound on the optimal value of the LP, namely the value $6y_1 + 10y_2 + 4y_3$. The "best" upper bound we can find in this way is the optimal value of the problem

$$\begin{array}{rrcl} \min & 6y_1 + 10y_2 + 4y_3 & & \\ & y_1 + \quad y_2 + \quad y_3 & \geq & 3 \\ & y_1 + \; 2y_2 - \quad y_3 & \geq & 2 \\ & y_1, y_2, y_3 & \geq & 0 \end{array}$$

The above is also a linear programming problem, called the *dual* of the original LP. In the next section we generalise the above argument to derive the dual of problems in standard form.

3.1 Duality in standard form

Consider the LP problem in standard form.

$$\begin{array}{rrl} z^* = & \max c^\top x & \\ & Ax \leq & b \\ & x \geq & 0 \end{array} \tag{3.2}$$

where $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and x is a vector of indeterminates in $\mathbb{R}^n$.

How could we prove that a certain feasible solution $\bar{x}$ is optimal? Let $V := c^\top \bar{x}$ be the objective value of $\bar{x}$. Suppose we could somehow prove that every feasible solution x satisfies the inequality $c^\top x \leq V$. This is equivalent to saying that no feasible solution of (3.2) can have value more than V, thus we would have proven that $\bar{x}$ is indeed the optimal solution.

Thus, we want to derive upper-bounds on the optimal value z^* of (3.2). In the previous section, this was accomplished as follows: we multiplied each inequality $a_i^\top x \leq b_i$ of $Ax \leq b$ by a nonnegative multiplier y_i ($i = 1, \ldots, m$), and then summed them up to obtain a new inequality.

In matrix notation, given a nonnegative vector $y \in \mathbb{R}^m$ of multipliers, any feasible solution x for (3.2) must satisfy the inequality

$$(y^\top A)x \leq y^\top b,$$

because it is obtained by multiplying each of the constraints in $Ax \leq b$ by a nonnegative number and then summing them up. Furthermore, assume that the vector y satisfies $A^\top y \geq c$. Since x is nonnegative, x satisfies

$$c^\top x \leq (y^\top A)x \leq y^\top b = b^\top y,$$

where the second inequality holds because y is nonnegative.

Hence, given any $y \in \mathbb{R}^m$ satisfying $A^\top y \geq c$ and $y \geq 0$, for every feasible solution x for (3.2) we have $c^\top x \leq b^\top y$, thus

$$z^* \leq b^\top y.$$

In other words, $b^\top y$ is an upper-bound to the optimal value z^*. What is "the best" (i.e. the tightest) possible upper bound that we can obtain with such technique? Clearly it is the smallest possible such value. Hence, we want to find a vector y satisfying $A^\top y \geq c$, $y \geq 0$ such that $b^\top y$ is as small as possible. This is, again, a linear programming problem!

Namely, we want to solve

$$\begin{aligned} d^* = \quad & \min b^\top y \\ & A^\top y \geq \quad c. \\ & y \geq 0 \end{aligned} \tag{3.3}$$

Problem (3.3) is said the *dual* of (3.2). The multipliers $y \in \mathbb{R}_m$ are the variables of the dual problem, and are called *dual variables*. We will refer to (3.2) as the *primal problem*, and the variables x of the primal problem are the *primal variables*.

We have shown the following.

Theorem 3.1 (Weak duality theorem)**.** *Given any feasible solution x^* for the primal LP (3.2) and a feasible solution y^* for its dual (3.3), we have*

$$c^\top x^* \leq b^\top y^*.$$

This immediately implies the two following corollaries.

Corollary 3.2. *Let x^* be a feasible solution for (3.2) and y^* be a feasible solution for (3.3). If $c^\top x^* = b^\top y^*$, then x^* is an optimal solution for (3.2) and y^* is optimal for (3.3).*

Corollary 3.3. *Consider the primal problem* (3.2) *and its dual* (3.3).

