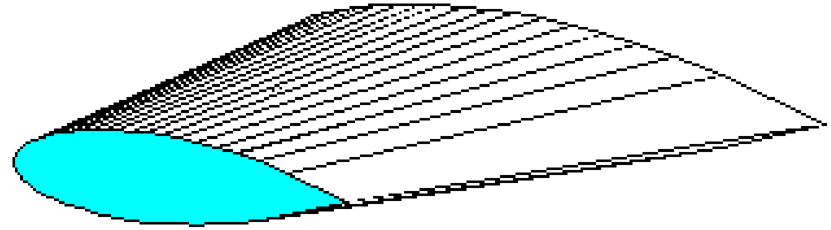
《数值分析》16

- 样条插值的算例
- 三次样条的概念
- 用一阶导数表示的样条
- 三次样条的极性

例1. 飞机机翼剖面图

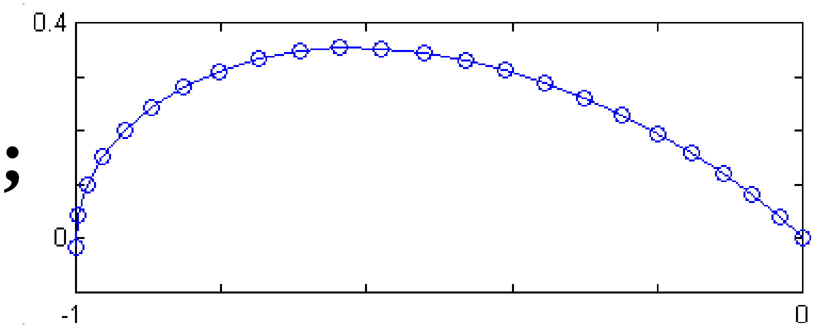


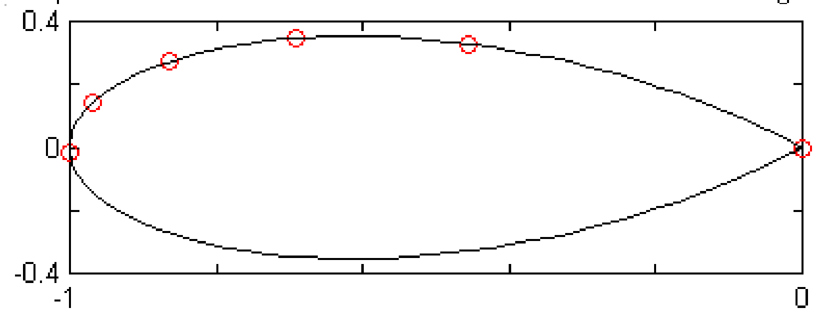
1. 数据采集

X	0	-0.4552	-0.6913	-0.8640	-0.9689	-0.9996
Y	0	0.3285	0.3467	0.2716	0.1408	-0.0160

2. 数据样条插值

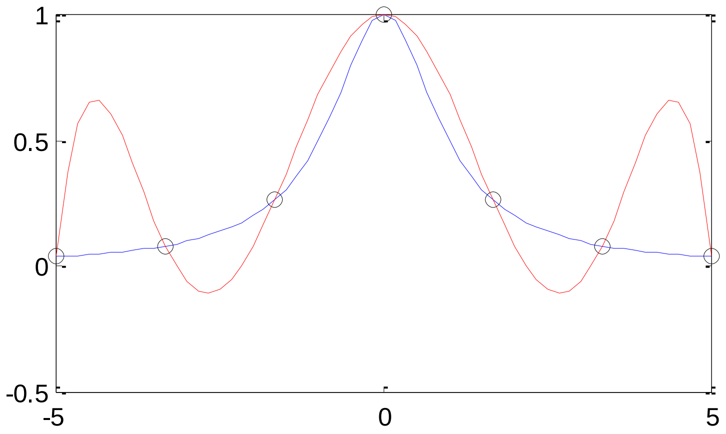
```
S=sqrt(diff(x).^2+diff(y).^2);
S=[0,S];Sk=cumsum(S);
sk=linspace(0,Sk(end),24);
xt=spline(Sk,x,sk);
yt=spline(Sk,y,sk);
```



