Econometrics 2022.4.22.

The FWL theorem:

$$y = X_1 \beta_1 + X_2 \beta_2 + \varepsilon \quad \text{—— ①}$$

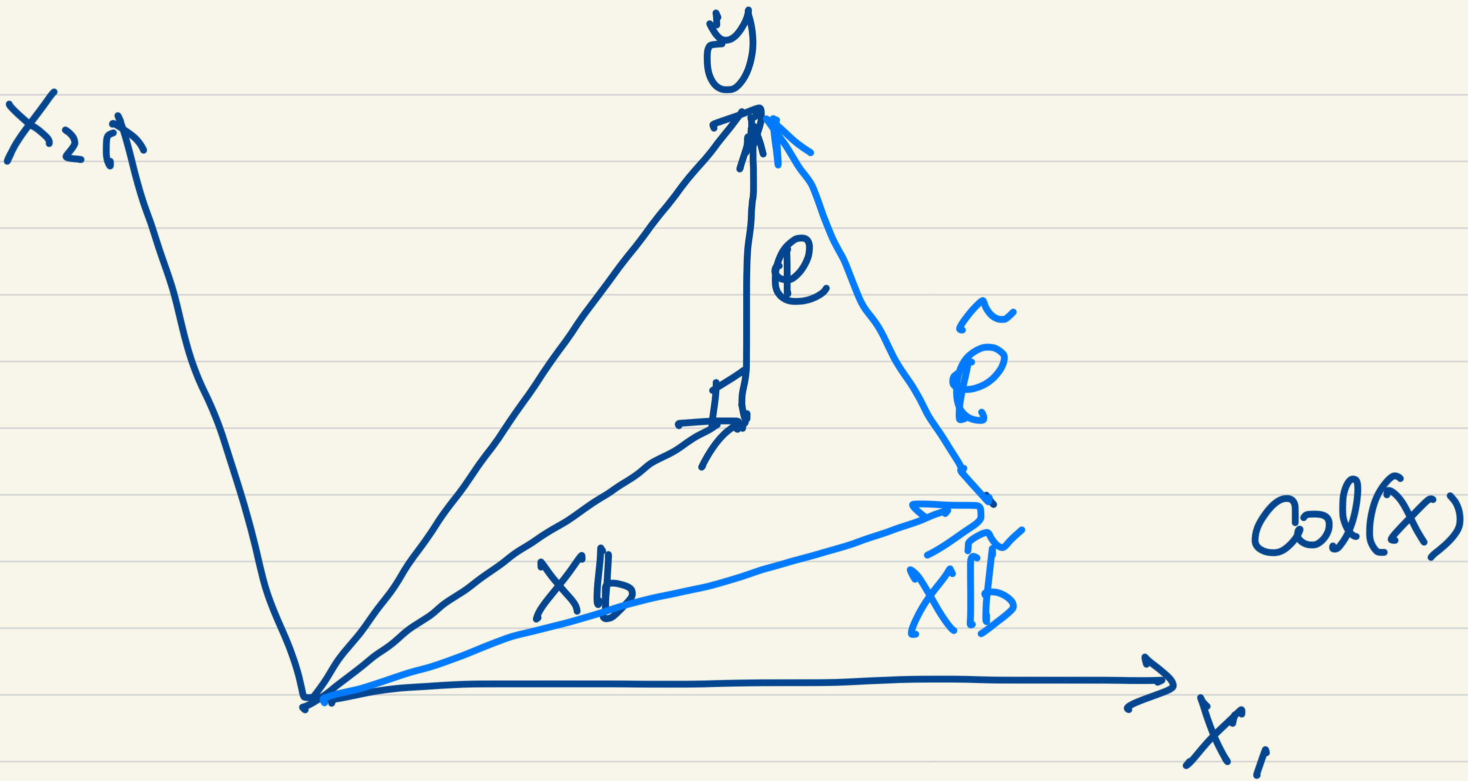
$$(M_1 y) = (M_1 X_2) \beta_2^* + \varepsilon_2^* \quad \text{—— ②}$$