(*i*) *If the primal problem is unbounded, then the dual is infeasible.*

(*ii*) *If the dual problem is unbounded, then the primal is infeasible.*

Proof: (*i*) Suppose the dual (3.3) has a feasible solution $\bar{y}$. Then by the weak duality theorem $c^\top x \leq b^\top \bar{y}$ for any feasible solution x for (3.2), contradicting the fact that (3.2) is unbounded. Now, suppose that the primal has a feasible solution $\bar{x}$, then by the weak duality theorem $c^\top \bar{x} \leq b^\top y$ for any feasible solution y for (3.3) contradicting the fact that (3.3) is unbounded. $\square$

So far we have only shown that the optimal value d^* of the dual is an upper-bound to the optimal primal value z^*. When do the two values coincide? The answer is somewhat surprising: always! This is known as the *strong duality* theorem.

Theorem 3.4 (Strong duality theorem)**.** *If (3.2) has a finite optimum x^*, then also (3.3) has a finite optimum y^*, and $c^\top x^* = b^\top y^*$.*

One important consequence of the Strong Duality Theorem is the following. If a feasible solution x^* is optimal for (3.2), one can give a **certificate** of this fact; namely, an optimal solution y^* for the dual (3.3). Once given such a certificate, one can verify that x^* is optimal by

- Checking that y^* is a feasible solution for (3.3);
- Checking that $c^\top x^* = b^\top y^*$.

Example. The dual of the (by now familiar) LP

$$\begin{array}{lrcrcr} \max & 3x_1 & + & 2x_2 & & \\ s.t. & x_1 & + & x_2 & \leq & 6 \\ & x_1 & + & 2x_2 & \leq & 10 \\ & x_1 & - & x_2 & \leq & 4 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

is

$$\begin{array}{lrcrcrcr} \min & 6y_1 & + & 10y_2 & + & 4y_3 & & \\ & y_1 & + & y_2 & + & y_3 & \geq & 3 \\ & y_1 & + & 2y_2 & - & y_3 & \geq & 2 \\ & & & & & y_1, y_2, y_3 & \geq & 0 \end{array}$$

As we have seen, the optimal solution for the primal problem is $x^* = (5, 1)$ with value 17. From the discussion at the beginning of the chapter, we can argue that the optimal solution for the dual is the vector $y^* = (\frac{5}{2}, 0, \frac{1}{2})$. Note that y^* is feasible and has value 17.

3.2 Dual of the dual

What is the dual of the dual? Let us consider the problem in standard form (3.2) and its dual (3.3). To compute the dual of (3.3), we write the problem again in standard form:

$$\begin{array}{rrl} -d^* = & \max -b^\top y & \\ & -A^\top y \leq & -c \ . \\ & y \geq & 0 \end{array}$$

Then its dual is

$$\begin{aligned} \min\ & -c^\top z \\ -Az \geq\ & -b. \\ z \geq\ & 0 \end{aligned}$$

Writing the problem in standard form again, we get the equivalent problem

$$\begin{aligned} \max\ & c^\top z \\ Az \leq\ & b. \\ z \geq\ & 0 \end{aligned}$$

which is identical to (3.2).

Therefore **the dual of the dual is the primal!**

In particular, this implies the following, stronger version of the LP strong duality theorem.

Theorem 3.5 (Strong Duality Theorem)**.** *If one of the problems (3.2) and (3.3) has a finite optimum, then both problems have a finite optimum. Given an optimal solution x^* for (3.2) and an optimal solution y^* for (3.3), then $c^\top x^* = b^\top y^*$.*

The above theorem, the fundamental theorem of LP (Theorem 2.1), and Corollary 3.3, imply that any primal/dual pair of LP problems satisfies one of the alternatives represented in the table below.