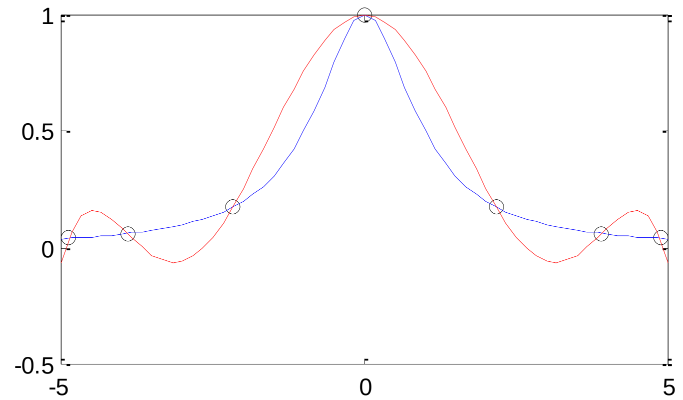


例2:龙格函数的插值逼近

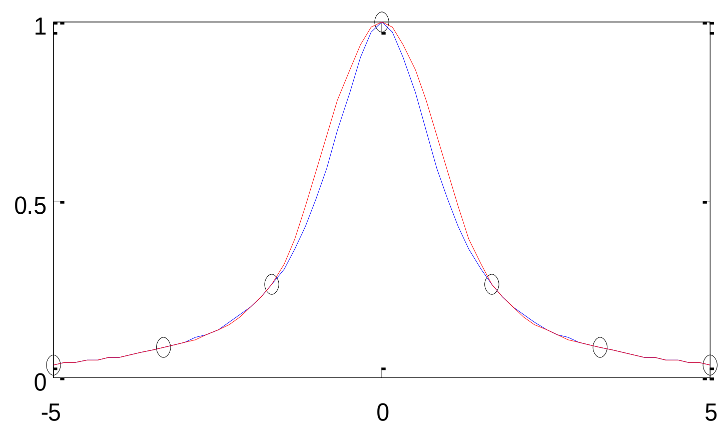
$$f(x)=\frac{1}{1+x^2}$$



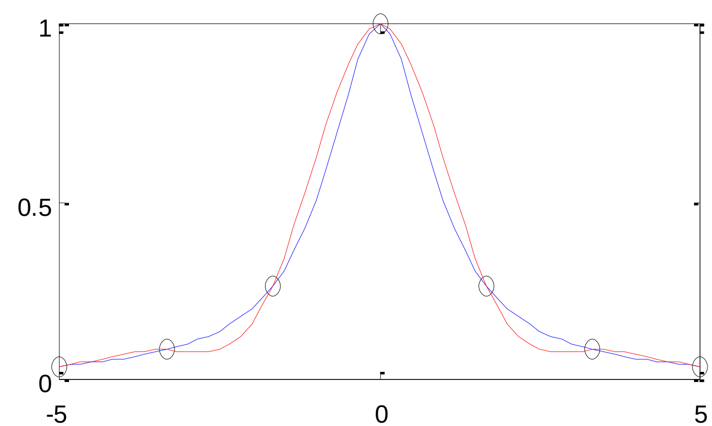
7结点等距插值



7结点切比雪夫插值



7结点埃尔米特插值



7结点样条插值

定义 5.4: 给定区间$[a,b]$上的一个分划:

$$a = x_0 < x_1 < \dots < x_n = b$$

已知 $f(x_j) = y_j \quad (j = 0,1,\cdots,n)$, 如果

$$S(x)=\begin{cases} S_1(x), x\in[x_0,x_1] \\ S_2(x), x\in[x_1,x_2] \\ \cdots\cdots\cdots\cdots \\ S_n(x), x\in[x_{n-1},x_n] \end{cases}$$

满足: (1) $S(x)$在 $[x_j, \ x_{j+1}]$上为三次多项式;

(2) $S''(x)$在区间$[a, \ b]$上连续;

(3) $S(x_j) = y_j \ (j = 0,1,\cdots,n)$.

