高级计量经济学
Assignment 4

Assume OLS assumptions A1 through A6. We then have

$$\mathbf{y} \mid \mathbf{X} \sim N(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}_n).$$

The parameters in the regression model are the elements of vector $\boldsymbol{\beta}$ and the scalar σ^2 (here we take σ^2 as a single quantity, i.e., as $\sigma^2 = \gamma$, rather than as the square of σ). The likelihood function of $(\boldsymbol{\beta}, \sigma^2)$ is the conditional density function $f(\mathbf{y} \mid \mathbf{X})$, i.e.,

$$L(\boldsymbol{\beta}, \sigma^2) = f(\mathbf{y} \mid \mathbf{X}) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left[-\frac{1}{2\sigma^2}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})\right].$$

The maximum likelihood (ML) estimator is the estimator that maximizes the (log) likelihood function. It is known that the ML estimator of $\boldsymbol{\beta}$ is the OLS estimator $\mathbf{b}$, and the ML estimator of σ^2 is

$$\frac{1}{n}\mathbf{e}'\mathbf{e} = \frac{\mathrm{SSR}}{n} = \frac{n-k}{n}s^2,$$

where $s^2 = \dfrac{1}{n-k}\mathbf{e}'\mathbf{e}$.

Prove this result in the following two steps:

1. Find $\mathbf{b}$ that maximizes the log likelihood function $\log L(\boldsymbol{\beta}, \sigma^2)$ for any given σ^2.
2. Find $\hat{\sigma}^2$ that maximizes the log likelihood $\log L(\mathbf{b}, \sigma^2)$ where $\mathbf{b}$ is obtained in step 1.

Solution

The log likelihood function is

$$\log L(\boldsymbol{\beta}, \sigma^2) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})\,.$$

For any fixed σ^2, the first term on the right-hand side is a constant, then the value of $\boldsymbol{\beta}$ that maximizes $\log L(\boldsymbol{\beta}, \sigma^2)$ is the one that maximizes the second term on the right-hand side, which is identical to the one that minimizes $(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})$. The latter is exactly the OLS estimator $\mathbf{b}$ of $\boldsymbol{\beta}$.

Now let $\boldsymbol{\beta} = \mathbf{b}$, and $\sigma^2 = \gamma$. The first order condition of the maximization of $\log L(\mathbf{b}, \gamma)$ with respect to γ is

$$\frac{\partial}{\partial\gamma}\log L(\mathbf{b}, \gamma) = -\frac{n}{2\gamma} + \frac{1}{2\gamma^2}\mathbf{e}'\mathbf{e} = 0\,,$$

which implies $\gamma = \dfrac{\mathbf{e}'\mathbf{e}}{n}$. Furthermore, it can be easily seen that $\dfrac{\partial}{\partial \gamma} \log L(\mathbf{b}, \gamma) > 0$ if $0 < \gamma < \dfrac{\mathbf{e}'\mathbf{e}}{n}$ and $\dfrac{\partial}{\partial \gamma} \log L(\mathbf{b}, \gamma) < 0$ if $\gamma > \dfrac{\mathbf{e}'\mathbf{e}}{n}$. Thus $\gamma = \dfrac{\mathbf{e}'\mathbf{e}}{n}$ maximizes $\log L(\mathbf{b}, \gamma)$.

We conclude that the ML estimator of $(\boldsymbol{\beta}, \sigma^2)$ is $(\mathbf{b}, \frac{1}{n}\mathbf{e}'\mathbf{e})$.