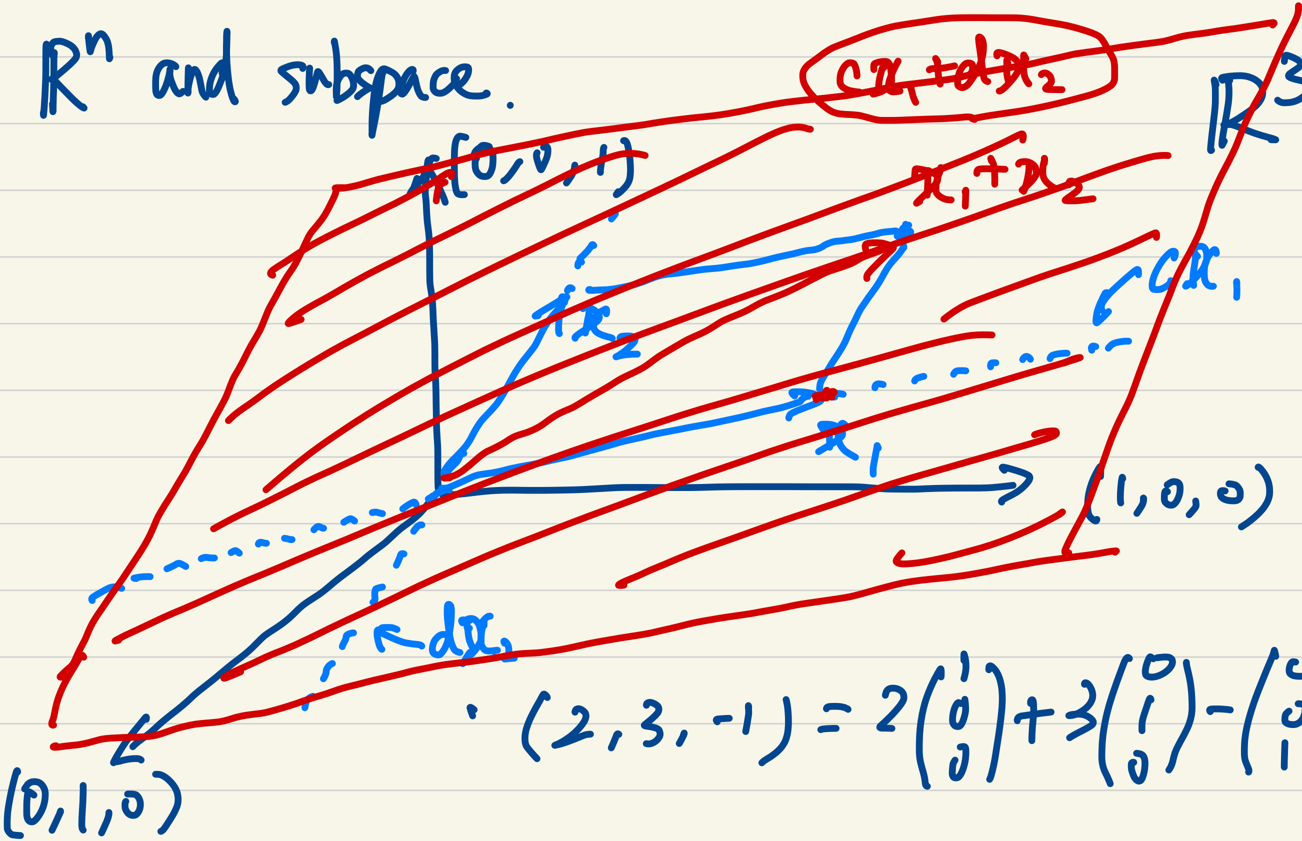
Econometrics 2022-4-1.

