Bethe Ansatz Solution of ASEP with Reflecting Boundaries

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1 Single Particle Solution

To investigate the ASEP model of N particles on L lattice sites with reflecting boundaries. Let us start with one single particle in such a closed lattice. The master equation of the particle can be written as

$$\frac{d}{dt}P(x,t) = \alpha P(x-1,t) + \beta P(x+1,t) - (\alpha+\beta)P(x,t) \tag{1a}$$

$$\frac{d}{dt}P(1,t) = \beta P(2,t) - \alpha P(1,t) \tag{1b}$$

$$\frac{d}{dt}P(L,t) = \alpha P(L-1,t) - \beta P(L,t) \tag{1c}$$

where α and β is hopping rate of particle to left and right, respectively. x denotes the position of the particle is confined to be the integer in the range of $1, 2, \cdots, L$. Eq. (1b) and (1c) are actually the special cases of master equation at the boundaries. By assuming Eq. (1a) is valid for the whole space, we can rewrite Eq. (1b) and (1c) as boundaries condition

$$\alpha P(0,t) = \beta P(1,t) \tag{2a}$$

$$\alpha P(L,t) = \beta P(L+1,t) \tag{2b}$$

The above equations use the technique so called "ghost coordinate", i.e. $x = 0, L+1$. But physically they are essentially the same as the master equation (1b) and (1c), which means the flux of particle are balanced in both direction, thus the reflecting boundaries. The advantage of using Eq. (2) is that it simplifies the calculation a lot.

To solve the case of single particle ASEP, we take the ansatz of separation of variables $P(x,t) = \phi(x)e^{\lambda t}$ and plug into the master equation, obtaining

$$\beta\phi(x+1) - (\alpha+\beta+\lambda)\phi(x) + \alpha\phi(x-1) = 0 \tag{3}$$

Given that x is an integer number, Eq. (3) is essentially a set of liner difference equations with the boundaries by substituting the ansatz of $P_x(t)$ into boundaries of Eq. (2):

$$\alpha\phi(0) = \beta\phi(1) \tag{4a}$$
$$\alpha\phi(L) = \beta\phi(L+1) \tag{4b}$$

The standard method to find the solution is again to take an ansatz $\phi(x) = Az^x$, where z is an arbitrary complex number. We arrive at the characteristic quadratic equation

$$\beta z^2 - (\alpha+\beta+\lambda)z + \alpha = 0 \tag{5}$$

The two roots fulfill $z_+ z_- = \frac{\alpha}{\beta}$. And the solution of (3) can be written as

$$\phi(x) = A_+ z_+^x + A_- z_-^x \tag{6}$$

By applying the boundaries Eq. (4) to Eq. (6) we can find all the eigenvalues and corresponding eigenvectors. The results are summarised as following

$$\begin{aligned} \lambda_0 &= 0;\ \phi_0(x) = const. \left(\frac{\alpha}{\beta}\right)^x; \\ \lambda_k &= -(\alpha+\beta) + 2\sqrt{\alpha\beta}\cos(\frac{k\pi}{L});\ k = 1,2,\ldots,L-1 \\ \phi_k(x) &= const. \left(\frac{\alpha}{\beta}\right)^{\frac{x}{2}} \left[\sin\left(\frac{k\pi}{L}x\right) - \sqrt{\frac{\beta}{\alpha}}\sin\left(\frac{k\pi}{L}(x-1)\right)\right]. \end{aligned} \tag{7}$$

The eigenvalue $\lambda_0 = 0$ and corresponding eigenvector represent the stationary mode $\phi_0(x)$. Define a scalar product between any two functions by

$$(\phi,\psi) = \sum_x \frac{\phi(x)\psi(x)}{\phi_0(x)} \tag{8}$$

Notice that definition Eq. (8) makes $\phi_0(x)$ identical to the stationary distribution $P^e(x)$, and remember here x is an integer. By properly choose the constant and when $L \to \infty$, one can check the orthogonality and completeness of the eigenfunctions.

$$\sum_{x=1}^{L} \phi_k(x)\phi_l(x) = \delta_{k,l} \tag{9}$$
$$\sum_{k=1}^{L} \phi_k(x)\phi_k(y) = \delta_{x,y} \tag{10}$$

So for arbitrary initial distribution of $P(x,0)$, we can always decompose it as

$$P(x,0) = \sum_k c_k \phi_k(x) \tag{11}$$

where c_k can be calculated by

$$c_k = \sum_x \phi_k(x) P(x,0) \tag{12}$$

Finally, the solution of single particle on reflecting lattice can be written as

$$P(x,t) = \sum_k \phi_k(x) e^{\lambda_k t} \sum_{x'} \phi_k(x') P(x',0) \tag{13}$$

For the special case that $P(x,0) = \delta_{x,y}$, solution (13) can be simplified to

$$P(x,t) = \sum_k \phi_k(x) \phi_k(y) e^{\lambda_k t} \tag{14}$$

With the complete solution of single particle, we can go further to systems of more than one particle. The idea is that the single particle solution works as building blocks for the N particle solutions. To start with that, we first illustrate the case $N = 2$ and the position of particles are denotes by x_1, x_2 with constraint $x_1 < x_2$.

