

《数值分析》8

- → 三对角矩阵的LU分解
- → 对称正定矩阵三角分解
- → MATLAB矩阵分解命令

常微分方程两点边值问题



P1:
$$\begin{cases} y'' + py' + qy = f(x), & x \in (0,1) \\ y(0) = \alpha, y(1) = \beta. \end{cases}$$

P2:
$$\begin{cases} [k(x)y']' + q(x)y = f(x), & x \in (0,1) \\ y(0) = \alpha, y(1) = \beta. \end{cases}$$

P3:
$$\begin{cases} X'' + \lambda X = 0, & x \in (0,L) \\ X(0) = 0, X(L) = 0. \end{cases}$$



二阶常微分方程:
$$\begin{cases} y'' + y + x = 0, & x \in (0,1) \\ y(0) = 0, y(1) = 0. \end{cases}$$

三对角方程组
$$-y_{j-1} + (2-h^2)y_j - y_{j+1} = x_j h^2$$

$$\begin{bmatrix} 2-h^2 & -1 & & & \\ -1 & 2-h^2 & -1 & & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2-h^2 & -1 \\ & & & -1 & 2-h^2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} = h^2 \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

$$= h^{2} \begin{bmatrix} x_{1} \\ x_{2} \\ \vdots \\ x_{n-1} \\ x_{n} \end{bmatrix}$$

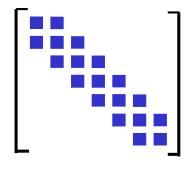


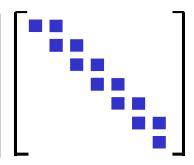
一般形式

$$\begin{bmatrix} b_1 & c_1 & & & & \\ a_2 & b_2 & c_2 & & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & a_n & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-1} \\ f_n \end{bmatrix}$$

三角分解:

$$A = LU$$





三对角矩阵

单位下三角阵 上三角阵

$$AX=F \rightarrow L\underline{U}X=F$$

$$(1) \quad L Y = F$$

$$\bigcirc UX = Y$$

LU分解



$$\begin{bmatrix} 2 & -1 \\ -1 & 3 & -2 \\ & -2 & 4 & -2 \\ & & -3 & 5 \end{bmatrix}$$

$$\begin{bmatrix}
2 & -1 \\
-1/2 & 5/2 & -2 \\
-4/5 & 12/5 & -2 \\
-3 & 5
\end{bmatrix}$$

$$\begin{bmatrix}
2 & -1 \\
-1/2 & 5/2 & -2 \\
-4/5 & 12/5 & -2 \\
-5/4 & 5/2
\end{bmatrix}$$

$$L = \begin{bmatrix} 1 \\ -1/2 & 1 \\ & -4/5 & 1 \\ & & -5/4 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 2 & -1 & & & \\ & 5/2 & -2 & & \\ & & 12/5 & -2 \\ & & & 5/2 \end{bmatrix}$$

三对角矩阵的三角分解 A = LU



$$A = \begin{bmatrix} b_1 & c_1 \\ a_2 & b_2 & \ddots \\ & \ddots & \ddots \\ & & a_n & b_n \end{bmatrix} \rightarrow \begin{bmatrix} \beta_1 & c_1 \\ \alpha_2 & \beta_2 & c_2 \\ & \ddots & \ddots \\ & & \alpha_n & \beta_n \end{bmatrix}$$

算法I:
$$\begin{cases} \beta_1 = b_1 \\ \alpha_k = a_k / \beta_{k-1}, \beta_k = b_k - \alpha_k c_{k-1} \end{cases}$$
 $(k = 2, 3, \dots, n)$

格式相对简单 因为三对角

下三角方程组 LY = f 上三角方程组 UX = Y

$$\begin{bmatrix} \mathbf{1} & & & \\ \boldsymbol{\alpha}_2 & \mathbf{1} & & \\ & \ddots & \ddots & \\ & & \boldsymbol{\alpha}_n & \mathbf{1} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{1} & & & \\ \alpha_2 & \mathbf{1} & & \\ & \ddots & \ddots & \\ & & \alpha_n & \mathbf{1} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \begin{bmatrix} \beta_1 & c_1 & & \\ & \beta_2 & \ddots & \\ & & \ddots & c_{n-1} \\ & & & \beta_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