		Primal		
		Fin. opt.	Infeasible	Unbounded
Dual	Fin. opt.	*Possible*	*NO*	*NO*
	Infeasible	*NO*	*Possible*	*Possible*
	Unbounded	*NO*	*Possible*	*NO*

We point out that it is indeed possible that both the primal and the dual problem are infeasible. For example, consider the following pair of primal/dual problems.

$$\begin{array}{lrcrcr} \max & 2x_1 & - & x_2 & & \\ & x_1 & - & x_2 & \leq & 1 \\ & -x_1 & + & x_2 & \leq & -2 \\ & \multicolumn{5}{l}{x_1, x_2 \geq 0} \end{array} \quad , \quad \begin{array}{lrcrcr} \min & y_1 & - & 2y_2 & & \\ & y_1 & - & y_2 & \geq & 2 \\ & -y_1 & + & y_2 & \geq & -1 \\ & \multicolumn{5}{l}{y_1, y_2 \geq 0} \end{array}$$

Observe that both problems are infeasible. Indeed, if we sum the first two constraints of the primal, we obtain the inequality $0 \leq -1$, which is never verified; if we sum the first two constraints of the dual, we obtain the inequality $0 \geq 1$.

3.3 Duality in other forms

3.3.1 Duality in standard equality form

The dual is defined for any LP problem, not just for standard form LPs. In principle, we could convert any LP problem in standard form and then compute the dual, but this is often cumbersome. We will give a general way of constructing the dual in the next section, but for now we derive the dual for problems in standard equality form.

Consider the following LP problem in standard equality form.

$$\begin{aligned} z^* = \quad & \max c^\top x \\ Ax = \quad & b \\ x \geq \quad & 0 \end{aligned} \tag{3.4}$$

where $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and x is a vector of indeterminates in $\mathbb{R}^n$.

What is the dual of (3.4)? We have two way of finding the dual: one is to repeat the argument we did for LPs earlier in standard form, with the necessary modifications. The other is to write (3.4) in standard form and then use the form of the dual given in (3.3).

Method 1 As before, we look for upper bounds to the optimal value z^* of (3.4). Note that any vector x satisfying $Ax = b$ also satisfies

$$(y^\top A)x = y^\top b,$$

for any vector $y \in \mathbb{R}^m$, but note that here y need not be nonnegative. Furthermore, suppose that y also satisfies $A^\top y \geq c$. Then, for any feasible solution x for (3.4), x satisfies the inequality

$$c^\top x \leq (y^\top A)x$$

because x is nonnegative. Combining the two above inequalities, we obtain, for any feasible solution x for (3.4) and any $y \in R^m$ such that $A^\top y \geq c$, the following inequality

$$c^\top x \leq b^\top y.$$

Thus any $y \in \mathbb{R}^m$ satisfying $A^\top y \geq c$, provides an upper-bound to the optimal value z^* of (3.4), namely

$$z^* \leq b^\top y.$$

The best possible upper-bound we can obtain in this way is the optimal value d^* of the LP problem

$$\begin{aligned} d^* = \quad & \min b^\top y \\ A^\top y \geq & \quad c. \end{aligned} \tag{3.5}$$

The problem (3.5) is the dual of (3.4).

Method 2 Let us write (3.4) in standard form by replacing each equality constraint $a_i^\top x = b_i$ with two inequality constraints $a_i^\top x \leq b_i$ and $-a_i^\top x \leq -b_i$. We get the problem

$$\begin{aligned} z^* = \quad & \max c^\top x \\ Ax \leq & \quad b \\ -Ax \leq & \quad -b \\ x \geq & \quad 0 \end{aligned}$$

Using (3.2), we can write the dual of the above problem:

$$\begin{aligned} \min b^\top y^+ - b^\top y^- \\ A^\top y^+ - A^\top y^- \geq c \\ y^+, y^- \geq 0. \end{aligned}$$

If we make a change of variables $y = y^+ - y^-$, y is now a vector of free variables in $\mathbb{R}^m$. Note that we get exactly (3.5).

3.3.2 Duality for general problems

So far we have derived the dual problem only for problems in standard equality form. One can, in fact, associate a dual problem to any linear programming problem, whether it is a maximization or minimization problem, regardlessly of the particular form of the linear constraints.

