Econometrics 1

Lecture 6: Linear Regression (1)
Linear regression with one regressor

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The linear regression model

Linear relationship between X and Y

- A school district cuts the size of its elementary school classes. What is the effect on its students' test score?
- This question is about the unknown effect of changing one variable, X (class size), on another variable, Y (student test score).
- Linear regression (with one regressor) is a model investigating the linear relationship between X and Y.

Class size and test score

- Relative change, or the effect of changing *X* on *Y*:

$$\beta_{ClassSize} = \frac{change\ in\ TestScore}{change\ in\ ClassSize} = \frac{\Delta TestScore}{\Delta ClassSize}$$

$$\Delta TestScore = \beta_{ClassSize} \times \Delta ClassSize$$

This is the definition of the slope of a straight line relating test scores and class size:

$$TestScore = \beta_0 + \beta_{ClassSize} \times ClassSize$$

Incorporating other factors

- This relation may not hold for all districts. Therefore we must incorporate other factors influencing test scores.

$$TestScore = \beta_0 + \beta_{ClassSize} \times ClassSize + \text{other factors}$$

- In a more general expression, *ClassSize* becomes *X*, and *TestScore* becomes *Y*.

The linear regression model

- The linear regression model with one regressor

$$Y_i = \beta_0 + \beta_1 X_i + u_i$$

coefficients

error term

dependent variable

independent variable / regressor

The linear regression model

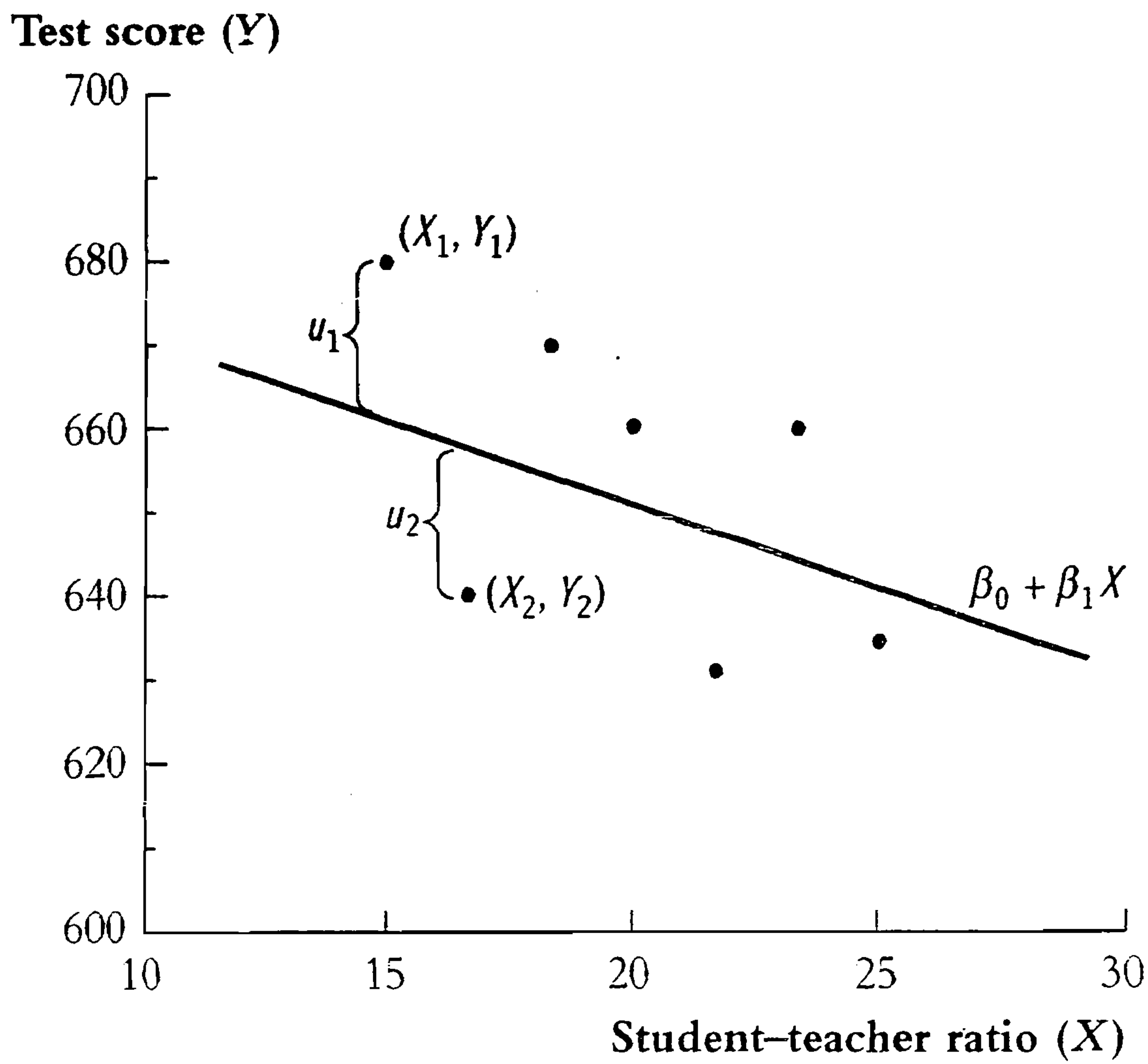
- The linear regression model with one regressor

$$Y_i = \boxed{\beta_0 + \beta_1 X_i} + u_i$$

population regression line / population regression function

FIGURE 4.1 Scatterplot of Test Score vs. Student–Teacher Ratio (Hypothetical Data)

The scatterplot shows hypothetical observations for seven school districts. The population regression line is $\beta_0 + \beta_1 X$. The vertical distance from the i^{th} point to the population regression line is $Y_i - (\beta_0 + \beta_1 X_i)$, which is the population error term u_i for the i^{th} observation.

