Econometrics. 2022-4-15

$y = X\beta + \varepsilon$

Partitioned Regression (cont.)

$$\underset{(n\times k)}{X} = [\underset{n\times s}{X_1} \quad \underset{n\times(k-s)}{X_2}] \Rightarrow y = [X_1 \quad X_2]\begin{bmatrix}\beta_1 \\ \beta_2\end{bmatrix} + \varepsilon$$

$$= X_1\beta_1 + X_2\beta_2 + \varepsilon.$$

long regression

The normal equations:

$$X'Xb = X'y \Rightarrow \begin{bmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_2 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} X_1'y \\ X_2'y \end{bmatrix} \begin{matrix} \text{— ①} \\ \text{— ②} \end{matrix}$$

(i) X_1 and X_2 are orthogonal.

$$X_1'X_2 = \mathbb{O}, \quad X_2'X_1 = \mathbb{O}.$$

$\mathbb{a}'\mathbb{b} = 0 \Leftrightarrow \mathbb{b}'\mathbb{a} = 0$

$\mathbb{a}$ and $\mathbb{b}$ are orthogonal

Homework

$x_1 \neq c x_2$.

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} X_1'X_1 & \mathbb{O} \\ \mathbb{O} & X_2'X_2 \end{bmatrix} \begin{bmatrix} \mathbb{b}_1 \\ \mathbb{b}_2 \end{bmatrix} = \begin{bmatrix} X_1'y \\ X_2'y \end{bmatrix}.$$

$$\Leftrightarrow \begin{cases} X_1'X_1\mathbb{b}_1 = X_1'y \\ X_2'X_2\mathbb{b}_2 = X_2'y \end{cases} \Rightarrow \begin{cases} \mathbb{b}_1 = (X_1'X_1)^{-1}X_1'y \\ \mathbb{b}_2 = (X_2'X_2)^{-1}X_2'y \end{cases}.$$

$$y = X_1\beta_1^* + \varepsilon_1^*, \quad y = X_2\beta_2^* + \varepsilon_2^*$$

short regressing

(ii) X_1 and X_2 are not orthogonal.

① $\Rightarrow \mathbb{b}_1 = (X_1'X_1)^{-1}X_1'y - (X_1'X_1)^{-1}X_1'X_2\mathbb{b}_2$

Substitute it into ②.

$$X_2'X_1\mathbb{b}_1 + X_2'X_2\mathbb{b}_2 = X_2'y$$

$$\underline{X_2'X_1(X_1'X_1)^{-1}X_1'y - X_2'X_1(X_1'X_1)^{-1}X_1'X_2\mathbb{b}_2 + X_2'X_2\mathbb{b}_2 = X_2'y}$$

$$[X_2'X_2 - X_2'X_1(X_1'X_1)^{-1}X_1'X_2]\mathbb{b}_2 = [X_2' - X_2'X_1(X_1'X_1)^{-1}X_1']y$$

I

$$X_2'\underline{[I - X_1(X_1'X_1)^{-1}X_1']}X_2\mathbb{b}_2 = X_2'\underline{[I - X_1(X_1'X_1)^{-1}X_1']}y$$

Residual maker: $M = I - X(X'X)^{-1}X'$. $(y = X\beta + \varepsilon)$

From the short regression $y = X_1\beta_1^* + \varepsilon_1^*$.

residual.

$$M_1 = I - X_1(X_1'X_1)^{-1}X_1'.$$

$$\Rightarrow \boxed{X_2'M_1X_2 b_2 = X_2'M_1 y}$$

M_1 is symmetric and idempotent.

$M_1 = M_1 \cdot M_1$
$= M_1' \cdot M_1$

$$\Rightarrow \quad b_2 = (X_2'M_1X_2)^{-1}(X_2'M_1 y)$$

$$= (X_2'M_1'M_1X_2)^{-1}(X_2'M_1'M_1 y)$$

$$= \left[(M_1X_2)'M_1X_2\right]^{-1}(M_1X_2)'M_1 y$$

$$M_1 X_2 = M_1 \begin{bmatrix} x_{s+1} & x_{s+2} & \cdots & x_k \end{bmatrix}$$

$$= \begin{bmatrix} M_1 x_{s+1} & M_1 x_{s+2} & \cdots & M_1 x_k \end{bmatrix}$$

$$\left\{ \begin{aligned} x_{s+1} &= X_1 \beta^*_{s+1} + \varepsilon^*_{s+1} \xrightarrow{\text{residual}} M_1 x_{s+1}. \\ x_{s+2} &= X_1 \beta^*_{s+2} + \varepsilon^*_{s+2} \longrightarrow M_1 x_{s+2} \\ &\vdots \\ x_k &= X_1 \beta^*_k + \varepsilon^*_{s+2} \longrightarrow M_1 x_k. \end{aligned} \right.$$

Let $X_2^* = M_1 X_2$, $y^* = M_1 y$, we have

$$\boxed{b_2 = (X_2^{*\prime} X_2^*)^{-1} X_2^{*\prime} y^* .}$$

OLS solution

$b = (X'X)^{-1} X' y$

b_2 is the OLS solution of

$$y^* = X_2^* \beta_2^{**} + \varepsilon_2^{**}$$

coefficient

[Frisch-Waugh-Lovell Theorem]

In the linear regression $y = X_1\beta_1 + X_2\beta_2 + \varepsilon$, the least squares solution b_2 is the set of coefficients obtained when the residual from a regression of y on X_1 "y^*" alone are regressed on the set of residuals obtained when each column of X_2 is regressed on X_1 "X_2^*"

The FWL Theorem says that the fitted values $\mathbb{b}_2$ in the following two models are identical.

(S1) $y = X_1\beta + X_2\beta_2 + \varepsilon$ (the long regression)

$\Rightarrow$ OLS coefficients $\underline{\mathbb{b}_2 = (X_2' M_1 X_2)^{-1}(X_2' M_1 y)}$.

(S2) Step 1. Fit the models and calculate the residuals

$y = X_1\beta_1^* + \varepsilon_1^* \Rightarrow e_y = M_1 y$

columns in X_2

$x_{s+1} = X_1\beta_{s+1}^* + \varepsilon_{s+1}^* \qquad e_{s+1} = M_1 x_{s+1}$

$\vdots$

$x_k = X_1\beta_k^* + \varepsilon_k^* \qquad e_k = M_1 x_k$

$\mathbb{b}_2^* = (X_2^{*\prime} X_2^*)^{-1} \cdot X_2^{*\prime} y^*$

Step 2. Put $y^* = e_y$, $X_2^* = [e_{s+1} \cdots e_k]$

$\Rightarrow$ The OLS coefficients of $y^* = X_2^*\beta_2^{**} + \varepsilon_2^*$.

$\Rightarrow \mathbb{b}_2 = \mathbb{b}_2^*$