Econometrics 1 *Applied Econometrics with R*

Supplement: Review of Probability

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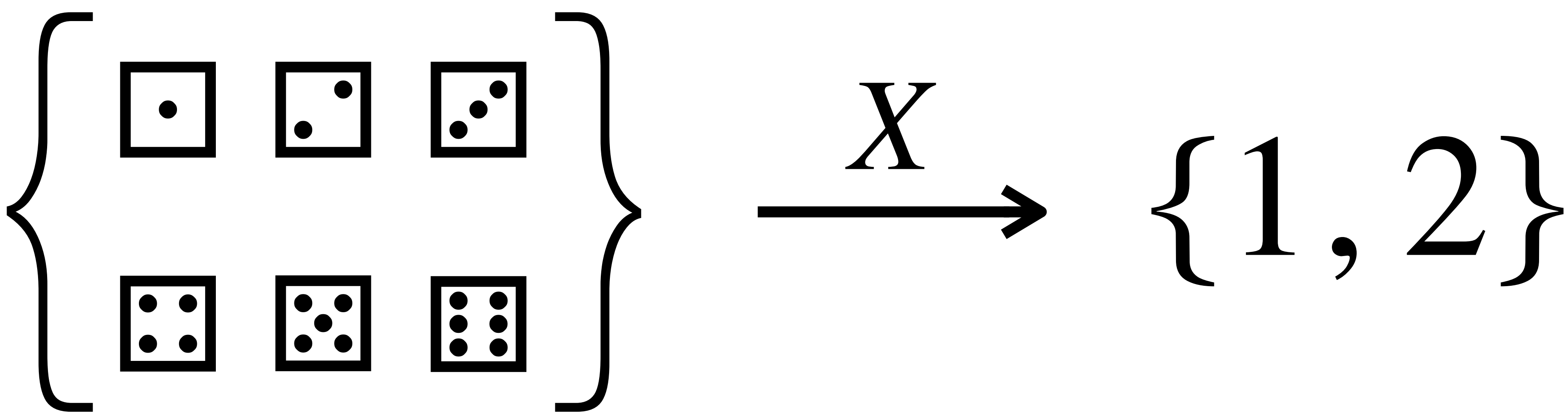
Review of Probability

Some basic definitions

- *Randomness*: something you cannot control.
- *Outcome*: potential results of a random experiment or process.
- *Probability*: the frequency, or the proportion of time, that an outcome occurs in the long run.
- *Sample space*: the set of all possible outcomes.
- *Event*: a subset of the sample space.

Random variable

- A random variable (r.v.) is a *mapping* from the sample space to a set of values.
- If the values of the r.v. is a discrete/continuous set, the r.v. is called a discrete/continuous random variable.



$$X(\text{⚀}) = X(\text{⚁}) = X(\text{⚂}) = 1, \quad X(\text{⚃}) = X(\text{⚄}) = X(\text{⚅}) = 2$$

Probability and random variable

- Probabilities are defined for events (on sample space).

$$\Pr(\text{⚀}) = 1/6, \qquad \Pr(\{\text{⚃}, \text{⚄}, \text{⚅}\}) = 1/2$$

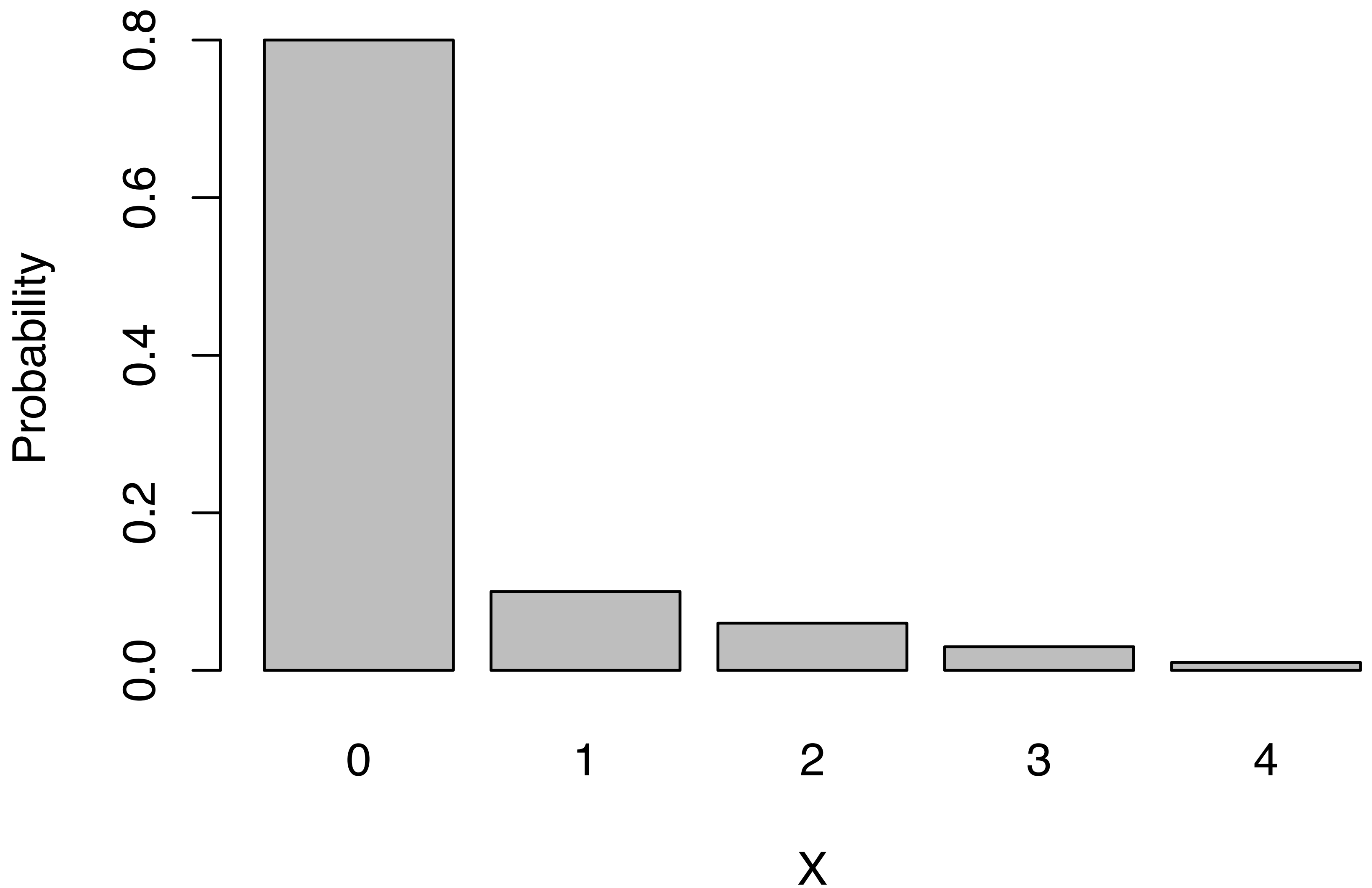
- The probabilities of a random variable

$$\Pr(X = 1) = \Pr(\{\text{⚀}, \text{⚁}, \text{⚂}\}) = 1/2$$
$$\Pr(X = 2) = \Pr(\{\text{⚃}, \text{⚄}, \text{⚅}\}) = 1/2$$

- Usually we just write $\Pr(X = 1) = 1/2$.

Probability distribution of a discrete r.v.

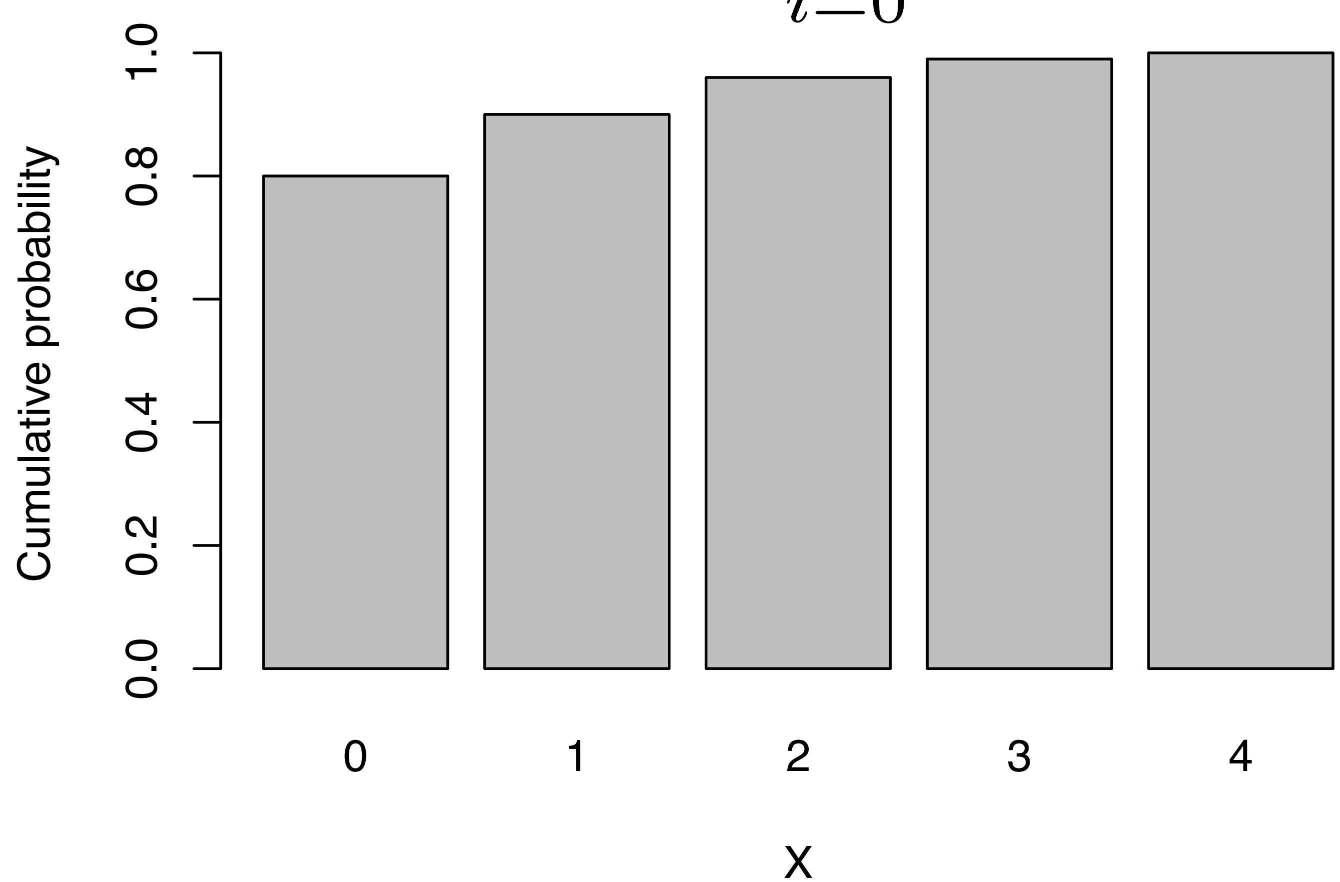
- The probability distribution of a discrete random variable is a list of all possible values of the variable and the probability that each value will occur.



