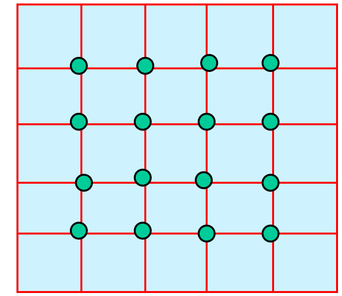
《数值分析》9

- 大型稀疏矩阵的背景
- Jacobi迭代与Seidel迭代
- 迭代法的矩阵表示
- 迭代法数值实验

对象:
1. 迭代法求解线性方程组（间接法）
2. 但不仅限于此

边值问题: $\begin{cases} u_{xx}+u_{yy}=0, \quad 0<x,y<1 \\ u(0,y)=u(x,0)=u(x,1)=0 \\ u(1,y)=\sin\pi y \end{cases}$



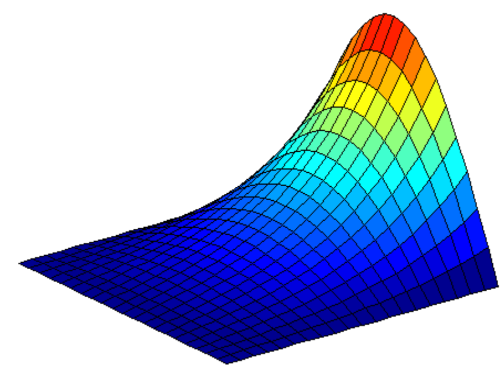
令 $h = 1/(n+1)$， $x_i = ih,\ y_j = jh \quad (i, j = 0,1, \cdots; n+1)$

记 $u_{i,j} = u(x_i, y_j),\quad (i, j = 0,1, \cdots; n+1)$

$$\frac{u_{i+1,j}-2u_{ij}+u_{i-1,j}}{h^2}+\frac{u_{i,j+1}-2u_{ij}+u_{i,j-1}}{h^2}=0$$

$$u_{i,j-1}+u_{i-1,j}-4u_{ij}+u_{i+1,j}+u_{i,j+1}=0$$

$$\Rightarrow \begin{bmatrix} u_{1,j-1}\\ u_{2,j-1}\\ u_{3,j-1}\\ u_{4,j-1}\end{bmatrix}+\begin{bmatrix} u_{0,j}\\ u_{1,j}\\ u_{2,j}\\ u_{3,j}\end{bmatrix}-4\begin{bmatrix} u_{1,j}\\ u_{2,j}\\ u_{3,j}\\ u_{4,j}\end{bmatrix}+\begin{bmatrix} u_{2,j}\\ u_{3,j}\\ u_{4,j}\\ u_{5,j}\end{bmatrix}+\begin{bmatrix} u_{1,j+1}\\ u_{2,j+1}\\ u_{3,j+1}\\ u_{4,j+1}\end{bmatrix}=0$$

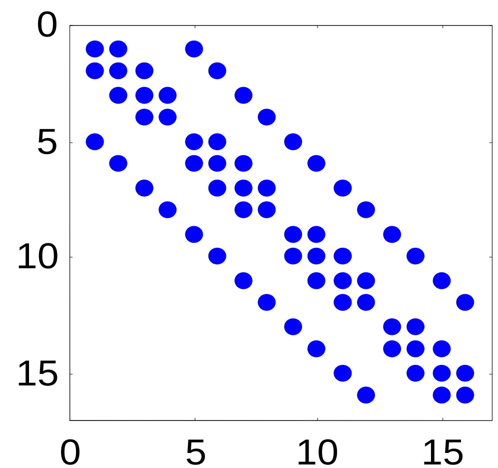


(n=4)

$$U_1=\begin{bmatrix}u_{11}\\u_{21}\\u_{31}\\u_{41}\end{bmatrix}\quad U_2=\begin{bmatrix}u_{12}\\u_{22}\\u_{32}\\u_{42}\end{bmatrix}\quad U_3=\begin{bmatrix}u_{13}\\u_{23}\\u_{33}\\u_{43}\end{bmatrix}\quad U_4=\begin{bmatrix}u_{14}\\u_{24}\\u_{34}\\u_{44}\end{bmatrix}\quad \rightarrow \quad U=\begin{bmatrix}U_1\\U_2\\U_3\\U_4\end{bmatrix}$$