2 Solution of Two Particles

Firstly, we shall write down the master equation, which looks as following

$$\begin{aligned} \frac{dP(x_1,x_2;t)}{dt} =& \alpha P(x_1-1,x_2;t) + \beta P(x_1+1,x_2;t) \\ &+ \alpha P(x_1,x_2-1;t) + \beta P(x_1,x_2+1;t) \\ &- 2(\alpha+\beta) P(x_1,x_2;t) \end{aligned} \tag{15}$$

We take the same eigenfunction expansion as for the single particle case:

$$P(x_1,x_2,t) = \sum_k \Psi_k(x_1,x_2) e^{\Lambda_k t} \tag{16}$$

Plug into the master equation Eq. (15) we have

$$\begin{aligned} \Lambda\Psi(x_1,x_2) =& \alpha\Psi(x_1-1,x_2) + \beta\Psi(x_1+1,x_2) \\ &+ \alpha\Psi(x_1,x_2-1) + \beta\Psi(x_1,x_2+1) \\ &- 2(\alpha+\beta)\Psi(x_1,x_2) \end{aligned} \tag{17}$$

And the reflecting boundaries write as

$$\alpha\Psi(0, x_2) = \beta\Psi(1, x_2) \tag{18a}$$

$$\alpha\Psi(x_1, L) = \beta\Psi(x_1, L+1) \tag{18b}$$

For the case of more than one particle, we need to take into account the exclusion effect, i.e., one site can be occupied by at most one particle. This can be also written as a boundary condition as

$$\alpha\Psi(x, x) + \beta\Psi(x+1, x+1) = (\alpha+\beta)\Psi(x, x+1) \tag{19}$$

Notice that the exclusive condition must hold for any x. The notation of $\Psi(x, x)$ may looks a little bit weird, but keep in mind that it is a boundary condition that denotes the limiting situation $x_1 = x_2$.

Before delve into the Bethe Ansatz solution, we want to remark here another equivalent general form of eigenfunctions specific for single particle on lattice with the reflecting boundaries:

$$\psi_s(x) = A\left(\frac{\alpha}{\beta}\right)^x \tag{20a}$$

$$\psi(x) = \left(\frac{\alpha}{\beta}\right)^{\frac{x}{2}}\left(A_+e^{ipx} + A_-e^{-ipx}\right) \tag{20b}$$

A, A_+, A_- are amplitude coefficients that will be fixed by the boundary conditions and/or normalization, p is the wave vector of excited eigenmodes, in case of single particle case above, $p = \frac{k\pi}{L}$. Here however, we separate the stationary and the non-stationary cases, we will show later that Eq. (20) will be very useful for searching the solution of more than one particle ASEP.

The idea to construct the N particle solution is inspired by the standard Coordinate Bethe Ansatz (CBA). However, instead of using the plain plane wave function as building blocks, we use the general form of single particle eigenfunctions with unfixed amplitude coefficients. The example of Ansatz for $\Psi(x_1, x_2)$ reads

$$\Psi(x_1, x_2) = \psi_1(x_1)\psi_2(x_2) + \tilde{\psi}_2(x_1)\tilde{\psi}_1(x_2) \tag{21}$$

Here $\psi_n(x)$ is draw from of Eq. (20), either stationary Eq. (20a) or non-stationary Eq.(20b). We classify ψ_1, $\tilde{\psi}_1$ as one class and ψ_2, $\tilde{\psi}_2$ as the other class. Basically, the number of this class equals to the number of particles. For one class, if it is in non-stationary modes, $\tilde{\psi}_n(x)$ share the same wave vector p_n with $\psi_n(x)$ but different amplitude coefficients $A_{n\pm}$. And if the class is in the stationary mode, then functions in the class are basically differentiated by amplitude coefficient A. It is important that ψ_n

and $\tilde{\psi}_n$ have different amplitude coefficients. The main idea is to tune these amplitude coefficients so that $\Psi(x_1, x_2)$ satisfies the reflecting boundaries Eq. (18) and exclusive condition Eq. (19). In the following subsections, we will discuss several different cases separately.

2.1 Stationary Solution

Let us first check the case that $\Psi(x_1, x_2)$ are constructed all by stationary eigenmodes. Namely $\psi_1(x) = A_1 \left(\frac{\alpha}{\beta}\right)^x$, $\psi_2(x) = A_2 \left(\frac{\alpha}{\beta}\right)^x$, $\tilde{\psi}_1(x) = \tilde{A}_1 \left(\frac{\alpha}{\beta}\right)^x$, $\tilde{\psi}_2(x) = \tilde{A}_2 \left(\frac{\alpha}{\beta}\right)^x$. Plug in to (21) we obtain

$$P^e(x_1, x_2) = \Psi(x_1, x_2) = A \left(\frac{\alpha}{\beta}\right)^{x_1 + x_2} \tag{22}$$

where $A = A_1\tilde{A}_1 + A_2\tilde{A}_2$. One can readily check that Eq. (22) is exactly the stationary eigenfunction of the two particle system as expected. First, we can easily obtain the corresponding eigenvalue is $\Lambda_0 = 0$ by simply plug it in to Eq. (17). And then one can check the reflecting boundaries Eq. (18) and the exclusive condition Eq. (19) are both fulfilled.

