Appendix: Other MSD-based algorithms

Similarly as SUnSAL-TV-MSD, we propose to apply MSD on other unmixing models, including the SUnSAL, CLSUnSAL, ADSpLRU, and JSpBLRU models, and then name the resulting algorithms as CLSUnSAL-MSD, CLSUnSAL-MSD, ADSpLRU-MSD, and JSpBLRU-MSD, respectively. In the following, we present the pseudocode of each algorithm. To improve the readability, we first recall two functions, which will be needed in the CLSUnSAL, ADSpLRU, and JSpBLRU based algorithms. Let $\textbf{vect-soft}(\cdot, \tau)$ be a nonlinear function defined by

$$\textbf{vect-soft}(\mathbf{x}, \tau) = \mathbf{x}\frac{\max\{\|\mathbf{x}\|_2 - \tau, 0\}}{\max\{\|\mathbf{x}\|_2 - \tau, 0\} + \tau}$$

for any vector $\mathbf{x}$ and $\tau > 0$. Let $\mathbf{X} = \mathbf{U\Sigma V}^T$ be the singular value decomposition (SVD) of $\mathbf{X}$. Then the singular value threshold operation $\mathbf{SVT}_\alpha$ on $\mathbf{X}$ is defined by

$$\mathbf{SVT}_\alpha(\mathbf{X}) = \mathbf{U}\textbf{soft}(\mathbf{\Sigma}, \alpha)\mathbf{V}^T, \ \forall\ \alpha > 0.$$

Appendix A SUnSAL-MSD

Consider the SUnSAL model

$$\min_{\mathbf{X}} \frac{1}{2}\|\mathbf{AX} - \mathbf{Y}\|_F^2 + \lambda\|\mathbf{X}\|_{1,1} + \iota_{\mathbb{R}+}(\mathbf{X}) \tag{33}$$

where $\lambda \geq 0$ is a regularization parameter. With variable replacement, we rewrite (33) as

$$\begin{aligned}&\min_{\mathbf{X}} \frac{1}{2}\|\mathbf{AX} - \mathbf{Y}\|_F^2 + \lambda\|\mathbf{V}\|_{1,1} + \iota_{\mathbb{R}+}(\mathbf{V})\\ &\text{s.t. } \mathbf{X} = \mathbf{V}\end{aligned}$$

and define a function $\mathcal{L}_1$ as

$$\mathcal{L}_1(\mathbf{X}, \mathbf{A}, \mathbf{V}, \mathbf{D}) = \frac{1}{2}\|\mathbf{AX} - \mathbf{Y}\|_F^2 + \lambda\|\mathbf{V}\|_{1,1} + \iota_{\mathbb{R}+}(\mathbf{V}) + \frac{\mu}{2}\|\mathbf{X} - \mathbf{V} - \mathbf{D}\|_F^2$$

where $\mu > 0$ is a penalty parameter. Then we propose to apply MSD and minimize $\mathcal{L}_1(\mathbf{X}, \mathbf{A}, \mathbf{V}, \mathbf{D})$ with respect to $\mathbf{X}$ and $\mathbf{V}$ and update $\mathbf{D}$ at $(k+1)$th iteration as the following framework:

$$\begin{cases}\mathbf{X}^{k+\frac{1}{2}} = \arg\min_{\mathbf{X}} \mathcal{L}_1(\mathbf{X}, \mathbf{A}^k, \mathbf{V}^k, \mathbf{D}^k)\\ [\mathbf{X}^{k+1}, \mathbf{A}^{k+1}, s^{k+1}] = \mathrm{MSD}(\mathbf{X}^{k+\frac{1}{2}}, \mathbf{A}^k, s^k)\\ \mathbf{D}^{k+\frac{1}{2}} = \mathbf{D}^k(s^{k+1}, :)\\ \mathbf{V}^{k+1} = \arg\min_{\mathbf{V}} \mathcal{L}_1(\mathbf{X}^{k+1}, \mathbf{A}^{k+1}, \mathbf{V}, \mathbf{D}^{k+\frac{1}{2}})\\ \mathbf{D}^{k+1} = \mathbf{D}^{k+\frac{1}{2}} - \mathbf{X}^{k+1} + \mathbf{V}^{k+1}.\end{cases}$$

Next we compute $\mathbf{X}$, $\mathbf{V}$, and $\mathbf{D}$ in detail.

- For $\mathbf{X}^{k+\frac{1}{2}}$ subproblem, we consider the optimization model:

$$\min_{\mathbf{X}} \frac{1}{2}\|\mathbf{A}^k\mathbf{X} - \mathbf{Y}\|_F^2 + \frac{\mu}{2}\|\mathbf{X} - \mathbf{V}^k - \mathbf{D}^k\|_F^2$$

and it is easy to obtain the solution

$$\mathbf{X}^{k+\frac{1}{2}} = ((\mathbf{A}^k)^T\mathbf{A}^k + \mu\mathbf{I})^{-1}((\mathbf{A}^k)^T\mathbf{Y} + \mu(\mathbf{V}^k + \mathbf{D}^k)).$$

• For $\mathbf{X}^{k+1}$, $\mathbf{A}^{k+1}$, and s^{k+1} subproblem, we apply MSD in Algorithm 1 on $\mathbf{X}^k$ and $\mathbf{A}^k$, and s^k, similarly as in SUnSAL-TV-MSD. Then we update $\mathbf{D}^{k+\frac{1}{2}}$ according to the obtained support set s^{k+1}.