Consider an $m \times n$ matrix A with rows $a_1^\top, \ldots, a_m^\top \in \mathbb{R}^n$ and columns $A_1, \ldots, A_n \in \mathbb{R}^m$, and let $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$.

If the primal is a maximization problem, then the dual is a minimization problem, and viceversa. The dual problem will have a variable for every constraint of the primal, except for the nonnegativity or nonpositivity constraint, and a constraint for every variable of the primal. The coefficient of a primal variable in the objective function will be the right-hand-side of the corresponding dual constraint, and the right-hand-side of a primal constraint will be the objective function coefficient of the corresponding dual variable.

The "recipe" to go from to construct the dual of an LP problem is given by the following table.

$\max c^\top x$	$\min b^\top y$	
$a_i^\top x \leq \; b_i,$	$y_i \geq \; 0,$	$i = 1, \ldots, h;$
$a_i^\top x \geq \; b_i,$	$y_i \leq \; 0,$	$i = h+1, \ldots, k;$
$a_i^\top x = \; b_i,$	y_i free,	$i = k+1, \ldots, m.$
$x_j \geq \; 0,$	$A_j{}^\top y \geq \; c_j,$	$j = 1, \ldots, p;$
$x_j \leq \; 0,$	$A_j{}^\top y \leq \; c_j,$	$j = p+1, \ldots, q;$
x_j free	$A_j{}^\top y = \; c_j,$	$j = q+1, \ldots, n;$

The above table can be read in both directions: from left to right if we want to write the dual of a maximization problem, from right to left if we want to write the dual of a minimization problem.

For example, in a maximization problem, constraints of the type "$\geq$" correspond dual variables restricted to be "≤ 0". In a minimization problem, instead, to constraints of the type "$\geq$" correspond dual variables restricted to be "≥ 0". Note that equality constraints in the primal correspond to free variables in the dual, and free variables in the primal correspond to equality constraints in the dual.

Example. Consider the LP in the example of Section 2.2

$$\begin{array}{rlcr}
\min & 3x_1 - 2x_2 - x_3 & & \\
 & -x_1 + x_2 + 2x_3 & = & 4 \\
 & 2x_1 \qquad\quad - x_3 & \geq & -2 \\
 & -2x_1 - x_2 + x_3 & \leq & 1 \\
 & \quad x_1 \leq 0,\ x_3 \geq 0 & &
\end{array}$$

The problem is a minimization problem, thus the dual is a maximization problem. The problem has three constraints (besides the nonpositivity constraint $x_1 \leq 0$ and the nonnegativity constraint $x_3 \geq 0$), thus the dual will have three variables, y_1, y_2, y_3. The problem has three variables, thus the dual will have three constraints (besides the nonnegativity and nonpositivity constraints on some of the variables). The dual is the following

$$\begin{array}{rlcr}
\max & 4y_1 - 2y_2 + y_3 & & \\
 & -y_1 + 2y_2 - 2y_3 & \geq & 3 \\
 & y_1 \qquad\quad - y_3 & = & -2 \\
 & 2y_1 - y_2 + y_3 & \leq & -1 \\
 & \quad y_2 \geq 0,\ y_3 \leq 0 & &
\end{array}$$

One can verify that, if we write the dual of the above problem, we end up with the original one.

3.4 An economics interpretation of the dual

When one formulates a real-world problem as an LP, the variables, constraints, and objective function of the LP typically have a clear interpretation in terms of the original problem. It is often the case that also the variables, constraints, and objective function of the dual can be interpreted and give further information on the original problem. A typical case when such an interpretation of the dual is possible is in the case of resource allocation problems. Here we give an example of such a problem.

A chip's manufacturer produces four types of memory chips in one their state-of-the-art factories. These chips are sold to electronics corporations for their devices. The main resources used in the chip's production are labor and silicon wafers.