The prefactor A can be fixed by normalization. However, it is not a trivial work because of the constraint $x_1 < x_2$. We will discuss in detail in general case of N particles later.

2.2 Non-stationary Modes

For a simple two particles system, there are two different types of non-stationary modes. The first type is constructed by one class of Eq. (20a) and one class of Eq. (20b). The second type is constructed all by two classes of Eq. (20b) with different p.

Let us start with the first type. Without loss of generality, we choose $\{\psi_1(x), \tilde{\psi}_1(x)\}$ to be the class characterized by Eq. (20a), then $\Psi(x_1, x_2)$ can be written as

$$\begin{aligned}\Psi(x_1, x_2) =& A_1 \left(\frac{\alpha}{\beta}\right)^{x_1} \left(\frac{\alpha}{\beta}\right)^{\frac{x_2}{2}} \left(A_{2+}e^{ip_2x_2} + A_{2-}e^{-ip_2x_2}\right) \\ &+ \tilde{A}_1 \left(\frac{\alpha}{\beta}\right)^{x_2} \left(\frac{\alpha}{\beta}\right)^{\frac{x_1}{2}} \left(\tilde{A}_{2+}e^{ip_2x_1} + \tilde{A}_{2-}e^{-ip_2x_1}\right)\end{aligned} \tag{23}$$

Plug Eq. (23) in to the master equation, we will find the corresponding eigenvalue. Plug it in to the reflecting boundaries and exclusive condition,

A_1, $\tilde{A}_1$, $A_{2\pm}$ and $\tilde{A}_{2\pm}$ can be tuned to fulfill these conditions. Consistency condition will give us the Bethe equation about wave vector p_2, we will show the details of this procedure in the following text.

We first insert the solution to the master equation Eq. (17), obtaining the corresponding eigenvalue:

$$\Lambda = -(\alpha + \beta) + 2\sqrt{\alpha\beta}\cos(p_2) \tag{24}$$

p_2 is the wave vector that will be determined later.

Accordingly, the reflecting condition Eq. (18) gives us

$$\frac{A_{2+}}{A_{2-}} = -\frac{\left(\alpha - \sqrt{\alpha\beta}e^{-ip_2}\right)e^{-ip_2L}}{\left(\alpha - \sqrt{\alpha\beta}e^{ip_2}\right)e^{ip_2L}} \tag{25a}$$

$$\frac{\tilde{A}_{2+}}{\tilde{A}_{2-}} = -\frac{\alpha - \sqrt{\alpha\beta}e^{-ip_2}}{\alpha - \sqrt{\alpha\beta}e^{ip_2}} \tag{25b}$$

We now check the exclusive condition Eq. (19). Simply substitute Eq. (23) into the condition. In order to fulfill the exclusive condition, we find that

$$\frac{A_1 A_{2+}}{\tilde{A}_1\tilde{A}_{2+}} = -\frac{\alpha e^{ip_2} - (\alpha+\beta)\sqrt{\frac{\alpha}{\beta}} + \sqrt{\alpha\beta}}{\alpha e^{ip_2} - (\alpha+\beta)e^{ip_2} + \sqrt{\alpha\beta}} \tag{26a}$$

$$\frac{A_1 A_{2-}}{\tilde{A}_1\tilde{A}_{2-}} = -\frac{\alpha e^{-ip_2} - (\alpha+\beta)\sqrt{\frac{\alpha}{\beta}} + \sqrt{\alpha\beta}}{\alpha e^{-ip_2} - (\alpha+\beta)e^{-ip_2} + \sqrt{\alpha\beta}} \tag{26b}$$

Finally, use the consistency condition that

$$\frac{\tilde{A}_{2+}}{\tilde{A}_{2-}}\frac{A_{2-}}{A_{2+}} = \frac{\tilde{A}_1\tilde{A}_{2+}}{A_1A_{2+}}\frac{A_1A_{2-}}{\tilde{A}_1\tilde{A}_{2-}} \tag{27}$$

We obtain the Bethe equation

$$e^{i2p_2L} = 1 \tag{28}$$

Solve the equation we get $p_2 = \frac{k\pi}{L}$. Notice that it is exactly the same as the spectrum of single particle case. With the solution of p_2 we can resubstitute it into Eq. (25) and Eq. (26) to get the corresponding eigenfunctions. The results of this type of non-stationary eigenvalues and corresponding eigenfunctions are summarized as following

$$\Lambda_k = -(\alpha+\beta) + 2\sqrt{\alpha\beta}\cos(\frac{k\pi}{L});\ k = 1, 2, \ldots, L-1 \tag{29a}$$

$$\Psi_k(x_1, x_2) = A\left[\frac{\alpha}{\beta}\left(\frac{\alpha}{\beta}\right)^{x_1}\phi_k(x_2) + \left(\frac{\alpha}{\beta}\right)^{x_2}\phi_k(x_1)\right] \tag{29b}$$

Here $\phi_k(x)$ is exactly single particle non-stationary eigenfunction and A is a constant normalization coefficient.