• For $\mathbf{V}^{k+1}$ subproblem, we have

$$\min_{\mathbf{V}} \lambda\|\mathbf{V}\|_{1,1} + \iota_{\mathbb{R}+}(\mathbf{V}) + \frac{\mu}{2}\|\mathbf{X}^{k+1} - \mathbf{V} - \mathbf{D}^{k+\frac{1}{2}}\|_F^2.$$

It follows that

$$\mathbf{V}^{k+1} = \max(\mathbf{soft}(\mathbf{X}^{k+1} - \mathbf{D}^{k+\frac{1}{2}}, \frac{\lambda}{\mu}), \mathbf{0}).$$

• Finally, we update Lagrange multiplier as

$$\mathbf{D}^{k+1} = \mathbf{D}^{k+\frac{1}{2}} - \mathbf{X}^{k+1} + \mathbf{V}^{k+1}.$$

The estimated abundance matrix $\hat{\mathbf{W}}$ is obtained with zero initialization and update $\hat{\mathbf{W}}(s^{k+1}, :) = \mathbf{X}^{k+1}$. We summarize the above procedure in Algorithm 3.

Algorithm 3: Pseudocode of SUnSAL-MSD

Input: $\mathbf{\Phi} \in \mathbb{R}^{L\times m}$ and $\mathbf{Y} \in \mathbb{R}^{L\times N}$
Initialize: $\hat{\mathbf{W}} = \mathbf{0} \in \mathbb{R}^{m\times N}$, $\mathbf{A}^0 = \mathbf{\Phi}$, $\mathbf{V}^0$, $\mathbf{D}^0$, $s^0 = \{1, \cdots, m\}$, and set $k = 0$
Select parameters: λ, μ, and ϵ
Repeat
  Compute $\mathbf{X}^{k+\frac{1}{2}} = ((\mathbf{A}^k)^T\mathbf{A}^k + \mu\mathbf{I})^{-1}((\mathbf{A}^k)^T\mathbf{Y} + \mu(\mathbf{V}^k + \mathbf{D}^k))$
  Compute $\mathbf{X}^{k+1}$, $\mathbf{A}^{k+1}$, and s^{k+1} by Algorithm 1
  Compute $\mathbf{D}^{k+\frac{1}{2}} = \mathbf{D}^k(s^{k+1}, :)$
  Compute $\mathbf{V}^{k+1} = \max(\mathbf{soft}(\mathbf{X}^{k+1} - \mathbf{D}^{k+\frac{1}{2}}, \frac{\lambda}{\mu}), \mathbf{0})$
  Compute $\mathbf{D}^{k+1} = \mathbf{D}^{k+\frac{1}{2}} - \mathbf{X}^{k+1} + \mathbf{V}^{k+1}$
until convergence
Update $\hat{\mathbf{W}}(s^{k+1}, :) = \mathbf{X}^{k+1}$
Output: $\hat{\mathbf{W}}$

Appendix B CLSUnSAL-MSD

We consider the optimization model

$$\min_{\mathbf{X}} \frac{1}{2}\|\mathbf{A}\mathbf{X} - \mathbf{Y}\|_F^2 + \lambda\|\mathbf{X}\|_{2,1} + \iota_{\mathbb{R}+}(\mathbf{X}) \tag{34}$$

where $\lambda \geq 0$ is a regularization parameter. Rewrite (34) and we have

$$\begin{aligned}&\min_{\mathbf{X}} \frac{1}{2}\|\mathbf{V}_1 - \mathbf{Y}\|_F^2 + \lambda\|\mathbf{V}_2\|_{2,1} + \iota_{\mathbb{R}+}(\mathbf{V}_3)\\ &\text{s.t. } \mathbf{A}\mathbf{X} = \mathbf{V}_1, \mathbf{X} = \mathbf{V}_2, \mathbf{X} = \mathbf{V}_3.\end{aligned}$$

Define

$$\mathcal{L}_2(\mathbf{X},\mathbf{A},\mathbf{V},\mathbf{D}) = \frac{1}{2}\|\mathbf{V}_1-\mathbf{Y}\|_F^2 + \lambda\|\mathbf{V}_2\|_{2,1} + \iota_{\mathbb{R}+}(\mathbf{V}_3) + \frac{\mu}{2}(\|\mathbf{A}\mathbf{X}-\mathbf{V}_1-\mathbf{D}_1\|_F^2 + \|\mathbf{X}-\mathbf{V}_2-\mathbf{D}_2\|_F^2 + \|\mathbf{X}-\mathbf{V}_3-\mathbf{D}_3\|_F^2)$$

