

《数值分析》8

- 三对角矩阵的 LU 分解
- 对称正定矩阵三角分解
- *MATLAB*矩阵分解命令

常微分方程两点边值问题

$$\text{P1: } \begin{cases} y'' + py' + qy = f(x), & x \in (0,1) \\ y(0) = \alpha, y(1) = \beta. \end{cases}$$

$$\text{P2: } \begin{cases} [k(x)y']' + q(x)y = f(x), & x \in (0,1) \\ y(0) = \alpha, y(1) = \beta. \end{cases}$$

$$\text{P3: } \begin{cases} X'' + \lambda X = 0, & x \in (0,L) \\ X(0) = 0, X(L) = 0. \end{cases}$$

二阶常微分方程:
$$\begin{cases} y'' + y + x = 0, & x \in (0,1) \\ y(0) = 0, y(1) = 0. \end{cases}$$

令 $h = 1/(n+1)$, $x_j = jh$, $y_j = y(x_j)$ ($j = 0, 1, \dots, n+1$)

$$\frac{y_{j+1} - 2y_j + y_{j-1}}{h^2} + y_j + x_j = 0 \quad (j = 1, 2, \dots, n)$$

三对角方程组
$$-y_{j-1} + (2-h^2)y_j - y_{j+1} = x_j h^2$$

$$\begin{bmatrix} 2-h^2 & -1 & & & \\ -1 & 2-h^2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2-h^2 & -1 \\ & & & -1 & 2-h^2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} = h^2 \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

一般形式

$$\begin{bmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & a_n & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-1} \\ f_n \end{bmatrix}$$

三角分解:

$$A = LU$$

$$\begin{bmatrix} \blacksquare & & & & \\ \blacksquare & \blacksquare & & & \\ & \blacksquare & \blacksquare & & \\ & & \blacksquare & \blacksquare & \\ & & & \blacksquare & \blacksquare \\ & & & & \blacksquare & \blacksquare \end{bmatrix} = \begin{bmatrix} \blacksquare & & & & \\ \blacksquare & \blacksquare & & & \\ & \blacksquare & \blacksquare & & \\ & & \blacksquare & \blacksquare & \\ & & & \blacksquare & \blacksquare \\ & & & & \blacksquare & \blacksquare \end{bmatrix} \begin{bmatrix} \blacksquare & & & & \\ & \blacksquare & & & \\ & & \blacksquare & & \\ & & & \blacksquare & \\ & & & & \blacksquare & \blacksquare \end{bmatrix}$$

三对角矩阵

单位下三角阵

上三角阵

$$AX=F \rightarrow \underline{LU}X=F$$

$$\textcircled{1} LY=F$$

$$\textcircled{2} UX=Y$$

LU分解

$$\begin{aligned}
 & \begin{bmatrix} 2 & -1 & & \\ -1 & 3 & -2 & \\ & -2 & 4 & -2 \\ & & -3 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 & & \\ -1/2 & 5/2 & -2 & \\ & -2 & 4 & -2 \\ & & -3 & 5 \end{bmatrix} \\
 & \begin{bmatrix} 2 & -1 & & \\ -1/2 & 5/2 & -2 & \\ & -4/5 & 12/5 & -2 \\ & & -3 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 & & \\ -1/2 & 5/2 & -2 & \\ & -4/5 & 12/5 & -2 \\ & & -5/4 & 5/2 \end{bmatrix} \\
 L = & \begin{bmatrix} 1 & & & \\ -1/2 & 1 & & \\ & -4/5 & 1 & \\ & & -5/4 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 2 & -1 & & \\ & 5/2 & -2 & \\ & & 12/5 & -2 \\ & & & 5/2 \end{bmatrix}
 \end{aligned}$$

三对角矩阵的三角分解 $A = L U$

$$A = \begin{bmatrix} b_1 & c_1 & & \\ a_2 & b_2 & \ddots & \\ & \ddots & \ddots & \\ & & a_n & b_n \end{bmatrix} \Rightarrow \begin{bmatrix} \beta_1 & c_1 & & \\ \alpha_2 & \beta_2 & c_2 & \\ & \ddots & \ddots & \\ & & \alpha_n & \beta_n \end{bmatrix}$$

算法I:
$$\begin{cases} \beta_1 = b_1 \\ \alpha_k = a_k / \beta_{k-1}, \beta_k = b_k - \alpha_k c_{k-1} \end{cases}$$

$(k = 2, 3, \dots, n)$

格式相对简单
因为三对角

下三角方程组 $LY = f$

上三角方程组 $UX = Y$

$$\begin{bmatrix} 1 & & & \\ \alpha_2 & 1 & & \\ & \ddots & \ddots & \\ & & \alpha_n & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

$$\begin{bmatrix} \beta_1 & c_1 & & \\ & \beta_2 & \ddots & \\ & & \ddots & c_{n-1} \\ & & & \beta_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