The factory's problem for the next month is

$$\begin{array}{lrcll} \text{maximise} & 15x_1 + 24x_2 + 32x_3 + 40x_4 & & & \\ \text{subject to} & x_1 + \ \ 2x_2 + \ \ 8x_3 + \ \ 7x_4 & \leq & 2000 & \text{(Labour)} \\ & 6x_1 + \ \ 8x_2 + 12x_3 + 15x_4 & \leq & 15000 & \text{(Silicon wafers)} \\ & x_1,\ x_2,\ x_3,\ x_4 \geq 0. & & & \end{array}$$

where variable x_j represents the number of chips produced, in thousands, the objective function coefficients represent the unit profit (in £) of the four types of chips, 2000 is the number of hours of labour available, and 15000 is the number of silicon wafers available. Therefore, the objective function's coefficients represent thousand £ per unit of each variable $x_1, \dots, x_4$ (so, for example, $x_2 = 1$ corresponds to producing 1000 chips of type 2, which requires 2 hours of labour, 8 wafers, and gives a profit of £24000, that is, $24x_2$ thousand £.)

Mango Inc., a giant consumer electronics corporation, urgently needs as many units as possible of a new type of memory chip, due to a stronger-than-expected demand for their new smart phone. Mango Inc. would like the chip manufacturer to sell all its resources (labour and wafers) to Mango for the production of this new chip. Mango Inc. intends to determine prices to offer for each of the resources in order to convince the manufacturer to sell them, while minimising the total sum paid.

Let y_1 denote the price (in thousand £) that the buyer intends to pay per hour of labour, and let y_2 denote the price (in thousand £) that Mango Inc. intends to pay for each silicon wafer.

Clearly these prices must be greater than or equal to zero, that is, $y_1, y_2 \geq 0$.

In order to persuade the manufacturer to sell, the buyer must offer the prices such that the manufacturer is not tempted to retain its resources to produce chips of type 1. So the buyer must offer prices such that the total value of the resources used in producing chips of type 1 is at least the amount of profit the manufacturer could attain.

That is

$$y_1 + 6y_2 \geq 15.$$

Similarly for chips of type 2, Mango Inc. must pitch his prices so that the manufacturer is at least as well off selling as he would be by not selling:

$$2y_1 + 8y_2 \geq 24.$$

Each type of chip generates such a constraint. On the other hand, the buyer wants to pay the minimum amount possible for the entire amount of resources, that is, it wants to minimise $2000y_1 + 15000y_2$.

Mango Inc.'s problem is therefore

$$\begin{array}{lrcl} \text{minimise} & 2000y_1 + 15000y_2 & & \\ \text{subject to} & y_1 + 6y_2 & \geq & 15 \\ & 2y_1 + 8y_2 & \geq & 24 \\ & 8y_1 + 12y_2 & \geq & 32 \\ & 7y_1 + 15y_2 & \geq & 40 \\ & y_1,\, y_2 & \geq & 0 \end{array}$$

The manufacturer's and the Mango Inc.'s problems are dual to each other. Under this interpretation, the dual values represent prices of resources, which explains why dual values are often also called *shadow prices*. Of course, we know that the total amount that the buyer will have to pay is the same as the amount of profit that the manufacturer would achieve by carrying on its usual production.

3.5 Optimality conditions

3.5.1 Standard form

Let us look more closely at the proof of the weak duality theorem (Theorem 3.1). Consider the LP problem (P) in standard form, and its dual (D).

$$\begin{array}{rcll} \max \sum_{j=1}^{n} c_j x_j & & & \\ \sum_{j=1}^{n} a_{ij} x_j \leq & b_i & i = 1, \ldots, m & \\ x_j \geq & 0 & j = 1, \ldots, n & \end{array} \quad (P) \qquad ; \qquad \begin{array}{rcll} \min \sum_{i=1}^{m} b_i y_i & & & \\ \sum_{i=1}^{m} a_{ij} y_i \geq & c_j & j = 1, \ldots, n & \\ y_i \geq & 0 & i = 1, \ldots, m & \end{array} \quad (D)$$

Let x^* and y^* be feasible solutions for (P) and (D) respectively.