We now turn to the second type of non-stationary eigenmodes. Notice that, the number of remaining unknown eigenfunctions and eigenvalues is in principle $L(L-1)/2 - L$. We will show that there are all contained in this class. We first write down the Ansatz:

$$\begin{aligned}\Psi(x_1,x_2) = \left(\frac{\alpha}{\beta}\right)^{\frac{x_1+x_2}{2}} & \left[\left(A_{1+}e^{ip_1x_1} + A_{1-}e^{-ip_1x_1}\right)\left(A_{2+}e^{ip_2x_2} + A_{2-}e^{-ip_2x_2}\right)\right. \\ & \left. + \left(\tilde{A}_{1+}e^{ip_1x_2} + \tilde{A}_{1-}e^{-ip_1x_2}\right)\left(\tilde{A}_{2+}e^{ip_1x_1} + \tilde{A}_{2-}e^{-ip_2x_1}\right)\right]\end{aligned} \tag{30}$$

Plug it in to the master equation, we get the corresponding eigenvalues

$$\Lambda = \sum_{n=1}^{2}\left[-(\alpha+\beta) + 2\sqrt{\alpha\beta}\cos(p_n)\right] \tag{31}$$

And plug it in to the reflecting boundaries, we obtain

$$\frac{A_{1+}}{A_{1-}} = -\frac{\alpha - \sqrt{\alpha\beta}e^{-ip_1}}{\alpha - \sqrt{\alpha\beta}e^{ip_1}} \tag{32a}$$

$$\frac{\tilde{A}_{1+}}{\tilde{A}_{1-}} = -\frac{\left(\alpha - \sqrt{\alpha\beta}e^{-ip_1}\right)e^{-ip_1L}}{\left(\alpha - \sqrt{\alpha\beta}e^{ip_1}\right)e^{ip_1L}} \tag{32b}$$

$$\frac{\tilde{A}_{2+}}{\tilde{A}_{2-}} = -\frac{\alpha - \sqrt{\alpha\beta}e^{-ip_2}}{\alpha - \sqrt{\alpha\beta}e^{ip_2}} \tag{32c}$$

$$\frac{A_{2+}}{A_{2-}} = -\frac{\left(\alpha - \sqrt{\alpha\beta}e^{-ip_2}\right)e^{-ip_2L}}{\left(\alpha - \sqrt{\alpha\beta}e^{ip_2}\right)e^{ip_2L}} \tag{32d}$$

To ease the notation, we define the function $a(p,p') = \sqrt{\alpha\beta}e^{i(p+p')} - (\alpha + \beta)e^{ip} + \sqrt{\alpha\beta}$. Then the exclusive condition gives

$$\frac{A_{1+}A_{2+}}{\tilde{A}_{1+}\tilde{A}_{2+}} = -\frac{a(p_1,p_2)}{a(p_2,p_1)} \tag{33a}$$

$$\frac{A_{1+}A_{2-}}{\tilde{A}_{1+}\tilde{A}_{2-}} = -\frac{a(p_1,-p_2)}{a(-p_2,p_1)} \tag{33b}$$

$$\frac{A_{1-}A_{2+}}{\tilde{A}_{1-}\tilde{A}_{2+}} = -\frac{a(-p_1,p_2)}{a(p_2,-p_1)} \tag{33c}$$

$$\frac{A_{1-}A_{2-}}{\tilde{A}_{1-}\tilde{A}_{2-}} = -\frac{a(-p_1,-p_2)}{a(-p_2,-p_1)} \tag{33d}$$

The similar consistency condition of Eq. (27) gives the following Bethe equation:

$$e^{i2p_1L} = \frac{a(p_1,p_2)}{a(p_2,p_1)}\frac{a(p_2,-p_1)}{a(-p_1,p_2)} \tag{34a}$$

$$e^{i2p_2L} = \frac{a(p_2,p_1)}{a(p_1,p_2)}\frac{a(p_1,-p_2)}{a(-p_2,p_1)} \tag{34b}$$

Now it would be interesting to interpret the Bethe Equation and compare with the well know Bethe Equation of priodic boundary case. And we will show later this interpretation is useful to derive N particles Bethe Equation. We can consider Eq. (25) as a reflector that reflects the particle and change the direction of wave vector, i.e., $p_n \leftrightarrow -p_n$. On the other hand, Eq. (26) can be interpreted as a permutator that permutates two neighboring particles $n \leftrightarrow (n+1)$. Let us say particle 1 starts from the left side of the lattice and then permutates with all the particle at right side until reaches the right boundary (of course in case of two particles, there are only one particle at the right side), and then reflects by the boundary, become a particle traveling in the oppsite direction, and then permutates with all the left side particles until reaches the left side boundary, and then reflects again, which recovers the initial state. A schematic of this process is shown in Fig. 1. In this sense, the particle works as if it is on a lattice with periodic boundary. Use this interpretation and the well know result of periodic Bethe Equation, once can easily recover exactly Eq. (34).