算法II:
$$\begin{cases} y_1 = f_1 \\ y_k = (f_k - \alpha_k y_{k-1}), \quad (k = 2, \dots, n) \end{cases}$$

算法III:
$$\begin{cases} x_n = y_n / \beta_n \\ x_k = (y_k - c_k x_{k+1}) / \beta_k, \quad (k = n-1, \dots, 1) \end{cases}$$

```
function f=triGauss(a,b,c,f)
```

```
%TriDiag(a,b,c)systems
```

```
n=length(b);
```

```
for k=1:n-1
```

```
    d=a(k)/b(k);
```

```
    b(k+1)=b(k+1)-d*c(k);
```

```
    f(k+1)=f(k+1)-d*f(k);
```

```
end
```

```
f(n)=f(n)/b(n);
```

```
for k=n-1:-1:1
```

```
    f(k)=(f(k)-c(k)*f(k+1))/b(k);
```

```
end
```

$$\begin{bmatrix} b_1 & c_1 & & \\ a_2 & b_2 & c_2 & \\ & \ddots & \ddots & \ddots \\ & & a_n & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

对称正定矩阵的Cholesky分解

$A^T = A$, A 的各阶顺序主子式大于零.

A 的 LU 分解

$$A = \begin{bmatrix} 1 & & & \\ m_{21} & 1 & & \\ \vdots & \ddots & \ddots & \\ m_{n1} & \cdots & m_{n,n-1} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ & u_{22} & \ddots & \vdots \\ & & \ddots & u_{n-1,n} \\ & & & u_{nn} \end{bmatrix}$$

$u_{ii} > 0$ ($i = 1, \cdots, n$), 有如下关系

$$\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ & u_{22} & \cdots & u_{2n} \\ & & \ddots & \vdots \\ & & & u_{nn} \end{bmatrix} = \begin{bmatrix} u_{11} & & & \\ & u_{22} & & \\ & & \ddots & \\ & & & u_{nn} \end{bmatrix} \begin{bmatrix} 1 & r_{12} & \cdots & r_{1n} \\ & 1 & \ddots & \vdots \\ & & \ddots & r_{n-1,n} \\ & & & 1 \end{bmatrix}$$

$$A = LU \rightarrow A = LDR \rightarrow A^T = R^T D L^T$$

$$A^T = A \rightarrow R^T D L^T = LDR \rightarrow R^T = L \rightarrow R = L^T$$

(1) 对称矩阵的 LDL^T 分解: $A = LDL^T$

(2) 对称矩阵的 LL^T 分解

由 $u_{kk} > 0$, 记 $D^{\frac{1}{2}} = \text{diag}(\sqrt{u_{11}}, \dots, \sqrt{u_{nn}})$ 得

$$A = \tilde{L} \tilde{L}^T \quad \text{其中} \quad \tilde{L} = L D^{\frac{1}{2}}$$

在不引起符号混乱时仍记为: $A = LL^T$

例 设 $A \in R^{n \times n}$ 为对称正定矩阵, $x, y \in R^n$. 定义

$$(x, y)_A = x^T A y$$

证明: $\|x\|_A = \sqrt{x^T A x}$ 是 R^n 上的向量范数.

证: 由于 A 对称正定, 故存在非奇异上三角矩阵 R (类似非奇异下三角矩阵 L 的形式), 使

$$A = R^T R$$

所以, $x^T A x = x^T (R^T R) x = (Rx)^T (Rx)$

$$\|x\|_A = \sqrt{x^T A x} = \sqrt{(Rx)^T (Rx)} = \|Rx\|_2$$

故, $\|x\|_A$ 是 R^n 上的向量范数

MATLAB中矩阵分解命令

(1) 特征值分解命令: $[P \ D]=\text{eig}(A)$

特征值矩阵 $D = \text{diag}[\lambda_1, \lambda_2, \dots, \lambda_n]$

特征向量矩阵 $P = [\alpha_1, \alpha_2, \dots, \alpha_n]$

(2) LU 分解命令: $[L \ U \ P]=\text{lu}(A)$

L 下三角阵, U 上三角阵, P 初等矩阵 $PA = LU$

(3) Cholesky分解命令: $R=\text{chol}(A)$

R 是上三角矩阵 $R^T R = A$

(4)QR分解 $[Q \ R]=qr(A)$

Q正交矩阵,R上三角矩阵 $QR=A$

(5)A的奇异值分解: $[U,S,V] = SVD(A)$

概念:设 $A \in C^{n \times m}$, $A^H A$ 的特征值为 $\lambda_1, \lambda_2, \dots, \lambda_n$

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$$

令 $\lambda_i = \sigma_i^2$, 称实数 $\sigma_1, \sigma_2, \dots, \sigma_n$ 为 A 的奇异值