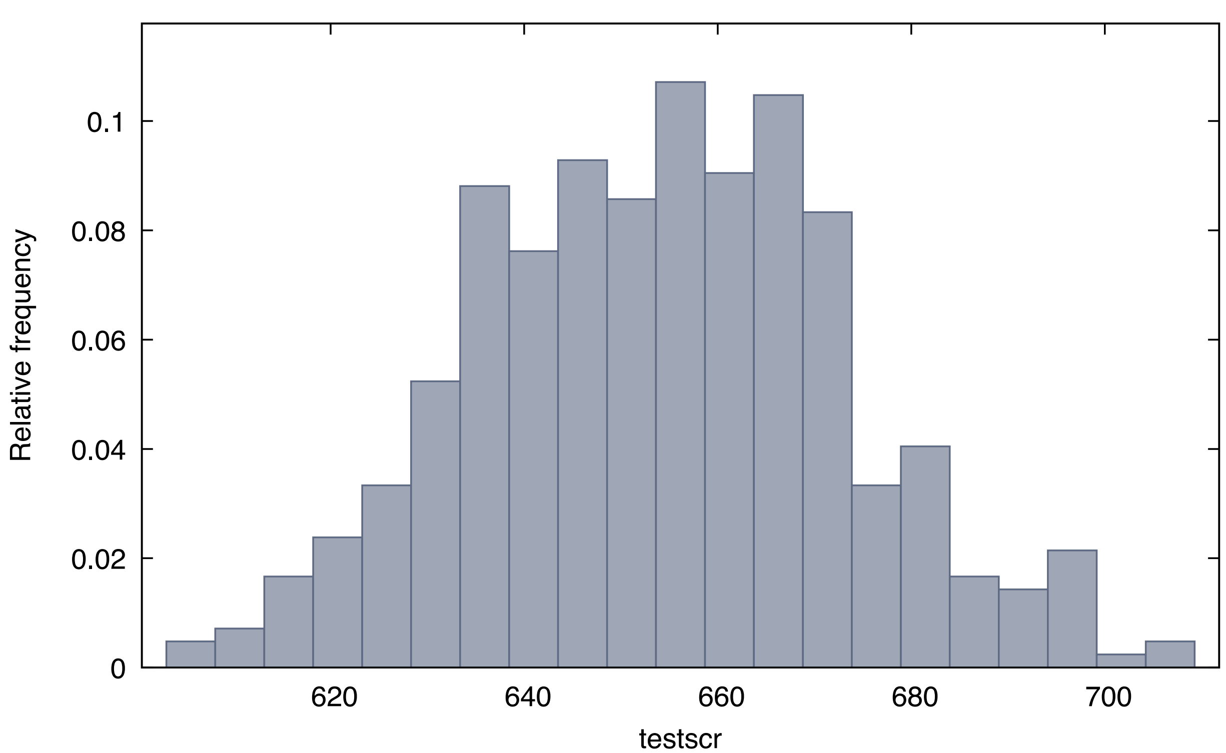


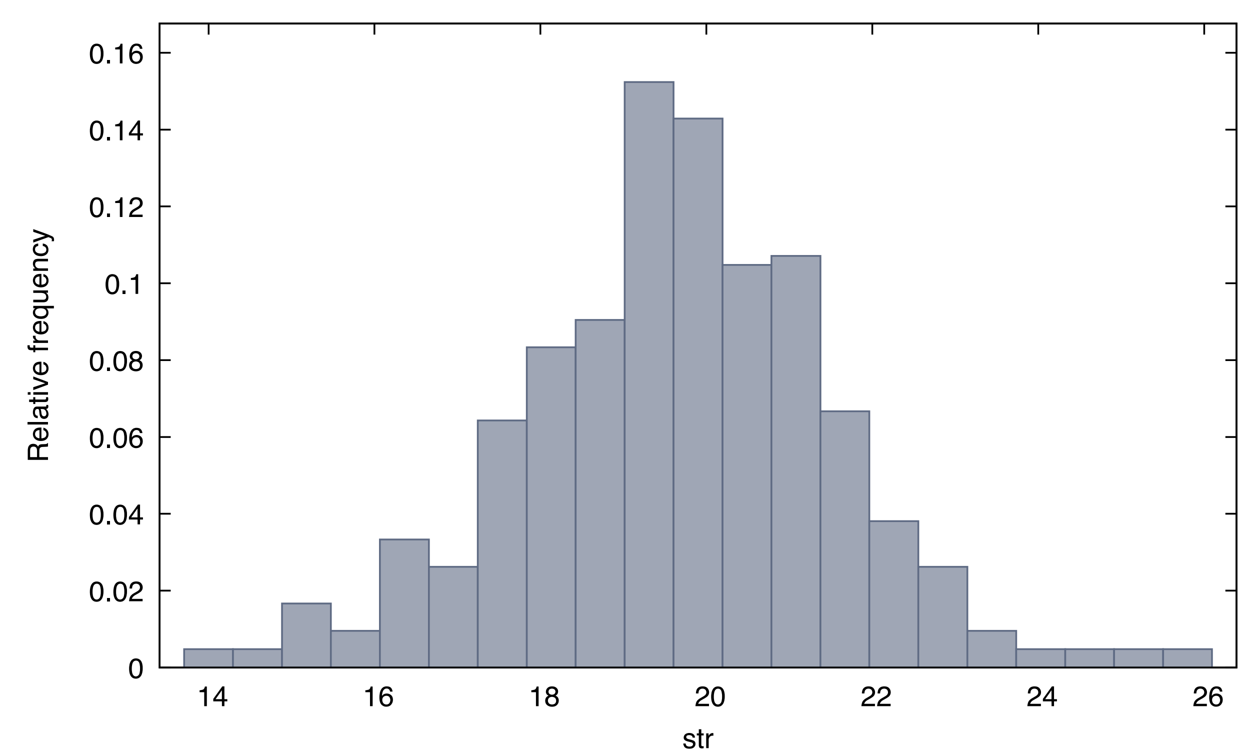
A test score data in California: the STAR dataset

- The file `caschool.xlsx`
- The California Standardized Testing and Reporting (STAR) dataset (1998-1999).
- Average test scores on 420 districts in California.
- For details, see `californiatestscores.docx`

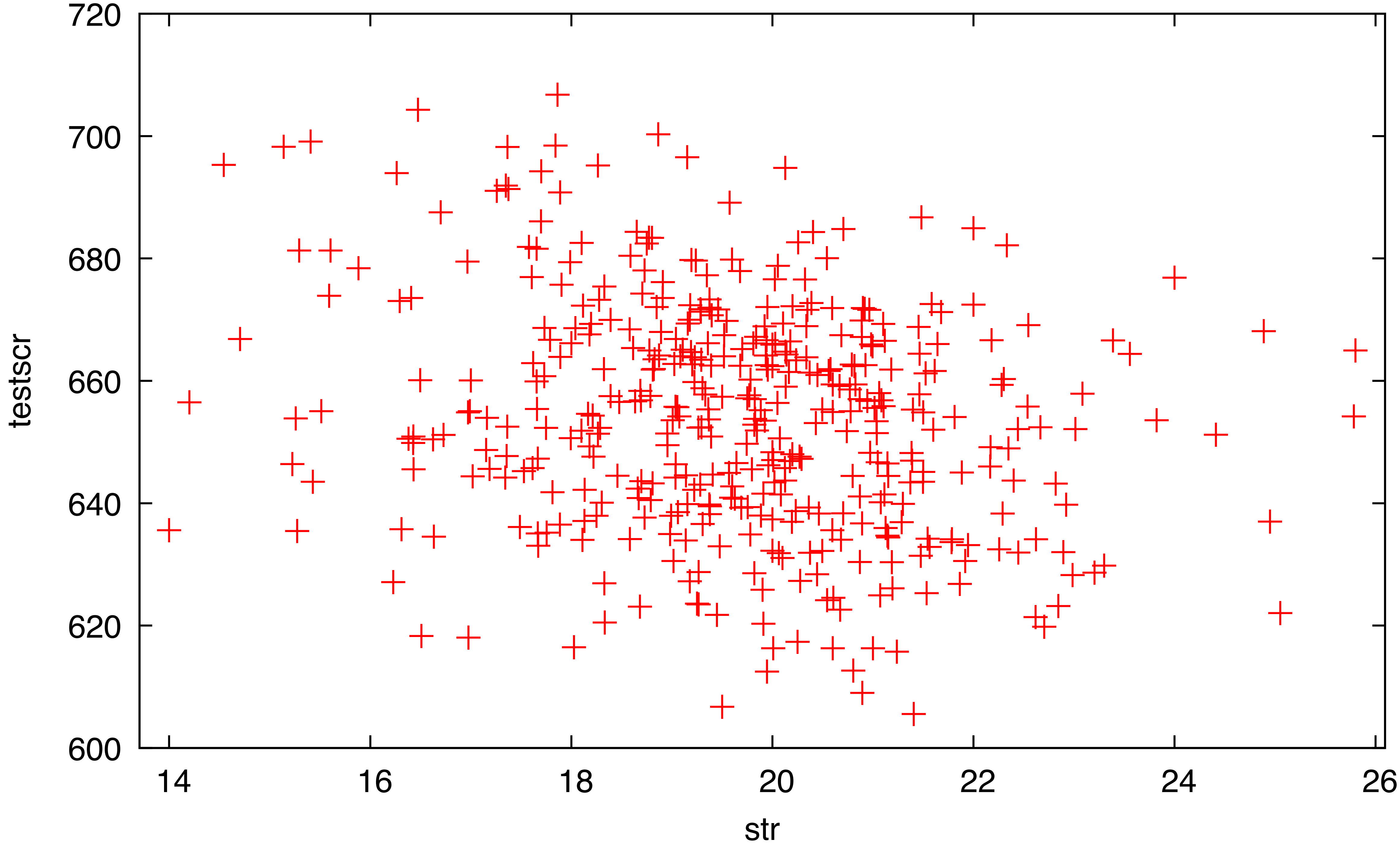
Average test score v.s. student-teacher ratio

- “testscr”: the average test score (of reading and math)
- “str”: the student-teacher ratio (No. of student / No. of teachers)





