Econometrics. 2022.5.6

Statistical Properties of the OLS estimator.

$$E[(W-\theta)^2] = \underbrace{Var[W]}_{\text{variance}} + (\underbrace{E[W]-\theta}_{\text{bias}})^2$$

MSE: mean squared error

variance.

bias^2

→ minimum variance unbiased estimator. (MVUE).

→ OLS estimator $\mathbb{b}$ is minimum variance linear unbiased estimator

best

⇒ BLUE.

Finite sample properties of the OLS estimator.
(small).

Bias : Def. C2.

- $E[\mathbb{b}] = \beta \Leftarrow \mathbb{b}$ is an unbiased estimator of β.

The OLS solution is

A.2

$| \; y = X\beta + \varepsilon$ (A.1)

$$\begin{aligned}\mathbb{b} &= (X'X)^{-1}X'y \\ &= (X'X)^{-1}X'(X\beta+\varepsilon) \\ &= \underbrace{(X'X)^{-1}}\,\underbrace{X'X}\,\beta + (X'X)^{-1}X'\varepsilon \\ &= \beta + (X'X)^{-1}X'\varepsilon .\end{aligned}$$

$$E[\mathbb{b} \mid X] = E[\beta + \underline{(X'X)^{-1}X\varepsilon} \mid \underline{X}]$$

A.3 : $E[\varepsilon \mid X] = 0.$

$$= \beta + (X'X)^{-1}X \cdot E[\varepsilon \mid X]$$

$$= \beta$$

$$E[E[\mathbb{b} \mid X]] = \underset{\sim\sim\sim\sim\sim\sim\sim}{E[\mathbb{b}] = \beta}.$$

$\mathbb{b}$ is unbiased.

- The efficiency of $\mathbb{b}$.

Efficient estimator. Def. C. 3.

$\hat{\theta}_1$ and $\hat{\theta}_2$ are estimators of θ

$\Rightarrow$ $\hat{\theta}_1$ is more efficient than $\hat{\theta}_2$ if

$Var[\hat{\theta}_2] - Var[\hat{\theta}_1]$ is positive (semi) definite.

E.g. $x_1, \cdots, x_n$ are i.i.d. $E[x_i] = \mu$, $Var[x_i] = \sigma^2$

$$\left.\begin{aligned} \bar{x} &= \frac{1}{n}\sum_{i=1}^{n} x_i, \\ x_1 & \end{aligned}\right\} \text{are estimators of } \mu. \Rightarrow \begin{aligned} &E[\bar{x}] = E[x_1] = \mu. \\ &Var[\bar{x}] = \frac{1}{n}\sigma^2 \\ &Var[x_1] = \sigma^2. \end{aligned}$$

$$\mathbb{b} = \beta + \underbrace{(X'X)^{-1}X'}_{= A}\varepsilon = \beta + A\varepsilon.$$

$$\begin{aligned}
\mathrm{Var}[\mathbb{b}|X] &= E\left[(\mathbb{b} - E[\mathbb{b}|X])(\mathbb{b} - E[\mathbb{b}|X])' \,|\, X\right] \quad \text{(def.)} \\
&= E\left[(\mathbb{b} - \beta)(\mathbb{b} - \beta)' \,|\, X\right] \\
&= E\left[A\varepsilon\varepsilon'A' \,|\, X\right]. \\
&= A\,E[\varepsilon\varepsilon'|X]\,A' \quad \leftarrow A.4 : E[\varepsilon\varepsilon'|X] = \sigma^2 I \\
&= A(\sigma^2 I)A' \\
&= \sigma^2 (X'X)^{-1}\underbrace{X' \cdot X}(X'X)^{-1}
\end{aligned}$$

$$\mathrm{Var}[\mathbb{b}|X] = \sigma^2 (X'X)^{-1}.$$

- Linear estimator of β $\stackrel{\text{def}}{=}$ a linear function of $\mathbb{y}$

$$\Updownarrow$$

$\exists\, \underset{(k \times n)}{C}$: $C\mathbb{y}$ is a linear estimator of β.

C depends on X.

Let $\mathbb{b}_0 = C\mathbb{y}$,

then $E[\mathbb{b}_0 \mid X] = E[C\mathbb{y} \mid X] = E[C(X\beta + \varepsilon) \mid X]$

$$= E[CX\beta + C\varepsilon \mid X]$$

$$= CX\beta + C\underbrace{E[\varepsilon \mid X]}_{= 0} = CX\beta.$$

$\Rightarrow \mathbb{b}_0$ is unbiased iff $\underset{\sim}{CX = I}$.

$\mathbb{b} = Ay \Rightarrow Var[\mathbb{b}|X] = \sigma^2 AA'$.

$\mathbb{b}_0 = Cy \Rightarrow Var[\mathbb{b}_0|X] = \sigma^2 CC'$

Let $D = C - A \Rightarrow DX = (C-A)X = \underset{\overset{\shortparallel}{I}}{CX} - \underset{\overset{\shortparallel}{I}}{AX} = \mathbb{O}$.

$DA' = \underline{DX}(X'X)^{-1} = \mathbb{O}$

$Var[\mathbb{b}_0|X] - Var[\mathbb{b}|X]$

$= \sigma^2 CC' - \sigma^2 AA'$

$= \sigma^2 (D+A)(D+A)' - \sigma^2 AA'$

$= \sigma^2 DD' + \underset{\mathbb{O}}{\sigma^2 DA'} + \underset{\mathbb{O}}{\sigma^2 AD'} + \cancel{\sigma^2 AA'} - \cancel{\sigma^2 AA'}$

$= \sigma^2 \underset{\sim}{DD'}$

positive semi-definite.

$\Rightarrow \mathbb{b}$ is BLUE.