where $\mu > 0$ is a penalty parameter with $\mathbf{D}^T = [\mathbf{D}_1^T, \mathbf{D}_2^T, \mathbf{D}_3^T]^T$ and $\mathbf{V}^T = [\mathbf{V}_1^T, \mathbf{V}_2^T, \mathbf{V}_3^T]^T$. We propose to apply MSD at each iteration according to the following framework

$$\begin{cases} \mathbf{X}^{k+\frac{1}{2}} = \arg\min_{\mathbf{X}} \mathcal{L}_2(\mathbf{X}, \mathbf{A}^k, \mathbf{V}^k, \mathbf{D}^k) \\ [\mathbf{X}^{k+1}, \mathbf{A}^{k+1}, s^{k+1}] = \mathrm{MSD}(\mathbf{X}^{k+\frac{1}{2}}, \mathbf{A}^k, s^k) \\ \mathbf{D}_1^{k+\frac{1}{2}} = \mathbf{D}_1^k, \mathbf{D}_i^{k+\frac{1}{2}} = \mathbf{D}_i^k(s^{k+1}, :), i = 2, 3 \\ \mathbf{V}^{k+1} = \arg\min_{\mathbf{V}} \mathcal{L}_2(\mathbf{X}^{k+1}, \mathbf{A}^{k+1}, \mathbf{V}, \mathbf{D}^{k+\frac{1}{2}}) \\ \mathbf{D}_1^{k+1} = \mathbf{D}_1^{k+\frac{1}{2}} - \mathbf{A}^{k+1}\mathbf{X}^{k+1} + \mathbf{V}_1^{k+1} \\ \mathbf{D}_i^{k+1} = \mathbf{D}_i^{k+\frac{1}{2}} - \mathbf{X}^{k+1} + \mathbf{V}_i^{k+1}, i = 2, 3. \end{cases} \tag{35}$$

Next we consider each subproblem in detail.

• For $\mathbf{X}^{k+\frac{1}{2}}$ subproblem, we have

$$\min_{\mathbf{X}} \|\mathbf{A}^k\mathbf{X} - \mathbf{V}_1^k - \mathbf{D}_1^k\|_F^2 + \|\mathbf{X} - \mathbf{V}_2^k - \mathbf{D}_2^k\|_F^2 + \|\mathbf{X} - \mathbf{V}_3^k - \mathbf{D}_3^k\|_F^2.$$

It is easy to obtain that

$$\mathbf{X}^{k+\frac{1}{2}} = ((\mathbf{A}^k)^T\mathbf{A}^k + 2\mathbf{I})^{-1}((\mathbf{A}^k)^T(\mathbf{V}_1^k + \mathbf{D}_1^k) + \mathbf{V}_2^k + \mathbf{D}_2^k + \mathbf{V}_3^k + \mathbf{D}_3^k).$$

• For $\mathbf{X}^{k+1}$, $\mathbf{A}^{k+1}$, and s^{k+1} subproblem, we apply MSD in Algorithm 1 on $\mathbf{X}^k$, $\mathbf{A}^k$, and s^k. Then we update $\mathbf{D}$ according to the support set s^{k+1}.

• For $\mathbf{V}^{k+1}$ subproblem, we equivalently divide it into three subproblems.

◦ For $\mathbf{V}_1^{k+1}$, we have

$$\min_{\mathbf{V}_1} \frac{1}{2}\|\mathbf{V}_1 - \mathbf{Y}\|_F^2 + \frac{\mu}{2}\|\mathbf{A}^{k+1}\mathbf{X}^{k+1} - \mathbf{V}_1 - \mathbf{D}_1^{k+\frac{1}{2}}\|_F^2.$$

It follows that

$$\mathbf{V}_1^{k+1} = \frac{1}{\mu+1}(\mathbf{Y} + \mu(\mathbf{A}^{k+1}\mathbf{X}^{k+1} - \mathbf{D}_1^{k+\frac{1}{2}})).$$

◦ For $\mathbf{V}_2^{k+1}$, we get

$$\min_{\mathbf{V}_2} \lambda\|\mathbf{V}_2\|_{2,1} + \frac{\mu}{2}\|\mathbf{X}^{k+1} - \mathbf{V}_2 - \mathbf{D}_2^{k+\frac{1}{2}}\|_F^2.$$

Its solution is

$$\mathbf{V}_2^{k+1} = \textbf{vect-soft}(\mathbf{X}^{k+1} - \mathbf{D}_2^{k+\frac{1}{2}}, \frac{\lambda}{\mu}).$$

◦ For $\mathbf{V}_3^{k+1}$, we obtain

$$\min_{\mathbf{V}_3} \iota_{\mathbb{R}+}(\mathbf{V}_3) + \frac{\mu}{2}\|\mathbf{X}^{k+1} - \mathbf{V}_3 - \mathbf{D}_3^{k+\frac{1}{2}}\|_F^2.$$

It is easy to calculate

$$\mathbf{V}_3^{k+1} = \max(\mathbf{X}^{k+1} - \mathbf{D}_3^{k+\frac{1}{2}}, \mathbf{0}).$$

- Finally, we update Lagrange multiplier according to (35). The procedure is summarized in Algorithm 4.