$$\begin{cases}BU_1+U_2=F_1\\U_1+BU_2+U_3=F_2\\\qquad U_2+BU_3+U_4=F_3\\\qquad\qquad U_3+BU_4=F_4\end{cases}\qquad B=\begin{bmatrix}-4&1&&\\1&-4&1&\\&1&-4&1\\&&1&-4\end{bmatrix}$$

$$AU=F\qquad A=\begin{bmatrix}B&I&&\\I&B&I&\\&I&B&I\\&&I&B\end{bmatrix}$$



例4.1 $\begin{cases} 9x_1 - x_2 - x_3 = 7 \\ -x_1 + 10x_2 - x_3 = 8 \\ -x_1 - x_2 + 15x_3 = 13 \end{cases}$

特点：系数矩阵主对角元均不为零

$$\Leftrightarrow \begin{cases} x_1 = (7 + x_2 + x_3)/9 \\ x_2 = (8 + x_1 + x_3)/10 \\ x_3 = (13 + x_1 + x_2)/15 \end{cases}$$

取 $X^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

计算格式 $X^{(1)} = B\,X^{(0)} + f$

$$\begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 & 1/9 & 1/9 \\ 1/10 & 0 & 1/10 \\ 1/15 & 1/15 & 0 \end{bmatrix} \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{bmatrix} + \begin{bmatrix} 7/9 \\ 8/10 \\ 13/15 \end{bmatrix}$$

计算格式: $X^{(k+1)}=BX^{(k)}+f$

$X^{(0)}$	$X^{(1)}$	$X^{(2)}$	$X^{(3)}$	$X^{(4)}$	········
0	0.7778	0.9630	0.9929	0.9987	
0	0.8000	0.9644	0.9935	0.9988	
0	0.8667	0.9778	0.9952	0.9991	

准确解 → X^*

1.0000
1.0000
1.0000

雅可比迭代法

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases} \iff \sum_{j=1}^{n} a_{ij}x_j = b_i$$

$$(i = 1,2,\ldots,n)$$

$$x_i^{(k+1)} = \frac{1}{a_{ii}}[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}]$$

$$(i = 1,2,\ldots n;\ k=1,2,\ldots\ldots)$$

取初始向量$X^{(0)}=[x_1^{(0)}\ x_2^{(0)}\ \cdots\ x_n^{(0)}]^T$, 迭代计算

迭代法适用于解大型稀疏方程组

(万阶以上的方程组, 系数矩阵中零元素占很大比例, 而非零元按某种模式分布)

背景：电路分析、边值问题的数值解和数学物理方程

问题: (1)如何构造迭代格式？

(2)迭代格式是否收敛？

(3)收敛速度如何？

(4)如何进行误差估计？

对比：
迭代法方程求根的迭代法

高斯-赛德尔迭代法

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \quad (i = 1,2,\ldots,n)$$

$$x_i^{(k+1)} = \frac{1}{a_{ii}}[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}]$$

$$(i = 1,2,\ldots n;\ k = 1,2,\ldots\ldots)$$

取初始向量$x^{(0)}=[x_1^{(0)}\ x_2^{(0)}\ \cdots\ x_n^{(0)}]^T$, 迭代计算

例 $$\begin{cases} 9x_1 - x_2 - x_3 = 7 \\ -x_1 + 10x_2 - x_3 = 8 \\ -x_1 - x_2 + 15x_3 = 13 \end{cases} \Longleftrightarrow \begin{cases} x_1 = (7 + x_2 + x_3)/9 \\ x_2 = (8 + x_1 + x_3)/10 \\ x_3 = (13 + x_1 + x_2)/15 \end{cases}$$

$$x_1^{(k+1)} = (7 + x_2^{(k)} + x_3^{(k)})/9$$

$$x_2^{(k+1)} = (8 + x_1^{(k+1)} + x_3^{(k)})/10$$

$$x_3^{(k+1)} = (13 + x_1^{(k+1)} + x_2^{(k+1)})/15$$

$$\begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1/10 & 1 & 0 \\ -1/15 & -1/15 & 1 \end{bmatrix} \begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ x_3^{(k+1)} \end{bmatrix} = \begin{bmatrix} 0 & 1/9 & 1/9 \\ 0 & 0 & 1/10 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{bmatrix} + \begin{bmatrix} 7/9 \\ 8/10 \\ 13/15 \end{bmatrix}$$