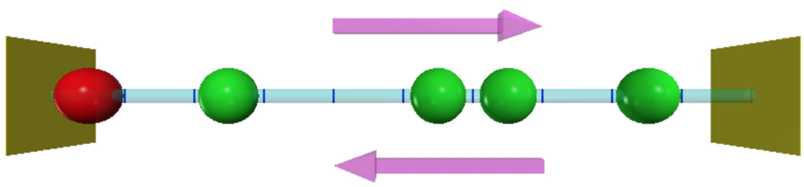

Figure 1: Intepretation of Bethe Equation.

By solving the Bethe equation, one can get p_1 and p_2 and thus all amplitude coefficients up to a constant normalization factor. Then Eq. (31) gives the eigenvalues and Eq. (30) gives the eigenfunctions. Unfortunately, it might not possible to solve the Bethe equation analytically. So we resort to numerical solutions. We have verified the resulting eigenvalues and eigenfunctions by benchmark with the results from brute force diagonalizing of the transition matrix for small system size $L \leq 10$. Notice that those roots that $p_n = 0$ or $p_n = \pi$ have to be filtered out because they correspond to the first type of non-stationary eigenmodes which is not compatible by Eq. (30)

To summarize, the complete solution of two particle hopping system with reflecting boundaries was found. The solution is shown in the form of eigenfunction expansion, i.e. Eq. (17). The stationary eigenfunction with eigenvalue $\Lambda_0 = 0$ is listed as Eq. (22), while the two types of non-stationary eigenvalues and corresponding eigenfunctions are listed in Eq. (24), (31) and Eq. (23), (30), respectively. They can be fully determined by solving the Bethe Equation Eq. (28) and Eq. (34). Part of them are analytically shown in Eq. (29).

Finally, there are several remarks we want to make here. Firstly, as one can see from Eq. (29) that the eigenvalues of two particle system always contain the eigenvalues of single partile system. We will show later this can be generalised that the eigenvalues of $N+1$ particle system always contain the eigenvalues of N particle system for $N < L/2$. Secondly, the relaxation time of the system is realted to the largest non-zero eigenvalue Λ_1. Eq. (29) hints $\Lambda_1 = -(\alpha+\beta) + 2\sqrt{\alpha\beta}\cos\left(\frac{\pi}{L}\right)$. However, since there is no analytical solution for eigenvalues of the second kind, it will be difficult to rigously prove that. Nummerical evidences will be provided to verify this is indeed true.

In next section, we will generalise the solution to the N particles system. It is actually quite straight forward after we have done the two particles case.

3 General Solution of N particles ASEP

As before, we first write down the master equation of a N particles system.

$$\begin{aligned}\frac{dP(x_1,\cdots,x_N;t)}{dt} =& \sum_{j=1}^{N}\left[\alpha P(\cdots,x_j-1,\cdots;t) + \beta P(\cdots,x_j+1,\cdots;t)\right.\\ &\left.-(\alpha+\beta)P(\cdots,x_j,\cdots;t)\right]\end{aligned} \tag{35}$$

Similarly, after the eigenfunction expansion the reflecting boundaries write as

$$\alpha\Psi(0,x_2,\cdots,x_N) = \beta\Psi(1,x_2,\cdots,x_N) \tag{36a}$$

$$\alpha\Psi(x_1,\cdots,x_{N-1},L) = \beta\Psi(x_1,\cdots,x_{N-1},L+1) \tag{36b}$$

The exclusive condition for N particles case is more tricky. In principle, one has to consider to case of three body collision and four body collision and so on. Luckily, in the simple model of ASEP, one can prove that these more than two body exclusive conditions are not new but just linear recombination of two body exclusive condition. So we can write the exclusive condition of a N particles system as

$$\alpha\Psi(\cdots,x,x,\cdots)+\beta\Psi(\cdots,x+1,x+1,\cdots)=(\alpha+\beta)\Psi(\cdots,x,x+1,\cdots) \tag{37}$$

The reason that the exclusive condition can be written in such a simple way is rooting from the Yang-Baxter Equation, which encodes the integrability of the ASEP system.

3.1 Stationary Solution

Intuitively, we construct the N particles stationary solution as

$$P^e(x_1,x_2,\cdots,x_N)=\Psi(x_1,x_2,\cdots,x_N)=A\prod_{j=1}^{N}\left(\frac{\alpha}{\beta}\right)^{x_j} \tag{38}$$

One can plug in the master equation check that the corresponding eigenvalue $\Lambda_0=0$, and also verify the exclusive condition as well as the reflecting boundaries are fulfilled by insert the solution in to Eq. (37) and Eq. (36) separately.