Average test score v.s. student-teacher ratio



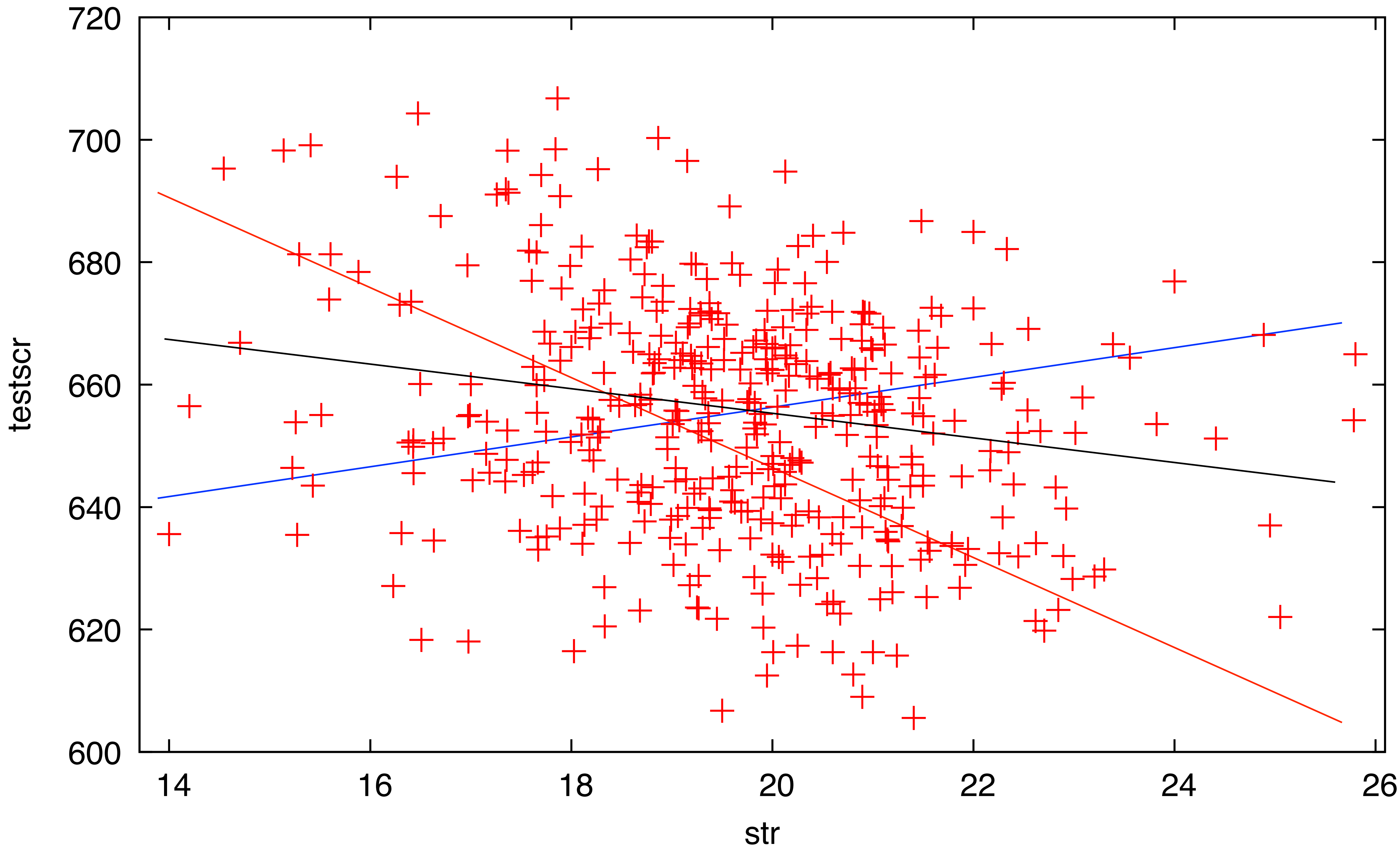
Estimation

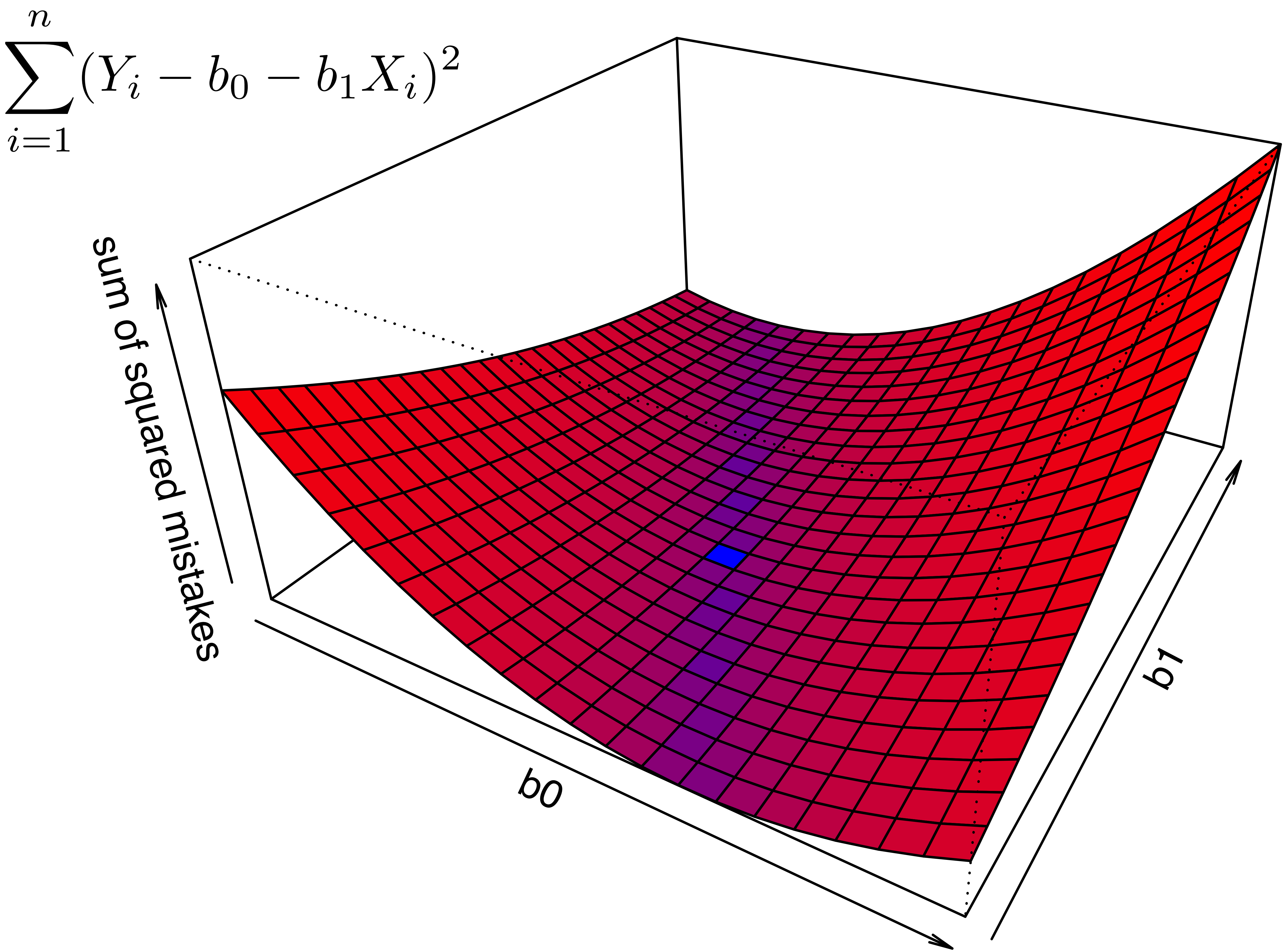
Estimating the coefficients

- $\overline{Y}$ is an estimator of the population mean.
- Similarly, we need estimators of the coefficients β_0 and β_1 .
- The ordinary least squares (OLS) estimators $\hat{\beta}_0$ and $\hat{\beta}_1$ are the ones that minimize

$$\sum_{i=1}^{n}(Y_i - b_0 - b_1 X_i)^2$$

How to determine the sample regression line $\hat{\beta}_0 + \hat{\beta}_1 X$?





$\sum_{i=1}^{n}(Y_i - b_0 - b_1 X_i)^2$
sum of squared mistakes
b0
b1

The OLS estimator, predicted values, and residuals

- The OLS estimators of the slope and the intercept are

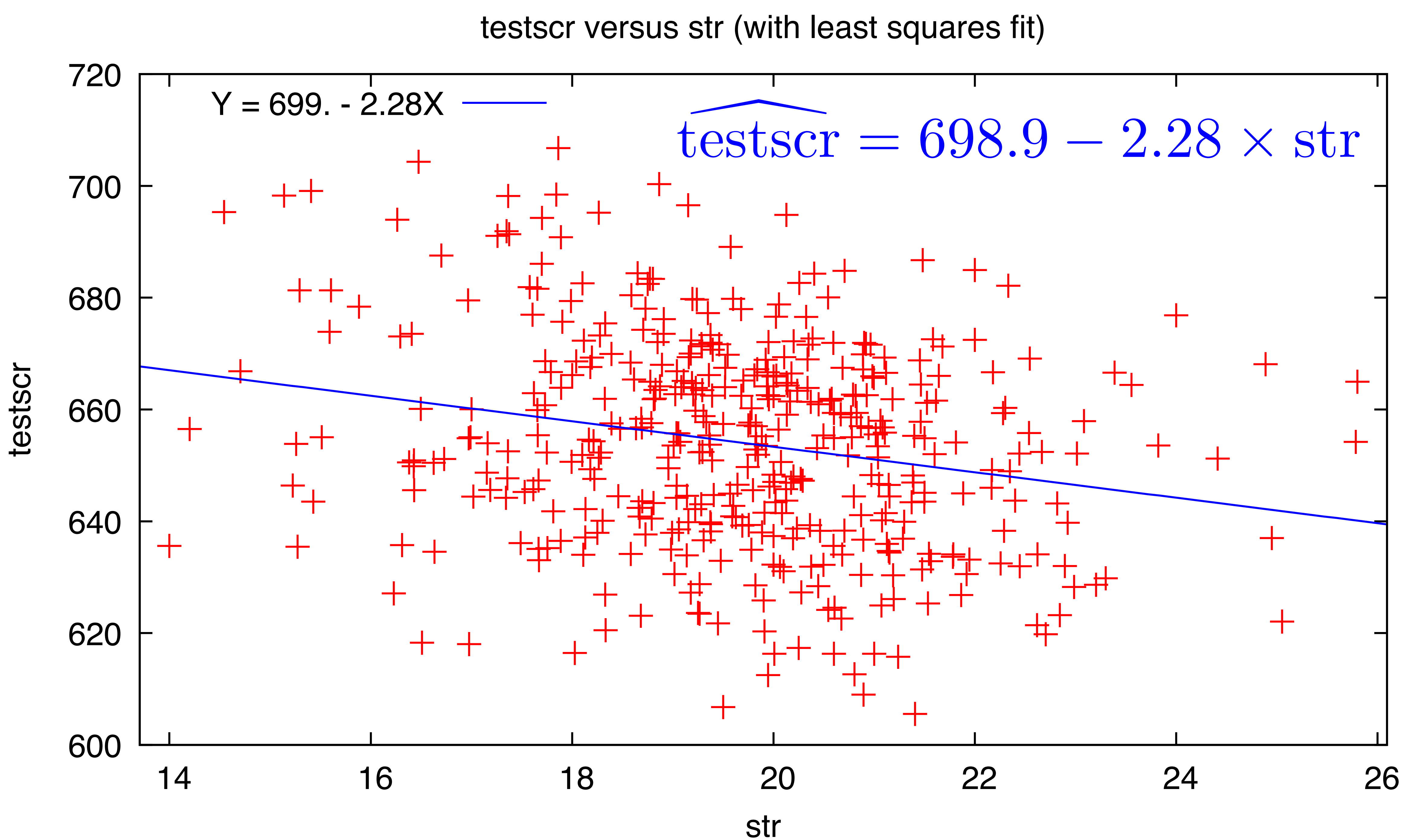
$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (X_i - \overline{X})(Y_i - \overline{Y})}{\sum_{i=1}^n (X_i - \overline{X})^2} = \frac{s_{XY}}{s_X^2}$$

