Bethe Ansatz Solution of ASEP with Reflecting Boundaries

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To investigate the ASEP model of N particles on L lattice sites with reflecting boundaries. Let us start with one single particle in such a closed lattice. The master equation of the particle can be written as

$$\frac{d}{dt}P_x(t) = \alpha P_{x-1}(t) + \beta P_{x+1}(t) - (\alpha + \beta)P_x(t) \tag{1a}$$

$$\frac{d}{dt}P_1(t) = \beta P_2(t) - \alpha P_1(t) \tag{1b}$$

$$\frac{d}{dt}P_L(t) = \alpha P_{L-1}(t) - \beta P_L(t) \tag{1c}$$

where α and β is hopping rate of particle to left and right, respectively. x denotes the position of the particle and Eq. (1b) and (1c) are actually the special cases of master equation at the boundaries. By assuming Eq. (1a) is valid for the whole space, we can rewrite Eq. (1b) and (1c) as boundaries condition

$$\alpha P_0(t) = \beta P_1(t) \tag{2a}$$

$$\alpha P_L(t) = \beta P_{L+1}(t) \tag{2b}$$

The above equations use the technique so called "ghost coordinate", i.e. $x = 0, L + 1$. But physically they are essentially the same as the master equation (1b) and (1c), which means the flux of particle are balanced in both direction, thus the reflecting boundaries. The advantage of using Eq. (2) is that it simplifies the calculation a lot.

To solve the case of single particle ASEP, we take the ansatz of separation of variables $P_x(t) = \phi(x)e^{\lambda t}$ and plug into the master equation, obtaining

$$\beta \phi(x + 1) - (\alpha + \beta + \lambda)\phi(x) + \alpha \phi(x - 1) = 0 \tag{3}$$

1

Given that x is an integer number, Eq. (3) is essentially a set of liner difference equations with the boundaries by substituting the ansatz of $P_x(t)$ into boundaries of Eq. (2):

$$\alpha\phi(0) = \beta\phi(1) \tag{4a}$$
$$\alpha\phi(L) = \beta\phi(L+1) \tag{4b}$$

The standard method to find the solution is again to take an ansatz $\phi(x) = Az^x$, where z is an arbitrary complex number. We arrive at the characteristic quadratic equation

$$\beta z^2 - (\alpha + \beta + \lambda)z + \alpha = 0 \tag{5}$$

The two roots fulfill $z_1 z_2 = \frac{\alpha}{\beta}$. And the solution of (3) can be written as

$$\phi(x) = C_1 z_1^x + C_2 z_2^x \tag{6}$$

By applying the boundaries Eq. (4) to Eq. (6) we can find all the eigenvalues and corresponding eigenvectors. The results are summarised as following

$$\lambda_k = \begin{cases} -(\alpha + \beta) + 2\sqrt{\alpha\beta}\cos(\frac{k\pi}{L}); & k = 1, 2, \ldots, L-1 \\ 0; & k = L \end{cases} \tag{7a}$$

$$\phi_k(x) = \begin{cases} const. \left(\frac{\alpha}{\beta}\right)^{\frac{x}{2}} \left[\sin\left(\frac{k\pi}{L}x\right) - \sqrt{\frac{\beta}{\alpha}}\sin\left(\frac{k\pi}{L}(x-1)\right)\right]; & k = 1, 2, \ldots, L-1 \\ const. \left(\frac{\alpha}{\beta}\right)^x; & k = L \end{cases} \tag{7b}$$

The eigenvalue $\lambda_L = 0$ and corresponding eigenvector represent the stationary mode. By properly choose the constant and when $L \to \infty$, one can check the orthogonality and completeness of the eigenfunctions. Need to double check!

$$\sum_{x=1}^{L} \phi_k(x)\phi_l(x) = \delta_{k,l} \tag{8}$$

$$\sum_{k=1}^{L} \phi_k(x)\phi_k(y) = \delta_{x,y} \tag{9}$$

So for arbitrary initial distribution of $P(x,0)$, we can always decompose it as

$$P_x(0) = \sum_k c_k \phi_k(x) \tag{10}$$

2

where c_k can be calculated by

$$c_k = \sum_x \phi_k(x) P_x(0) \tag{11}$$

Finally, the solution of single particle on reflecting lattice can be written as

$$P_x(t) = \sum_k \phi_k(x) e^{\lambda_k t} \sum_y \phi_k(y) P_y(0) \tag{12}$$

For the special case that $P_x(0) = \delta_{x,y}$, solution (12) can be simplified to

$$P_x(t) = \sum_k \phi_k(x) \phi_k(y) e^{\lambda_k t} \tag{13}$$

With the complete solution of single particle, we can go further to systems of more than one particle. For the purpose of illustrating the path of solution searching, we go just one step to show the solution of two particles on L lattice sites. The position of particles are denotes by x_1, x_2 with constraint $x_1 < x_2$.