高级计量经济学
Assignment 3

1. Let $x_1, x_2, \ldots, x_n$ be i.i.d. random observations of random variable x with $E[x] = \mu$. Which of the following estimators of μ are unbiased? Which are consistent?

(1) $\hat{\mu}_1 = \dfrac{1}{n+1}\sum_{i=1}^{n} x_i$

(2) $\hat{\mu}_2 = \dfrac{1.01}{n}\sum_{i=1}^{n} x_i$

(3) $\hat{\mu}_3 = 0.01x_1 + \dfrac{0.99}{n-1}\sum_{i=2}^{n} x_i$

2. Let $s^2 = \dfrac{\mathbf{e}'\mathbf{e}}{n-k}$ where $\mathbf{e}$ is the OLS residual vector of the linear regression model, n is the number of observations, and k is the number of variables in $\mathbf{X}$.

Prove that $E[s^2 \mid \mathbf{X}] = \sigma^2$ under assumptions A.1 — A.4.

Solution

1. With Khinchine's weak law of large numbers (Theorem D.5), one has $\operatorname{plim}\bar{x}_n = \mu$.

$\hat{\mu}_1 = \dfrac{n}{n+1}\dfrac{1}{n}\sum_{i=1}^{n} x_i = \dfrac{n}{n+1}\bar{x}_n \xrightarrow{p} \mu \quad \Rightarrow \quad \hat{\mu}_1$ is biased but consistent.

$\hat{\mu}_2 = 1.01\dfrac{1}{n}\sum_{i=1}^{n} x_i = 1.01\bar{x}_n \xrightarrow{p} 1.01\mu \quad \Rightarrow \quad \hat{\mu}_2$ is biased and inconsistent.

$\hat{\mu}_3 = 0.01x_1 + 0.99\bar{x}_{n-1} \xrightarrow{p} 0.01x_1 + 0.99\mu \quad \Rightarrow \quad \hat{\mu}_2$ is unbiased but inconsistent. The unbiasedness comes from $E[\hat{\mu}_3] = 0.01E[x_1] + 0.99E[\bar{x}_{n-1}] = 0.01\mu + 0.99\mu = \mu$.

Hence, $\hat{\mu}_3$ is unbiased, and $\hat{\mu}_1$ is consistent.

■

2. From assumption A.1 we have

$$\mathbf{e} = \mathbf{M}\mathbf{y} = \mathbf{M}(\mathbf{X}\beta + \varepsilon) = \mathbf{M}\mathbf{X}\beta + \mathbf{M}\varepsilon = \mathbf{M}\varepsilon$$

where $\mathbf{M}$ is the residual maker, and the last equality comes from the fact $\mathbf{M}\mathbf{X} = \mathbf{0}$. Then,

$$E[s^2 \mid \mathbf{X}] = E\left[\frac{\mathbf{e}'\mathbf{e}}{n-k} \,\middle|\, \mathbf{X}\right] = \frac{1}{n-k} E[\mathbf{e}'\mathbf{e} \mid \mathbf{X}]$$

$$= \frac{1}{n-k} E[\varepsilon'\mathbf{M}'\mathbf{M}\varepsilon \mid \mathbf{X}] = \frac{1}{n-k} E[\varepsilon'\mathbf{M}\varepsilon \mid \mathbf{X}]$$

Since $\varepsilon'\mathbf{M}\varepsilon = \sum_{j=1}^{n}\sum_{i=1}^{n} \varepsilon_i M_{ij} \varepsilon_j$ where M_{ij} denotes the (i, j) element of $\mathbf{M}$, and from assumption A.4 that $E[\varepsilon_i^2 \mid \mathbf{X}] = \sigma^2$ and $E[\varepsilon_i \varepsilon_j \mid \mathbf{X}] = 0$ for $i \neq j$, one has

$$E[\varepsilon'\mathbf{M}\varepsilon \mid \mathbf{X}] = E\left[\sum_{j=1}^{n}\sum_{i=1}^{n} \varepsilon_i M_{ij} \varepsilon_j \,\middle|\, \mathbf{X}\right] = \sum_{j=1}^{n}\sum_{i=1}^{n} M_{ij} E[\varepsilon_i \varepsilon_j \mid \mathbf{X}] = \sum_{i=1}^{n} M_{ii}\sigma^2 = \sigma^2 \cdot \mathrm{tr}(\mathbf{M})$$

where $\mathrm{tr}(\mathbf{M})$ is the trace of $\mathbf{M}$.

From assumption A.2 we have $\mathbf{P} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ (and note that the existence of $\mathbf{M}$ also requires A.2), then

$$\mathrm{tr}(\mathbf{M}) = \mathrm{tr}(\mathbf{I_n} - \mathbf{P}) = \mathrm{tr}(\mathbf{I}_n) - \mathrm{tr}(\mathbf{P}) = n - \mathrm{tr}(\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}').$$

Since $\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$,

$$\mathrm{tr}(\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}') = \mathrm{tr}((\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}) = \mathrm{tr}(\mathbf{I}_k) = k,$$

one has $\mathrm{tr}(\mathbf{M}) = n - k$. Hence,

$$E[s^2 \mid \mathbf{X}] = \frac{1}{n-k} \cdot \sigma^2 \cdot \mathrm{tr}(\mathbf{M}) = \frac{1}{n-k} \cdot \sigma^2 \cdot (n-k) = \sigma^2.$$

■

Note: this result indicates that $s^2 = \dfrac{\mathbf{e}'\mathbf{e}}{n-k}$ is an unbiased estimator of σ^2, and that the maximum likelihood estimator $\hat{\sigma}^2 = \dfrac{\mathbf{e}'\mathbf{e}}{n}$ with assumption A.6 is biased (but consistent).