We have

$$\sum_{j=1}^{n} c_j x_j^* \leq \sum_{j=1}^{n} (\sum_{i=1}^{m} a_{ij} y_i^*) x_j^* = \sum_{i=1}^{m} (\sum_{j=1}^{n} a_{ij} x_j^*) y_i^* \leq \sum_{i=1}^{m} b_i y_i^*. \tag{3.6}$$

where the first inequality follows from the facts that $\sum_{i=1}^{m} a_{ij} y_i \geq c_j$ and that $x_j \geq 0$ for $j = 1, \ldots, n$, and the second inequality follows from the facts that $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ and that $y_i \geq 0$ for $i = 1, \ldots, m$.

By strong duality, x^* and y^* are optimal for (P) and (D), respectively, if and only if $\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$. This is the case if and only if equality holds throughout in the chain of inequalities (3.6). Hence, x^* and y^* are optimal if and only if

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{j=1}^{n} (\sum_{i=1}^{m} a_{ij} y_i^*) x_j^*, \quad \text{and} \quad \sum_{i=1}^{m} (\sum_{j=1}^{n} a_{ij} x_j^*) y_i^* = \sum_{i=1}^{m} b_i y_i^*.$$

The first condition holds if and only if

$$\sum_{j=1}^{n} (\sum_{i=1}^{m} a_{ij} y_i^* - c_j) x_j^* = 0. \tag{3.7}$$

The second condition holds if and only if

$$\sum_{i=1}^{m} (\sum_{j=1}^{n} a_{ij} x_j^* - b_i) y_i^* = 0. \tag{3.8}$$

Note that, since x^* and y^* are feasible for (P) and (D), they satisfy $x_j^* \geq 0$ and $\sum_{i=1}^{m} a_{ij} y_i^* - c_j \geq 0$ for all $j = 1, \ldots, n$, and $y_i^* \geq 0$, $\sum_{j=1}^{n} a_{ij} x_j^* - b_i \leq 0$ for all $i = 1, \ldots, m$. Since all the terms of (3.7) are positive, the sum is zero if and only if each term is zero. Similarly, since all the terms of (3.8) are negative, the sum is zero if and only if each term is zero.

Therefore the last two equations hold if and only if

$$(\sum_{i=1}^{m} a_{ij} y_i^* - c_j) x_j^* = 0, \quad \text{for } j = 1, \ldots, n, \tag{3.9}$$

and

$$(\sum_{j=1}^{n} a_{ij} x_j^* - b_i) y_i^* = 0, \quad \text{for } i = 1, \ldots, m. \tag{3.10}$$

We have proved the following.

Theorem 3.6 (Complementary slackness theorem)**.** *Given the problem (P) in standard form and feasible solutions x^* and y^* for (P) and its dual (D), respectively, x^* and y^* are optimal if and only if the following* complementary slackness conditions *hold*

$$\begin{array}{llll} \forall\, j \in \{1,\ldots,n\}, & x_j^* = 0 & \textbf{or} & \sum_{i=1}^m a_{ij}y_i^* - c_j = 0, \\ \forall\, i \in \{1,\ldots,m\}, & y_i^* = 0 & \textbf{or} & \sum_{j=1}^n a_{ij}x_j^* - b_i = 0. \end{array} \qquad (CS)$$

3.5.2 General form

The complementary slackness theorem holds for problems in general form.