$\mathbb{R}^n$ and subspace.



$$(2,3,-1) = 2\begin{pmatrix}1\\0\\0\end{pmatrix} + 3\begin{pmatrix}0\\1\\0\end{pmatrix} - \begin{pmatrix}0\\0\\1\end{pmatrix}$$

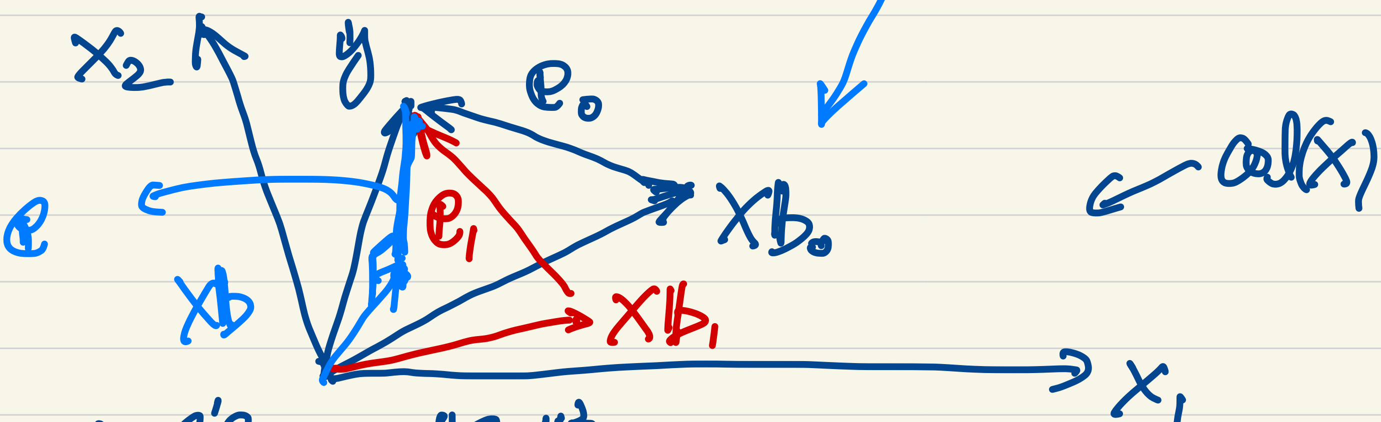
$$\mathbf{y} = X\mathbf{b}_0 + \mathbf{e}_0 = [x_1\ x_2\ \cdots\ x_k]\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_k \end{bmatrix} + \mathbf{e}_0$$

$$\mathbf{y} = X\mathbf{b}_1 + \mathbf{e}_1$$

$$= b_1 x_1 + b_2 x_2 + \cdots + b_k x_k .$$

$\mathbb{R}^n$

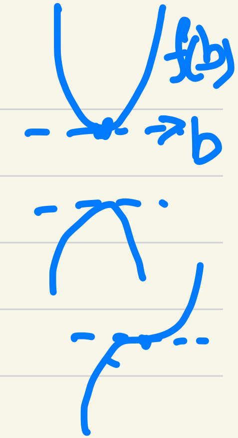
- The column space of X : $\mathrm{col}(X)$. ← k 维.



$$\min_{\mathbf{b}_0} \mathbf{e}_0'\mathbf{e}_0 = \min_{\mathbf{b}_0} \|\mathbf{e}_0\|^2$$

Least square solutions.

$$\min_{b_0} e_0' e_0 = \min_{b_0} (y - Xb_0)'(y - Xb_0).$$



$$(y-Xb_0)'(y-Xb_0) = (y' - b_0'X')(y - Xb_0)$$

$$= y'y - \underbrace{y'Xb_0}_{= \square\square = \text{scalar}} - b_0'X'y + b_0'X'Xb_0$$

A^T

$$= y'y - 2\underbrace{y'Xb_0}_{= (X'y)'b_0} + b_0'\underbrace{X'X}_{= A}b_0.$$

$(X'y)' b_0 \longrightarrow X'y$

$A \Rightarrow$ symmetric.