迭代法解线性方程组

$$\begin{cases} 9x_1 - x_2 - x_3 = 7 \\ -x_1 + 10x_2 - x_3 = 8 \\ -x_1 - x_2 + 15x_3 = 13 \end{cases}$$

雅可比迭代法实验数据

0.7778	0.8000	0.8667
0.9630	0.9644	0.9719
0.9929	0.9935	0.9952
0.9987	0.9988	0.9991
0.9998	0.9998	0.9998
1.0000	1.0000	1.000

赛德尔迭代法实验数据

0.7778	0.8778	0.9770
0.9839	0.9961	0.9987
0.9994	0.9998	0.9999
1.0000	1.0000	1.0000
1.0000	1.0000	1.0000

总结：雅可比迭代法的矩阵表示

将方程组$AX=b$ 的系数矩阵 A 分解

$$A=D-U-L$$

$$D=\begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix} \quad L=-\begin{bmatrix} 0 & & & \\ a_{21} & 0 & & \\ \vdots & \ddots & \ddots & \\ a_{n1} & \cdots & a_{n,n-1} & 0 \end{bmatrix} \quad U=-\begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ & \ddots & \ddots & \vdots \\ & & 0 & a_{n-1,n} \\ & & & 0 \end{bmatrix}$$

$$AX=b \quad => \quad DX^{(k+1)}=(U+L)X^{(k)}+b$$

$$X^{(k+1)}=D^{-1}(U+L)X^{(k)}+D^{-1}b$$

记$B_J=D^{-1}(U+L)$ $\qquad$ $X^{(k+1)}=B_JX^{(k)}+f_J$

雅可比迭代矩阵

$$B_J=\begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix}^{-1}\begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & 0 & \cdots & -a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ -a_{n1} & -a_{n2} & \cdots & 0 \end{bmatrix}$$

$$B_J=\begin{bmatrix} 0 & -a_{12}/a_{11} & \cdots & -a_{1n}/a_{11} \\ -a_{21}/a_{22} & 0 & \cdots & -a_{2n}/a_{22} \\ \cdots & \cdots & \cdots & \cdots \\ -a_{n1}/a_{nn} & -a_{n2}/a_{nn} & \cdots & 0 \end{bmatrix} \qquad f_J=\begin{bmatrix} b_1/a_{11} \\ b_2/a_{22} \\ \vdots \\ b_n/a_{nn} \end{bmatrix}$$

高斯–赛德尔迭代法的矩阵表示

$$a_{ii}x_i^{(k+1)} = [b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}]$$

$$\sum_{j=1}^{i} a_{ij}x_j^{(k+1)} = b_i - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)} \quad (i = 1,2,\ldots,n)$$

$$\begin{bmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+2)} \\ \vdots \\ x_n^{(k+1)} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} - \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ & 0 & \ddots & \vdots \\ & & \ddots & a_{n-1,n} \\ & & & 0 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ \vdots \\ x_n^{(k)} \end{bmatrix}$$

$$(D-L)X^{(k+1)} = b + UX^{(k)}$$

$$X^{(k+1)} = (D-L)^{-1}b + (D-L)^{-1}UX^{(k)}$$

记 $B_{G\text{-}S}=(D-L)^{-1}U$, $f_{G\text{-}S}=(D-L)^{-1}b$

高斯–赛德尔迭代格式: $X^{(k+1)}=B_{G\text{-}S}X^{(k)}+f_{G\text{-}S}$

$$B_{G-S}=\begin{bmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}^{-1}\begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ & 0 & \ddots & \vdots \\ & & \ddots & a_{n-1,n} \\ & & & 0 \end{bmatrix}$$

$$f_{G-S}=\begin{bmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}^{-1}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

总结：矩阵分裂导出的迭代法

$A = M - N$ (要求M为可逆矩阵)