Theorem 3.7 (Complementary slackness theorem)**.** *Let (P) be a linear programming problem in the variables $x_1,\ldots,x_n$, with m constraints of the form $\sum_{j=1}^n a_{ij}x_i \le b_i$, $\sum_{j=1}^n a_{ij}x_i \ge b_i$, or $\sum_{j=1}^n a_{ij}x_i = b_i$ $(i = 1,\ldots,m)$ plus possibly nonnegativity or nonpositivity constraints on the variables.*

Given a feasible solution x^* for (P) and a feasible solution y^* for its dual (D), x^* and y^* are optimal if and only if the following complementary slackness conditions *hold:*

$$\begin{array}{llll} \forall\, j \in \{1,\ldots,n\}, & x_j^* = 0 & \textbf{or} & \sum_{i=1}^m a_{ij}y_i^* - c_j = 0, \\ \forall\, i \in \{1,\ldots,m\}, & y_i^* = 0 & \textbf{or} & \sum_{j=1}^n a_{ij}x_j^* - b_i = 0. \end{array} \qquad (CS)$$

Observe that, if a primal variable x_j is free, then the corresponding dual constraint is the equation $\sum_{i=1}^m a_{ij}y_i - c_j = 0$, while if a dual variable y_i is free, the corresponding primal constraint is the equation $\sum_{j=1}^n a_{ij}x_i = b_i$. Therefore, the complementary slackness conditions corresponding to free variables are always trivially satisfied.

Note that the above theorem also implies the following: suppose we are given a primal feasible solution x^* and we want decide if it is optimal. By the complementary slackness theorem, this is the case if and only if there exists a dual feasible solution such that x^*, y^* satisfy the complementary slackness conditions.

Example Consider the following LP problem from the example of Section 2.2

$$\begin{array}{lrcl} \min & 3x_1 - 2x_2 - x_3 & & \\ & -x_1 + x_2 + 2x_3 & = & 4 \\ & 2x_1 \qquad\quad - x_3 & \ge & -2 \\ & -2x_1 - x_2 + x_3 & \le & 1 \\ & x_1 \le 0,\ x_3 \ge 0 & & \end{array}$$

and its dual

$$\begin{array}{rl} \max & 4y_1 - 2y_2 + y_3 \\ & -y_1 + 2y_2 - 2y_3 \geq 3 \\ & y_1 - y_3 = -2 \\ & 2y_1 - y_2 + y_3 \leq -1 \\ & y_2 \geq 0,\ y_3 \leq 0 \end{array}$$

We have mentioned in Section 2.2, without proving it, that the optimal solution is the point $x^* = (-1, 3, 0)$. We now use the Complementary Slackness conditions to prove optimality of x^*. First, one can easily verify that x^* satisfies the primal constraints, thus it is feasible. To prove that x^* is optimal, we compute a feasible dual solution y^* that is in complementary slackness with x^*.

The complementary slackness conditions are

$$\begin{array}{ccc} x_1^* = 0 & \text{or} & -y_1^* + 2y_2^* - 2y_3^* = 3 \\ x_2^* = 0 & \text{or} & y_1^* - y_3^* = -2 \\ x_3^* = 0 & \text{or} & 2y_1^* - y_2^* + y_3^* = -1 \\ y_1^* = 0 & \text{or} & -x_1^* + x_2^* + 2x_3^* = 4 \\ y_2^* = 0 & \text{or} & 2x_1^* - x_3^* = -2 \\ y_3^* = 0 & \text{or} & -2x_1^* - x_2^* + x_3^* = 1 \end{array}$$

The third, forth and fifth conditions are already satisfied because $x_3^* = 0$ and x^* satisfies as equality the first and second primal constraints. The remaining conditions are satisfied if and only if y^* satisfies

$$\begin{array}{l} -y_1^* + 2y_2^* - 2y_3^* = 3 \\ y_1^* - y_3^* = -2 \\ y_3^* = 0 \end{array}$$

The only solution to these equations is $y^* = (-2, \frac{1}{2}, 0)$. We have to verify that y^* is indeed feasible for the dual problem, and this is easily checked to hold. Hence $(-1, 3, 0)$ is an optimal primal solution while $(-2, \frac{1}{2}, 0)$ is an optimal dual solution. Observe that, as expected in light of the Strong Duality Theorem, the value of x^* in the primal (namely, -9) equals the value of y^* in the dual.