$$\hat{\beta}_0 = \overline{Y} - \hat{\beta}_1 \overline{X}$$

- The OLS predicted value: $\hat{Y}_i = \boxed{\hat{\beta}_0 + \hat{\beta}_1 X_i}$
- The residuals: $\hat{u}_i = Y_i - \hat{Y}_i$

sample regression line/
sample regression function

Average test score v.s. student-teacher ratio



Why use the OLS estimator

- OLS is the dominating method used in practice.
- Under certain assumptions, the OLS estimator is *unbiased* and *consistent*.
- With some further assumptions, the OLS estimator is also *efficient* among a class of unbiased estimators.

  $\Rightarrow$ Gauss-Markov Theorem (Section 5.5)

For the definitions of unbiasedness, consistency, and efficiency, read Chapter 3.

Measures of fit

The R^2

- The R^2 — correlation of determination, the fraction of the sample variance of Y_i explained by X_i.

- Recall that $Y_i = \hat{Y}_i + \hat{u}_i$

$$R^2 = \frac{\sum_{i=1}^n (\hat{Y}_i - \overline{Y})^2}{\sum_{i=1}^n (Y_i - \overline{Y})^2} = \frac{ESS}{TSS}$$

(**e**xplained **s**um of **s**quares)

(**t**otal **s**um of **s**quares)

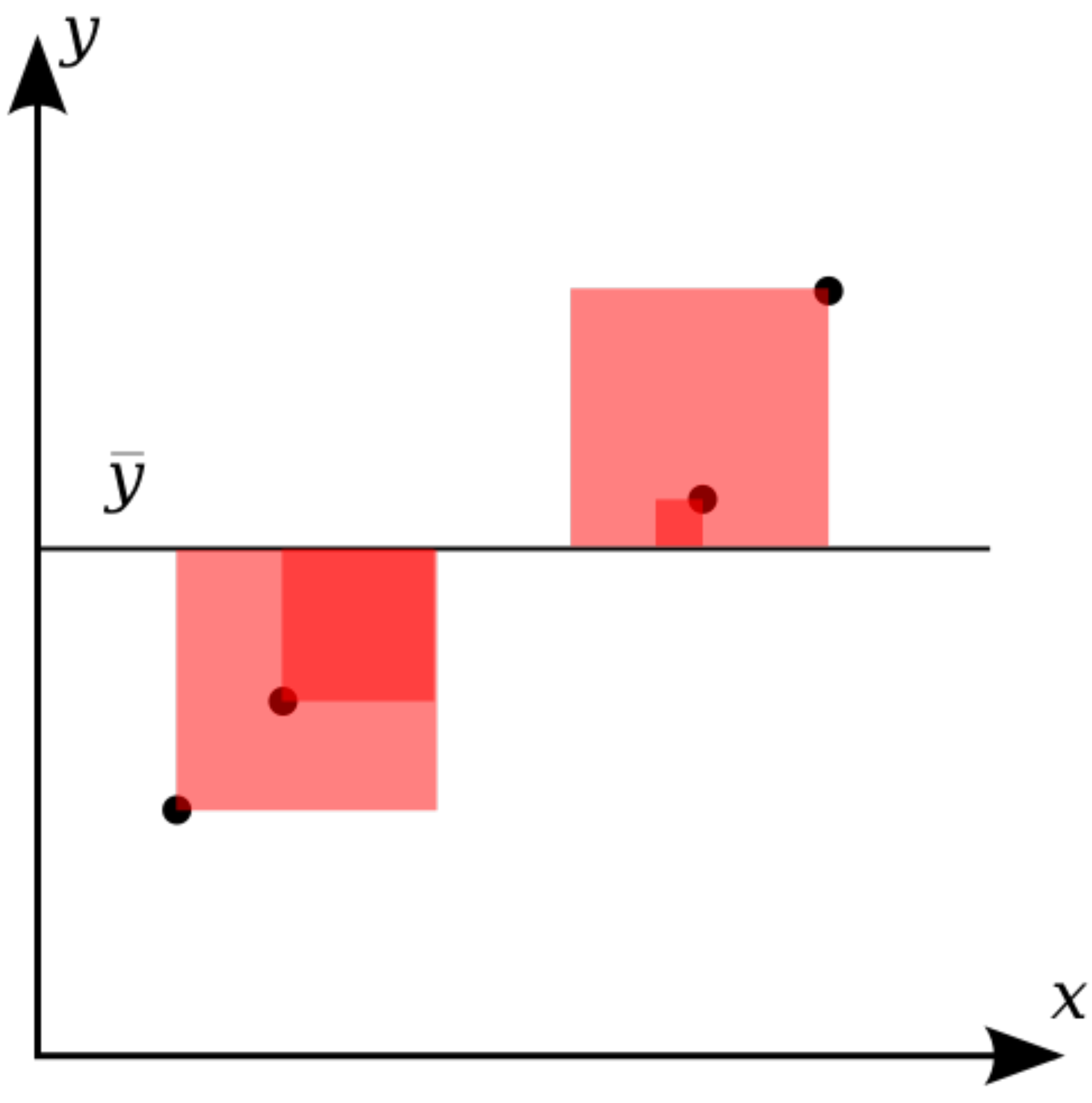
$$= 1 - \frac{\sum_{i=1}^n \hat{u}_i^2}{\sum_{i=1}^n (Y_i - \overline{Y})^2} = 1 - \frac{SSR}{TSS}$$

(**s**um of **s**quared **r**esiduals)

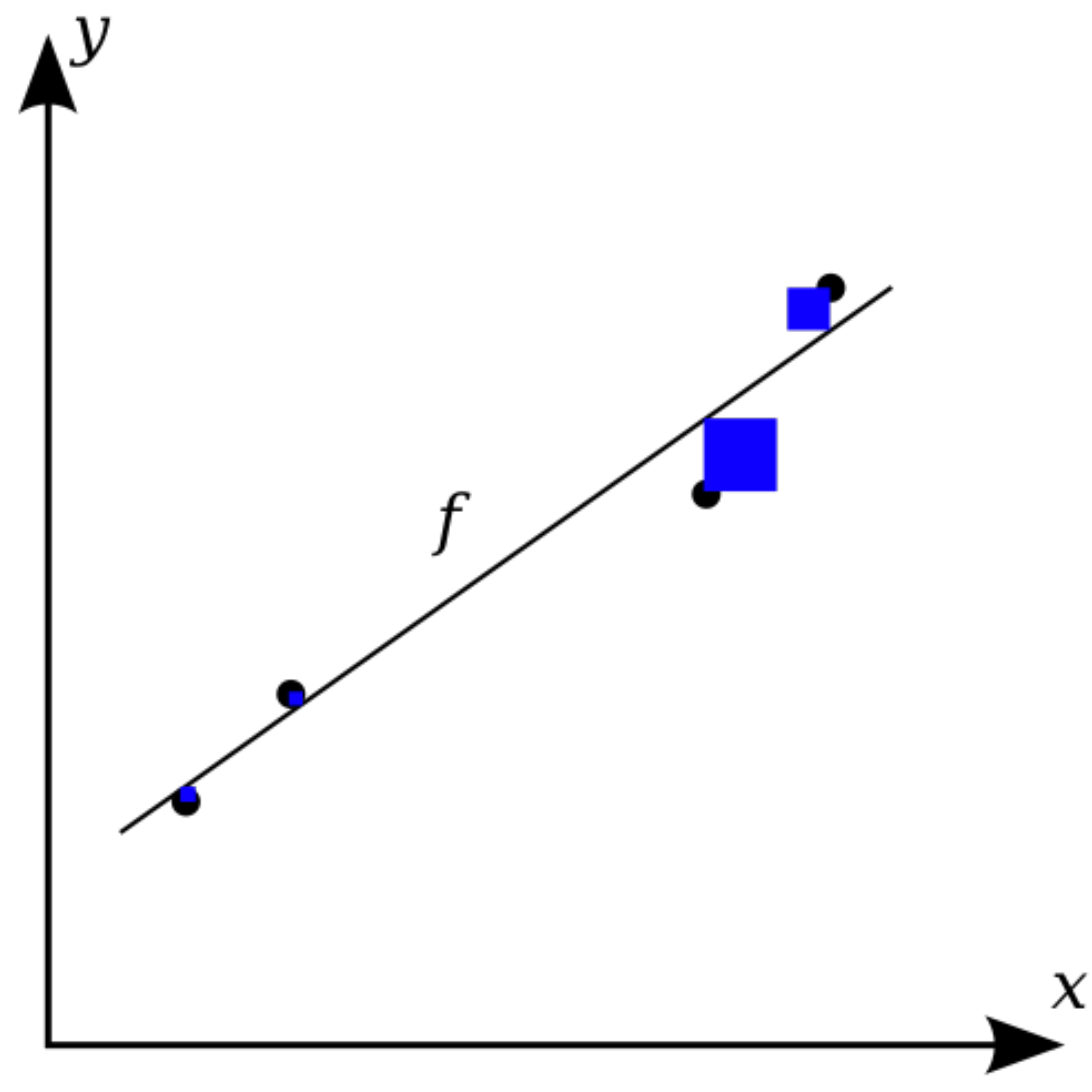
Read Appendix 4.3 if you want to know why the second equality holds.