Algorithm 4: Pseudocode of CLSUnSAL-MSD

Input: $\mathbf{\Phi} \in \mathbb{R}^{L\times m}$ and $\mathbf{Y} \in \mathbb{R}^{L\times N}$
Initialize: $\hat{\mathbf{W}} = \mathbf{0} \in \mathbb{R}^{m\times N}$, $\mathbf{A}^0 = \mathbf{\Phi}$, $\mathbf{V}^0$, $\mathbf{D}^0$, $s^0 = \{1, \cdots, m\}$, and set $k = 0$
Select parameters: λ, μ, and ϵ
Repeat
  Compute $\mathbf{X}^{k+\frac{1}{2}} = ((\mathbf{A}^k)^T\mathbf{A}^k + 2\mathbf{I})^{-1}((\mathbf{A}^k)^T(\mathbf{V}_1^k + \mathbf{D}_1^k) + \mathbf{V}_2^k + \mathbf{D}_2^k + \mathbf{V}_3^k + \mathbf{D}_3^k)$
  Compute $\mathbf{X}^{k+1}$, $\mathbf{A}^{k+1}$, and s^{k+1} by Algorithm 1
  Compute $\mathbf{D}_1^{k+\frac{1}{2}} = \mathbf{D}_1^k$, $\mathbf{D}_i^{k+\frac{1}{2}} = \mathbf{D}_i^k(s^{k+1}, :), i = 2, 3$
  Compute $\mathbf{V}_1^{k+1} = \frac{1}{\mu+1}(\mathbf{Y} + \mu(\mathbf{A}^{k+1}\mathbf{X}^{k+1} - \mathbf{D}_1^{k+\frac{1}{2}}))$
    $\mathbf{V}_2^{k+1} = \textbf{vect-soft}(\mathbf{X}^{k+1} - \mathbf{D}_2^{k+\frac{1}{2}}, \frac{\lambda}{\mu})$
    $\mathbf{V}_3^{k+1} = \max(\mathbf{X}^{k+1} - \mathbf{D}_3^{k+\frac{1}{2}}, \mathbf{0})$
  Compute $\mathbf{D}_1^{k+1} = \mathbf{D}_1^{k+\frac{1}{2}} - \mathbf{A}^{k+1}\mathbf{X}^{k+1} + \mathbf{V}_1^{k+1}$
    $\mathbf{D}_i^{k+1} = \mathbf{D}_i^{k+\frac{1}{2}} - \mathbf{X}^{k+1} + \mathbf{V}_i^{k+1}$, $i = 2, 3$
until convergence
Update $\hat{\mathbf{W}}(s^{k+1}, :) = \mathbf{X}^{k+1}$
Output: $\hat{\mathbf{W}}$

Appendix C ADSpLRU-MSD

We reconsider the ADSpLRU model

$$\min_{\mathbf{X}} \frac{1}{2}\|\mathbf{A}\mathbf{X} - \mathbf{Y}\|_F^2 + \gamma\|\mathbf{Z} \odot \mathbf{X}\|_{1,1} + \tau\|\mathbf{X}\|_{\mathbf{b},*} + \iota_{\mathbb{R}+}(\mathbf{X}) \tag{36}$$

where $\odot$ denotes the Hadamard product, γ and τ are nonnegative regularization parameters. Then, we rewrite (36) as

$$\begin{aligned} &\min_{\mathbf{X}} \frac{1}{2}\|\mathbf{V}_1 - \mathbf{Y}\|_F^2 + \gamma\|\mathbf{Z} \odot \mathbf{V}_2\|_{1,1} + \tau\|\mathbf{V}_3\|_{\mathbf{b},*} + \iota_{\mathbb{R}+}(\mathbf{V}_4) \\ &\text{s.t. } \mathbf{A}\mathbf{X} = \mathbf{V}_1, \mathbf{X} = \mathbf{V}_2, \mathbf{X} = \mathbf{V}_3, \mathbf{X} = \mathbf{V}_4. \end{aligned}$$

Let $\mathbf{V}^T = [\mathbf{V}_1^T, \mathbf{V}_2^T, \mathbf{V}_3^T, \mathbf{V}_4^T]^T$ and define

$$\begin{aligned} &\mathcal{L}_3(\mathbf{X}, \mathbf{A}, \mathbf{V}, \mathbf{D}) = \frac{1}{2}\|\mathbf{V}_1 - \mathbf{Y}\|_F^2 + \gamma\|\mathbf{Z} \odot \mathbf{V}_2\|_{1,1} + \tau\|\mathbf{V}_3\|_{\mathbf{b},*} + \iota_{\mathbb{R}+}(\mathbf{V}_4) \\ &+ \frac{\mu}{2}(\|\mathbf{A}\mathbf{X} - \mathbf{V}_1 - \mathbf{D}_1\|_F^2 + \|\mathbf{X} - \mathbf{V}_2 - \mathbf{D}_2\|_F^2 + \|\mathbf{X} - \mathbf{V}_3 - \mathbf{D}_3\|_F^2 + \|\mathbf{X} - \mathbf{V}_4 - \mathbf{D}_4\|_F^2). \end{aligned}$$