Cumulative distribution function

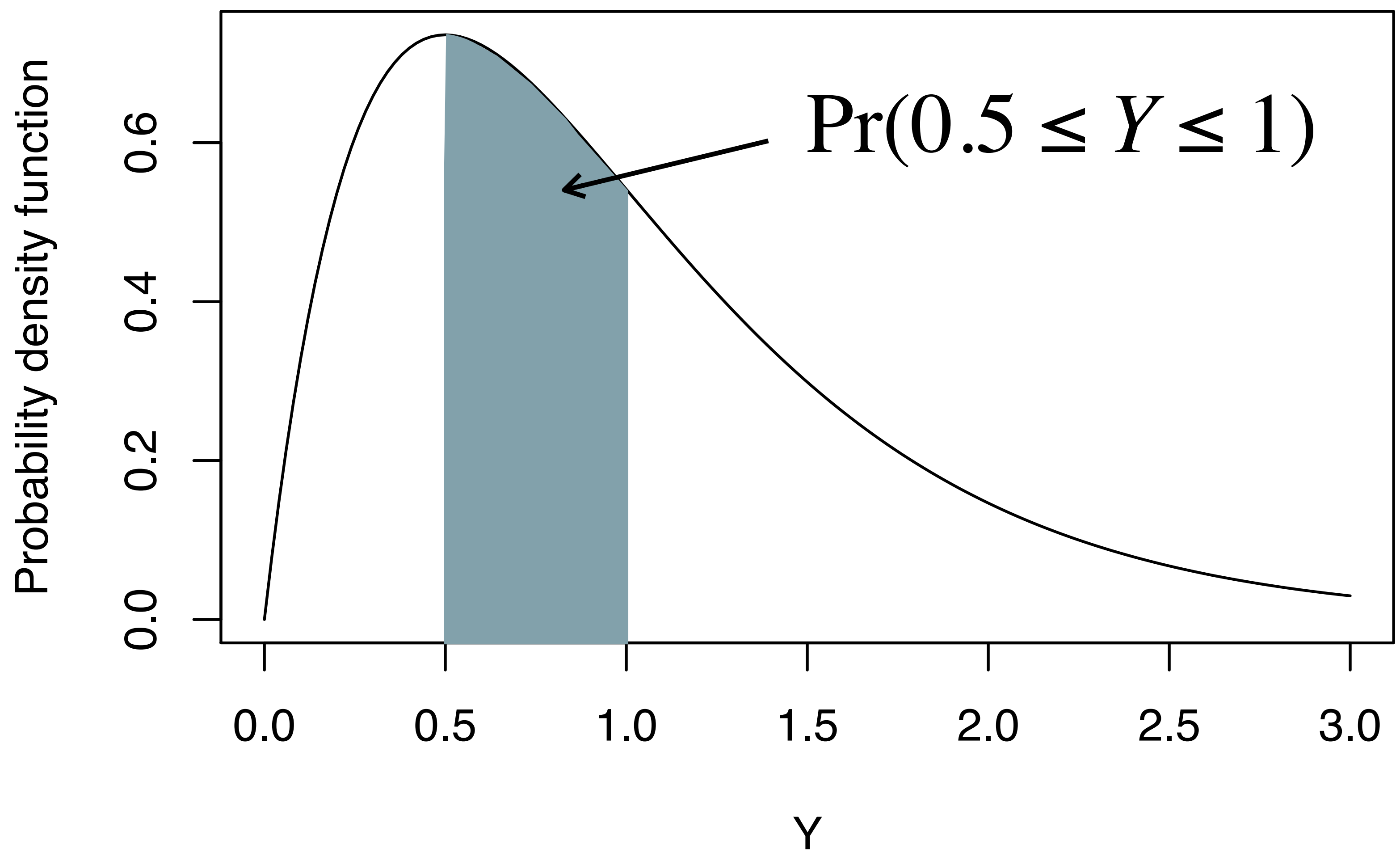
- The cumulative distribution function (c.d.f) is a function describing the probability that the random variable is *less than or equal to* a particular value.

$$\Pr(X \leq n) = \sum_{i=0}^{n} \Pr(X = i)$$



Probability distribution of a continuous r.v.

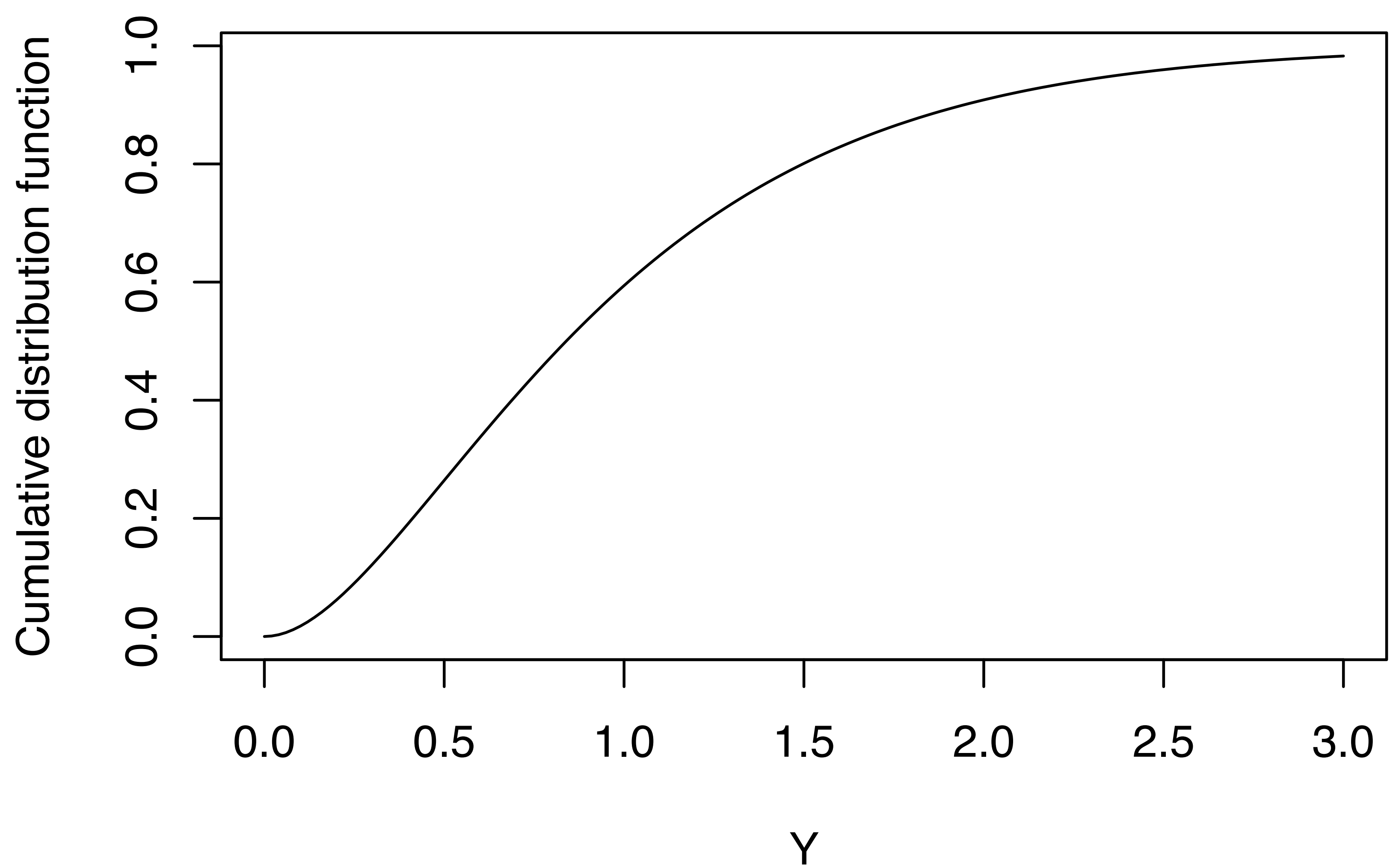
- The probability distribution of a continuous random variable is described by a *probability density function* (p.d.f.), where the area under it between any two points is the probability that the random variable falls between those two points.