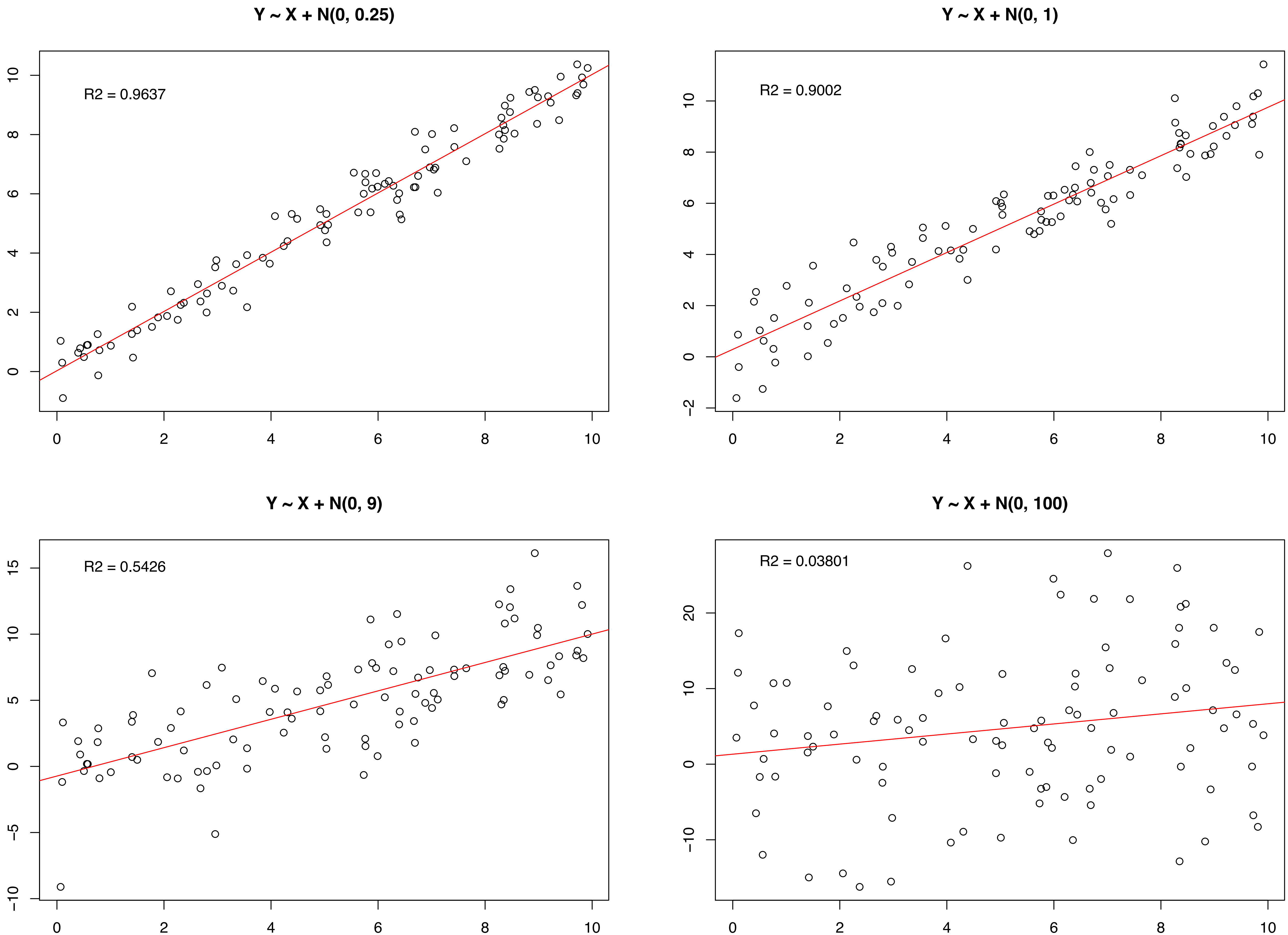
A graphical explanation of SSR

Simple average of Y_i



OLS regression





Y ~ X + N(0, 0.25)
R2 = 0.9637
0 2 4 6 8 10
Y ~ X + N(0, 1)
R2 = 0.9002
−2 0 2 4 6 8 10
Y ~ X + N(0, 9)
R2 = 0.5426
−10 −5 0 5 10 15
Y ~ X + N(0, 100)
R2 = 0.03801
−10 0 10 20

How to read R^2

- R^2 measures how well the OLS regression line fits the data.

- The value of R^2 ranges between 0 and 1. A high R^2 indicates that the regressor (X_i) is good at predicting Y_i, while a low R^2 indicates that the regressor (X_i) is not very good at predicting Y_i.

- A low R^2 does **not** imply that *this regression* is either "good" or "bad", it **does** tell us that other important factors influence the dependent variable.

The standard error of the regression

- The standard error of the regression (*SER*) is an estimator of the standard deviation of the regression error u_i.

$$SER = s_{\hat{u}}, \quad \text{where } s_{\hat{u}}^2 = \frac{1}{n-2}\sum_{i=1}^{n}\hat{u}_i^2 = \frac{SSR}{n-2}$$

- *SER* measures the magnitude of a typical deviation from the regression line.

- *SER* has the same units of the dependent variable.

OLS regression in gretl

- From the menu:

  > Model > Ordinary least squares >

- Scripts:

```
ols testscr const str
```

dependent variable

regressors

Regression results in gretl

```
Model 1: OLS, using observations 1-420
Dependent variable: testscr

              coefficient   std. error   t-ratio    p-value
  ---------------------------------------------------------
  const        698.933       9.46749      73.82     6.57e-242 ***
  str           -2.27981     0.479826     -4.751    2.78e-06  ***

Mean dependent var   654.1565   S.D. dependent var   19.05335
Sum squared resid    144315.5   S.E. of regression   18.58097
R-squared            0.051240   Adjusted R-squared   0.048970
F(1, 418)            22.57511   P-value(F)           2.78e-06
Log-likelihood      -1822.250   Akaike criterion     3648.499
Schwarz criterion    3656.580   Hannan-Quinn         3651.693
```

