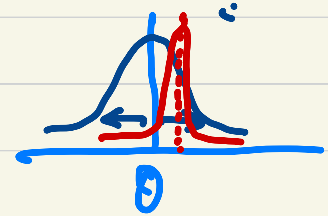
Econometrics. 2022.5.13.

The Gauss-Markov Theorem.

$\boxed{\text{When Assumptions A.1-A.4 hold, the least squares estimator } \mathbf{b} = (X'X)^{-1}X'y \text{ is the minimum variance linear unbiased estimator (or BLUE) of } \beta.}$

- A.1 : $y = X\beta + \varepsilon$. The model is correctly specified.
- A.4 : homoskadastisity & no autocorrelation.
- linear estimator.
- trade-off of bias and variance.



Asymptotic (large sample) properties.

$\|x-c\| > \varepsilon$

$|x_i - c_i| > \varepsilon$.

- $n \to \infty$

- Convergence in probability (Def. D1)

> A random variable x_n converges in probability to a constant c (a random variable z) if
>
> $$\lim_{n\to\infty} \text{Prob}(|x_n - c| > \varepsilon) = 0$$
>
> $<$ $=1$
>
> for any positive ε. $\Leftrightarrow$ $\text{plim}\, x_n = c$

Let $x_n = \begin{cases} 0. & \text{with prob. } 1-\frac{1}{n} \to 1 \\ n. & \text{"} \quad \frac{1}{n} \to 0. \end{cases}$ $\text{plim}\, x_n = 0$.

Consistent estimator (Def D.2).

An estimator $\hat{\theta}_n$ of a parameter θ is a consistent estimator of θ iff $\operatorname{plim} \hat{\theta}_n = \theta$.

Theorem D.4

The mean of a random sample $\bar{x} = \frac{\sum x_i}{n}$ from any population with finite μ and σ^2 is a consistent estimator of μ.

The OLS estimator is consistent. $\Leftrightarrow$ $\underline{\text{plim } \mathbb{b} = \beta}$

$$\mathbb{b} = (X'X)^{-1}X'y$$ (A.1, A.2)

$$= (X'X)^{-1}X'(X\beta + \varepsilon)$$

$$= \beta + \underbrace{(X'X)^{-1}}\cdot\underbrace{(X\varepsilon)}$$

$$\text{plim } \mathbb{b} = \text{plim } \beta + \text{plim } \_\_\_\_$$

$$= \beta + \text{plim } \_\_\_\_$$

Therem D.14

If $\text{plim } X_n = A$, $\text{plim } Y_n = B$, then

$$\text{plim } X_n Y_n = AB.$$

$\cdot\ (X'X)^{-1}$.

$$\underset{(k\times n)}{X'}\ \underset{(n\times k)}{X} = \begin{array}{c} \\ x_s' \\ \\ \end{array}\begin{bmatrix} \\ \underbrace{\qquad\qquad\qquad\qquad}_{n\text{ obs.}} \\ \\ \end{bmatrix} \begin{array}{c} x_t \\ \begin{bmatrix} & \\ n\text{ obs.}\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. & \\ & \end{bmatrix} \end{array} x_{i,t}.$$

The (s, t) element of $X'X$ is $\underline{\sum_{i=1}^{n} x_{is}\cdot x_{it}}$.

$\frac{1}{n}X'X$

A.S.a $(x_i, \varepsilon_i), i=1,\cdots,n$ is a sequance of i.i.d. observations.

Theorem D.5. Law of Large numbers.

If x_i, $i=1,\cdots,n$ is i.i.d. random sample with $E[x_i]=\mu$. then

$$\operatorname{plim} \bar{x}_n = \cancel{\operatorname{plim}}\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \mu.$$

$$X'X = \begin{bmatrix} \cdots & x_i & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ x_i' \\ \vdots \end{bmatrix} = \sum_{i=1}^{n} x_i x_i'$$

$$[0 \cdots 0 \cdots 0] \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \sum 0 \cdot 0$$

$$\frac{1}{n}X'X = \frac{1}{n}\sum_{i=1}^{n} x_i x_i' \xrightarrow{\;P\;} E[x_i x_i'] .$$

Let $E[x_i x_i'] = Q$ ← positive definite, invertible.

We have $\operatorname{plim}\left(\frac{1}{n}X'X\right) = Q$.

$$\Rightarrow \quad \operatorname{plim}\left(\frac{1}{n}X'X\right)^{-1} = Q^{-1} .$$

$X'\varepsilon$. (A.3)

$$E[X'\varepsilon] = E_X[E[X'\varepsilon|X]] = E_X[X' \cdot \underbrace{E[\varepsilon|X]}_{=0}]$$

$$= E_X[X'0] = 0.$$

$$\Rightarrow E[x_i\varepsilon_i] = 0.$$

$$X'\varepsilon = \sum_{i=1}^{n} x_i\varepsilon_i$$

$$\text{plim}\left(\frac{1}{n}X'\varepsilon\right) = \text{plim}\left(\frac{1}{n}\sum_{i=1}^{n} x_i\varepsilon_i\right) = E[x_i\varepsilon_i] = 0.$$

$$\operatorname{plim}(\mathbb{b}) = \beta + \operatorname{plim}\left(\frac{1}{n}X'X\right)^{-1} \cdot \operatorname{plim}\left(\frac{1}{n}X'\varepsilon\right)$$

$$= \beta + Q^{-1} \cdot \mathbb{0}$$

$$= \beta \, .$$

Under assumptions A.1 ~ A.3 and A.5.a,

$\mathbb{b}$ is a consistent estimator of β.