则称 $S(x)$为三次样条插值函数.

n个三次多项式(每个三次多项式是4个待定系数), 待定系数共$4n$个！！

当$x \in [x_j, x_{j+1}]$ ($j = 0,1,\ldots n-1$)时

$$S_j(x) = a_j + b_j x + c_j x^2 + d_j x^3$$

由样条定义,可建立方程$(4n-2)$个！！ Why?

插值条件: $S(x_j) = y_j$ ($j = 0,1,\cdots;n$)

连续性条件: $S(x_j+0) = S(x_j-0)$ ($j = 1,\cdots;n-1$)

$S'(x_j+0) = S'(x_j-0)$ ($j = 1,\cdots;n-1$)

$S''(x_j+0) = S''(x_j-0)$ ($j = 1,\cdots;n-1$)

方程数少于未知数个数？？

(1)自然边界条件: $S''(x_0)=0, S''(x_n)=0$

(2)周期边界条件: $S'(x_0)=S'(x_n), \ S''(x_0)=S''(x_n)$

(3)固定边界条件: $S'(x_0)=f'(x_0), \ S'(x_n)=f'(x_n)$

例 5.7：已知$f(-1) = 1, f(0) = 0, f(1) = 1$.验证下面分段三次多项式是自然样条函数.

$$S(x)=\begin{cases}\frac{1}{2}x^3+\frac{3}{2}x^2, & x\in[-1,0]\\ -\frac{1}{2}x^3+\frac{3}{2}x^2, & x\in[0,1]\end{cases}$$

证:显然 $S(-1)=1 \quad S(1)=1 \quad S(0-)=S(0+)=0$

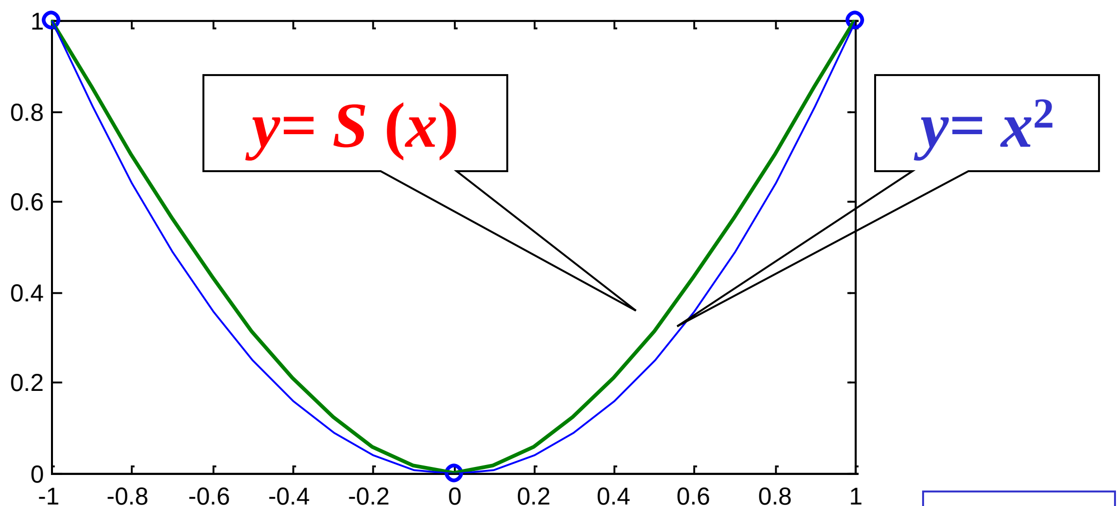
求导数得

$$S'(x)=\begin{cases}\frac{3}{2}x^2+3x,\ x\in[-1,0]\\ -\frac{3}{2}x^2+3x,\ x\in[0,1]\end{cases}\qquad S''(x)=\begin{cases}3x+3,\ x\in[-1,0]\\ -3x+3,\ x\in[0,1]\end{cases}$$