算法II:
$$\begin{cases} y_1 = f_1 \\ y_k = (f_k - \alpha_k y_{k-1}), (k = 2, \dots n) \end{cases}$$

算法III:
$$\begin{cases} x_n = y_n / \beta_n \\ x_k = (y_k - c_k x_{k+1}) / \beta_k, (k = n-1, \dots, 1) \end{cases}$$



```
function f=triGauss(a,b,c,f)
%TriDiag(a,b,c)systems
n=length(b);
for k=1:n-1
d=a(k)/b(k);
```

$$\begin{bmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & \ddots & \ddots & \ddots \\ & & a_n & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

f(n)=f(n)/b(n);



对称正定矩阵的Cholesky分解

 $A^{T} = A$, A 的各阶顺序主子式大于零.

$$A$$
的 $LU分解$
$$A = \begin{bmatrix} 1 & & & & \\ m_{21} & 1 & & & \\ \vdots & \ddots & \ddots & & \\ m_{n1} & \cdots & m_{n,n-1} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ & u_{22} & \ddots & \vdots \\ & & \ddots & u_{n-1,n} \\ & & & u_{nn} \end{bmatrix}$$

 $u_{ii} > 0$ ($i = 1, \dots, n$),有如下关系

$$\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ & u_{22} & \cdots & u_{2n} \\ & & \ddots & \vdots \\ & & u_{nn} \end{bmatrix} = \begin{bmatrix} u_{11} & & & & \\ & u_{22} & & \\ & & \ddots & \vdots \\ & & & u_{nn} \end{bmatrix} \begin{bmatrix} 1 & r_{12} & \cdots & r_{1n} \\ & 1 & \ddots & \vdots \\ & & \ddots & \vdots \\ & & & \ddots & r_{n-1,n} \\ & & & 1 \end{bmatrix}$$



$$A = LU \rightarrow A = LDR \rightarrow A^T = R^TDL^T$$

$$A^T = A \rightarrow R^T D L^T = L D R \rightarrow R^T = L \rightarrow R = L^T$$

- (1) 对称矩阵的 LDL^{T} 分解: $A = LDL^{T}$
- (2) 对称矩阵的 LL^{T} 分解

由
$$u_{kk} > 0$$
,记 $D^{\frac{1}{2}} = diag(\sqrt{u_{11}}, \dots, \sqrt{u_{nn}})$ 得 $A = \tilde{L}\tilde{L}^T$ 其中 $\tilde{L} = LD^{\frac{1}{2}}$

在不引起符号混乱时仍记为: $A = LL^T$



例 设 $A \in \mathbb{R}^{n \times n}$ 为对称正定矩阵, $x, y \in \mathbb{R}^n$.定义

$$(x, y)_A = x^T A y$$

证明: $||x||_A = \sqrt{x^T A x}$ 是 R^n 上的向量范数.

证:由于A对称正定,故存在非奇异上三角矩阵R(类似非奇异下三角矩阵L的形式),使

$$A = R^T R$$

所以, $x^TAx = x^T (R^TR) x = (Rx)^T (Rx)$

$$||x||_A = \sqrt{x^T A x} = \sqrt{(Rx)^T (Rx)} = ||Rx||_2$$

故, $\|x\|_A$ 是 R^n 上的向量范数

MATLAB中矩阵分解命令



(1)特征值分解命令: [P D]=eig(A)

特征值矩阵 $D = diag[\lambda_1, \lambda_2, \dots, \lambda_n]$

特征向量矩阵 $P = [\alpha_1, \alpha_2, \dots, \alpha_n]$

(2) LU分解命令: $[L \ U \ P] = lu(A)$

L下三角阵, U上三角阵, P初等矩阵 PA = LU

(3)Cholesky分解命令: R=chol(A)

R是上三角矩阵 $R^TR = A$



(4)QR分解 [Q R]=qr(A)

Q正交矩阵,R上三角矩阵 QR=A

(5)A的奇异值分解: [U,S,V] = SVD(A)

概念:设 $A \in \mathbb{C}^{n \times m}$, $A^H A$ 的特征值为 $\lambda_1, \lambda_2, \dots, \lambda_n$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$$

令 $\lambda_i = \sigma_i^2$, 称实数 $\sigma_1, \sigma_2, \dots, \sigma_n$ 为 A 的奇异值