$$① \overset{OLS}{\Rightarrow} b_2 \, , \quad e$$

$$\qquad\quad \| \qquad\quad \|$$

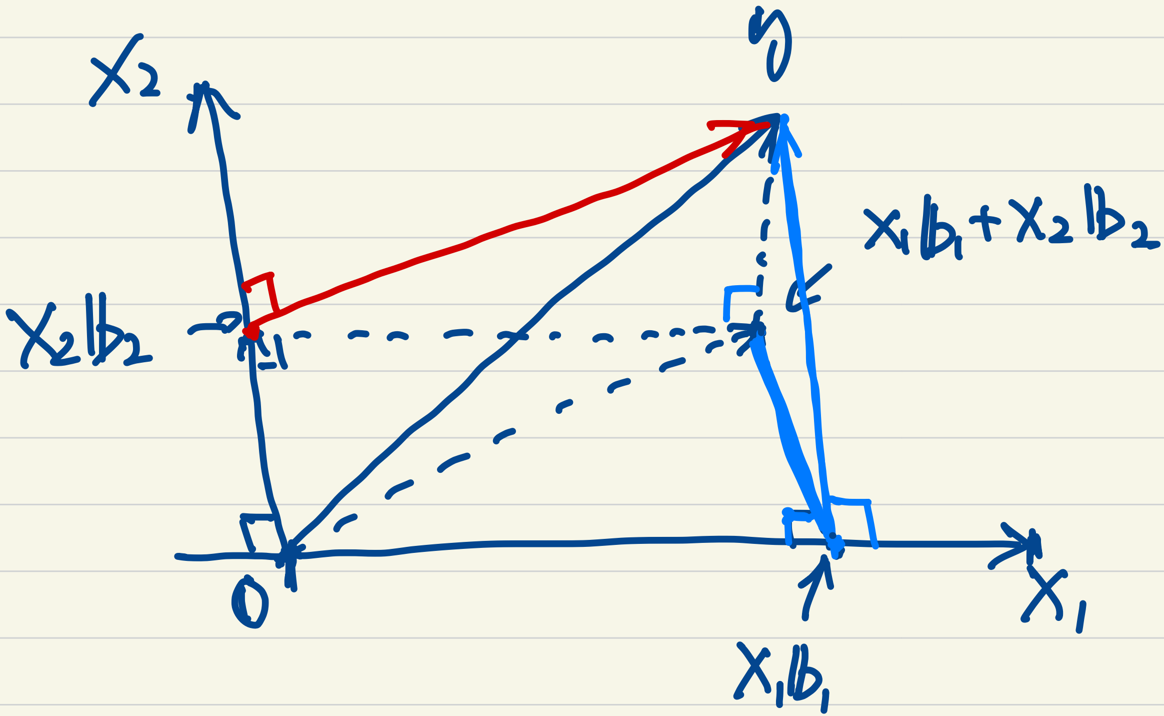
$$② \overset{OLS}{\Rightarrow} b_2^* \, , \quad e_2^*$$

↑ Prove this in homework.