The c.d.f. of a continuous random variable

- Given the p.d.f. $f(y)$ of random variable Y, the c.d.f. $F(y)$ of Y is defined by

$$F(y) := \Pr(Y \leq y) = \int_0^y f(u)du$$



- Expected value (or the mean) μ_X

$$\mathrm{E}(X) = \sum x_i \Pr(X = x_i)$$

$$\mathrm{E}(Y) = \int y\, f_Y(y)\, dy$$

- Variance σ_X^2

$$\mathrm{var}(X) = \mathrm{E}[(X - \mu_X)^2] = \sum (x_i - \mu_X)^2 \Pr(X = x_i)$$

$$\mathrm{var}(Y) = \mathrm{E}[(Y - \mu_Y)^2] = \int (y - \mu_Y)^2 f_Y(y)\, dy$$

- Standard deviation

$$\sigma_X = \sqrt{\mathrm{var}(X)}$$

Two discrete random variables

- Joint probability $\Pr(X = x, Y = y)$
- Conditional probability

$$\Pr(Y = y \mid X = x) = \frac{\Pr(X = x, Y = y)}{\Pr(X = x)}$$

- X and Y are independent if $\Pr(Y = y \mid X = x) = \Pr(Y = y)$
- X and Y are independent if and only if

$$\Pr(X = x, Y = y) = \Pr(X = x)\Pr(Y = y)$$

Two discrete random variables

- Marginal distribution

$$\Pr(Y = y) = \sum_{i=1}^{n} \Pr(X = x_i, Y = y)$$

		X				Marginal probability of Y
		1	2	3	4	
Y	1	0.04	0.04	0.08	0.04	0.2
	2	0.01	0.03	0.2	0.06	0.3
	3	0.01	0.02	0.1	0.17	0.3
	4	0.04	0.01	0.12	0.03	0.2
Marginal probability of X		**0.1**	**0.1**	**0.5**	**0.3**	

Two discrete random variables

- Covariance σ_{XY}

$$\begin{aligned}\operatorname{cov}(X,Y) &= \mathrm{E}[(X-\mu_X)(Y-\mu_Y)] \\ &= \sum_i \sum_j (x_i-\mu_X)(y_j-\mu_Y)\Pr(X=x_i, Y=y_j)\end{aligned}$$

- Correlation

$$\operatorname{corr}(X,Y) = \frac{\operatorname{cov}(X,Y)}{\sqrt{\operatorname{var}(X)\operatorname{var}(Y)}} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

Correlation and dependence

- X and Y are said to be uncorrelated if $\mathrm{corr}(X, Y) = 0$.

- X and Y are independent $\Rightarrow$ X and Y are uncorrelated

$$
\begin{aligned}
\mathrm{cov}(X, Y) &= \sum_i \sum_j (x_i - \mu_X)(y_j - \mu_Y)\mathrm{Pr}(X = x_i, Y = y_j) \\
&= \sum_i \sum_j (x_i - \mu_X)(y_j - \mu_Y)\mathrm{Pr}(X = x_i)\mathrm{Pr}(Y = y_j) \\
&= \sum_i (x_i - \mu_X)\mathrm{Pr}(X = x_i) \sum_j (y_j - \mu_Y)\mathrm{Pr}(Y = y_j) \\
&= (\mathrm{E}(X) - \mu_X)(\mathrm{E}(Y) - \mu_Y) = 0
\end{aligned}
$$

Correlation and dependence

- X and Y are said to be uncorrelated if $\operatorname{corr}(X, Y) = 0$.
- X and Y are uncorrelated $\not\Rightarrow$ X and Y are independent
- Let X and Z be independent random variables such that

X			
Value	**-1**	**0**	**1**
Prob	**0**	**1/2**	**1/2**

Z			
Value	**-1**	**0**	**1**
Prob	**1/2**	**0**	**1/2**

and let $Y = XZ$. Verify that X and Y are uncorrelated and dependent.

Important Distributions

The normal distribution

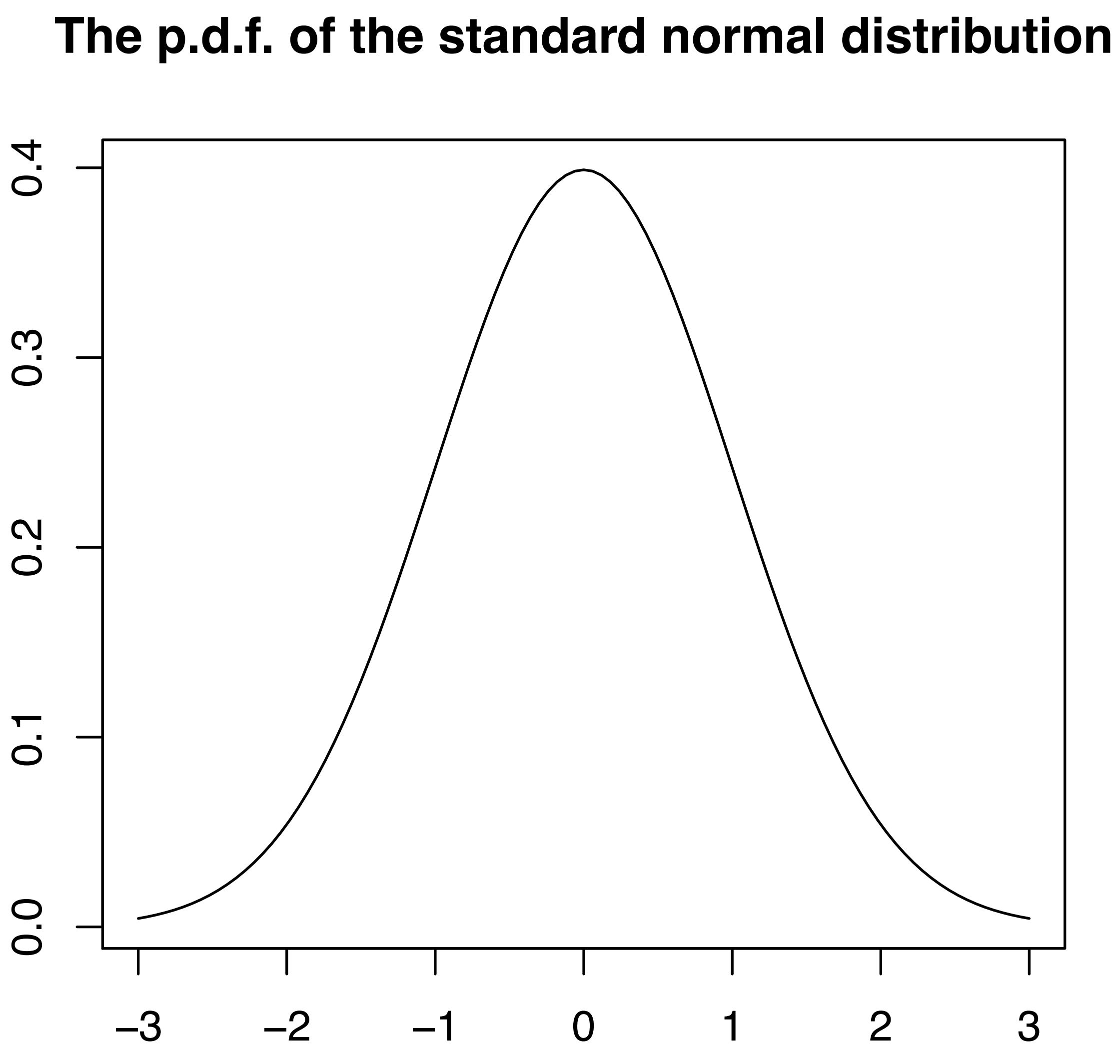
- The p.d.f. of a normal distribution with mean μ and variance σ^2, i.e. $N(\mu, \sigma^2)$

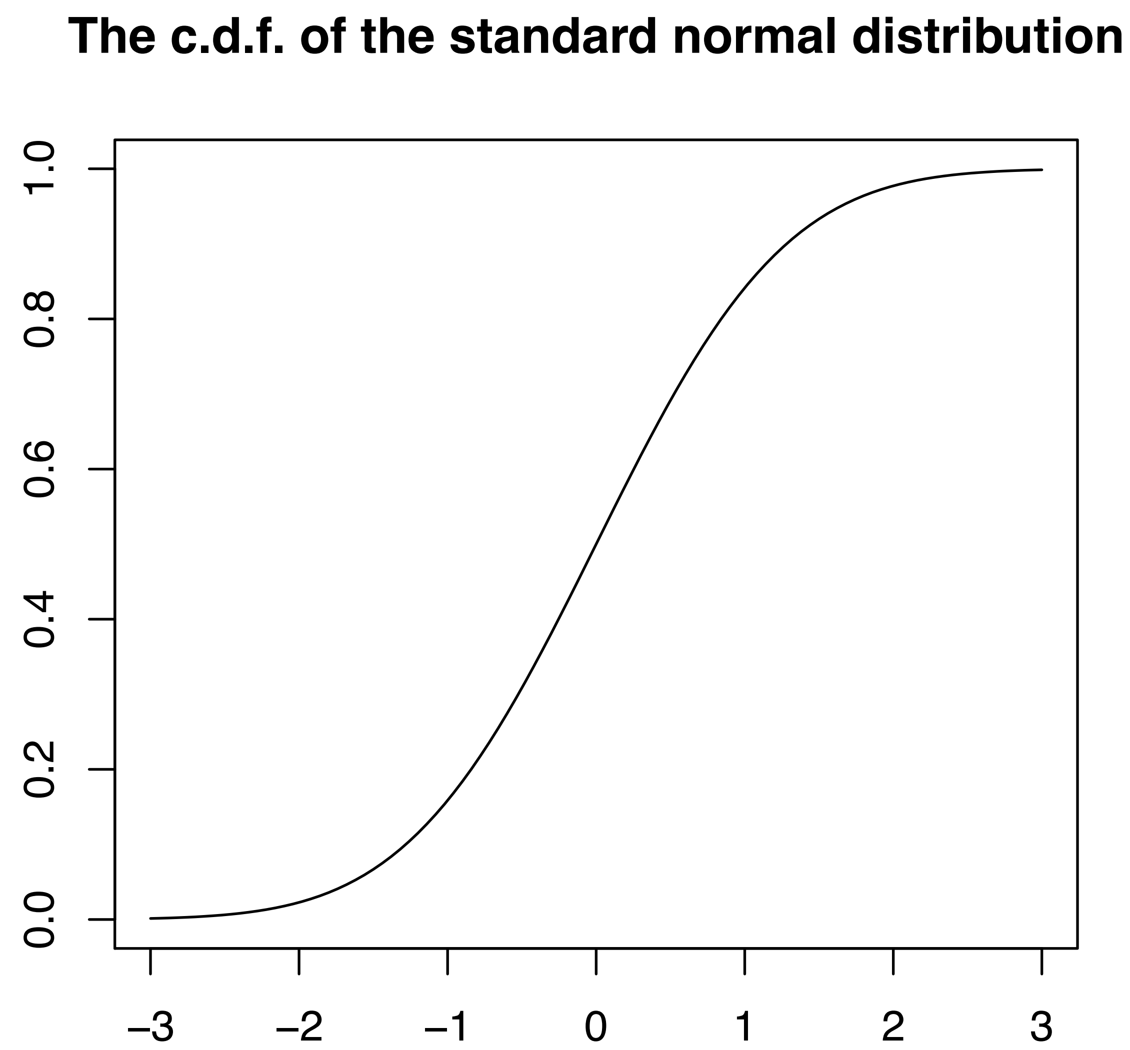
$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\Big[-\frac{1}{2}\Big(\frac{x-\mu}{\sigma}\Big)^2\Big]$$

- The standard normal distribution is $N(0, 1)$. A standard normal random variable is usually denoted as Z, whose c.d.f is denoted by

$$\Pr(Z \leq z) = \Phi(z)$$

The normal distribution





The normal distribution and normal r.v.