where $\mu > 0$ is a penalty parameter and $\mathbf{D}^T = [\mathbf{D}_1^T, \mathbf{D}_2^T, \mathbf{D}_3^T, \mathbf{D}_4^T]^T$. The framework of ADSpLRU-MSD is proposed as below

$$\begin{cases} \mathbf{X}^{k+\frac{1}{2}} = \arg\min_{\mathbf{X}} \mathcal{L}_3(\mathbf{X}, \mathbf{A}^k, \mathbf{V}^k, \mathbf{D}^k) \\ [\mathbf{X}^{k+1}, \mathbf{A}^{k+1}, s^{k+1}] = \mathrm{MSD}(\mathbf{X}^{k+\frac{1}{2}}, \mathbf{A}^k, s^k) \\ \mathbf{D}_1^{k+\frac{1}{2}} = \mathbf{D}_1^k, \mathbf{D}_i^{k+\frac{1}{2}} = \mathbf{D}_i^k(s^{k+1}, :), i = 2, 3, 4 \\ \mathbf{V}^{k+1} = \arg\min_{\mathbf{V}} \mathcal{L}_3(\mathbf{X}^{k+1}, \mathbf{A}^{k+1}, \mathbf{V}, \mathbf{D}^{k+\frac{1}{2}}) \\ \mathbf{D}_1^{k+1} = \mathbf{D}_1^{k+\frac{1}{2}} - \mathbf{A}^{k+1}\mathbf{X}^{k+1} + \mathbf{V}_1^{k+1} \\ \mathbf{D}_i^{k+1} = \mathbf{D}_i^{k+\frac{1}{2}} - \mathbf{X}^{k+1} + \mathbf{V}_i^{k+1}, i = 2, 3, 4. \end{cases} \tag{37}$$

In the following, we compute each subproblem in detail.

• For $\mathbf{X}^{k+\frac{1}{2}}$ subproblem, we consider

$$\min_{\mathbf{X}} \|\mathbf{A}^k\mathbf{X} - \mathbf{V}_1^k - \mathbf{D}_1^k\|_F^2 + \|\mathbf{X} - \mathbf{V}_2^k - \mathbf{D}_2^k\|_F^2 + \|\mathbf{X} - \mathbf{V}_3^k - \mathbf{D}_3^k\|_F^2 + \|\mathbf{X} - \mathbf{V}_4^k - \mathbf{D}_4^k\|_F^2.$$

This is a least squares problem and its solution is

$$\mathbf{X}^{k+\frac{1}{2}} =((\mathbf{A}^k)^T\mathbf{A}^k + 3\mathbf{I})^{-1}((\mathbf{A}^k)^T(\mathbf{V}_1^k + \mathbf{D}_1^k) + \mathbf{V}_2^k + \mathbf{D}_2^k + \mathbf{V}_3^k + \mathbf{D}_3^k + \mathbf{V}_4^k + \mathbf{D}_4^k).$$

• For $\mathbf{X}^{k+1}$, $\mathbf{A}^{k+1}$, and s^{k+1} subproblem, we apply MSD in Algorithm 1 on $\mathbf{X}^k$, $\mathbf{A}^k$, and s^k. Then we update $\mathbf{D}$ according to the support set s^{k+1}.

• For $\mathbf{V}^{k+1}$ problem, notice that it can be divided it into four subproblems.

∘ For $\mathbf{V}_1^{k+1}$ subproblem, we consider the optimization problem

$$\min_{\mathbf{V}_1} \frac{1}{2}\|\mathbf{V}_1 - \mathbf{Y}\|_F^2 + \frac{\mu}{2}\|\mathbf{A}^{k+1}\mathbf{X}^{k+1} - \mathbf{V}_1 - \mathbf{D}_1^{k+\frac{1}{2}}\|_F^2.$$

Simple calculation gives that

$$\mathbf{V}_1^{k+1} = \frac{1}{\mu+1}(\mathbf{Y} + \mu(\mathbf{A}^{k+1}\mathbf{X}^{k+1} - \mathbf{D}_1^{k+\frac{1}{2}})).$$

∘ For $\mathbf{V}_2^{k+1}$ subproblem, we have

$$\min_{\mathbf{V}_2} \gamma\|\mathbf{Z}^{k+1} \odot \mathbf{V}_2\|_{1,1} + \frac{\mu}{2}\|\mathbf{X}^{k+1} - \mathbf{V}_2 - \mathbf{D}_2^{k+\frac{1}{2}}\|_F^2$$

and its solution is

$$\mathbf{V}_2^{k+1} = \mathbf{soft}(\mathbf{X}^{k+1} - \mathbf{D}_2^{k+\frac{1}{2}}, \frac{\gamma}{\mu}\mathbf{Z}^{k+1})$$

where $\mathbf{Z}^{k+1} = [z_{ij}^{k+1}]$ with

$$z_{ij}^{k+1} = \frac{1}{|(\mathbf{X}^{k+1} - \mathbf{D}_2^{k+\frac{1}{2}})_{i,j}| + \varepsilon}$$

and ε being a small constant to avoid singularities.