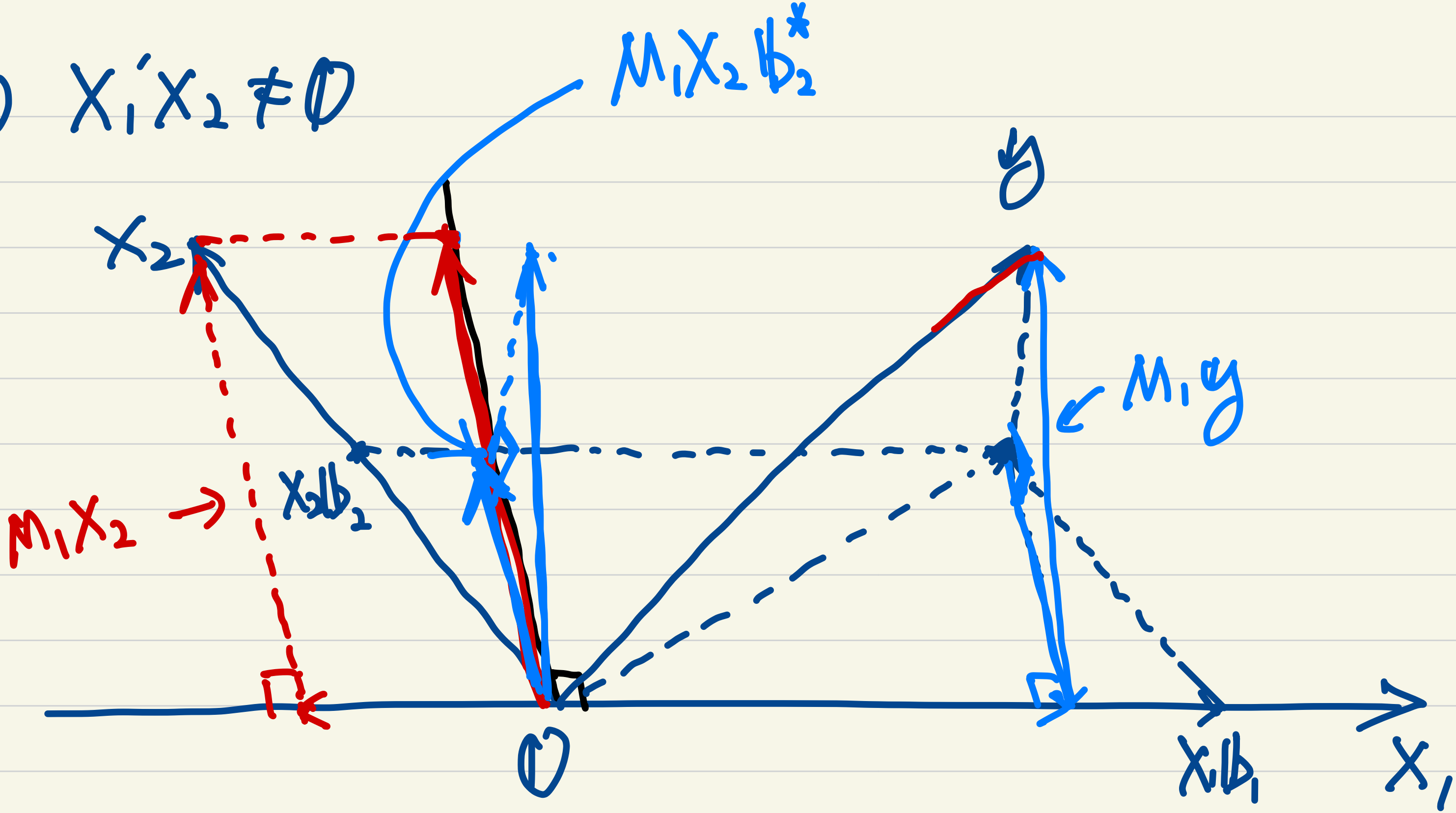


y
X_2
e
$\tilde{e}$
$X\hat{b}$
$X\tilde{b}$
col(X)
X_1

(1) $X_1' X_2 = 0$



(ii) $X_1'X_2 \neq 0$



Deviation from the mean (centering).

Consider the regression model

$$y = i\beta_1 + X_2\beta_2 + \varepsilon, \qquad i = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

$$\Rightarrow \mathbf{b}_2 = (X_2' M_1 X_2)^{-1} X_2' M_1 y.$$

Since $X_1 = i$, then

$$M_1 = I - X_1(X'X)^{-1}X'$$

$$= I - i(\underbrace{i'i})^{-1} i'$$

$$= I - \frac{1}{n} i i' = I - \frac{1}{n}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} = M^0$$

$$i'i = [1 \cdots 1]\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \sum_{i=1}^{n} 1 = n.$$

