Econometrics 2022-4-8.

$$y = X\beta + \varepsilon.$$

$$\min_{b_0} \underline{e_0' e}$$

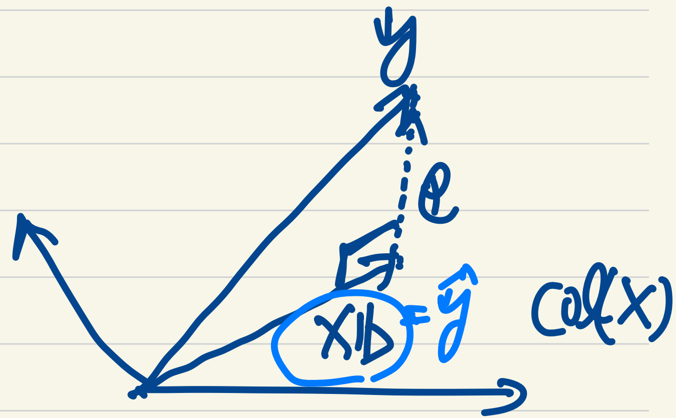
where $e_0 = y - Xb_0$

OLS solution b satisfies

$$X'Xb = X'y$$

$$\Rightarrow \boxed{b = (X'X)^{-1}X'y}$$

$$y = \underline{Xb} + e$$



Homework 1.

$y_i = \alpha + \beta x_i + \varepsilon_i$

$$y_i = \beta_1 \underbrace{X_{1,i}}_{x_{ik}} + \beta_2 \underbrace{X_{2,i}} + \varepsilon_i .$$

variable

obs.

3. FOCs : $\frac{\partial S}{\partial b_1} = 0$, $\frac{\partial S}{\partial b_2} = 0$

$$b_1 \sum x_{1,i}^2 + b_2 \sum x_{1,i} x_{2,i} = \sum x_{1,i} y_i$$

$$b_1 \sum x_{1,i} x_{2,i} + b_2 \sum x_{2,i}^2 = \sum x_{2,i} y_i$$

$A = \sum x_{1,i}^2$, $B = \sum x_{1,i} x_{2,i}$, $C = \sum x_{1,i} y_i$, $D = \sum x_{2,i}^2$, $E = \sum x_{2,i} y_i$

$$\begin{cases} A b_1 + B b_2 = C \\ B b_1 + D b_2 = E \end{cases}$$

SOCs : Hessian matrix.

$$H = \begin{bmatrix} \frac{\partial^2 S}{\partial b_1^2} & \frac{\partial^2 S}{\partial b_1 \partial b_2} \\ \frac{\partial^2 S}{\partial b_2 \partial b_1} & \frac{\partial^2 S}{\partial b_2^2} \end{bmatrix}$$ is positive definite.

" $z' H z > 0$ for all nonzero z ".

4. (i) $z'Hz = \ldots\ldots \overset{?}{>} 0$

(ii) For symmetric A (2×2), $z'Az = a_{11}z_1^2 + 2a_{12}z_1z_2 + a_{22}z_2^2$.

A is positive definite $\Leftrightarrow$ $a_{11} > 0$ and $a_{11}a_{22} - a_{12}^2 > 0$.

(iii) For symmetric A (2×2)

A is positive definite $\Leftrightarrow$ All eigenvalues of A are positive.

$|A - \lambda I| = 0$

Frisch-Waugh-Lovell Theorem

$$\mathbf{b} = (X'X)^{-1}X'y.$$

$$\mathbf{e} = y - X\mathbf{b} = y - X(X'X)^{-1}X'y = [I - X(X'X)^{-1}X']y$$

Define $M = I - X(X'X)^{-1}X$. then

$$\mathbf{e} = My.$$

We call M a <u>residual maker</u>.

OLS predicted value $\hat{y} = X b = y - e$

$$= y - My = (I - M) y.$$

Define $P = I - M = X(X'X)^{-1}X'$, and call it

$$\begin{array}{ccc} n\times k & \begin{pmatrix} k\times n \\ n\times k \end{pmatrix} & k\times n \end{array}$$

<u>projection matrix</u>.

$$\Rightarrow \quad y = Xb + e = \hat{y} + e = \underbrace{Py}_{\text{projection}} + \underbrace{My}_{\text{residual}}$$

projection residual.

M and P are both $n \times n$, symmetric and idempotent. [A is idempotent iff $A = A^2$].

$$
\begin{aligned}
P' = [X(X'X)^{-1}X']' &= (X')'[(X'X)^{-1}]'X' \\
&= X[(X'X)']^{-1}X' \\
&= X[X' \cdot (X')']^{-1}X' \\
&= X(X' \cdot X)^{-1}X' = P
\end{aligned}
$$

$M = I - P \Rightarrow M' = (I-P)' = I' - P' = I - P = M.$

$$P^2 = X(X'X)^{-1}X' \cdot X(X'X)^{-1}X'$$

$$= X(X'X)^{-1}(X'X)(X'X)^{-1}X'$$

$$= X(X'X)^{-1}X' = P.$$

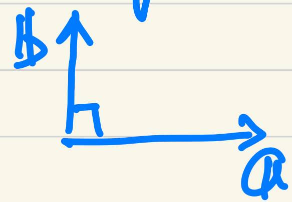
$$M^2 = (I-P)^2 = I^2 - I \cdot P - P \cdot I + P^2$$

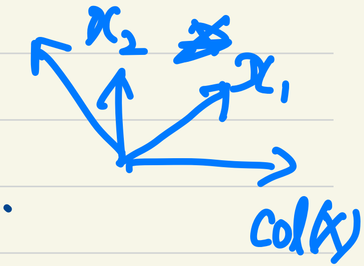
$$= I - 2P + P^2$$

$$= I - 2P + P = I - P = M.$$

M and P are orthogonal, i.e. $MP = PM = \mathbf{0}$

[Vectors a and b are orthogonal iff $a'b = b'a = 0$.