- Some special probabilities for $X \sim N(\mu, \sigma^2)$

$$\Pr(\mu - \sigma \leq X \leq \mu + \sigma) \approx 0.683$$
$$\Pr(\mu - 2\sigma \leq X \leq \mu + 2\sigma) \approx 0.954$$
$$\Pr(\mu - 3\sigma \leq X \leq \mu + 3\sigma) \approx 0.997$$

$$\Pr(\mu - 1.96\sigma \leq X \leq \mu + 1.96\sigma) \approx 0.95$$

- Standardization of normal random variables

$$Z = (X - \mu)/\sigma$$

The chi-squared distribution

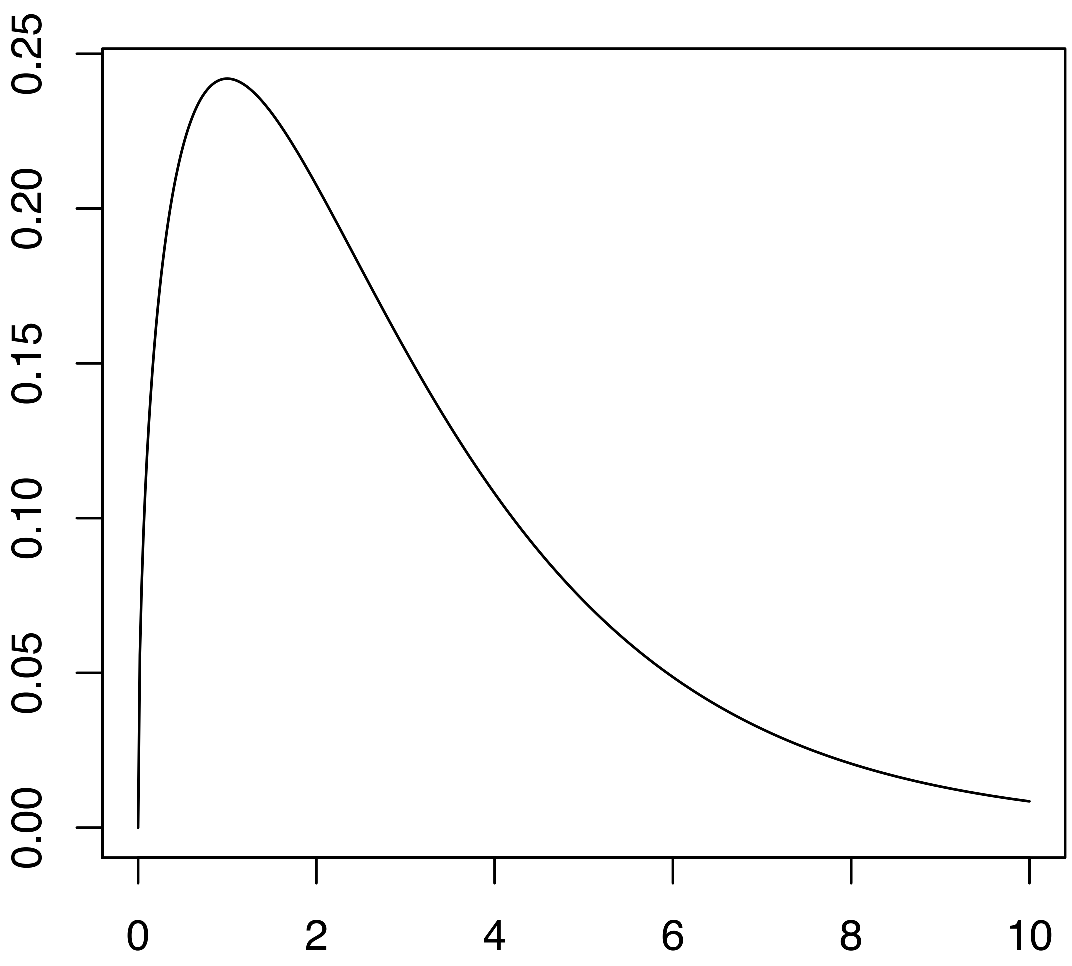
- The chi-squared distribution with m degrees of freedom, denoted by χ_m^2, is the distribution of a sum of the squares of m independent standard normal random variables.

- Let Z_1, Z_2, Z_3 be independent standard normal random variables. The c.d.f. of the chi-squared distribution with degree of freedom 3 is then

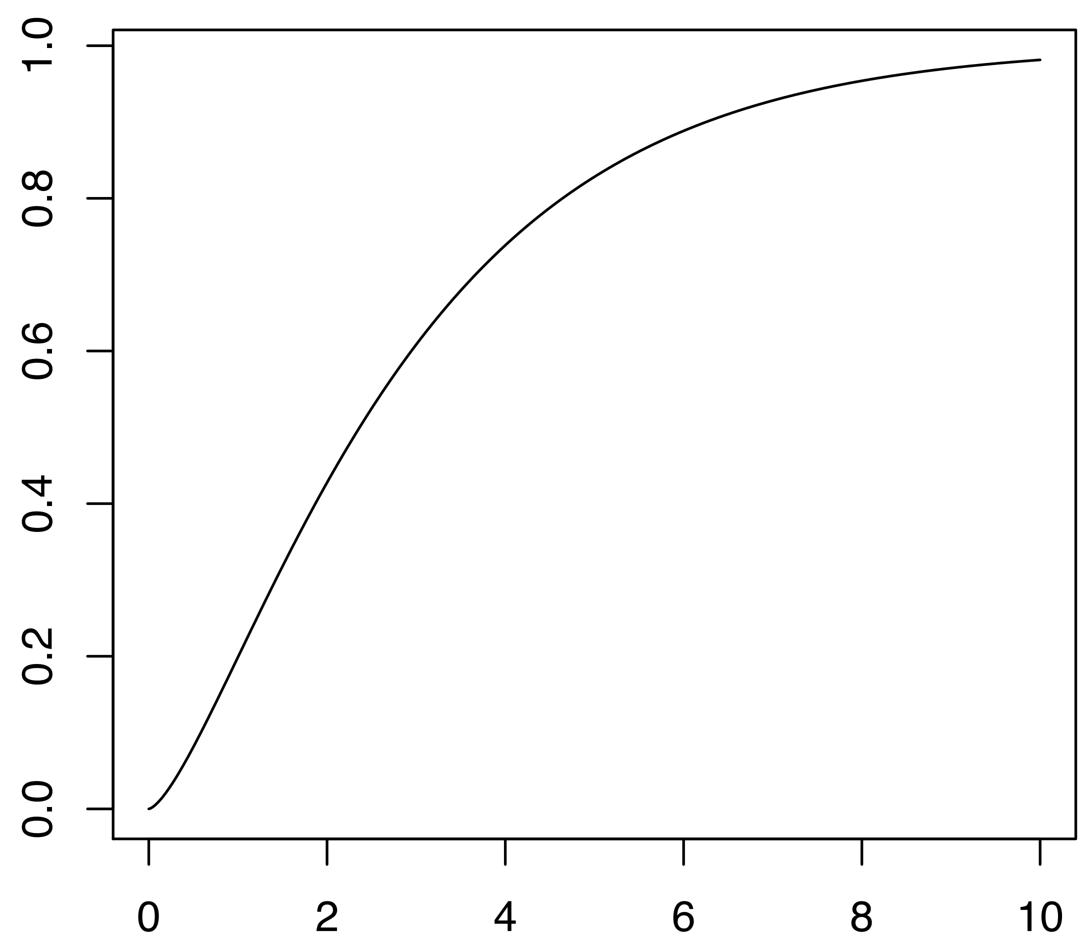
$$F_{\chi_3^2}(z) = \Pr(Z_1^2 + Z_2^2 + Z_3^2 \leq z)$$

