Householder transformation

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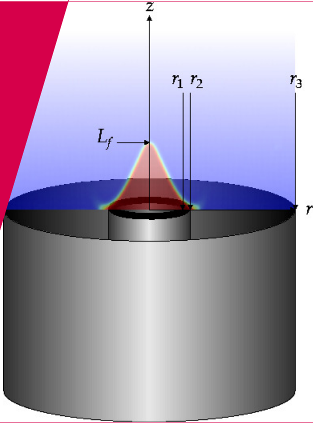

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Summary

Householder transformation

Definition (Householder transformation)

Let $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \neq \mathbf{0}$

Then the matrix $\mathbf{H} := \mathbf{I} - 2\dfrac{\mathbf{x}\mathbf{x}^T}{\mathbf{x}^T\mathbf{x}}$ is called a *Householder transformation*

Properties

- $\mathbf{H}$ is symmetric

  Proof

- $\mathbf{H}$ is orthogonal

  Proof

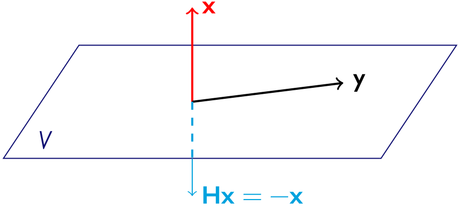
Interpretation as a mapping

Let $\mathbf{y} \in \mathbb{R}^n$. Then $\mathbf{H}\mathbf{y} = \left(\mathbf{I} - 2\frac{\mathbf{x}\mathbf{x}^T}{\mathbf{x}^T\mathbf{x}}\right)\mathbf{y} = \mathbf{y} - 2\frac{\mathbf{x}\mathbf{x}^T\mathbf{y}}{\mathbf{x}^T\mathbf{x}} = \mathbf{y} - 2\frac{\mathbf{x}^T\mathbf{y}}{\mathbf{x}^T\mathbf{x}}\mathbf{x}$

Let's make some special choices for $\mathbf{y}$:

- If $\mathbf{y} = \mathbf{x}$, then $\mathbf{H}\mathbf{x} = \mathbf{x} - 2\frac{\mathbf{x}^T\mathbf{x}}{\mathbf{x}^T\mathbf{x}}\mathbf{x} = \mathbf{x} - 2\mathbf{x} = -\mathbf{x}$
- If $\mathbf{y}$ is orthogonal to $\mathbf{x}$, then $(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T\mathbf{y} = 0$, so that $\mathbf{H}\mathbf{y} = \mathbf{y}$

Let V be the space of all vectors orthogonal to $\mathbf{x}$, so $V = \mathbf{x}^\perp = \{\mathbf{y} \in \mathbb{R}^n \mid (\mathbf{x}, \mathbf{y}) = 0\}$



$\mathbf{H}$ is the *reflection* in the hyperplane V — it mirrors vectors in V

Using Householder to transform a matrix

We want to use Householder transformations to obtain $\mathbf{A} = \mathbf{QR}$

The idea is to transform the matrix column by column

$$\underbrace{\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}}_{\mathbf{A}} \xrightarrow{\mathbf{H}_1} \underbrace{\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}}_{\mathbf{H}_1\mathbf{A}} \xrightarrow{\mathbf{H}_2} \underbrace{\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}}_{\mathbf{H}_2\mathbf{H}_1\mathbf{A}}$$

Define $\mathbf{R} := \mathbf{H}_2\mathbf{H}_1\mathbf{A}$
Then $\mathbf{R}$ is upper triangular and $\mathbf{A} = \mathbf{H}_1^T\mathbf{H}_2^T\mathbf{R}$

Define $\mathbf{Q} := \mathbf{H}_1^T\mathbf{H}_2^T$
Then $\mathbf{Q}$ is orthogonal and $\mathbf{A} = \mathbf{QR}$

Choosing Householder transformations

Recall: $\mathbf{H} := \mathbf{I} - 2\dfrac{\mathbf{x}\mathbf{x}^T}{\mathbf{x}^T\mathbf{x}}$

Suppose that you have two vectors, $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n$
How should you choose $\mathbf{x}$ to find the Householder transformation that maps $\mathbf{u}$ to $\mathbf{v}$?

Note that $\mathbf{H}$ is orthogonal, so we should have $\|\mathbf{u}\|_2 = \|\mathbf{v}\|_2$

Theorem (Choosing Householder transformations)

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ with $\|\mathbf{u}\|_2 = \|\mathbf{v}\|_2$

Take $\mathbf{x} = \mathbf{u} - \mathbf{v}$ and define $\mathbf{H} := \mathbf{I} - 2\dfrac{\mathbf{x}\mathbf{x}^T}{\mathbf{x}^T\mathbf{x}}$

Then $\mathbf{Hu} = \mathbf{v}$

Proof

Choosing Householder transformations (cont.)