◦ For $\mathbf{V}_3^{k+1}$ subproblem, we obtain

$$\min_{\mathbf{V}_3} \; \tau\|\mathbf{V}_3\|_{\mathbf{b},*} + \frac{\mu}{2}\|\mathbf{X}^{k+1} - \mathbf{V}_3 - \mathbf{D}_3^{k+\frac{1}{2}}\|_F^2$$

and its solution is

$$\mathbf{V}_3^{k+1} = \mathbf{SVT}_{\frac{\tau}{\mu}\mathbf{b}}(\mathbf{X}^{k+1} - \mathbf{D}_3^{k+\frac{1}{2}})$$

where $\mathbf{b} = [b_1, \cdots, b_r]$ with

$$b_i = \frac{1}{\sigma_i + \varepsilon}$$

and σ_i is the ith singular value of $\mathbf{X}^{k+1} - \mathbf{D}_3^{k+\frac{1}{2}}$.

◦ For $\mathbf{V}_4^{k+1}$ subproblem, we have

$$\min_{\mathbf{V}_4} \; \iota_{\mathbb{R}+}(\mathbf{V}_4) + \frac{\mu}{2}\|\mathbf{X}^{k+1} - \mathbf{V}_4 - \mathbf{D}_4^{k+\frac{1}{2}}\|_F^2$$

and clearly we obtain

$$\mathbf{V}_4^{k+1} = \max(\mathbf{X}^{k+1} - \mathbf{D}_4^{k+\frac{1}{2}}, \mathbf{0}).$$

• The Lagrange multipliers are updated according to (37). Moreover, the procedures are summarized in Algorithm 5.

Algorithm 5: Pseudocode of ADSpLRU-MSD

Input: $\mathbf{\Phi} \in \mathbb{R}^{L\times m}$ and $\mathbf{Y} \in \mathbb{R}^{L\times N}$

Initialize: $\hat{\mathbf{W}} = \mathbf{0} \in \mathbb{R}^{m\times N}$, $\mathbf{A}^0 = \mathbf{\Phi}$, $\mathbf{V}^0$, $\mathbf{D}^0$, $s^0 = \{1, \cdots, m\}$, and set $k = 0$

Select parameters: γ, τ, μ, and ϵ

Repeat

  Compute $\mathbf{X}^{k+\frac{1}{2}} = (\mathbf{A}^k(\mathbf{A}^k)^T + 3\mathbf{I})^{-1}((\mathbf{A}^k)^T(\mathbf{V}_1^k + \mathbf{D}_1^k) + \mathbf{V}_2^k + \mathbf{D}_2^k + \mathbf{V}_3^k + \mathbf{D}_3^k + \mathbf{V}_4^k + \mathbf{D}_4^k)$

  Compute $\mathbf{X}^{k+1}$, $\mathbf{A}^{k+1}$, and s^{k+1} by Algorithm 1

  Compute $\mathbf{D}_1^{k+\frac{1}{2}} = \mathbf{D}_1^k$, $\mathbf{D}_i^{k+\frac{1}{2}} = \mathbf{D}_i^k(s^{k+1}, :)$, $i = 2, 3, 4$.

  Compute $\mathbf{V}_1^{k+1} = \frac{1}{\mu+1}(\mathbf{Y} + \mu(\mathbf{A}^{k+1}\mathbf{X}^{k+1} - \mathbf{D}_1^{k+\frac{1}{2}}))$

    $\mathbf{V}_2^{k+1} = \mathbf{soft}(\mathbf{X}^{k+1} - \mathbf{D}_2^{k+\frac{1}{2}}, \frac{\gamma}{\mu}\mathbf{Z}^{k+1})$

    $\mathbf{V}_3^{k+1} = \mathbf{SVT}_{\frac{\tau}{\mu}\mathbf{b}}(\mathbf{X}^{k+1} - \mathbf{D}_3^{k+\frac{1}{2}})$

    $\mathbf{V}_4^{k+1} = \max(\mathbf{X}^{k+1} - \mathbf{D}_4^{k+\frac{1}{2}}, \mathbf{0})$

  Compute $\mathbf{D}_1^{k+1} = \mathbf{D}_1^{k+\frac{1}{2}} - \mathbf{A}^{k+1}\mathbf{X}^{k+1} + \mathbf{V}_1^{k+1}$

    $\mathbf{D}_i^{k+1} = \mathbf{D}_i^{k+\frac{1}{2}} - \mathbf{X}^{k+1} + \mathbf{V}_i^{k+1}$, $i = 2, 3, 4$

until convergence

Update $\hat{\mathbf{W}}(s^{k+1}, :) = \mathbf{X}^{k+1}$

Output: $\hat{\mathbf{W}}$

Appendix D JSpBLRU-MSD

We consider the JSpBLRU model

$$\min_{\mathbf{X}} \frac{1}{2}\|\mathbf{A}\mathbf{X}-\mathbf{Y}\|_F^2+\lambda\sum_{j=1}^{J}\|\mathbf{X}_j\|_{\mathbf{z}_j,2,1}+\tau\|\mathbf{X}\|_{\mathbf{b},*}+\iota_{\mathbb{R}+}(\mathbf{X}) \tag{38}$$