The chi-squared distribution

The p.d.f. of Chi–squared distribution with d.f. = 3



The c.d.f. of Chi–squared distribution with d.f. = 3



The Student *t* distribution

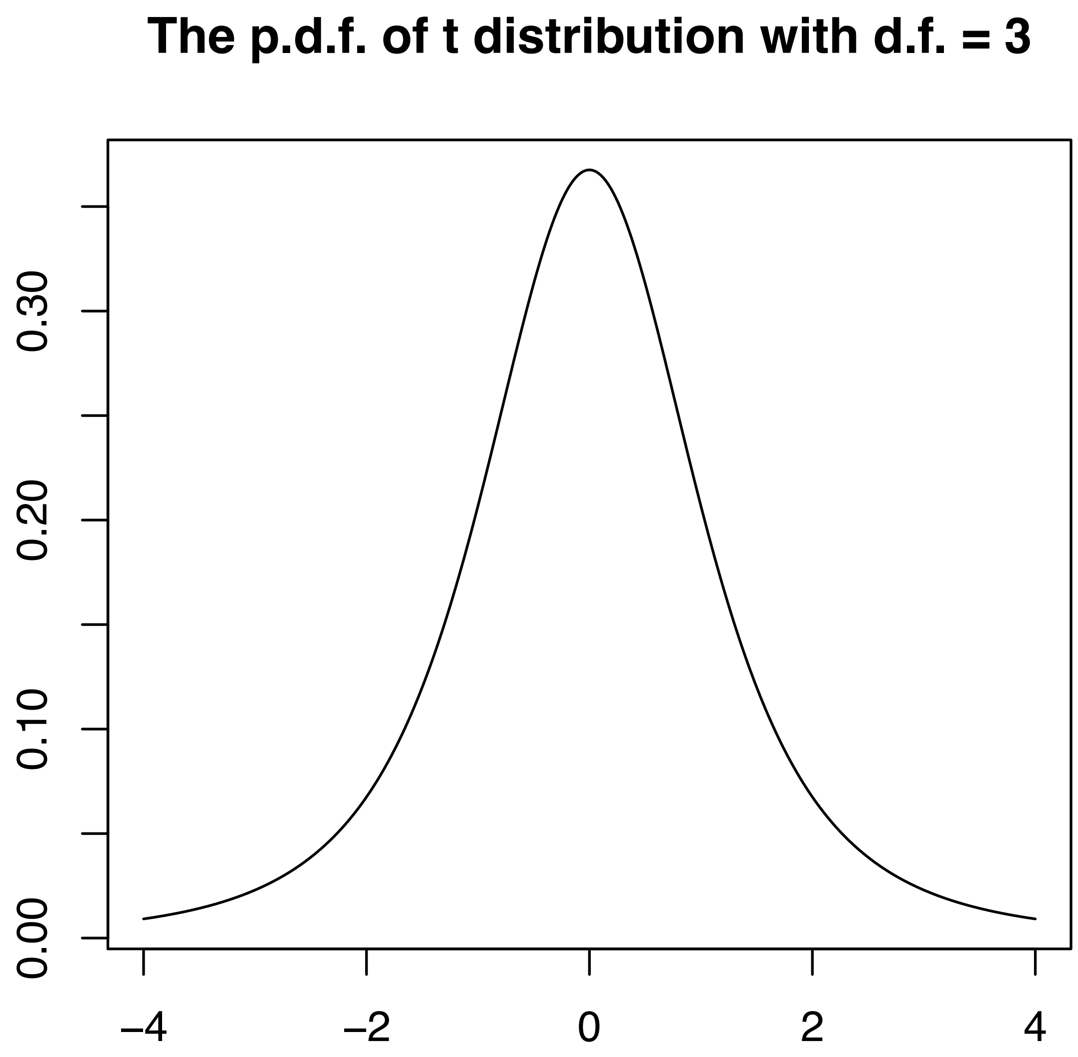
- The Student *t* distribution with m degrees of freedom, denoted by t_m, is defined to be the distribution of the ratio of a standard normal r.v., divided by the squared root of an independently distributed chi-squared r.v. with m degrees of freedom divided by m.

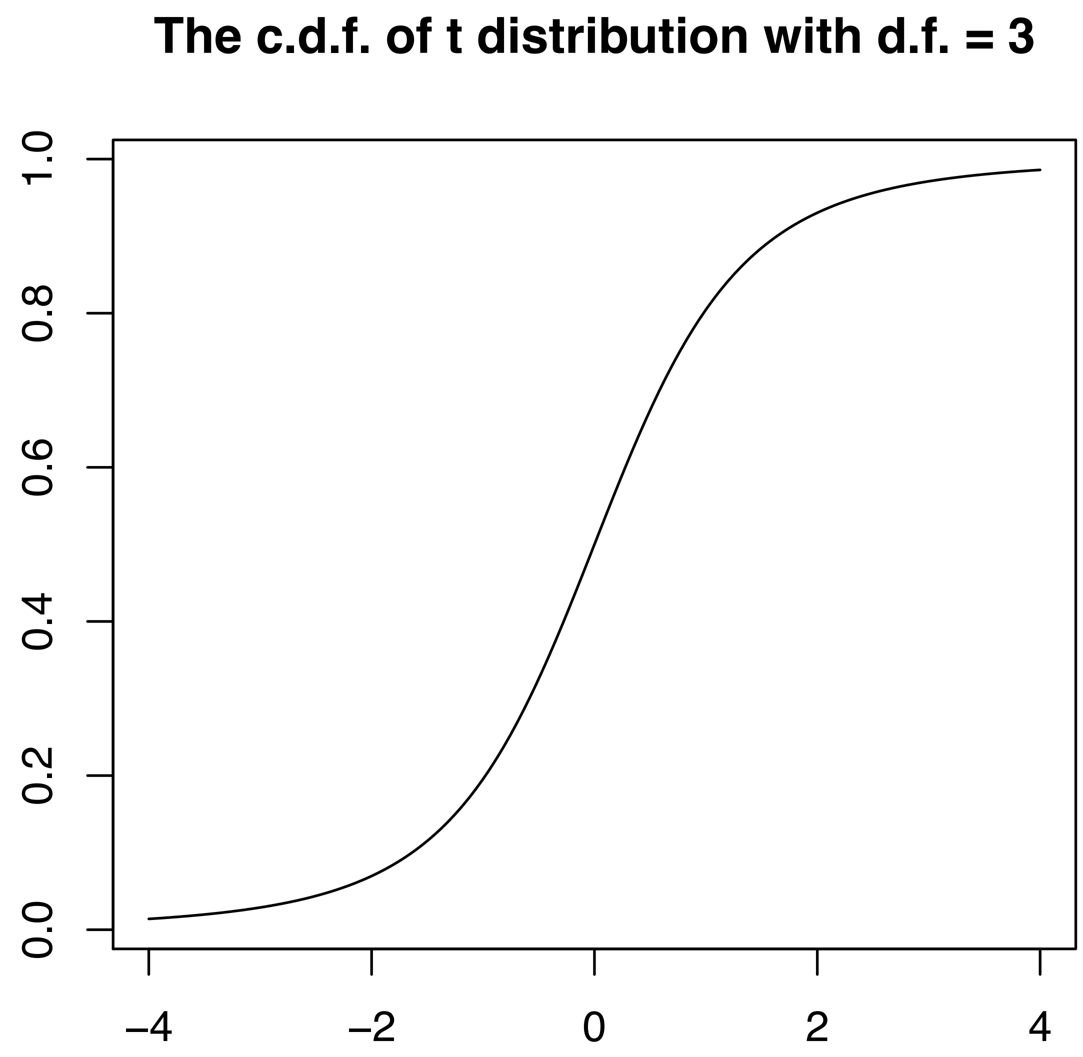
- Let Z be a standard normal r.v. and W be a r.v. with a chi-squared distribution with d.f. = m, then the r.v.

$$Z \Big/ \sqrt{W/m}$$

has a Student *t* distribution with d.f. = m.

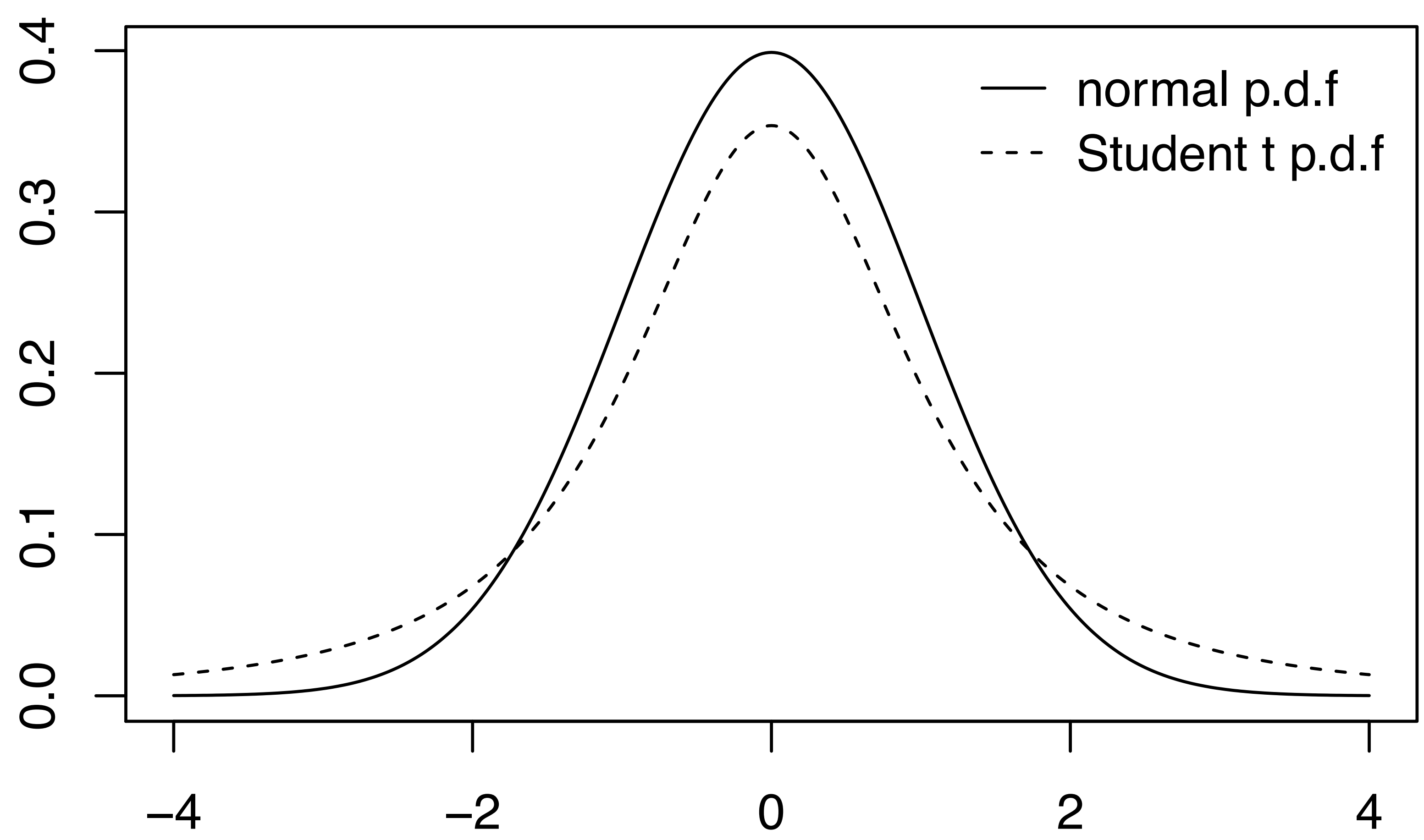
The Student *t* distribution





The *t* distribution v.s. normal distribution

- When the degree of freedom of a *t* distribution is small ($m < 20$), the *t* distribution has a "fatter tail" then a standard normal distribution.