Let $\mathbf{A} \in \mathbb{R}^{n\times n}$
Write $\mathbf{A} = \left(\mathbf{a}_1 \mid \mathbf{a}_2 \mid \cdots \mid \mathbf{a}_n\right)$
We choose $\mathbf{H}_1$ such that

$$\mathbf{u} = \mathbf{a}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} \xrightarrow{\mathbf{H}_1} \mathbf{v} = \begin{pmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \alpha \mathbf{e}_1$$

$\mathbf{u}$ and $\mathbf{v}$ must have same length, so

$$\|\mathbf{u}\|_2 = \|\mathbf{a}_1\|_2 = \|\mathbf{v}\|_2 = |\alpha|$$

Conclusion:

$$\alpha = \pm\|\mathbf{a}_1\|_2, \quad \mathbf{v} = \pm\|\mathbf{a}_1\|_2 \cdot \mathbf{e}_1$$

We need to take $\mathbf{H}_1 = \mathbf{I} - 2\dfrac{\mathbf{x}\mathbf{x}^T}{\mathbf{x}^T\mathbf{x}}$ with

$$\mathbf{x} = \mathbf{u} - \mathbf{v} = \mathbf{a}_1 \mp \|\mathbf{a}_1\|_2 \mathbf{e}_1$$

Let us choose the sign such that no cancellation occurs in computing $\mathbf{x}$:

$$\boxed{\mathbf{x} := \begin{cases} \mathbf{a}_1 + \|\mathbf{a}_1\|_2 \mathbf{e}_1 & \text{if } a_{11} \geq 0 \\ \mathbf{a}_1 - \|\mathbf{a}_1\|_2 \mathbf{e}_1 & \text{if } a_{11} < 0 \end{cases}}$$

Choosing Householder transformations (cont.)

After the first Householder transformation:

$$\mathbf{H}_1\mathbf{A} = \left(\begin{array}{c|cccc} * & * & * & \cdots & * \\ \hline 0 & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & * & * & \cdots & * \end{array}\right)$$

Let $\mathbf{B}$ be the $(n-1) \times (n-1)$ matrix that you get by deleting the first row and column from $\mathbf{H}_1\mathbf{A}$

Write $\mathbf{B} = \left(\mathbf{b}_1 \mid \mathbf{b}_2 \mid \cdots \mid \mathbf{b}_{n-1}\right)$

Transform the first column $\mathbf{b}_1$ in $\beta\mathbf{e}_1$ using the Householder transformation

$$\hat{\mathbf{H}}_2 = \mathbf{I} - 2\frac{\hat{\mathbf{x}}_2\hat{\mathbf{x}}_2^T}{\hat{\mathbf{x}}_2^T\hat{\mathbf{x}}_2} \in \mathbb{R}^{(n-1)\times(n-1)}$$

with $\hat{\mathbf{x}}_2 \in \mathbb{R}^{n-1}$

Define $\mathbf{H}_2 := \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \hat{\mathbf{H}}_2 \end{pmatrix} \in \mathbb{R}^{n\times n}$

After the second transformation:

$$\mathbf{H}_2\mathbf{H}_1\mathbf{A} = \left(\begin{array}{cc|ccc} * & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ \hline 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & * & \cdots & * \end{array}\right)$$

Transform $\mathbf{A}$ column by column into an upper triangular matrix

In general, we need $n-1$ Householder transformations to get

$$\mathbf{H}_{n-1}\cdots\mathbf{H}_2\mathbf{H}_1\mathbf{A} = \mathbf{R}$$

and hence

$$\begin{aligned} \mathbf{A} &= \mathbf{H}_1^T\mathbf{H}_2^T\cdots\mathbf{H}_{n-1}^T\mathbf{R} \\ &= \mathbf{H}_1\mathbf{H}_2\cdots\mathbf{H}_{n-1}\mathbf{R} \\ &=: \mathbf{QR} \end{aligned}$$

Example

Let $\mathbf{A} = \begin{pmatrix} 1 & -7 & 0 \\ 2 & -20 & 5 \\ 2 & 1 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -15 \\ -27 \\ 3 \end{pmatrix}$

Find a QR decomposition for $\mathbf{A}$ using Householder transformations
Solve the linear system $\mathbf{Ax} = \mathbf{b}$ using the QR decomposition

Solution

Summary

In this video we have seen how you can

- Find a Householder transformation to map one given vector to another given vector
- Compute the QR decomposition of a matrix using Householder transformations
- Solve a linear system using the QR decomposition

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