We now try to fix the parameter A by normalization. Let us denote $q:=\frac{\alpha}{\beta}$, then we can write A as following

$$A^{-1}=\sum_{\Omega}q^{\sum_j x_j}=\sum_{x_1<x_2<\cdots<x_N}q^{\sum_j x_j} \tag{39}$$

Let us do a variable change so that

$$\begin{aligned}\sum_j x_j &= E_0+E\\ E_0 &= 1+2+\cdots+N=\frac{N(N+1)}{2}\end{aligned}$$

We can derive that E is a integer in the range of $0,1,\cdots,N(L-N)$. So Eq. (39) can be rewrite as

$$A^{-1}=q^{E_0}\sum_{E=0}^{N(L-N)}g(E)q^E \tag{40}$$

where $g(E)$ is the number of partitions of positive integer E to N parts with each of size at most $L-N$. From the number partition theory, we identify

$$\sum_{E=0}^{N(L-N)}g(E)q^E=\binom{L}{N}_q=\frac{[L]_q!}{[L-N]_q![N]_q!} \tag{41}$$

where $[N]_q = 1 + q + q^2 + \cdots + q^{N-1}$ is called a q number, and Eq. (41) is called the q binomial coefficient[]. So we finally arrive at

$$P^e(x_1, x_2, \cdots, x_N) = q^{-\frac{N(N+1)}{2}} \binom{L}{N}_q^{-1} \prod_{j=1}^{N} q^{x_j} \tag{42}$$

In [], G. M. Schütz use a quantum group formalism obtained the same result with a different notation. We emphasize here that our method is much more easily to understand and no prerequisite knowledge of quantum mechanics and group theory is needed.

With the equilibrium N particle distribution, we can readily calculate the equilibrium distribution of any tagged particle. Denote he distribution of the nth particle $p_n(x)$, we have

$$\begin{aligned} p_n(x) &= \sum_{0<x_1<\cdots<x_{n-1}\leq x-1} P^e(x_1, x_2, \cdots, x_N) \sum_{x<x_{n+1}<\cdots<x_N\leq L} P^e(x_1, x_2, \cdots, x_N) \\ &= q^{(N+1-n)(x-n)} \binom{x-1}{n-1}_q \binom{L-x}{N-n}_q \Big/ \binom{L}{N}_q \end{aligned} \tag{43}$$

Finally, the equilibrium density profile can be obtain by summing up $p_n(x)$

$$\rho(x) = \sum_{n=1}^{N} p_n(x) \tag{44}$$

3.2 Non-stationary Modes

Inspired by the calculation of two particles case, we try to find the Bethe solution of the N particles system by taking the following Ansatz:

$$\Psi(x_1, x_2, \cdots, x_N) = \sum_{\sigma\in\mathcal{S}_N} \prod_{n=1}^{N} \psi_n^\sigma(x_{\sigma(n)}) \tag{45}$$

where $\mathcal{S}_N$ is the group of permutations of N elements and ψ_n^σ is the n^{th} class of eigenfunction of Eq. (20), either stationary or non-stationary. The subscript n means all functions in the n^{th} class share the same p_n, the superscript σ means the amplitude coefficients A_n^σ or $A_{n\pm}^\sigma$ are not the same for different permutations.

We again insert the solution to the master equation Eq. (35), notice that the amplitude coefficients are irrelevant with the eigenvalues. So we obtain the simple form of corresponding eigenvalue

$$\Lambda = \sum_{n=1}^{N} \lambda_n \tag{46}$$

λ_n here is a function and its expression depends on whether ψ_n^σ is stationary or non-stationary. For stationary $\lambda_n = 0$, for non-stationary $\lambda_n(p_n) = -(\alpha+\beta)+2\sqrt{\alpha\beta}\cos(p_n)$. Notice that, as we can see from the two particles example, the Bethe Equations will give the value of p_n and determine the eigenvalues. In general, they are more than one solution because Bethe Equations are nonlinear, and different solution can lead to different eigenvalues.

We now consider that the non-stationary eigenfunctions is constructed by N_s of Eq. (20a) classes and $N - N_s$ of Eq. (20b) classes. Notice that $N_s = 0$ corresponds to the second type of eigenfunctions in our discussion of the two particles system. We will start from this case first here.

Plug Eq. (45) in to the reflecting boundaries Eq. (36) we can obtain

$$\frac{A_{n+}^{\sigma|\sigma(1)=n}}{A_{n-}^{\sigma|\sigma(1)=n}} = -\frac{\alpha - \sqrt{\alpha\beta}e^{-ip_n}}{\alpha - \sqrt{\alpha\beta}e^{ip_n}} \tag{47a}$$

$$\frac{A_{n+}^{\sigma|\sigma(N)=n}}{A_{n-}^{\sigma|\sigma(N)=n}} = -\frac{\left(\alpha - \sqrt{\alpha\beta}e^{-ip_n}\right) e^{-ip_nL}}{\left(\alpha - \sqrt{\alpha\beta}e^{ip_n}\right) e^{ip_nL}} \tag{47b}$$

