高级计量经济学
Assignment 1

Consider the regression model

$$y_i = \beta_1 x_{1,i} + \beta_2 x_{2,i} + \varepsilon_i$$

where $x_{k,i}$ denotes the ith observation of the kth variable, $k \in \{1,2\}$. We assume that $\mathbf{x}_1 \neq \mathbf{0}$, $\mathbf{x}_2 \neq \mathbf{0}$, and $\mathbf{x}_2 \neq c\,\mathbf{x}_1$ for any real number c. Let the residual e_i be $e_i = y_i - (b_1 x_{1,i} + b_2 x_{2,i})$. Find the least squares fitted values (b_1, b_2) of parameters (β_1, β_2) following the steps below.

1. Let $S = \sum e_i^2$. Express the least squares problem as a minimization problem of S.
2. Find $\dfrac{\partial S}{\partial b_1}$, $\dfrac{\partial S}{\partial b_2}$, $\dfrac{\partial^2 S}{\partial b_1^2}$, $\dfrac{\partial^2 S}{\partial b_2^2}$, and $\dfrac{\partial^2 S}{\partial b_1 \partial b_2}$.
3. Derive the first and second order conditions.
4. Show that the second order condition is satisfied. You can use the Cauchy-Schwarz inequality if necessary.
5. Find the least squares solution by solving the first order condition.
6. Calculate $\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$ for this model, and confirm that the result coincides with the one you obtained in the previous step.

Solution

1. The least squares problem is

$$\min_{b_1, b_2} \sum \left[y_i - (b_1 x_{1,i} - b_2 x_{2,i})\right]^2$$

2.

$$\frac{\partial S}{\partial b_1} = -2\sum x_{1,i} y_i + 2b_1 \sum x_{1,i}^2 + 2b_2 \sum x_{1,i} x_{2,i}$$

$$\frac{\partial S}{\partial b_2} = -2\sum x_{2,i} y_i + 2b_1 \sum x_{1,i} x_{2,i} + 2b_2 \sum x_{2,i}^2$$

$$\frac{\partial^2 S}{\partial b_1^2} = 2\sum x_{1,i}^2, \qquad \frac{\partial^2 S}{\partial b_2^2} = 2\sum x_{2,i}^2, \qquad \frac{\partial^2 S}{\partial b_1 \partial b_2} = 2\sum x_{1,i} x_{2,i}$$

3. The first order conditions are

$$b_1 \sum x_{1,i}^2 + b_2 \sum x_{1,i}x_{2,i} = \sum x_{1,i}y_i,$$
$$b_1 \sum x_{1,i}x_{2,i} + b_2 \sum x_{2,i}^2 = \sum x_{2,i}y_i.$$

The second order condition is that

$$\mathbf{H} = \begin{bmatrix} 2\sum x_{1,i}^2 & 2\sum x_{1,i}x_{2,i} \\ 2\sum x_{1,i}x_{2,i} & 2\sum x_{2,i}^2 \end{bmatrix}$$

is positive definite.

4. The Cauchy-Schwarz inequality states that for two vectors $\mathbf{a}$ and $\mathbf{b}$,

$$|\mathbf{a}'\mathbf{b}| \leq \|\mathbf{a}\| \cdot \|\mathbf{b}\|$$

where the equality holds when $\mathbf{a} = c\mathbf{b}$ for some real number c. By assumption, we have

$$|\mathbf{x}_1'\mathbf{x}_2| < \|\mathbf{x}_1\| \cdot \|\mathbf{x}_2\|$$

which is equivalent to $\left| \sum x_{1,i}x_{2,i} \right| < \sqrt{(\sum x_{1,i}^2)(\sum x_{2,i}^2)}$, which implies

$$\left(\sum x_{1,i}x_{2,i}\right)^2 < (\sum x_{1,i}^2)(\sum x_{2,i}^2).$$

Approach 1:
The quadratic form of $\mathbf{H}$ satisfies

$$\begin{aligned} \mathbf{z}'\mathbf{H}\mathbf{z} &= 2(\sum x_{1,i}^2)\, z_1^2 + 4(\sum x_{1,i}x_{2,i})\, z_1 z_2 + 2(\sum x_{2,i}^2)\, z_2^2 \\ &= 2\|\mathbf{x}_1\|^2 z_1^2 + 4\mathbf{x}_1'\mathbf{x}_2 z_1 z_2 + 2\|\mathbf{x}_2\|^2 z_2^2 \\ &\geq 2\|\mathbf{x}_1\|^2 z_1^2 - 4\|\mathbf{x}_1\| \cdot \|\mathbf{x}_2\| \cdot |z_1| \cdot |z_2| + 2\|\mathbf{x}_2\|^2 z_2^2 \\ &= 2(\|\mathbf{x}_1\| z_1 - \|\mathbf{x}_2\| z_2)^2 \geq 0 \end{aligned}$$

for all nonzero vector $\mathbf{z}$. If both z_1 and z_2 are nonzero, the first inequality becomes strict by the Cauchy-Schwarz inequality, and thus $\mathbf{z}'\mathbf{H}\mathbf{z} > 0$. If $z_1 = 0$, then

$$\mathbf{z}'\mathbf{H}\mathbf{z} = 2\|\mathbf{x}_2\|^2 z_2^2 > 0$$

since $\mathbf{x_2} \neq \mathbf{0}$ and $z_2 \neq 0$. Similar argument applies to the case of $z_2 = 0$. Hence, $\mathbf{H}$ is positive definite.