显然 $S'(-1)=-\frac{3}{2}$ $S'(1)=\frac{3}{2}$ $S'(0-)=S'(0+)=0$

$S''(-1)=0$ $S''(1)=0$ $S''(0-)=S''(0+)=3$

所以,$S(x)$是满足插值条件且二阶导函数连续的分段三次多项式



分段Hermite插值公式导出的三次样条方法

已知函数表

x	x_0	x_1	……	x_n
$f(x)$	y_0	y_1	……	y_n

设 $f(x)$ 在各插值节点 x_j 处的一阶导数为 m_j (未知)

取 $x_{j+1}-x_j=h$，$(j=0,1,2,\cdots;n)$.当 $x\in[x_j, x_{j+1}]$时，分段Hermite插值

$$S(x)=(1+2\frac{x-x_j}{h})(\frac{x_{j+1}-x}{h})^2 y_j+(1+2\frac{x_{j+1}-x}{h})(\frac{x-x_j}{h})^2 y_{j+1}$$

$$+(x-x_j)(\frac{x_{j+1}-x}{h})^2 m_j+(x-x_{j+1})(\frac{x-x_j}{h})^2 m_{j+1}$$

由$S''(x)$连续：有等式：$S''(x_j+0)=S''(x_j-0)$

考虑 $S''(x)$ 在区间$[x_j, x_{j+1}]$和$[x_{j-1}, x_j]$上表达式.

当 $x\in[x_j, x_{j+1}]$时, $S(x)$ 由基函数组合而成

$$\alpha_j(x)=(1+2\frac{x-x_j}{h})(\frac{x_{j+1}-x}{h})^2$$

$$\alpha_{j+1}(x)=(1+2\frac{x_{j+1}-x}{h})(\frac{x-x_j}{h})^2$$

$$\beta_j(x)=(x-x_j)(\frac{x_{j+1}-x}{h})^2$$

$$\beta_{j+1}(x)=(x-x_{j+1})(\frac{x-x_j}{h})^2$$

$$
\begin{cases}
\alpha_j''(x_j)=[\frac{-8}{h^3}(x_{j+1}-x)+(1+2\frac{x-x_j}{h})\frac{2}{h^2}]_{x=x_j}=-\frac{6}{h^2} \\
\alpha_{j+1}''(x_j)=[-\frac{8}{h^3}(x-x_j)+(1+2\frac{x_{j+1}-x}{h})\frac{2}{h^2}]_{x=x_j}=\frac{6}{h^2}
\end{cases}
$$

$$
\begin{cases}
\beta_j''(x_j)=[\frac{4}{h^2}(x-x_{j+1})+(x-x_j)\frac{2}{h^2}]_{x=x_j}=-\frac{4}{h} \\
\beta_{j+1}''(x_j)=[\frac{4}{h^2}(x-x_j)+(x-x_{j+1})\frac{2}{h^2}]_{x=x_j}=-\frac{2}{h}
\end{cases}
$$

$$
\Rightarrow \quad S''(x_j+0)=\alpha_j''(x_j)y_j+\alpha_{j+1}''(x_j)y_{j+1} +\beta_j''(x_j)m_j+\beta_{j+1}''(x_j)m_{j+1}
$$

$$S''(x_j+0)=-\frac{6}{h^2}y_j+\frac{6}{h^2}y_{j+1}-\frac{4}{h}m_j-\frac{2}{h}m_{j+1}$$

同理, 有

$$S''(x_j-0)=\frac{6}{h^2}y_{j-1}-\frac{6}{h^2}y_j+\frac{2}{h}m_{j-1}+\frac{4}{h}m_j$$

联立得：

$$-\frac{6}{h^2}y_j+\frac{6}{h^2}y_{j+1}-\frac{4}{h}m_j-\frac{2}{h}m_{j+1}$$

$$=\frac{6}{h^2}y_{j-1}-\frac{6}{h^2}y_j+\frac{2}{h}m_{j-1}+\frac{4}{h}m_j$$

➔

$$m_{j-1}+4m_j+m_{j+1}=\frac{3}{h}(y_{j+1}-y_{j-1})$$

(j=1, 2, ·····; n-1)