where λ and τ are nonnegative regularization parameters, $\mathbf{z}_j$ and $\mathbf{b}$ are nonnegative weighting coefficients, which will be updated automatically. Rewrite (38) by some variable replacement and we obtain

$$\begin{aligned}&\min_{\mathbf{X}} \frac{1}{2}\|\mathbf{A}\mathbf{X}-\mathbf{Y}\|_F^2+\lambda\sum_{j=1}^{J}\|\mathbf{V}_{1,j}\|_{\mathbf{z}_j,2,1}+\tau\|\mathbf{V}_2\|_{\mathbf{b},*}+\iota_{\mathbb{R}+}(\mathbf{V}_3)\\&\text{s.t. } \mathbf{X}=\mathbf{V}_1,\mathbf{X}=\mathbf{V}_2,\mathbf{X}=\mathbf{V}_3.\end{aligned}$$

Define

$$\begin{aligned}\mathcal{L}_4(\mathbf{X},\mathbf{A},\mathbf{V},\mathbf{D})=&\frac{1}{2}\|\mathbf{A}\mathbf{X}-\mathbf{Y}\|_F^2+\lambda\sum_{j=1}^{J}\|\mathbf{V}_{1,j}\|_{\mathbf{z}_j,2,1}+\tau\|\mathbf{V}_2\|_{\mathbf{b},*}+\iota_{\mathbb{R}+}(\mathbf{V}_3)\\&+\frac{\mu}{2}(\|\mathbf{X}-\mathbf{V}_1-\mathbf{D}_1\|_F^2+\|\mathbf{X}-\mathbf{V}_2-\mathbf{D}_2\|_F^2+\|\mathbf{X}-\mathbf{V}_3-\mathbf{D}_3\|_F^2)\end{aligned}$$

where $\mu>0$ is a penalty parameter, $\mathbf{D}^T=[\mathbf{D}_1^T,\mathbf{D}_2^T,\mathbf{D}_3^T]^T$, and $\mathbf{V}^T=[\mathbf{V}_1^T,\mathbf{V}_2^T,\mathbf{V}_3^T]^T$. Then the framework of JSpBLRU-MSD is proposed as below

$$\begin{cases}\mathbf{X}^{k+\frac{1}{2}}=\arg\min_{\mathbf{X}}\mathcal{L}_4(\mathbf{X},\mathbf{A}^k,\mathbf{V}^k,\mathbf{D}^k)\\ [\mathbf{X}^{k+1},\mathbf{A}^{k+1},s^{k+1}]=\mathrm{MSD}(\mathbf{X}^{k+\frac{1}{2}},\mathbf{A}^k,s^k)\\ \mathbf{D}_i^{k+\frac{1}{2}}=\mathbf{D}_i^k(s^{k+1},:),i=1,2,3\\ \mathbf{V}^{k+1}=\arg\min_{\mathbf{V}}\mathcal{L}_4(\mathbf{X}^{k+1},\mathbf{A}^{k+1},\mathbf{V},\mathbf{D}^{k+\frac{1}{2}})\\ \mathbf{D}_i^{k+1}=\mathbf{D}_i^{k+\frac{1}{2}}-\mathbf{X}_i^{k+1}+\mathbf{V}_i^{k+1},i=1,2,3.\end{cases} \tag{39}$$

In the following, we consider each step in detail.

- For $\mathbf{X}^{k+\frac{1}{2}}$ subproblem, we have

$$\min_{\mathbf{X}} \frac{1}{2}\|\mathbf{A}^k\mathbf{X}-\mathbf{Y}\|_F^2+\frac{\mu}{2}\|\mathbf{X}-\mathbf{V}_1^k-\mathbf{D}_1^k\|_F^2+\frac{\mu}{2}\|\mathbf{X}-\mathbf{V}_2^k-\mathbf{D}_2^k\|_F^2+\frac{\mu}{2}\|\mathbf{X}-\mathbf{V}_3^k-\mathbf{D}_3^k\|_F^2$$

and it is easy to get

$$\mathbf{X}^{k+\frac{1}{2}}=((\mathbf{A}^k)^T\mathbf{A}^k+3\mu\mathbf{I})^{-1}((\mathbf{A}^k)^T\mathbf{Y}+\mu(\mathbf{V}_1^k+\mathbf{D}_1^k+\mathbf{V}_2^k+\mathbf{D}_2^k+\mathbf{V}_3^k+\mathbf{D}_3^k)).$$

- Then we update $\mathbf{X}^{k+1}$, $\mathbf{A}^{k+1}$, and s^{k+1} according to Algorithm 1. Moreover, we update $\mathbf{D}^{k+\frac{1}{2}}$ with s^{k+1}.
- For $\mathbf{V}^{k+1}$ subproblem, we decouple it into three subproblems.

