Proofs of Lemma III.2 and Proposition III.3 in "Electronic Component Parameter Recognition by a Nonparametric Approach with an Assignment Algorithm"

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Lemma III.2. *Let $p_{i,(k)} = \max_{j\in\{1,\dots,k\}} p_{i,j}$ and $\hat{p}_{i,(k)} = \max_{j\in\{1,\dots,k\}} \hat{p}_{i,j}$, where $\hat{p}_{i,j}$ is defined in (9). Let $\hat{p}_{i,h_i^0}$ been defined similar to p_{i,h_i^0} in Proposition III.1 with replacing $p_{i,j}$ by $\hat{p}_{i,j}$. If (C.1) holds and $p_{i,(k)} = p_{i,h_i^0}$ for each $i = 1,\dots,k$, we have $P(\hat{p}_{i,(k)} = \hat{p}_{i,h_i^0}) = 1$ as $n_i \to \infty$.*

Proof. Let $p_{i,h_i^0} = p_{i,1}$ without loss of generality for a fixed $i \in \{1,\dots,k\}$. It suffices to show that $P(\hat{p}_{i,h_i^0} = \hat{p}_{i,1}) = P(A\hat{\mathbf{p}}_i > 0) \to 1$ as $n_i \to \infty$, where

$$A = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 1 & \cdots & \cdots & 0 & -1 \end{pmatrix}_{(k-1)\times k}.$$

and $\hat{\mathbf{p}}_i$ defined in (9) is the MLE of $\mathbf{p}_i$ under (C.1). In addition, let $\hat{\Sigma}_n$ be the estimator of Σ with replacing the $p_{i,j}$ in (10) by $\hat{p}_{i,j}$. To simplify the notation, we skip the subscript i in the following derivation since the following arguments hold for any fixed $i \in \{1,\dots,k\}$.

By (9), we have $\mathrm{E}(A\hat{\mathbf{p}}) = A\mathbf{p} > 0$, where the inequality holds by the assumption of $p_{h^0} = p_1$. In addition, $\mathrm{Var}(A\hat{\mathbf{p}})$ can be approximated well by $B = n^{-1}A\Sigma A^\top$ for sufficiently large n. Furthermore, since there still exist unknown parameters in Σ, B is usually estimated by $\hat{B} = n^{-1}A\widehat{\Sigma}A^\top$, where $\widehat{\Sigma}$ is the mle of Σ. By the Strong Law of Large Numbers, $||\hat{\mathbf{p}} - \mathbf{p}||_2 = o_p(1)$ and thereby $||\widehat{\Sigma} - \Sigma||_F = o_p(1)$, where $||\cdot||_2$ denotes the L_2 norm of a vector and $||\cdot||_F$ denotes the Frobenius norm of a matrix. Therefore, $||\hat{B} - B||_F = o_p(n^{-1})$ as $n \to \infty$.

Additionally, since B is p.d. and by Cholesky decomposition, there exists a lower triangular matrix L with size $(k-1)\times(k-1)$ such that $B = LL^\top$, which indicates that $||LL^\top - \hat{B}||_F = o_p(n^{-1})$. Therefore, we have $P(A\hat{\mathbf{p}} > 0) \approx P(\mathbf{Z} > -L^{-1}\boldsymbol{\mu})$, where $\mathbf{Z}$ is a $(k-1)$-dimensional standard normal random vector and $\boldsymbol{\mu} = A\mathbf{p}$ is a $(k-1)\times 1$ positive vector. Notably,

$$L^{-1} = (I_{k-1} - H)(\mathrm{diag}(L))^{-1}, \tag{III.1}$$

where H is a $(k-1)\times(k-1)$ lower triangular matrix with (i,j) element being $-\ell_n(i,j)/\ell_n(i,i)$ for $j < i$ and $i = 2,\dots,k-1$, and $\ell_n(i,j)$ is the (i,j) component of L. Based on $B = LL^\top$ and $||B||_F = O(n^{-1})$, we have $\ell_n(i,j) = O(n^{-1/2})$ for $i \le j$. This together with (III.1) lead to $\inf(L^{-1}\boldsymbol{\mu}) = O(n^{1/2})$. Consequently, we have $P(A\hat{\mathbf{p}} > 0) \approx P(\mathbf{Z} > -L^{-1}\boldsymbol{\mu}) \to 1$ as $n \to \infty$. The proof is complete. □