The least square assumptions

The least squares assumptions

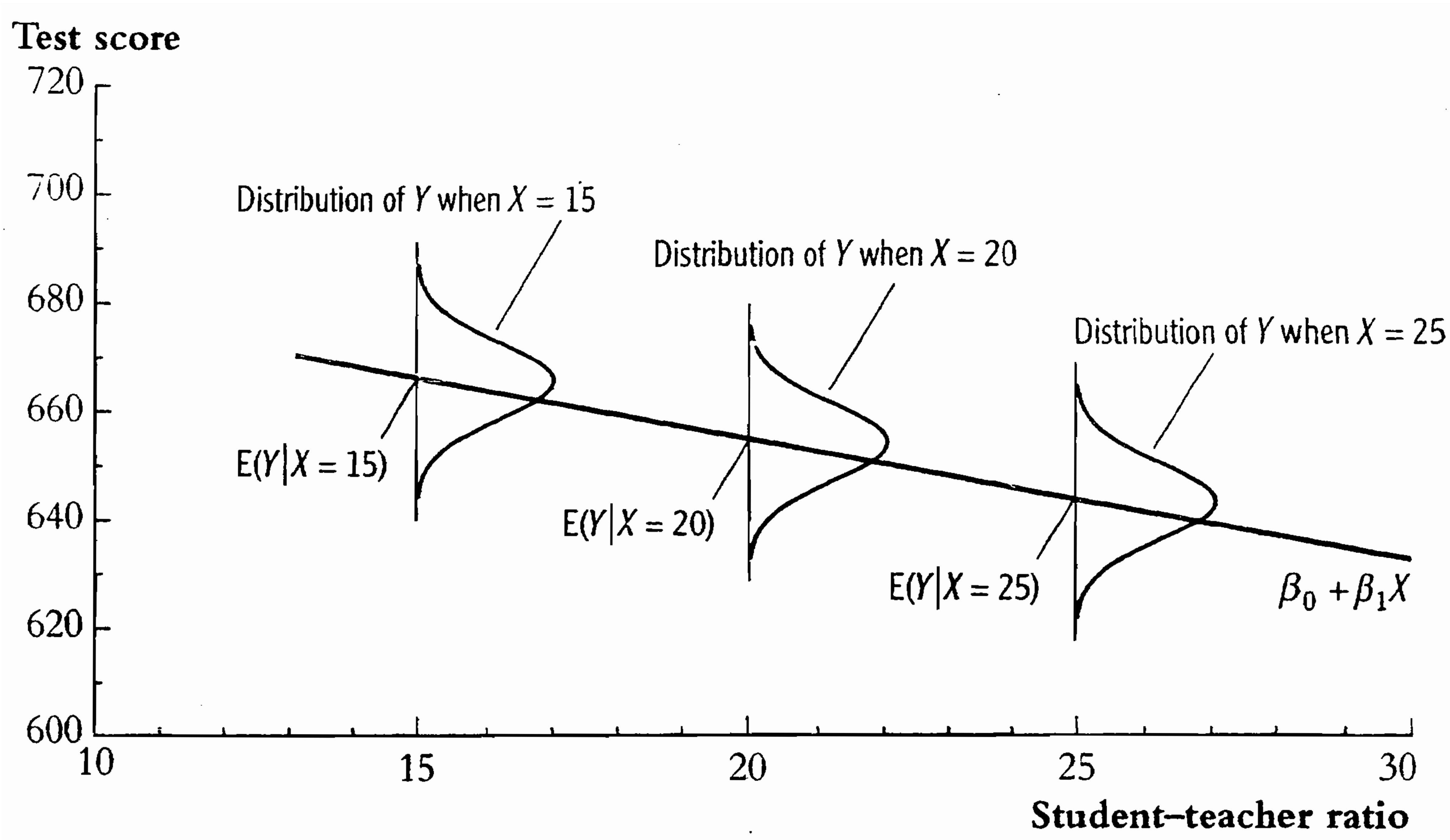
For the linear regression model

$$Y_i = \beta_0 + \beta_1 X_i + u_i, \quad i = 1, \ldots, n$$

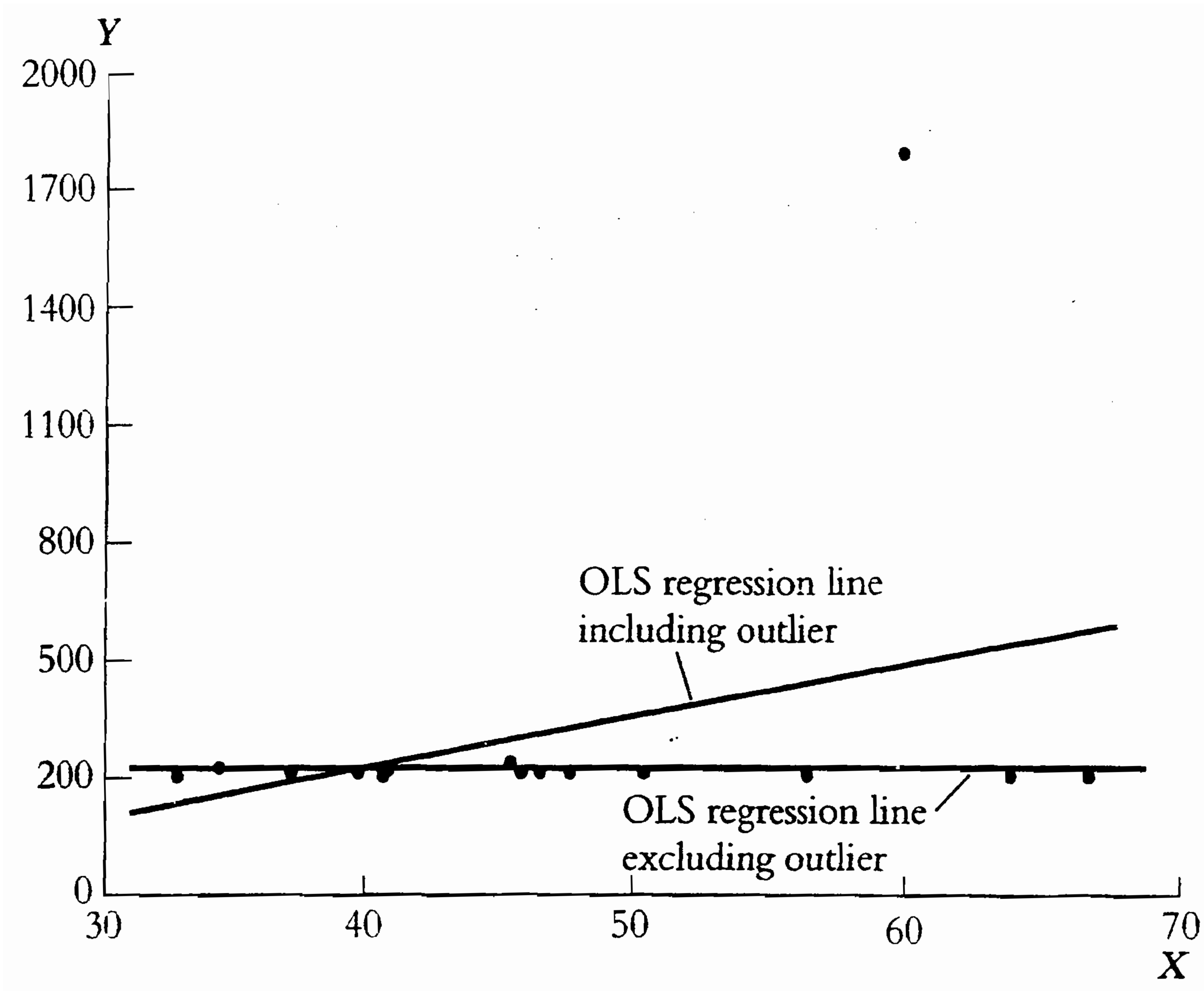
it is assumed that:

1. The error term u_i has conditional mean zero given X_i:
$$\mathrm{E}(u_i \mid X_i) = 0 \qquad (\Rightarrow \mathrm{corr}(X_i, u_i) = 0)$$
2. (X_i, Y_i), $i = 1, \ldots, n$, are i.i.d. draws from their joint distribution; and
3. Large outliers are unlikely: X_i and Y_i have nonzero finite fourth moments.

Implication of $\mathrm{E}(u_i \mid X_i) = 0$



Linear regression is sensitive to outliers



Hypothesis tests and confidence intervals

Large-sample distributions of $\hat{\beta}_0$ and $\hat{\beta}_1$

If the least square assumptions hold, then in large samples $\hat{\beta}_0$ and $\hat{\beta}_1$ have a jointly normal sampling distribution.

The large-sample distribution of $\hat{\beta}_1$ is $N(\beta_1, \sigma^2_{\hat{\beta}_1})$, where

$$\sigma^2_{\hat{\beta}_1} = \frac{1}{n}\frac{\mathrm{var}[(X_i - \mu_X)u_i]}{[\mathrm{var}(X_i)]^2}$$

The large-sample distribution of $\hat{\beta}_0$ is $N(\beta_0, \sigma^2_{\hat{\beta}_0})$, where

$$\sigma^2_{\hat{\beta}_0} = \frac{1}{n}\frac{\mathrm{var}(H_i u_i)}{[\mathrm{E}(H_i^2)]^2}, \text{ where } H_i = 1 - \left[\frac{\mu_X}{\mathrm{E}(X_i^2)}\right]X_i$$

Hypotheses concerning β_1

- Two-sided hypotheses

$$H_0 : \beta_1 = \beta_{1,0} \quad \text{vs.} \quad H_1 : \beta_1 \neq \beta_{1,0}$$

- The *t*-statistic

$$t = \frac{\hat{\beta}_1 - \beta_{1,0}}{SE(\hat{\beta}_1)}$$

where

$$SE(\hat{\beta}_1) = \sqrt{\hat{\sigma}^2_{\hat{\beta}_1}},\ \hat{\sigma}^2_{\hat{\beta}_1} = \frac{1}{n} \times \frac{\frac{1}{n-2}\sum_{i=1}^{n}(X_i - \overline{X})^2 \hat{u}_i^2}{[\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2]^2}$$