Approach 2:

For any symmetric 2×2 matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$, it can be shown that $\mathbf{A}$ is positive definite if and only if $a_{11} > 0$ and $a_{11}a_{22} - a_{12}^2 > 0$. We first prove this.

The quadratic form of $\mathbf{A}$ can be written as

$$\mathbf{z}'\mathbf{A}\mathbf{z} = a_{11}z_1^2 + 2a_{12}z_1z_2 + a_{22}z_2^2 = a_{11}\left(z_1 + \frac{a_{12}}{a_{11}}z_2\right)^2 + \left(a_{22} - \frac{a_{12}^2}{a_{11}}\right)z_2^2,$$

assuming that $a_{11} \neq 0$. Indeed, if $a_{11} = 0$, $\mathbf{z}'\mathbf{A}\mathbf{z} = a_{22}z_2^2$, which can be zero by choosing $z_2 = 0$.

[$\Rightarrow$] $\mathbf{z}'\mathbf{A}\mathbf{z} > 0$ implies both terms are positive, which implies $a_{11} > 0$ and $a_{11}a_{22} - a_{12}^2 > 0$.
[$\Leftarrow$] $a_{11} > 0$ and $a_{11}a_{22} - a_{12}^2 > 0$ implies $\mathbf{z}'\mathbf{A}\mathbf{z} \geq 0$. If $\mathbf{z}'\mathbf{A}\mathbf{z} = 0$, one has $z_1 + \frac{a_{12}}{a_{11}}z_2 = 0$ and $z_2 = 0$, which implies $\mathbf{z} = \mathbf{0}$. So for nonzero $\mathbf{z}$, the quadratic form is strictly positive.

Now we apply this property to $\mathbf{H}$. $\mathbf{H}$ is symmetric by definition. $2\sum x_{1,i}^2 > 0$ since $\mathbf{x}_1 \neq 0$. $(\sum x_{1,i}^2)(\sum x_{2,i}^2) - \left(\sum x_{1,i}x_{2,i}\right)^2 > 0$ by the Cauchy-Schwarz inequality and the assumption $\mathbf{x}_2 \neq c\,\mathbf{x}_1$. Hence, $\mathbf{H}$ is positive definite.

5. Solving the first equation in the first order conditions for b_1, one has

$$b_1 = \frac{\sum x_{1,i}y_i}{\sum x_{1,i}^2} - b_2\frac{\sum x_{1,i}x_{2,i}}{\sum x_{1,i}^2}.$$

Substitute this into the second equation, we obtain

$$b_2 = \frac{(\sum x_{2,i}y_i)(\sum x_{1,i}^2) - (\sum x_{1,i}y_i)(\sum x_{1,i}x_{2,i})}{(\sum x_{1,i}^2)(\sum x_{2,i}^2) - (\sum x_{1,i}x_{2,i})^2},$$

and therefore,

$$b_1 = \frac{(\sum x_{1,i}y_i)(\sum x_{2,i}^2) - (\sum x_{2,i}y_i)(\sum x_{1,i}x_{2,i})}{(\sum x_{1,i}^2)(\sum x_{2,i}^2) - (\sum x_{1,i}x_{2,i})^2}.$$

The denominator is not zero because of the Cauchy-Schwarz inequality and the assumptions about $\mathbf{x}_1$ and $\mathbf{x}_2$.

6. From the solution formula, we have

$$
\begin{aligned}
\mathbf{b} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} \\
&= \left(\begin{bmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,n} \\ x_{2,1} & x_{2,2} & \dots & x_{2,n} \end{bmatrix} \times \begin{bmatrix} x_{1,1} & x_{2,1} \\ x_{1,2} & x_{2,2} \\ \vdots & \vdots \\ x_{1,n} & x_{2,n} \end{bmatrix} \right)^{-1} \times \begin{bmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,n} \\ x_{2,1} & x_{2,2} & \dots & x_{2,n} \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \\
&= \begin{bmatrix} \sum x_{1,i}^2 & \sum x_{1,i}x_{2,i} \\ \sum x_{1,i}x_{2,i} & \sum x_{2,i}^2 \end{bmatrix}^{-1} \times \begin{bmatrix} \sum x_{1,i}y_i \\ \sum x_{2,i}y_i \end{bmatrix} \\
&= \frac{1}{(\sum x_{1,i}^2)(\sum x_{2,i}^2) - (\sum x_{1,i}x_{2,i})^2} \begin{bmatrix} \sum x_{2,i}^2 & -\sum x_{1,i}x_{2,i} \\ -\sum x_{1,i}x_{2,i} & \sum x_{1,i}^2 \end{bmatrix} \begin{bmatrix} \sum x_{1,i}y_i \\ \sum x_{2,i}y_i \end{bmatrix} \\
&= \frac{1}{(\sum x_{1,i}^2)(\sum x_{2,i}^2) - (\sum x_{1,i}x_{2,i})^2} \begin{bmatrix} (\sum x_{2,i}^2)(\sum x_{1,i}y_i) - (\sum x_{1,i}x_{2,i})(\sum x_{2,i}y_i) \\ (\sum x_{1,i}^2)(\sum x_{2,i}y_i) - (\sum x_{1,i}x_{2,i})(\sum x_{1,i}y_i) \end{bmatrix} \\
&= \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}
\end{aligned}
$$