设自然边界条件成立, 即

$$S''(x_0+0)=-\frac{6}{h^2}y_0+\frac{6}{h^2}y_1-\frac{4}{h}m_0-\frac{2}{h}m_1=0$$

$$S''(x_n-0)=\frac{6}{h^2}y_{n-1}-\frac{6}{h^2}y_n+\frac{2}{h}m_{n-1}+\frac{4}{h}m_n=0$$

自然样条的导数值满足:

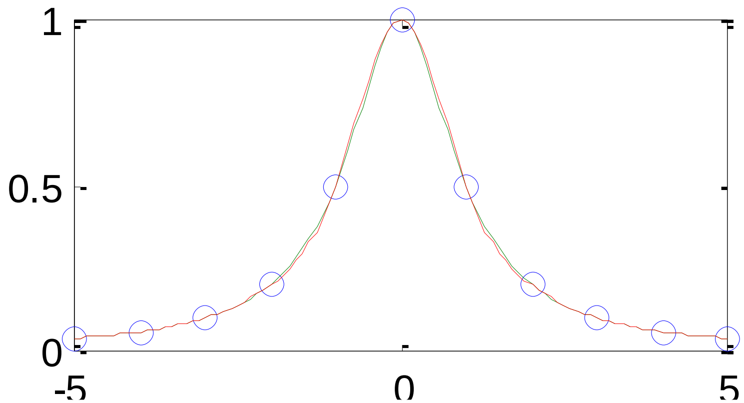
$$\boxed{2m_0+m_1=\frac{3}{h}[y_1-y_0]}\qquad \boxed{m_{n-1}+2m_n=\frac{3}{h}[y_n-y_{n-1}]}$$

$$m_{j-1}+4m_j+m_{j+1}=\frac{3}{h}(y_{j+1}-y_{j-1})$$

(j=1, 2, ·····; n-1)

MATLAB样条命令：yi = spline(x, y, xi)

```
x=-5:5;y=1./(1+x.^2);
xi=-5:0.1:5;f=1./(1+xi.^2);
yi=spline(x,y,xi); error=max(abs(yi-f))
plot(x,y,'o',xi,f,xi,yi,'r')
```



error =   0.0220

曲率比较：计算公式

$$K = \frac{|y''|}{(1+(y')^2)^{3/2}}$$

样条插值函数的极性

设$f(x)\in C^2[a, b]$, 对于$a = x_0 < x_1 < \dots < x_n = b$,有

$f(x_j)=y_j(j=0,1,\cdots,n)$.S($x$)是满足$S(x_j)=y_j(j=0,1,\cdots,n)$

的三次自然样条.则有

$$\|S''(x)\|\le\|f''(x)\|$$

证明: $\| f''(x) - S''(x) \|^2 = \int_a^b [f''(x) - S''(x)]^2 dx$

$$= \int_a^b [f''(x)]^2 dx - 2\int_a^b f''(x)S''(x)dx + \int_a^b [S''(x)]^2 dx$$

$$= \| f'' \|^2 - 2\int_a^b [f''(x) - S''(x)]S''(x)dx - \| S'' \|^2$$

$$\int_a^b [f''(x)-S''(x)]S''(x)dx = -\int_a^b [f'(x)-S'(x)]S'''(x)dx$$

$$= -\sum_{j=1}^{n} S_j'''[f(x)-S(x)]\Big|_{x_{j-1}}^{x_j} = 0$$

所以 $0 \le \| f''-S'' \|^2 = \| f'' \|^2 - \| S'' \|^2 \Rightarrow$

$$\| S'' \|^2 \le \| f'' \|^2$$

即 $$\int_a^b [S''(x)]^2 dx \le \int_a^b [f''(x)]^2 dx$$

样条函数S(*x*)在[*a*, *b*]上的**总曲率最小.**