W5&6

Row vector
Column vector

Definition: Row vector

An m dimensional row vector is a matrix of order $1 \times m$. It has one row and m columns:

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1m} \end{bmatrix}.$$

We usually drop the unchanging subscript, write the vector name in bold font, use lowercase and, for clarity, separate elements by commas:

$$\mathbf{a} = [a_1, \ldots, a_m].$$

Definition: Column vector

An m dimensional column vector is a matrix of order $m \times 1$. It has one column and m rows:

$$A = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}.$$

Like row vectors, we usually drop the unchanging subscript, write the vector name in bold font and use lowercase. Also, to save space, we will often write column vectors on one line and identify them as column vectors by using parentheses $(\cdots)$ rather than brackets $[\cdots]$:

$$\mathbf{a} = (a_1, \ldots, a_m).$$

Square diagonal matrix
Identity matrix

Definition: Square diagonal matrix

A square diagonal matrix of order $n \times n$ is a square matrix with all elements not on the diagonal equal to zero:

$$D = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_{nn} \end{bmatrix}.$$

Or, $D = [d_{ij}]$ where $d_{ij} = 0$ when $i \neq j$ for all $i, j = 1, 2, \ldots, n$.

Definition: Identity matrix

An identity matrix is a square diagonal matrix with all the diagonal elements equal to one:

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

For $I = [I_{ij}]$ with order $n \times n$, the elements are $I_{ij} = 1$ when $i = j$ and 0 otherwise for all $i, j = 1, 2, \ldots, n$.

matrix transpose

The matrix transpose has the following properties (assuming A and B have suitable orders):

- $(A+B)^T = A^T + B^T$
- $(A^T)^T = A$
- $(cA)^T = cA^T$ for scalar c
- $(A^p)^T = (A^T)^p$ for any $p \in \mathbb{N}$
- $(AB)^T = B^T A^T$.

inverse matrix

For a square matrix A, the inverse matrix A^{-1} satisfies the following property:

$$A^{-1}A = AA^{-1} = I.$$

Linear Equations

The system of m linear equations in the definition of a system of linear equations are compactly written in matrix form:

$$A\mathbf{x} = \mathbf{b}\,,$$

with column vectors

$$\mathbf{x} = (x_1\,, x_2\,, \dots, x_n) \quad \text{and} \quad \mathbf{b} = (b_1\,, b_2\,, \dots, b_m)\,,$$

and the $m \times n$ *coefficient matrix*

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}.$$

The matrix form is convenient for solving for unknowns $\mathbf{x}$, particularly when m and/or n are large.

Matrix Determinant

Definition: Determinant

For an $n \times n$ matrix A, the determinant of A, denoted by $\det(A)$, or $|A|$, is a real number which:

- for $n = 1$ is $|A| = a_{11}$
- for $n > 1$ is

$$\begin{aligned} |A| &= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} - \cdots + (-1)^{n+1}a_{1n}M_{1n} \\ &= \sum_{j=1}^{n} (-1)^{j+1} a_{1j} M_{1j}\,. \end{aligned}$$

where M_{1j} are minors of A.

Definition: Matrix minor

For the $n \times n$ matrix A, the minor M_{ij} is the determinant of the $(n-1) \times (n-1)$ order matrix obtained from A by omitting row i and column j.

Properties of matrix determinants

First we have some properties which tells when we can easily recognise that a matrix has determinant zero, and which tells us about how certain changes to a matrix will affect the determinant.

- If A has a row (or column) of zeros, then $|A| = 0$.
- If A has two identical rows (or columns) then $|A| = 0$.
- If two rows (or columns) of matrix A are swapped to obtain matrix B, then $|A| = -|B|$.
- If A' is obtained by multiplying a single row of A by the constant c, then the determinant changes by a factor of c, i.e. $|A'| = c|A|$.
- If A' is obtained from A by adding a multiple of one row to another then the determinant is unchanged, i.e. $|A'| = |A|$.
- $|AB| = |A|\,|B|$.
- $|A^T| = |A|$.
- For $n \times n$ matrix A and scalar c, $|cA| = c^n|A|$.
- $|I| = 1$

An important fact about the determinant is that it can give an easy method for finding out whether or not an inverse matrix exists

- A is invertible if $|A| \neq 0$.
- If A is invertible $|A^{-1}| = 1/|A|$.

The relationship between the determinant and the inverse also leads to useful results on solutions of matrix equations. The fact that A is invertible if $|A| \neq 0$ means that if $|A| = 0$:

- A is non-invertible
- A^{-1} does not exist
- there is no B such that $AB = I$ (or $BA = I$)
- there is not a unique solution to $A\mathbf{x} = \mathbf{b}$. Either:
    - $A\mathbf{x} = \mathbf{b}$ is inconsistent, or
    - $A\mathbf{x} = \mathbf{b}$ has an infinite number of solutions.

You can distinguish between the zero-solutions or infinite-solutions cases if you know that there is at least one solution. If $|A| = 0$ and you can find one solution, there must be an infinite number of solutions. This could be where we have:

$$A\mathbf{x} = \mathbf{0} \quad \text{or} \quad A\mathbf{x} = \mathbf{x}.$$

The latter case is very important, as it's a special case of the eigenvalue problem, which you will learn about in the next section.