The *F* distribution

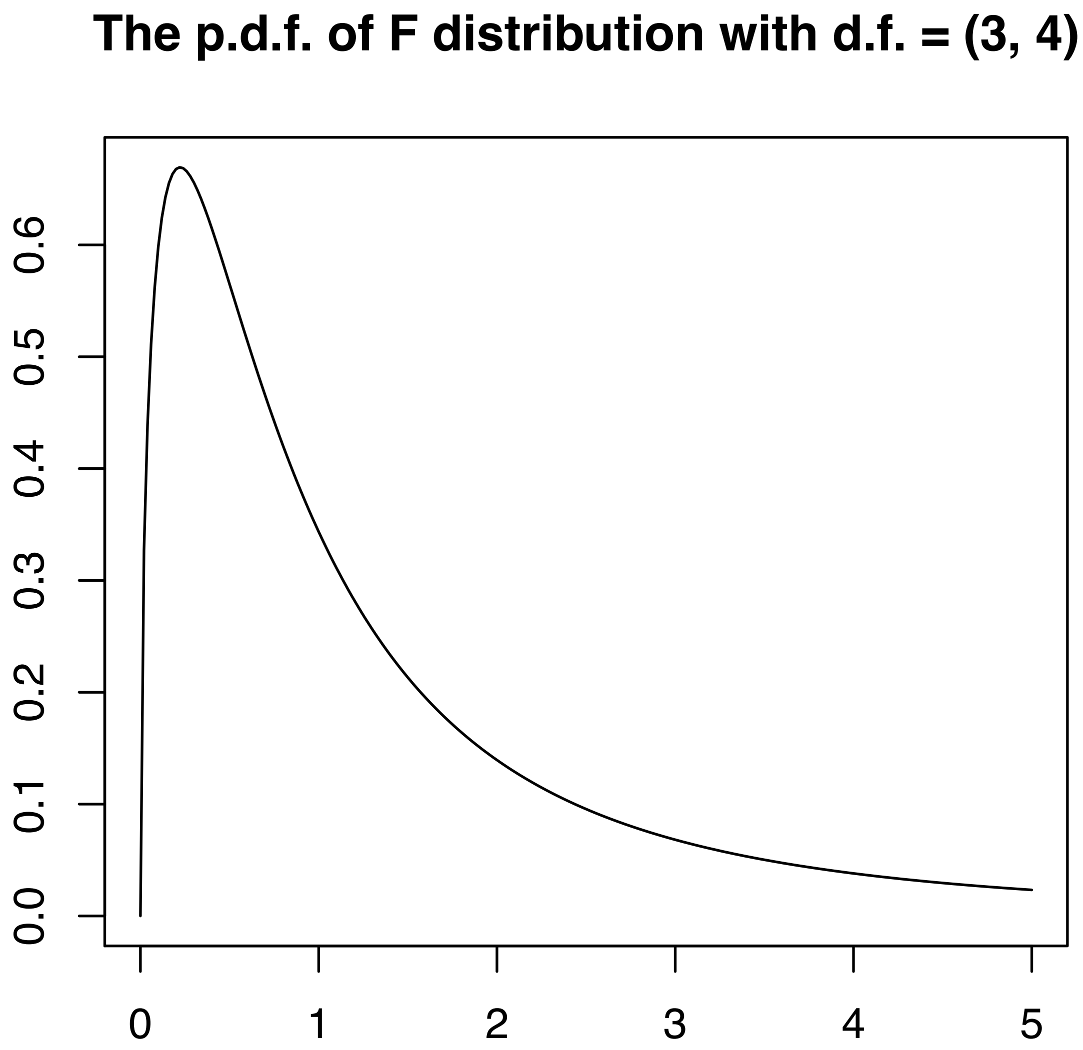
- The *F* distribution with m and n degrees of freedom, denoted by $F_{m,n}$, is defined to be the distribution of the ratio of a chi-squared r.v. with d.f. = m divided by m, to an independently distributed chi-squared r.v. with d.f. = n divided by n.

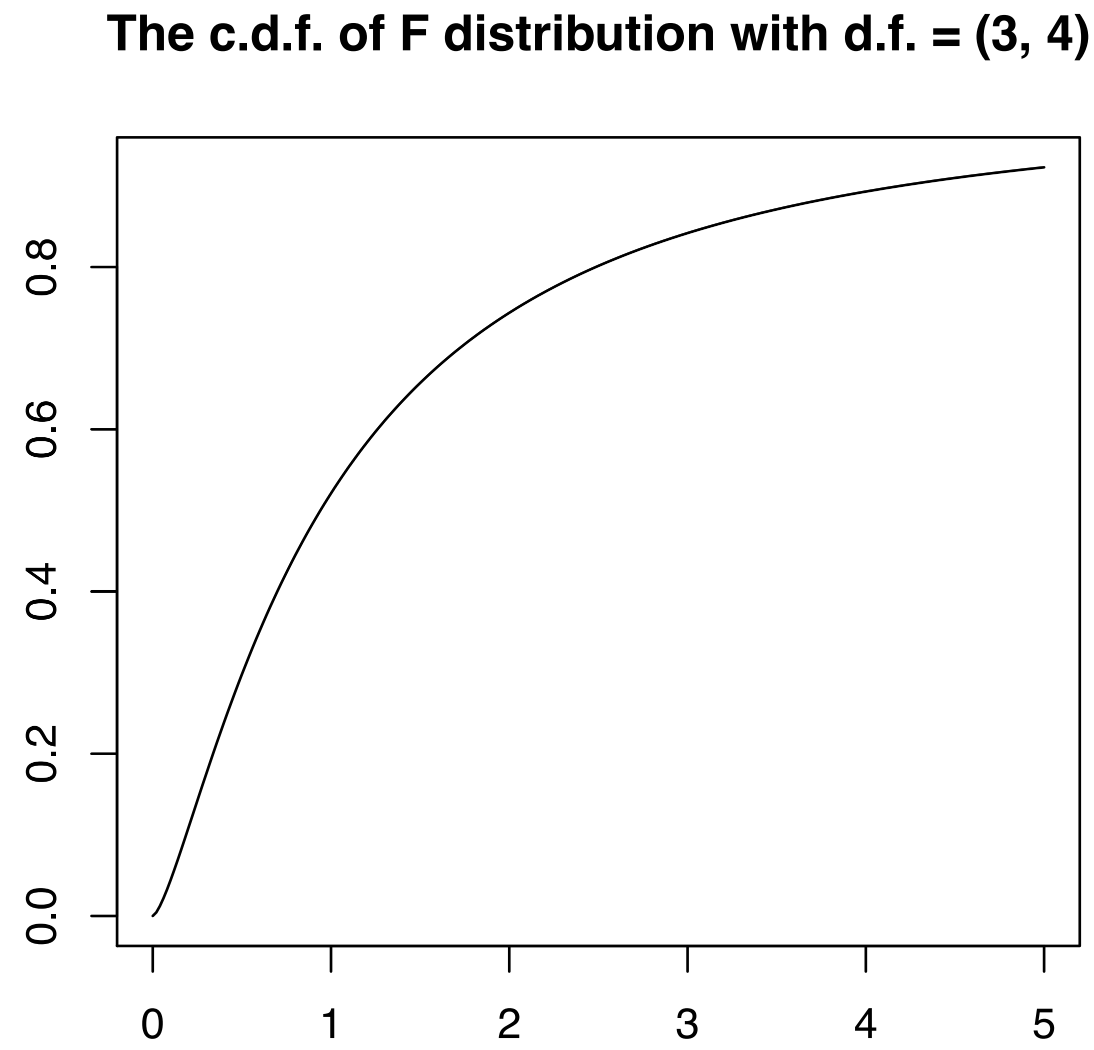
- Let $W \sim \chi^2_m$ and $V \sim \chi^2_n$, then the random variable

$$\frac{W/m}{V/n}$$

has a distribution of $F_{m,n}$.