- The *p*-value

$$p\text{-value} = 2\Phi(-|t^{act}|)$$

Confidence interval for β_1

- The 95% confidence interval for β_1 is

$$[\hat{\beta}_1 - 1.96\, SE(\hat{\beta}_1),\ \hat{\beta}_1 + 1.96\, SE(\hat{\beta}_1)]$$

Regression results in gretl

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Dependent variable: testscr

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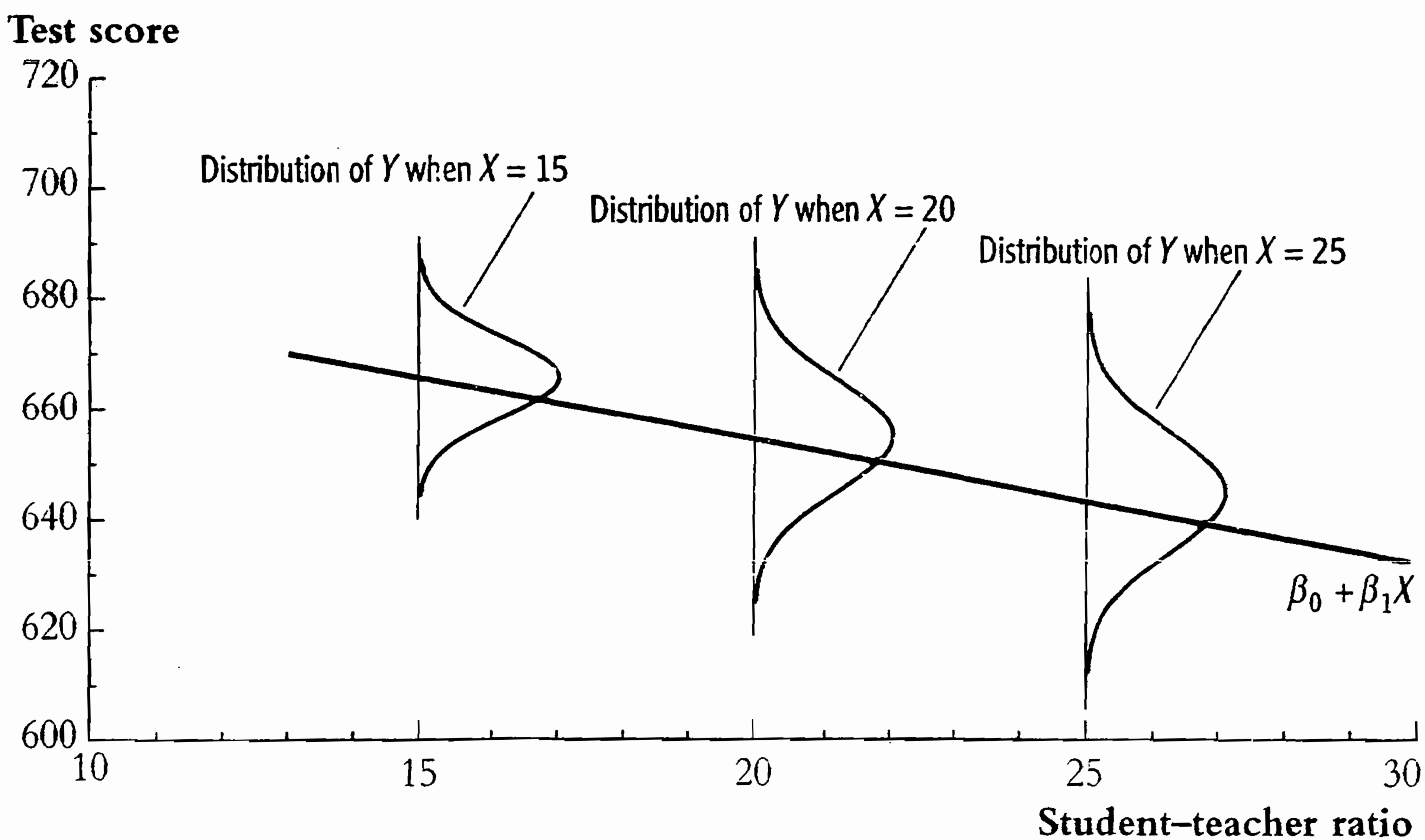
```

Heteroskedasticity and homoskedasticity

An example of heteroskedasticity

FIGURE 5.2 An Example of Heteroskedasticity

Like Figure 4.4, this shows the conditional distribution of test scores for three different class sizes. Unlike Figure 4.4, these distributions become more spread out (have a larger variance) for larger class sizes. Because the variance of the distribution of u given X, $\text{var}(u|X)$, depends on X, u is heteroskedastic.



Definition

The error term u_i is *homoskedastic* is the variance of the conditional distribution of u_i given X_i,

$$\mathrm{var}(u_i \mid X_i = x)$$

is constant for $i = 1, \ldots, n$ and in particular does not depend on x.

Otherwise, the error term is *heteroskedastic*.

Implications of homoskedasticity + least square assumptions

- The OLS estimators of coefficients are efficient among all estimators that are linear in $Y_1, \ldots, Y_n$. [BLUE]

- The standard errors of $\hat{\beta}_0$ and $\hat{\beta}_1$ reduce to simpler form, e.g.,

$$SE(\hat{\beta}_1) = \sqrt{\tilde{\sigma}^2_{\hat{\beta}_1}}$$

where

$$\tilde{\sigma}^2_{\hat{\beta}_1} = \frac{s^2_{\hat{u}}}{\sum_{i=1}^{n}(X_i - \overline{X})^2}$$

Standard errors of $\hat{\beta}_1$

- Homoskedasticity-only standard error

$$SE(\hat{\beta}_1) = \sqrt{\tilde{\sigma}^2_{\hat{\beta}_1}},\ \tilde{\sigma}^2_{\hat{\beta}_1} = \frac{s^2_{\hat{u}}}{\sum_{i=1}^n (X_i - \overline{X})^2}$$

- Heteroskedasticity-robust standard error (HC1)

$$SE(\hat{\beta}_1) = \sqrt{\hat{\sigma}^2_{\hat{\beta}_1}},\ \hat{\sigma}^2_{\hat{\beta}_1} = \frac{1}{n} \times \frac{\frac{1}{n-2}\sum_{i=1}^n (X_i - \overline{X})^2 \hat{u}_i^2}{[\frac{1}{n}\sum_{i=1}^n (X_i - \overline{X})^2]^2}$$

In practice

- If the errors are heteroskedastic but the homoskedastic-only formulas are used

  ⇒ *t*-statistic does not have a standard normal distribution, even in large samples

- If the errors are homoskedastic but the heteroskedastic-robust formulas are used

  ⇒ hypothesis tests and confidence intervals will be valid

- Always use heteroskedastic-robust standard errors

Practice in gretl

Heteroskedasticity-robust estimation in gretl

- Settings for the whole script

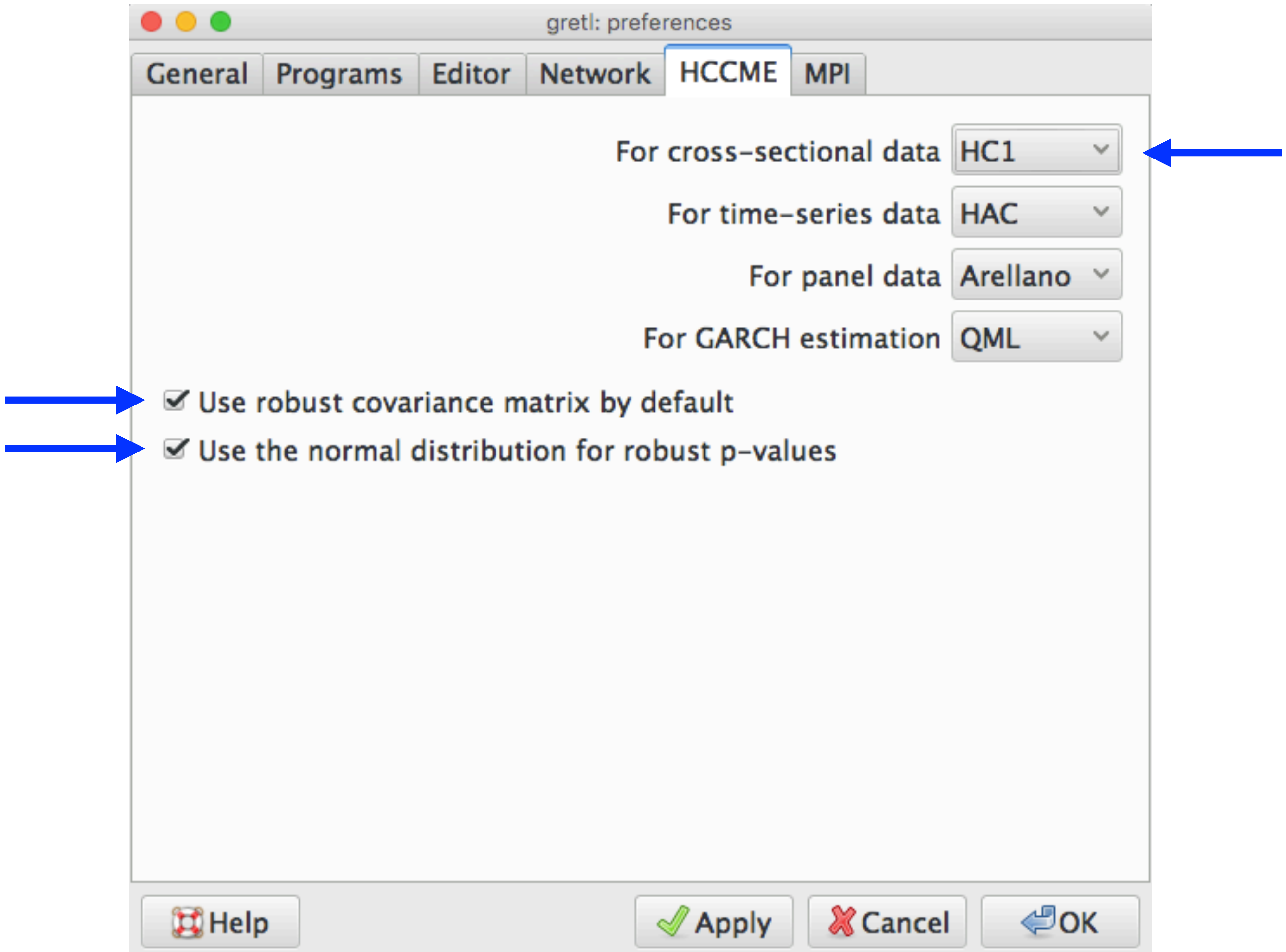
```
set force_hc on
set hc_version 1
    # 0 (the original White’s) is the default
set robust_z on
```

- For single resression