∘ For $\mathbf{V}_1^{k+1}$ subproblem, we obtain

$$\min_{\mathbf{V}_1} \ \lambda \sum_{j=1}^{J} \|\mathbf{V}_{1,j}\|_{\mathbf{z}_j,2,1} + \frac{\mu}{2}\|\mathbf{X}^{k+1} - \mathbf{V}_1 - \mathbf{D}_1^{k+\frac{1}{2}}\|_F^2.$$

Then we obtain the ith row of the jth column block of $\mathbf{V}_1^{k+1}$ as

$$(\mathbf{V}_{1,j}^{k+1})^{[i]} = \textbf{vect-soft}((\mathbf{X}_j^{k+1} - \mathbf{D}_{1,j}^{k+\frac{1}{2}})^{[i]}, \frac{\lambda}{\mu} z_{ij})$$

where $\mathbf{X}_j^{k+1}$ and $\mathbf{D}_{1,j}^{k+\frac{1}{2}}$ are the jth column blocks of $\mathbf{X}^{k+1}$ and $\mathbf{D}_1^{k+\frac{1}{2}}$, respectively,

$$z_{ij} = \frac{1}{\|(\mathbf{X}_j^{k+1} - \mathbf{D}_{1,j}^{k+\frac{1}{2}})^{[i]}\|_2 + \varepsilon}$$

and ε is a small positive constant to avoid singularities.

∘ For $\mathbf{V}_2^{k+1}$ subproblem, we have

$$\min_{\mathbf{V}_2} \ \tau\|\mathbf{V}_2\|_{\mathbf{b},*} + \frac{\mu}{2}\|\mathbf{X}^{k+1} - \mathbf{V}_2 - \mathbf{D}_2^{k+\frac{1}{2}}\|_F^2$$

with its solution

$$\mathbf{V}_2^{k+1} = \mathbf{SVT}_{\frac{\tau}{\mu}\mathbf{b}}(\mathbf{X}^{k+1} - \mathbf{D}_2^{k+\frac{1}{2}}).$$

where $\mathbf{b} = [b_1, \cdots, b_r]$ with $b_i = \frac{1}{\sigma_i+\varepsilon}$ and σ_i is the ith singular value of $\mathbf{X}^{k+1} - \mathbf{D}_2^{k+\frac{1}{2}}$.

∘ For $\mathbf{V}_3^{k+1}$ subproblem, we obtain

$$\min_{\mathbf{V}_3} \ \iota_{\mathbb{R}+}(\mathbf{V}_3) + \frac{\mu}{2}\|\mathbf{X}^{k+1} - \mathbf{V}_3 - \mathbf{D}_3^{k+\frac{1}{2}}\|_F^2$$

which follows that

$$\mathbf{V}_3^{k+1} = \max(\mathbf{X}^{k+1} - \mathbf{D}_3^{k+\frac{1}{2}}, \mathbf{0}).$$

• Finally, we update Lagrange multipliers according to (39) and summarize the procedures in Algorithm 6.

Algorithm 6: Pseudocode of JSpBLRU-MSD

Input: $\mathbf{\Phi} \in \mathbb{R}^{L\times m}$ and $\mathbf{Y} \in \mathbb{R}^{L\times N}$

Initialize: $\hat{\mathbf{W}} = \mathbf{0} \in \mathbb{R}^{m\times N}$, $\mathbf{A}^0 = \mathbf{\Phi}$, $\mathbf{V}^0$, $\mathbf{D}^0$, $s^0 = \{1, \cdots, m\}$, and set $k = 0$

Select parameters: λ, τ, μ, and ϵ

Repeat

  Compute $\mathbf{X}^{k+\frac{1}{2}} = ((\mathbf{A}^k)^T\mathbf{A}^k + 3\mu\mathbf{I})^{-1}((\mathbf{A}^k)^T\mathbf{Y} + \mu(\mathbf{V}_1^k + \mathbf{D}_1^k + \mathbf{V}_2^k + \mathbf{D}_2^k + \mathbf{V}_3^k + \mathbf{D}_3^k))$

  Compute $\mathbf{X}^{k+1}$, $\mathbf{A}^{k+1}$, and s^{k+1} by Algorithm 1

  Compute $\mathbf{D}_i^{k+\frac{1}{2}} = \mathbf{D}_i^k(s^{k+1}, :)$, $i = 1, 2, 3$

  Compute $(\mathbf{V}_{1,j}^{k+1})^{[i]} = \textbf{vect-soft}((\mathbf{X}_j^{k+1} - \mathbf{D}_{1,j}^{k+\frac{1}{2}})^{[i]}, \frac{\lambda}{\mu}z_{ij})$

  $\mathbf{V}_2^{k+1} = \mathbf{SVT}_{\frac{\tau}{\mu}\mathbf{b}}(\mathbf{X}^{k+1} - \mathbf{D}_2^{k+\frac{1}{2}})$

  $\mathbf{V}_3^{k+1} = \max(\mathbf{X}^{k+1} - \mathbf{D}_3^{k+\frac{1}{2}}, \mathbf{0})$

  Compute $\mathbf{D}_i^{k+1} = \mathbf{D}_i^{k+\frac{1}{2}} - \mathbf{X}^{k+1} + \mathbf{V}_i^{k+1}$, $i = 1, 2, 3$

until convergence

Update $\hat{\mathbf{W}}(s^{k+1}, :) = \mathbf{X}^{k+1}$

Output: $\hat{\mathbf{W}}$