And substitute the Ansatz in to the Exclusive condition Eq. (37) we get

$$\frac{A_{n\pm}^{\sigma} A_{(n+1)\pm}^{\sigma}}{A_{n\pm}^{\sigma|n\leftrightarrow n+1} A_{(n+1)\pm}^{\sigma|n\leftrightarrow n+1}} = -\frac{a(\pm p_n, \pm p_{n+1})}{a(\pm p_{n+1}, \pm p_n)} \tag{48a}$$

$$a(p,p') = \sqrt{\alpha\beta}e^{i(p+p')} - (\alpha+\beta)e^{ip} + \sqrt{\alpha\beta} \tag{48b}$$

Then one can either use the consistency condition or the interpretation we discussed in the two particles case and the well know periodic Bethe Equation, easily find the following set of Bethe Equations for the N particles system:

$$e^{i2p_nL} = \prod_{m\neq n}^{N} \frac{a(p_n,p_m)}{a(p_m,p_n)} \frac{a(p_m,-p_n)}{a(-p_n,p_m)} \tag{49}$$

By solving Eq. (49) we get all the p_n and then one can plug back in Eq. (46) and Eq. (45) for the corresponding eigenvalues and eigenfunctions.

Now let us come back to the case that $0 < N_s < N$. The calculation of the two particles system is a good illustration. We can easily find out there is nothing different except we will have only $N - N_s$ wave vectors and $2N - N_s$

amplitude coefficients. So one just have to use the relation similar to Eq. (26) together with Eq. (47) to build the Bethe Equation. Moreover, Eq. (28) tells us that the Bethe Equations we can obtain will be exact the same as the Bethe Equations of the second type $N - N_s$ particles system. So according to Eq. (46) for the eigenvalues, we can conclude that the eigenvalues of $N - N_s$ particles system are always contained in the eigenvalue set of N particle system. Notice that this is only true for $N < L/2$. For $N > L/2$, one can easily use the particle-hole duality which shows that the eigenvalue of N particles system should be the same as $L - N$ particles system.

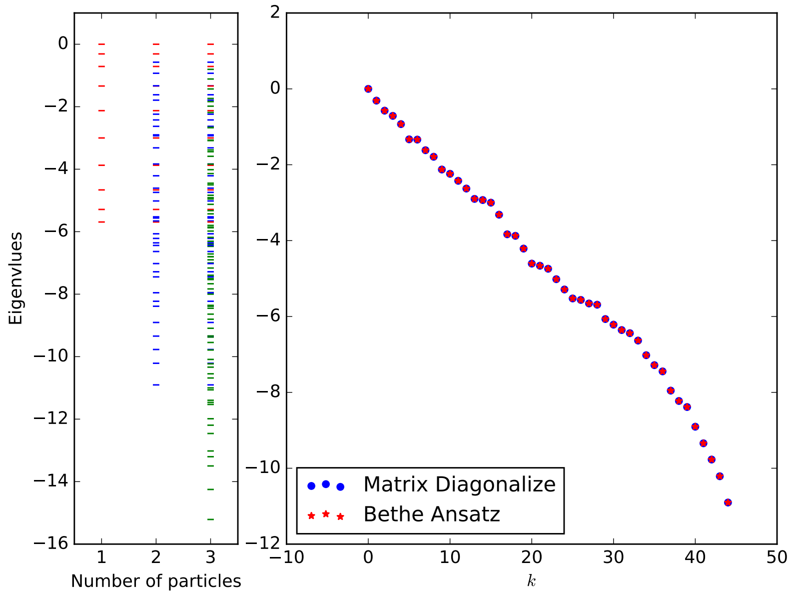

Figure 2: Left: embeding of eigenvalues; Right: benchmark of eigenvalues from Bethe Ansatz theory to directly diagonalize transition matrix, $N = 2$. $L = 10$, $\alpha = 2$ and $\beta = 1$ for both.

Finally, we remark that the Bethe Equations have to be solved numerically in most cases. However, there is a small set of non-stationary eigenvalue and eigenvectors we can obtain analytically, which correspond to the case with just one excitation mode. The results are summarized as following:

$$\Lambda_k = -(\alpha+\beta) + 2\sqrt{\alpha\beta}\cos(\frac{k\pi}{L});\ k = 1, 2, \dots, L-1 \tag{50a}$$

$$\Psi_k(x_1, x_2, \cdots, x_N) = A\sum_{n=1}^{N}\left(\frac{\alpha}{\beta}\right)^{n-1}\phi_k(x_n)\prod_{m\neq n}\left(\frac{\alpha}{\beta}\right)^{x_m} \tag{50b}$$

Notice that the set of eigenvalue is exact the single particle spectrum, and again $\phi_k(x)$ is exactly the single particle eigenfunction. Fortunately, the numerical evidence shows that the most interesting eigenmode, i.e. the slowest relaxation mode, is contained in this set. We will discuss it in next section.