The *F* distribution



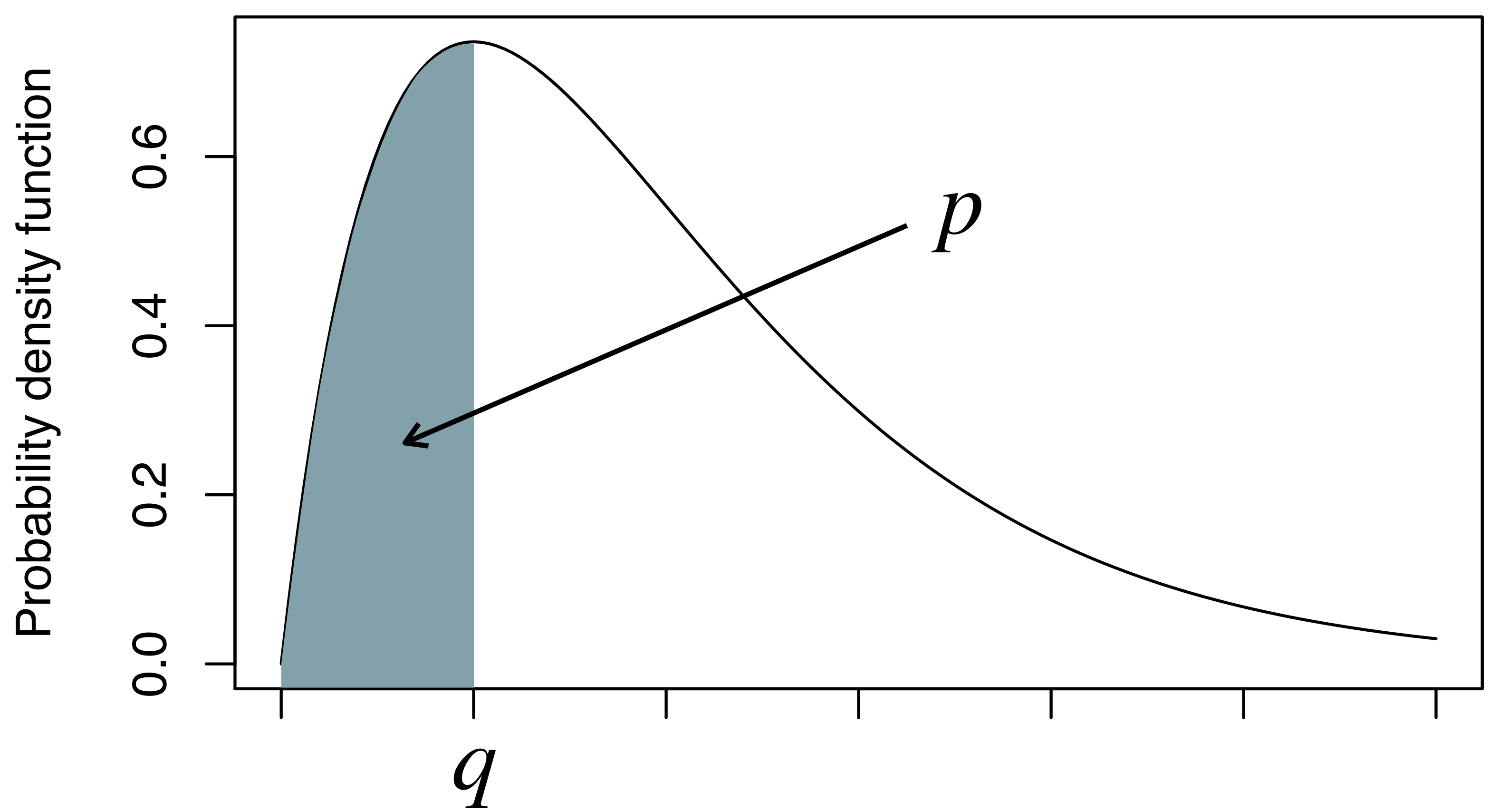


Quantile function

- The quantile function is the inverse of a c.d.f.

$$p = \Pr(X \leq q) = F_X(q) \quad \Rightarrow \quad q = F_X^{-1}(p)$$

- q is the point at which the cumulative probability of X is p.



R commands for probability distributions

- General form: `d***, p***, q***, r***`

- `d***` — probability density function
  `p***` — cumulative distribution function
  `q***` — quantile function
  `r***` — generate a random number

- Specific distributions:
  normal: `***` = norm
  chi-squared: `***` = chisq
  Student *t*: `***` = t
  F: `***` = f

Large Random Samples

Random sampling

- Simple random sampling:

  Randomly choose n objects (the sample) from the population, where each member of the population is equally likely to be included in the sample.

- i.i.d. (independent and identically distributed) random variables:

  independent — the outcome of X dose not depend on the outcome of Y

  identical — the distribution of X and Y are the same

Random sampling

- Let $Y_1, Y_2, \ldots, Y_n$ be a random sample, therefore they are i.i.d. random variables (before drawn).

- The sample average (or sample mean)

$$\overline{Y} = \frac{1}{n}(Y_1 + Y_2 + \cdots + Y_n) = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

  is also a random variable.

- The distribution of $\overline{Y}$ is called the *sampling distribution* of $\overline{Y}$.

Sampling distribution

- Let μ_Y be the mean, and σ_Y^2 be the variance of Y_i.
- Mean and variance of $\overline{Y}$

$$\mathrm{E}(\overline{Y}) = \mu_Y, \quad \mathrm{var}(\overline{Y}) = \frac{\sigma_Y^2}{n}$$

- Generally the sampling distribution of $\overline{Y}$ is complicated, but when the population distribution is normal, the sampling distribution is also normal.

$$Y_i \sim N(\mu_Y, \sigma_Y^2) \quad \Rightarrow \quad \overline{Y} \sim N(\mu_Y, \sigma_Y^2/n)$$

Large sample approximations of sampling distributions

- When the size n of the sample is small, the exact distribution of $\overline{Y}$ can be very complicated.
- When n is large (theoretically $n \to \infty$, in practice $n > 30$), we can use the following tools to approximate sampling distribution:

The law of large numbers: $\overline{Y} \xrightarrow{p} \mu_Y$

The central limit theorem: $\overline{Y} \xrightarrow{d} N(\mu_Y, \sigma_Y^2/n)$

The law of large numbers (LLN)

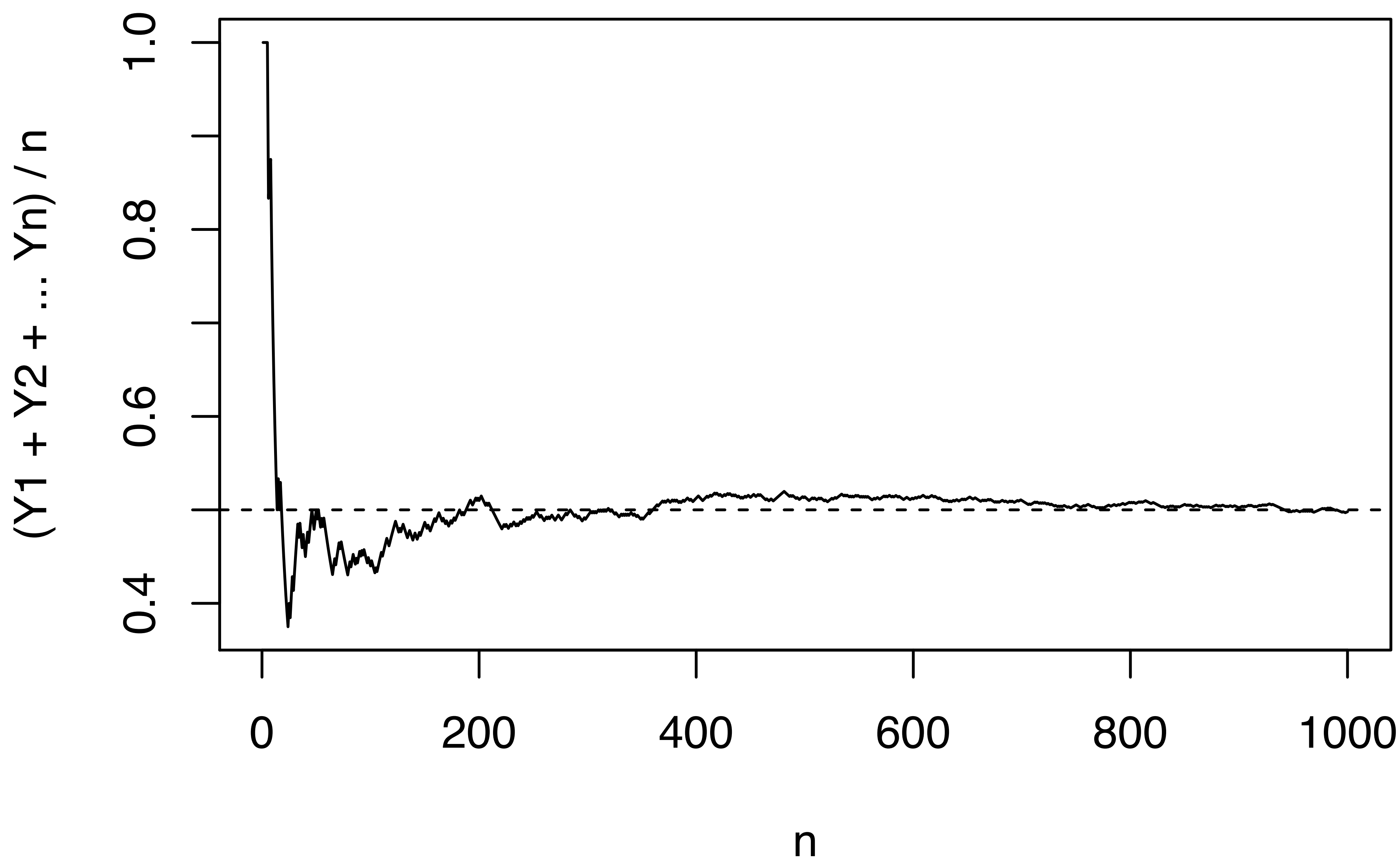
If the random sample are i.i.d., and the population variance is finite ($\sigma_Y^2 < \infty$), then the sample mean $\overline{Y}$ converges to the population mean μ_Y in probability as the sample size increases ($n \to \infty$).

- Converges in probability:
  The probability that $\mu_Y - c < \overline{Y} < \mu_Y + c$ becomes arbitrarily close to 1 as n increases for any constant $c > 0$.

- There are other versions of LLN.

Demonstrating the LLN

- The sample mean of n Bernoulli random variables (flipping a fair coin n times).



The central limit theorem (CLT)

If the random sample are i.i.d., and the population variance is finite ($\sigma_Y^2 < \infty$), then the distribution of $\overline{Y}$ becomes arbitrarily well approximated by the normal distribution $N(\mu_Y, \sigma_Y^2/n)$ as the sample size increases ($n \to \infty$).

- The central limit theorem does not require the population distribution to be normal.
- Verify the central limit theorem with R using an arbitrary population distribution.