$AX = b \rightarrow (M - N)X = b \rightarrow MX = NX + b$

$\rightarrow X^{(k+1)} = (M^{-1}N)X^{(k)} + M^{-1}b$

取 $M = D \rightarrow$ 雅可比迭代法

$A = D - (D - A) \rightarrow$

$X^{(k+1)} = D^{-1}[(D - A)X^{(k)} + b] \rightarrow$

$X^{(k+1)} = X^{(k)} + D^{-1}[b - AX^{(k)}]$

记 $r_k = b - AX^{(k)} \rightarrow X^{(k+1)} = X^{(k)} + D^{-1}r_k$

$A = D - U - L$ 取 $M = D - L \rightarrow$ 高斯-赛德尔迭代法

$$A = M - (M - A)$$

$$AX = b \rightarrow MX = (M - A)X + b$$

$$\rightarrow X^{(k+1)} = M^{-1}[(M - A)X^{(k)} + b]$$

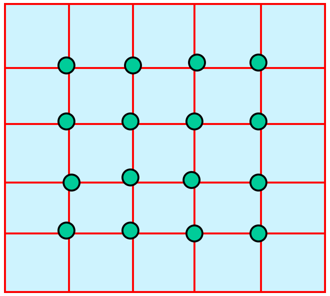
$$\rightarrow X^{(k+1)} = X^{(k)} + M^{-1}[b - AX^{(k)}]$$

总结：简单迭代法

$$X^{(k+1)} = X^{(k)} + \omega(b - AX^{(k)})$$

迭代矩阵 $B = I - \omega A$

平面温度场问题: $\begin{cases} u_{xx}+u_{yy}=0, \quad 0<x,y<1 \\ u(0,y)=u(x,0)=u(x,1)=0 \\ u(1,y)=\sin\pi y \end{cases}$



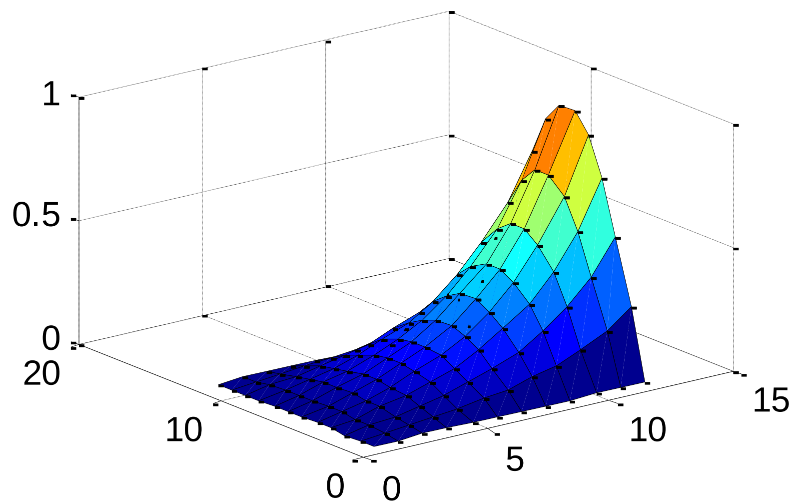
令 $h=1/(n+1)$， $x_i=ih,\ y_j=jh \quad (i,j=0,1,\cdots;n+1)$

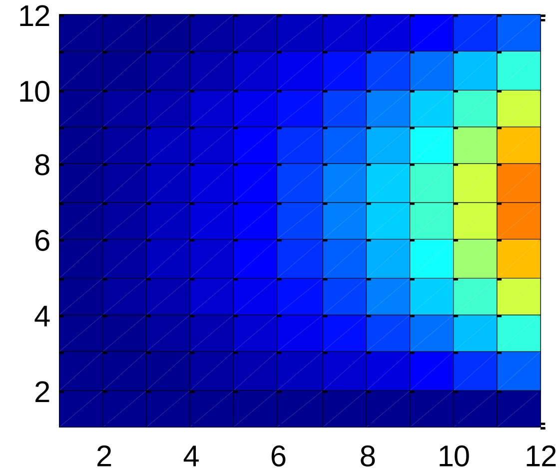
记 $u_{i,j}=u(x_i,y_j), \quad (i,j=0,1,\cdots;n+1)$

差分格式: $u_{i,j-1}+u_{i-1,j}-4u_{ij}+u_{i+1,j}+u_{i,j+1}=0$

矩阵形式: $AU=F$

$$A=\begin{bmatrix} B & I & & \\ I & B & I & \\ & I & B & I \\ & & I & B \end{bmatrix} \qquad B=\begin{bmatrix} -4 & 1 & & \\ 1 & -4 & 1 & \\ & 1 & -4 & 1 \\ & & 1 & -4 \end{bmatrix}$$





高斯–赛德尔迭代法实验(误差限10^{-8})：

结点数n^2	10^2	20^2	40^2
迭代次数	182	606	2077
CPU时间(s)	0.97	4.328	58.531
误差	0.0023	6.4274e-4	1.6814e-4