Proposition III.3. *Under (A.1), (A.2), (C.1), and (C.2), if $\hat{\sigma}^2_{c|d,i,h_i^0} \ge \hat{\sigma}^2_{c|d,i,j}$ and*

$$\Phi\Big(\frac{|\hat{\Delta}_{i,j}|}{2\hat{\sigma}_{c|d,i,h_i^0}}\Big) > \frac{\hat{r}_{i,j}}{\hat{r}_{i,j}+1}, \tag{III.2}$$

or if $\hat{\sigma}^2_{c|d,i,h_i^0} < \hat{\sigma}^2_{c|d,i,j}$ and

$$\Phi\Big(\frac{|\hat{\Delta}_{i,j}|}{2\hat{\sigma}_{c|d,i,j}}\Big) > \frac{\hat{r}_{i,j}+0.5}{\hat{r}_{i,j}+1}, \tag{III.3}$$

we have

$$P\Big(\min_{j\in\{1,\dots,k\}} \frac{\hat{p}_{i,h_i^0}}{\hat{p}_{i,j}} \times \frac{\hat{q}_{i,h_i^0}}{\hat{q}_{i,j}} > 1\Big) = 1, \text{ as } n_{i,j} \to \infty, \tag{III.4}$$

where $\Phi(\cdot)$ is the distribution function of $N(0,1)$, I_A denotes an indicator function of an event A, $\hat{p}_{i,h_i^0}$ and $\hat{p}_{i,j}$ are defined the same as in Lemma III.2, and $\hat{q}_{i,j}$, $\hat{\Delta}_{i,j} = \hat{\mu}_{c|d,i,h_i^0} - \hat{\mu}_{c|d,i,j}$, and $\hat{r}_{i,j} = \hat{p}_{i,j}/\hat{p}_{i,h_i^0}$ are the MLEs of $q_{i,j}$, $\Delta_{i,j} = \mu_{c|d,i,h_i^0} - \mu_{c|d,i,j}$, and $r_{i,j} = p_{i,j}/p_{i,h_i^0}$, respectively, for $i,j = 1,\dots,k$.

Proof. Without loss of generality, consider the case of $i = 1$ and let $q_{1,h_1^0} = q_{1,1}$. To simplify the illustration, we skip the subscripts $c|d$ and i in the following derivation since the following arguments hold for any fixed $i \in \{1,\dots,k\}$.

In the case of $n_c = 1$, the $d(\mathbf{X}_{c,i}, \boldsymbol{\mu}_{c|d,i,j})$ in (A.2) is simplified as $d(X_c, \mu_j) = |X_c - \mu_j|/\sigma_j$. If $\mathbf{X} = (\mathbf{X}_d^\top, X_c)^\top$ should be correctly assigned to θ_1, (1) indicates that $\mathbf{X}_d$ should be assigned to θ_1 and X_c is $N(\mu_1, \sigma_1^2)$ distributed. In this case and by (A.2), it suffices to show that

$$q_1 = P(X_c \to \theta_1 \mid E_1) > \max_{j=2,\dots,k} r_j q_j, \tag{III.5}$$

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where $r_j = p_j/p_1 > 0$.

On the LHS of (III.5), we need to consider the event of $\{d(X_c, \mu_1) < \min_{j=2,\ldots,k} d(X_c, \mu_j)\}$. For any $j \in \{2, \ldots, k\}$, since $\{d(X_c, \mu_1) < d(X_c, \mu_j)\} = \{a_j Z^2 + 2\sigma_1 \Delta_j Z + \Delta_j^2 > 0\}$, where $a_j = \sigma_1^2 - \sigma_j^2$, $\Delta_j = \mu_1 - \mu_j$, and Z is a standard normal random variable. If $a_j = 0$ (that is, $\sigma_1 = \sigma_j$), we have

$$P\Big(d(X_c, \mu_1) < d(X_c, \mu_j)\Big) = \Phi\Big(\frac{|\Delta_j|}{2\sigma_1}\Big) := \Phi_j, \quad \text{(III.6)}$$

where $\Phi(\cdot)$ is the distribution function of $N(0, 1)$. If $a_j > 0$ (that is, $\sigma_1 > \sigma_j$), we have

$$P\Big(d(X_c, \mu_1) < d(X_c, \mu_j)\Big) > \Phi_j. \quad \text{(III.7)}$$

If $a_j < 0$ (that is, $\sigma_1 < \sigma_j$), we have

$$P\Big(d(X_c, \mu_1) < d(X_c, \mu_j)\Big) > 2\Phi^* - 1, \quad \text{(III.8)}$$

where $\Phi_j^* = \Phi(|\Delta_j|/(2\sigma_j))$. Therefore, (III.6)-(III.8) reveal that