$$(X'X)' = X'(X')' = X' \cdot X$$

The first-order (necesary) condition (FOC)

$$\frac{\partial S(\mathbb{b}_0)}{\partial \mathbb{b}_0} = \begin{bmatrix} \frac{\partial S(\mathbb{b}_0)}{\partial b_{0,1}} \\ \frac{\partial S(\mathbb{b}_0)}{\partial b_{0,2}} \\ \vdots \\ \frac{\partial S(\mathbb{b}_0)}{\partial b_{0,k}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbb{0} \text{ — FOC.}$$

gradient

P.A-39. $\mathbb{a}'\mathbb{x} = \mathbb{x}'\mathbb{a} = \sum_{i=1}^{n} a_i x_i$

$$\frac{\partial(\mathbb{a}'\mathbb{x})}{\partial \mathbb{x}} = \begin{bmatrix} \frac{\partial(\mathbb{a}'\mathbb{x})}{\partial x_1} \\ \vdots \\ \frac{\partial(\mathbb{a}'\mathbb{x})}{\partial x_n} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \mathbb{a} \Rightarrow \frac{\partial(\mathbb{a}'\mathbb{x})}{\partial \mathbb{x}} = \mathbb{a}$$

P.A-40. $x'Ax = [x_1, \cdots, x_n] \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \\ a_{n1} & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j a_{ij}$$

If A is symmetric, then $\dfrac{\partial (x'Ax)}{\partial x} = 2Ax$

" not symmetric, then " $= (A + A')x$.

$$\Rightarrow \frac{\partial S(\mathbb{b}_0)}{\partial \mathbb{b}_0} = \frac{\partial}{\partial \mathbb{b}_0}\left(y'y - 2(X'y)'\mathbb{b}_0 + \mathbb{b}_0'(X'X)\mathbb{b}_0\right)$$

$$= -2X'y + 2X'X\mathbb{b}_0 \stackrel{\text{set}}{=} 0.$$

$$\Rightarrow \quad X'X\mathbb{b} = X'\mathbb{y}$$ least squares normal equation.

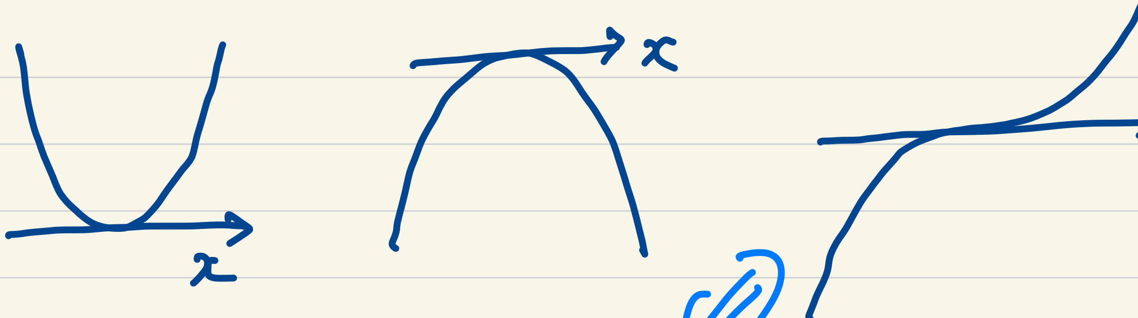
If $\text{rank}(X) = k$, then $\text{rank}(X'X) = k$ ← p. A-15.

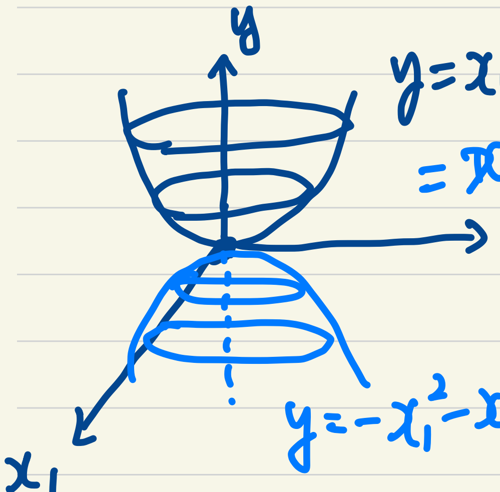
$\text{rank}(A) = \text{rank}(A')$
$= \text{rank}(A'A)$
$= \text{rank}(AA')$.

X is $(n \times k)$. $X'X = k \times k$.

$\Rightarrow X'X$ is invertible.

$$\Rightarrow \boxed{\mathbb{b} = (X'X)^{-1}X'\mathbb{y}}$$ least squares solution.