$$M_1 X_2 = M^0 X_2 = \left(I - \frac{1}{n} i i'\right) X_2$$

$$= X_2 - \frac{1}{n} i (i' X_2)$$

$$= X_2 - i [\bar{x}_2, \cdots, \bar{x}_k]$$

$$= [x_2, x_3, \cdots, x_k] - i [\bar{x}_2, \cdots, \bar{x}_k]$$

$$= [x_2 - i\bar{x}_2, \cdots, x_k - i\bar{x}_k].$$

$$i' X_2 = i' [x_2, \cdots, x_k]$$

$$= \left[\sum_{i=1}^{n} x_{i2}, \cdots, \sum_{i=1}^{n} x_{ik}\right]$$

$$\bar{x}_k = \frac{1}{n} \sum_{i=1}^{n} x_{ik}$$

$$M_1 y = M^0 y = \left(I - \frac{1}{n} i i'\right) y = y - i\bar{y}.$$

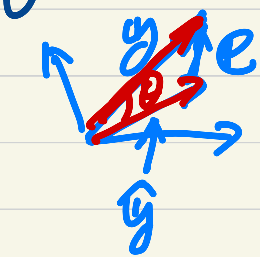
$\Rightarrow$ b_2 equals the OLS fitted value of

$$y - i\bar{y} = [x_2 - i\bar{x}_2, \cdots x_k - i\bar{x}_k] \cdot \beta^* + \varepsilon^*$$

Goodness-of-fit.

- Uncentered R^2 (coefficient of determination)

$$y = X\beta + \varepsilon = Xb + e = Py + My = \hat{y} + e$$



$$\Rightarrow \|y\|^2 = \|\hat{y}\|^2 + \|e\|^2$$

$$y'y = \hat{y}'\hat{y} + e'e$$

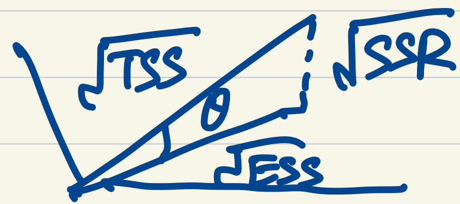
$$(TSS) = (ESS) + (SSR)$$

total sum of squares   explained sum of squares   sum of squared residuals

We define R_u^2 as

$$R_u^2 = \frac{ESS}{TSS} = \frac{TSS - SSR}{TSS} = 1 - \frac{SSR}{TSS} = \cos^2\theta$$

$$\Rightarrow \quad 0 \leq R_u^2 \leq 1$$



$$R_u^2 = \frac{ESS}{TSS} = \frac{\|Py\|^2}{\|y\|^2}$$

$$= 1 - \frac{SSR}{TSS} = 1 - \frac{\|My\|^2}{\|y\|^2}.$$

- R_u^2 is not invariant under some change of measure.

If $\underline{X = [\,\mathbb{i}\ \ X_2\,]}$, we can choose some α such that $y = \tilde{y} + \alpha\mathbb{i}$.

$$
\begin{aligned}
\tilde{y} + \alpha\mathbb{i} &= Xb + e \\
&= P(\tilde{y} + \alpha\mathbb{i}) + M(\tilde{y} + \alpha\mathbb{i}) \\
&= P\tilde{y} + \alpha \underbrace{P\mathbb{i}}_{=\mathbb{i}} + M\tilde{y} + \alpha \underbrace{M\mathbb{i}}_{=0} \\
&= \underbrace{P\tilde{y} + \alpha\mathbb{i}}_{\to \text{ESS}} + \underbrace{M\tilde{y}}_{\to \text{SSR}}
\end{aligned}
$$

$$R_u^2 = \frac{ESS}{TSS} = \frac{\|P\hat{y} + \alpha i\|^2}{\|\hat{y} + \alpha i\|^2} \xrightarrow{\alpha \to \infty} 1$$

- The centered R^2.

When $X = [i \;\; X_2]$, by the FWL theorem,

(a) $y = i b_1 + X_2 \boxed{b_2} + \boxed{e} \rightarrow R_u^2$

$\qquad\qquad\qquad\qquad\;\; \| \qquad\;\; \|$

(b) $M^0 y = M^0 X_2 \boxed{b_2} + \boxed{e} \rightarrow R_u^2$

Define

$$
\begin{aligned}
R_C^2 &= \overbrace{\frac{ESS}{TSS} = 1 - \frac{SSR}{TSS}}^{(b)} \\
&= 1 - \frac{\|My\|^2}{\|M^0 y\|^2} \\
&= 1 - \frac{e'e}{y'M^0 y} \\
&= 1 - \frac{\sum_i e_i^2}{\sum_i (y_i - \bar{y})^2}
\end{aligned}
$$