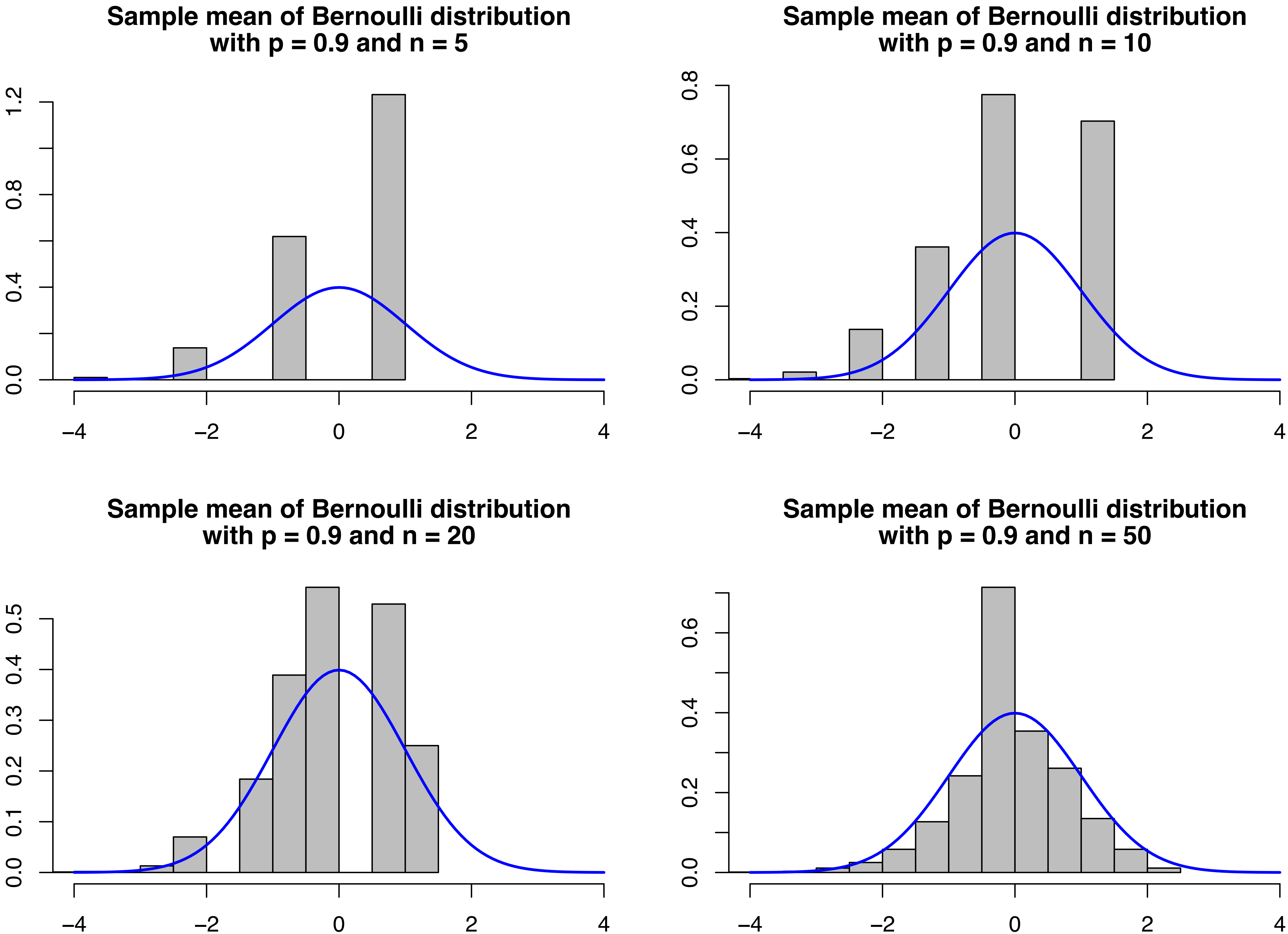
Idea for programming

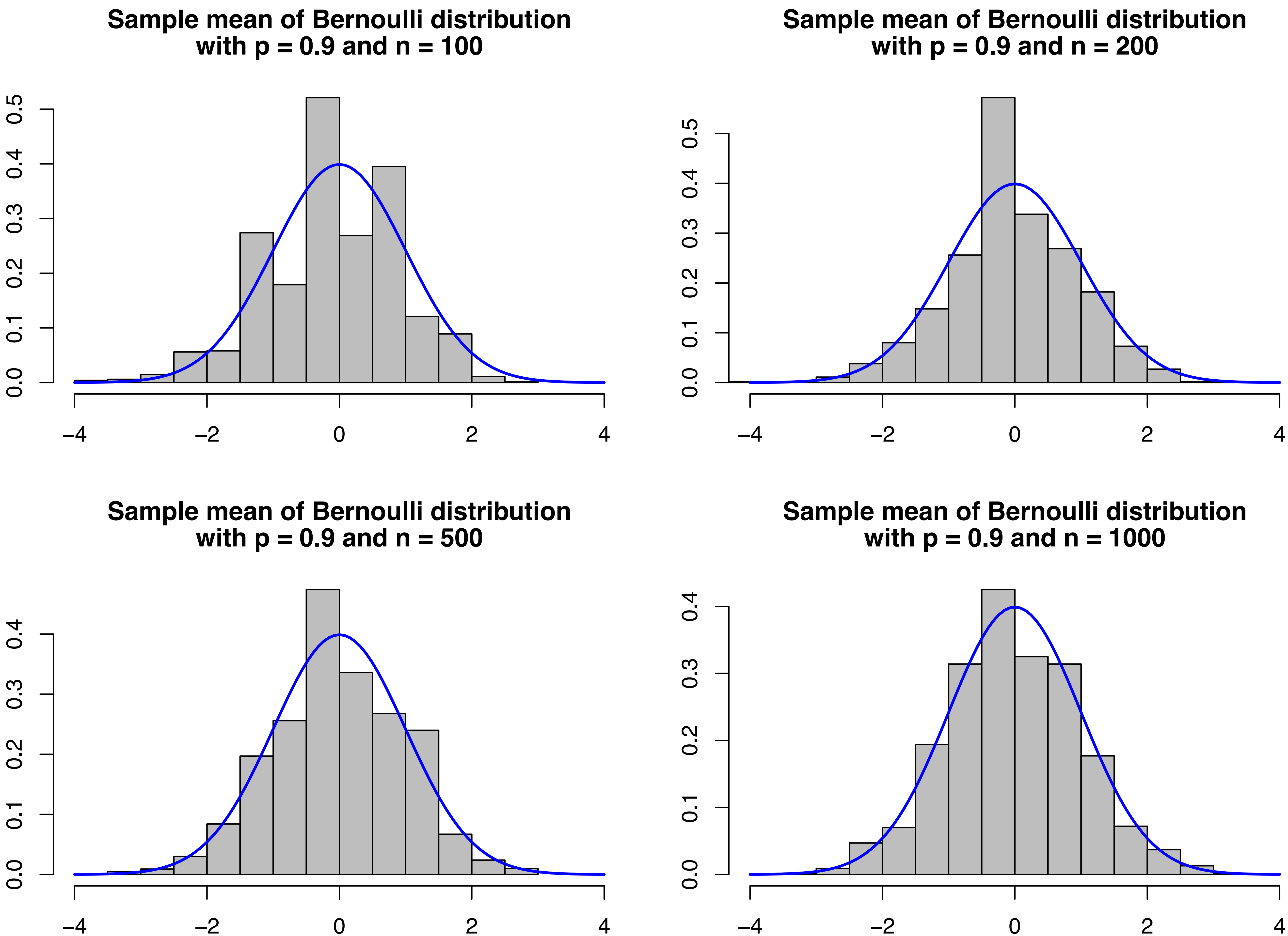
- The size n need to increase
- For each n:
    - We need a sufficient number of observations of sample mean
    - Draw the histogram of the sample means
    - Plot the density of the corresponding normal distribution

```
n <- c(2, 5, 10, 20, 50, 100, 200, 500, 1000) # sample size
k <- 2000    # number of samples for each size

p <- 0.9    # probability of success for Bernoulli distribution
pmean <- p    # population mean
pvar <- p * (1-p)    # population variance

for (i in 1:length(n)) {    # for each sample size
    smean <- rep(0, k)    # sample mean distribution (initialization)
    for (j in 1:k) {    # for each sample
        smean[j] <- mean(rbinom(n[i], 1, p))
            # calculate sample means
    }
    sd_smean <- (smean - pmean) / sqrt(pvar / n[i])
        # standardization
    hist(sd_smean, freq = FALSE, col = "grey",
         xlim = c(-4, 4))    # plot histogram
    curve(dnorm, add = TRUE, lwd = 2, col = "blue")
        # add standard normal density function
}
```


Sample mean of Bernoulli distribution
with p = 0.9 and n = 5
Sample mean of Bernoulli distribution
with p = 0.9 and n = 10
Sample mean of Bernoulli distribution
with p = 0.9 and n = 20
Sample mean of Bernoulli distribution
with p = 0.9 and n = 50
0.0
0.4
0.8
1.2
0.2
0.6
0.1
0.3
0.5
−4
−2
0
2
4


Sample mean of Bernoulli distribution
with p = 0.9 and n = 100
Sample mean of Bernoulli distribution
with p = 0.9 and n = 200
Sample mean of Bernoulli distribution
with p = 0.9 and n = 500
Sample mean of Bernoulli distribution
with p = 0.9 and n = 1000
0.0
0.1
0.2
0.3
0.4
0.5
−4
−2
0
2
4

Take home practice

- X and Y are two random variables, find the assumptions on X and Y that make each of the following equation valid

$$
\begin{aligned}
&1.\ E(X+Y) = E(X) + E(Y) \\
&2.\ E(XY) = E(X) \cdot E(Y) \\
&3.\ \mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)
\end{aligned}
$$

- Write a program to verify the Law of Large Numbers using i.i.d. chi-squared random variables.

References

1. Stock, J. H. and Watson, M. M., *Introduction to Econometrics*, 3rd Edition, Pearson, 2012.

2. DeGroot, M. H. and Schervish, M. J., *Probability and Statistics*, 4th Edition, Pearson, 2012.