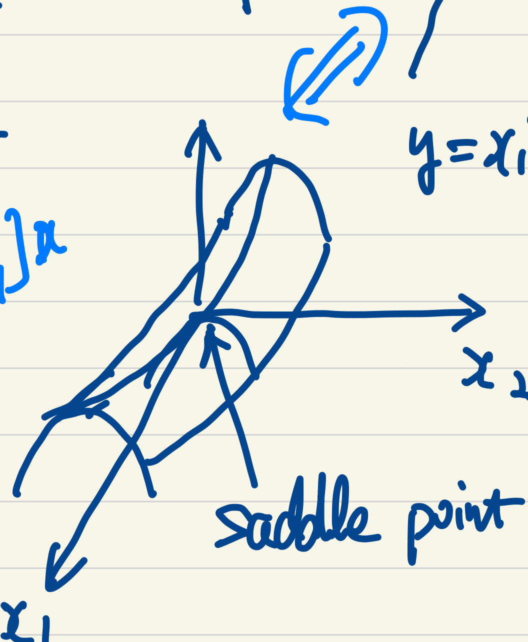




$y=x_1^2+x_2^2$

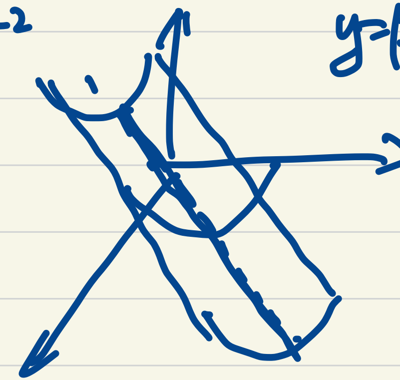
$=x'\begin{bmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & 1\end{bmatrix}x$

$y=-x_1^2-x_2^2$



$y=x_1^2-x_2^2$

saddle point



$y=(x_1-x_2)^2$

Second-order condition (SOC).

$$\frac{\partial^2 S(\mathbf{b}_0)}{\partial \mathbf{b}_0 \partial \mathbf{b}_0'} = \begin{bmatrix} \frac{\partial^2 S(\mathbf{b}_0)}{\partial b_{0,1}^{2}} & \cdots & \frac{\partial^2 S(\mathbf{b}_0)}{\partial b_{0,1}\partial b_{0,i}} & \cdots & \frac{\partial^2 S(\mathbf{b}_0)}{\partial b_{0,1}\partial b_{0,k}} \\ \vdots & \ddots & \frac{\partial^2 S(\mathbf{b}_0)}{\partial b_{0,i}^{2}} & & \vdots \\ \frac{\partial^2 S(\mathbf{b}_0)}{\partial b_{0,k}\partial b_{0,1}} & & \cdots & \ddots & \frac{\partial S(\mathbf{b}_0)}{\partial b_{0,k}^{2}} \end{bmatrix}$$

Hessian matrix

is positive definite.

正定.

p. A-41, A-42.

pA-35. For symmetric matrix A, the quadratic form is

$$q = x'Ax = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j a_{ij}.$$

- If $x'Ax \begin{cases} > 0 \\ \geq 0 \\ < 0 \\ \leq 0 \end{cases}$ for all nonzero x, then A

is $\begin{cases} \text{positive} \\ \text{negative} \end{cases}$ definite. semidefinite.

p.A-36. If A is $(n \times k)$, $n > k$, $\text{rank}(A) = k$, then $A'A$ is positive definite, AA' is positive semi-definite.

Proof. $\text{rank}(A) = k. \Rightarrow Ax \neq 0$

$$\Rightarrow x'\underline{A'A}x = (Ax)'(Ax) \quad \leftarrow y = Ax \neq 0$$

$$= y'y$$

$$= \sum_{i=1}^{n} y_i^2 > 0 \Rightarrow A'A \text{ is positive definite.}$$

(SOC) $\dfrac{\partial^2 S(\mathbb{b}_0)}{\partial \mathbb{b}_0 \partial \mathbb{b}'} = 2X'X$ is positive definite.

$\mathbb{b}$ minimizes $S(\mathbb{b}_0)$ $\leftarrow$ sum of squared residuals.