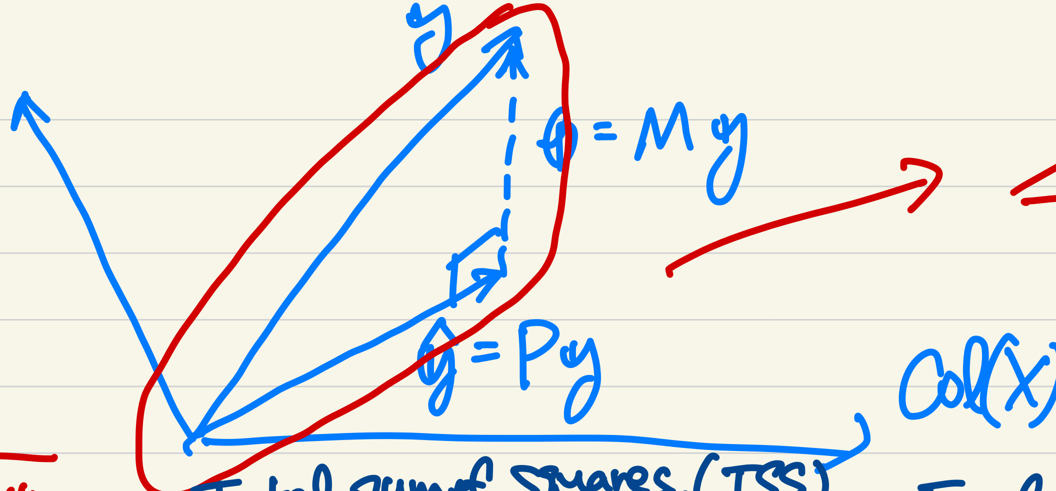


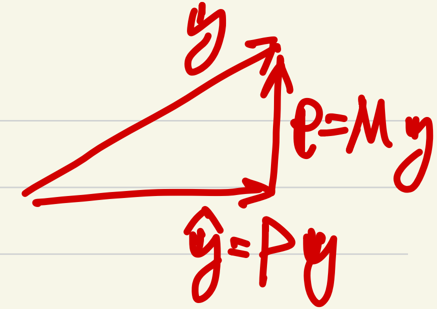


$$MP = M(I-M) = M - M^2 = M - M = \mathbf{0}.$$

$$PM = P(I-P) = P - P^2 = P - P = \mathbf{0}.$$

$$\Rightarrow PX = X, \quad MX = \mathbf{0}.$$





$\|y\| = \sqrt{y'y}$

Total sum of squares (TSS)

Explained sum of squares (ESS)

Sum of squared residuals (SSR).

$$\Rightarrow \boxed{y'y = \hat{y}'\hat{y} + e'e}$$

$$\begin{aligned} \text{RHS} &= (Py)'(Py) + (My)'(My) \\ &= y'P'Py + y'M'My \\ &= y'Py + y'My = y'(Py + My) = y'y = \text{LHS} \end{aligned}$$

Partitioned Regression.

$$y = X\beta + \varepsilon$$

Suppose $X = [X_1 \;\; X_2]$ where

$X = [x_1, x_2, \cdots, x_k]$, $X_1 = [x_1, x_2, \cdots, x_s]$,
$X_2 = [x_{s+1}, \cdots, x_k]$.

Then $y = X\beta + \varepsilon = [X_1 \; X_2]\begin{bmatrix}\beta_1 \\ \beta_2\end{bmatrix} + \varepsilon$

$= X_1\beta_1 + X_2\beta_2 + \varepsilon$

where $\beta_1 = \begin{bmatrix}\beta_1 \\ \beta_1 \\ \vdots \\ \beta_s\end{bmatrix}$, $\beta_2 = \begin{bmatrix}\beta_{s+1} \\ \vdots \\ \beta_k\end{bmatrix}$.

The normal equations are $\qquad [a, b]' = \begin{bmatrix} a' \\ b' \end{bmatrix}$

$$X'X\mathbb{b} = X'y$$

$$\begin{bmatrix} X_1' \\ X_2' \end{bmatrix} [X_1 \; X_2] \mathbb{b} = \begin{bmatrix} X_1' \\ X_2' \end{bmatrix} y$$

$$\begin{bmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_2 \end{bmatrix} \begin{bmatrix} \mathbb{b}_1 \\ \mathbb{b}_2 \end{bmatrix} = \begin{bmatrix} X_1'y \\ X_2'y \end{bmatrix}. \quad \begin{matrix} \text{--- } ① \\ \text{--- } ② \end{matrix}$$

① : $X_1'X_1\mathbb{b}_1 + X_1'X_2\mathbb{b}_2 = X_1'y$

$\Rightarrow \quad \mathbb{b}_1 = (X_1'X_1)^{-1}(X_1'y - X_1'X_2\mathbb{b}_2)$

$$\mathbf{b}_1 = (X_1'X_1)^{-1}X_1'y - \underbrace{(X_1'X_1)^{-1}X_1'X_2\mathbf{b}_2}_{\text{a correction vector.}}$$

$$y = X_1\beta_1 + \varepsilon_1$$

$$= (X_1'X_1)^{-1}X_1'(y - X_2\mathbf{b}_2)$$

$\Rightarrow$ next step : find $\mathbf{b}_2$. and $\mathbf{b}_1$.

{ X_1 and X_2 are orthogonal.

  " not orthogonal