```
ols yvar xvar --robust
```

(you still need to set the HC version)

Settings in the preferences of gretl



Regression results in gretl (homoskedasticity-only)

```
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Dependent variable: testscr

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  ---------------------------------------------------------
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Schwarz criterion    3656.580   Hannan-Quinn         3651.693
```

Regression results in gretl (heteroskedasticity-robust with normal distribution)

```
Model 1: OLS, using observations 1-420
Dependent variable: testscr
Heteroskedasticity-robust standard errors, variant HC1

             coefficient   std. error     z       p-value
  ---------------------------------------------------------
  const       698.933      10.3644      67.44    0.0000    ***
  str          -2.27981     0.519489    -4.389   1.14e-05 ***

Mean dependent var   654.1565   S.D. dependent var   19.05335
Sum squared resid    144315.5   S.E. of regression   18.58097
R-squared            0.051240   Adjusted R-squared   0.048970
F(1, 418)            19.25943   P-value(F)           0.000014
Log-likelihood      -1822.250   Akaike criterion     3648.499
Schwarz criterion    3656.580   Hannan-Quinn         3651.693
```

Exercises

1. Reproduce Table 4.1 using matrix.

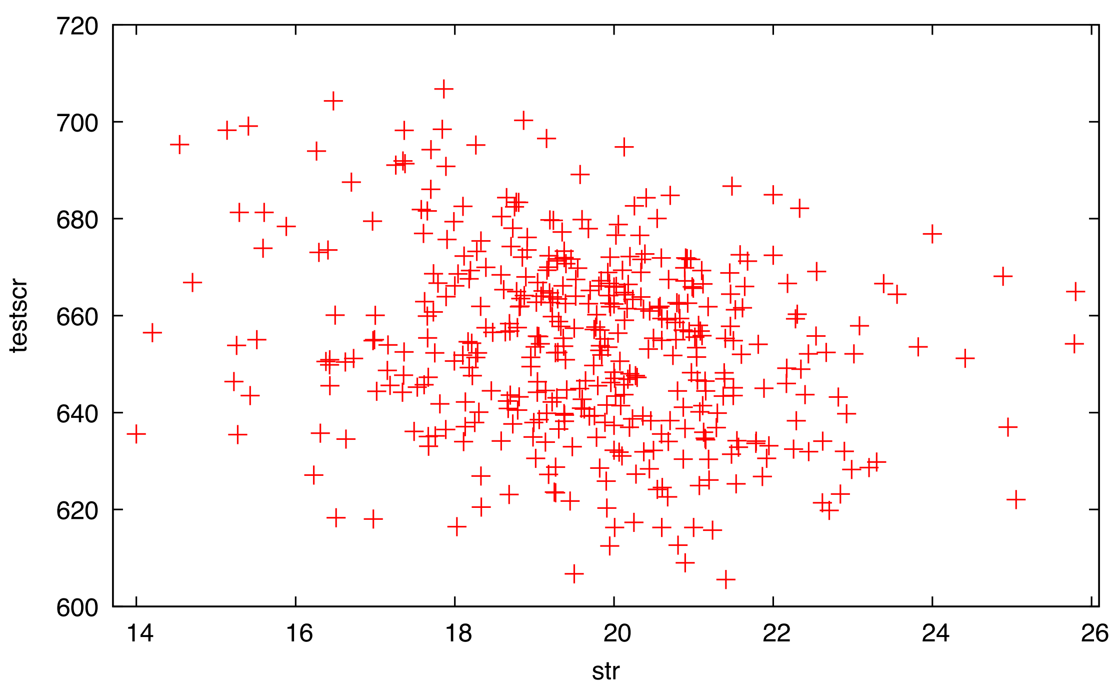
gretl: f

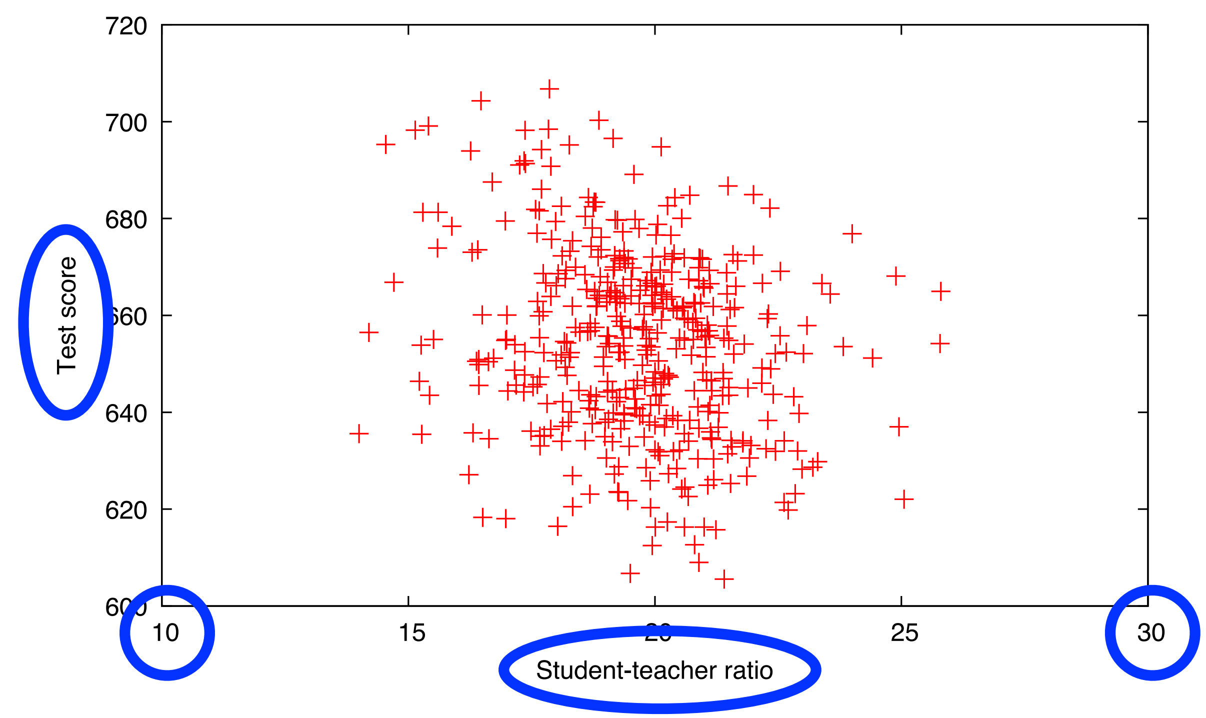
View Fill Transform

1, 1

	Average	Std	10% percentile	25% percentile	40% percentile	50% percentile	60% percentile	75% percentile	90% percentile
Student–teacher ratio	19.6	1.9	17.3	18.6	19.3	19.7	20.1	20.9	21.9
Test score	654.2	19.1	630.4	640.0	649.0	654.4	659.4	666.7	679.3

2. Learn command `gnuplot` (or `plot`) and reproduce Figure 4.2 with appropriate titles and ranges of axes.






```
gnuplot testscr str --output=display --fit=none
```

References

1. Stock, J. H. and Watson, M. M., *Introduction to Econometrics*, 3rd Edition, Pearson, 2012.

2. *Gretl User's Guide*