W3&4

1.Permutations and Combinations

	Without repetition	With repetition
Permutation	$P_r^n = \frac{n!}{(n-r)!}$	n^r
Combination	$\binom{n}{r} = \frac{n!}{(n-r)!r!}$	$\binom{r+(n-1)}{r} = \binom{r+(n-1)}{n-1}$

The binomial coefficient

The symbol $\binom{n}{r}$ is also called the *binomial coefficient*, and to read this aloud you would say 'n choose r'. The coefficient can be calculated using factorials:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Definition: Permutations

The number of permutations (arrangements) of n objects taken r at a time is

$$P_r^n = \frac{n!}{(n-r)!}$$

For permutations, order is *important*.

Definition: Combinations

The number of combinations of n objects taken r at a time is

$$C_r^n = \binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

For combinations, order is *not important*.

some properties of the binomial coefficient

1. $\binom{n}{r} = \binom{n}{n-r} \quad \text{if} \quad 0 \leq r \leq n$
2. $\binom{n}{0} = 1$
3. $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$
4. $\binom{n+m}{k} = \sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j}$

Definition: Permutations with repetition

The number of permutations of n objects taken r at a time, with repetition, is n^r.

Definition: Combinations with repetition

The number of combinations of n objects taken r at a time, with repetition, is

$$\binom{r+(n-1)}{r} = \binom{r+(n-1)}{n-1}$$

The following are equivalent scenarios where you might want to use combinations with repetition:

- The number of ways of choosing r things from a set of n, where repetition is allowed;
- The number of ways of allocating r identical objects to n containers;
- The number of non-negative integer solutions to $x_1 + x_2 + \ldots + x_n = r$.

2.Fundamentals of probility

Definition: Axioms of probability

The function $P(E)$is a probability if it satisfies

1. $0 \leq P(E) \leq 1$
2. $P(S) = 1$
3. If $E_1, E_2, \ldots$ are disjoint events $E_i \cap E_j = \emptyset$ if $i \neq j$:

$$P(E_1 \cup E_2 \cup \ldots) = \sum_{i=1}^{\infty} P(E_i).$$

Definition: Inclusion-exclusion principle for probability

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

Definition: De Morgan’s laws

$$(E \cup F)^c = E^c \cap F^c$$
$$(E \cap F)^c = E^c \cup F^c$$

the inclusion-exclusion principle

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Definition: Conditional probability

For two events A and B, if $P(B) > 0$, then the probability of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Definition: Independent events

Two events A and B are independent if A has no influence on the probability of B (and vice versa):

$$P(A|B) = P(A) \quad \text{and} \quad P(B|A) = P(B)$$

Therefore, $P(A \cap B) = P(A)P(B)$ if the events A and B are independent.

Bayes' theorem

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}.$$

- $P(B)$ is the prior;
- $P(A|B)$ is the likelihood;
- $P(B|A)$ is the posterior.

Definition: Law of Total Probability

For any event A,

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A|B_i)P(B_i),$$

so long as the B_is form a partition of the sample space S.

Definition: Generalised Bayes' theorem

For any events A and $B_i, i = 1, \ldots, n$:

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_{k=1}^{n} P(A|B_k)P(B_k)},$$

as long as the B_is form a partition of the sample space S.

3.Introduction to Discrete Random Variables

Definition: Discrete random variable and probability mass function

A random variable X that can take on a countable number of values is a discrete random variable, with probability mass function (pmf)

$$p(a) = P(X = a).$$

From the axioms of probability, we have that if X takes on the values $x_1, x_2, \ldots$, then

- $p(x_i) \geq 0$ for $i = 1, 2, \ldots$, and 0 otherwise;
- $\sum_{i=1}^{\infty} p(x_i) = 1$.

Definition: Expectation and Variance

The *expected value* or *expectation* of a discrete random variable X is:

$$E[X] = \sum_i x_i p(x_i).$$

The *variance* is:

$$Var(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2.$$

Bernoulli Random Variable

$$P(X = 1) = p; \quad P(X = 0) = 1 - p; \quad E[X] = p; \quad Var(X) = p(1 - p)$$

Binomial random variable

The number of successes k in n Bernoulli trials:

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}; \quad E[X] = np; \quad Var(X) = np(1 - p)$$

For something to be a binomial random variable, it needs to satisfy the following four conditions:

- There is a binary outcome (e.g. heads and tails, correct and incorrect).
- The trials are independent of each other.
- There is a fixed number of trials n.
- There is a constant probability of 'success' p, where a success is observing one of the binary outcomes.

Poisson Random Variable

The number of events in an interval, occurring at rate $\lambda > 0$:

$$P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}; \quad E[X] = \lambda; \quad Var(X) = \lambda$$