$$\begin{aligned} &P\Big(d(X_c, \mu_1) < d(X_c, \mu_j)\Big) \\ &\qquad \geq \Phi_j I_{\{\sigma_1^2 \geq \sigma_j^2\}} + \Big(2\Phi_j^* - 1\Big) I_{\{\sigma_1^2 < \sigma_j^2\}}, \end{aligned} \quad \text{(III.9)}$$

which tends to 1 for sufficiently large $|\Delta_j|/\max(\sigma_1, \sigma_j)$, $j = 2, \ldots, k$. Consequently, if $\min_{\ell=1,\ldots,k} |\Delta_j|/\sigma_\ell$ is sufficiently large, $q_1 = P(d(X_c, \mu_1) < \min_{j=2,\ldots,k} d(X_c, \mu_j))$ also approaches to 1.

On the RHS of the inequality of (III.5), we need to consider the event of $\{\min_{\ell \in \{1,\ldots,k\}, \ell \neq j} d(X_c, \mu_\ell) > d(X_c, \mu_j)\}$ under the consideration that $\mathbf{X}_d$ should be assigned to θ_1 but now is incorrectly assigned to θ_j, and X_c is $N(\mu_1, \sigma_1^2)$ distributed. Note that $q_j \leq q_j^1 = P(d(X_c, \mu_1) > d(X_c, \mu_j))$ and (III.9) leads to

$$q_j^1 \leq \Big(1 - \Phi_j\Big) I_{\{\sigma_1^2 \geq \sigma_j^2\}} + 2\Big(1 - \Phi_j^*\Big) I_{\{\sigma_1^2 < \sigma_j^2\}}. \quad \text{(III.10)}$$

By (III.5), (III.9), and (III.10), the desired result $\min_{j=2,\ldots,k}\{p_1 q_1/(p_j q_j)\} > 1$ holds in the population perspective if $q_1 > r_j q_j^1$ for all $j = 2, \ldots, k$, which holds if

$$\Phi_j > \frac{r_j}{r_j + 1} I_{\{\sigma_1^2 \geq \sigma_j^2\}} \;\text{ or }\; \Phi_j^* > \frac{r_j + 0.5}{r_j + 1} I_{\{\sigma_1^2 < \sigma_j^2\}}, \quad \text{(III.11)}$$

for all $j = 2, \ldots, k$, if the parameters μ_j, σ_j, and p_j for all $j = 1, \ldots, k$ are given.

The final step of the proof is to consider the estimation of μ_j, σ_j, and p_j, $j = 1, \ldots, k$. By Lemma III.2 and (C.2), $\hat{p}_j$, $\hat{\mu}_j$, and $\hat{\sigma}_j$ are the MLEs of p_j, μ_j, and σ_j, respectively, for $j = 1, \ldots, k$. Consequently, we have $|\hat{p}_j - p_j| = o_p(1)$, $|\hat{\mu}_j - \mu_j| = o_p(1)$, and $|\hat{\sigma}_j - \sigma_j| = o_p(1)$ for $j = 1, \ldots, k$. By the invariant property of MLEs, we also have $|\hat{d}(X_c, \hat{\mu}_j) - d(X_c, \mu_j)| = o_p(1)$, $|\hat{r}_j - r_j| = o_p(1)$, $|\hat{\Delta}_j - \Delta_j| = o_p(1)$, $|\hat{\Phi}_j - \Phi_j| = o_p(1)$, and $|\hat{\Phi}_j^* - \Phi_j^*| = o_p(1)$ as $n_j \to \infty$, where $\hat{d}(X_c, \hat{\mu}_j) = |X_c - \hat{\mu}_j|/\hat{\sigma}_j$, $\hat{\Phi}_j = \Phi\Big(|\hat{\Delta}_j|/(2\hat{\sigma}_1)\Big)$, and $\hat{\Phi}_j^* = \Phi\Big(|\hat{\Delta}_j|/(2\hat{\sigma}_j)\Big)$ for $j = 1, \ldots, k$. Therefore, if (III.11) holds, (III.2) and (III.3) follow with probability 1, and thereby $P\Big(\frac{\hat{p}_1}{\hat{p}_j} \times \frac{\hat{q}_1}{\hat{q}_j} > 1\Big) = 1$, as $n_j \to \infty$, for all $j = 2, \ldots, k$. $\square$