4 Relaxation Time

The largest non-zero eigenvalue we found and verified by numerical results is

$$\Lambda_1 = -(\alpha + \beta) + 2\sqrt{\alpha\beta}\cos(\frac{\pi}{L}) \tag{51}$$

If $L \gg 1$, we can expand the cos term, obtain

$$\Lambda_1 = -(\sqrt{\beta} - \sqrt{\alpha})^2 - \frac{\sqrt{\alpha\beta}\pi^2}{L^2} \tag{52}$$

And the relaxation time can be calculated as

$$\tau = -\frac{1}{\Lambda_1} \tag{53}$$

There are several information we can read form Eq. (52) and (53). Firstly, we can see the scaling $\tau \approx L^2$, which means the dynamical exponent of the system is 2. Secondly, as we can see from Eq. (52), the bigger difference between α and β, the smaller relaxation time we will get. This is also consistent as one would expect. If we map back to the polymer model, the result here can be compared with the prediction of Rouse theory. Unlike the prediction from Rouse theory that relaxation time does not depend on external force, we have here that stronger external force decreases the relaxation time. This point highlight the fundamental difference between the infinite extensible bead spring model and the rigid bead rod model which is of course finite extensible.

In Fig. 3 shows all the eigenvalues of a system with lattice size $L = 10$, calculated by brute force diagonalizing the transition matrix. Lattice The number of particles on the lattice is ranging from 1 to 9. The difficult to calculate larger lattice size L lies on that the dimension grow as $\binom{L}{N}$, and if we take $N = L/2$, the dimension of the matrix will became a extremely large number very soon.

As we can see in Fig. 3 that all eigenvalues of case $N = 1$ are contained in the set of eigenvalues $N = 2$, and all eigenvalues of case $N = 2$ are contained in the set of $N = 3$ and so on until reach the largest set $N = L/2$. This means the characteristic polynomial of $N = k + 1$ always contains the factor of the characteristic polynomial $N = k$. This is predict exactly by our theory.

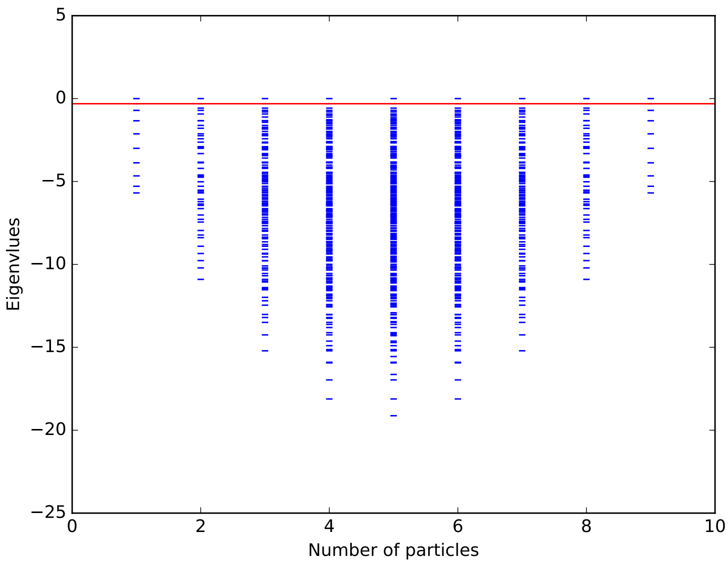

Figure 3: Eigenvalues calculated by numerically diagonalize the transition matrix.

5 Summary

We use the modified Bethe Ansatz methods solve the ASEP model with reflecting boundaries. The stationary distribution was solved exactly and the one correspond to the relaxation time of the system was postulated. Numerical evidence is provided for later statement.

A Derivation of the exclusion condition

To derivate the exclusion condition, which is a little bit confusing at a first sight, we use the two particles example and then generalise to N particle case. First, let us rewrite the master equation of this two particle system:

$$\begin{aligned}\frac{dP(x_1, x_2; t)}{dt} =& \alpha P(x_1 - 1, x_2; t) + \beta P(x_1 + 1, x_2; t) \\ &+ \alpha P(x_1, x_2 - 1; t) + \beta P(x_1, x_2 + 1; t) \\ &- 2(\alpha + \beta) P(x_1, x_2; t)\end{aligned} \tag{54}$$

We assume the above equation is valid for the whole space. However, this is actually not true when the two particles are sitting on the neighboring sites. Let us now consider this special case separately, remember that $x_2 = x_1 + 1$. The master equation of this special case can be written as

$$\begin{aligned}\frac{dP(x_1, x_2; t)}{dt} =& \alpha P(x_1 - 1, x_2; t) + \beta P(x_1, x_2 + 1; t) \\ &- (\alpha + \beta) P(x_1, x_2; t)\end{aligned} \tag{55}$$

Now, let us do a subtraction, i.e. (54) $-$ (55), obtain

$$\alpha P(x_1, x_2 - 1; t) + \beta P(x_1 + 1, x_2; t) = (\alpha + \beta) P(x_1, x_2; t) \tag{56}$$

Let us then denote $x := x_1$ and plug in $x_2 = x_1 + 1 = x + 1$ in the above equation. We finally arrive at

$$\alpha P(x, x; t) + \beta P(x + 1, x + 1; t) = (\alpha + \beta) P(x, x + 1; t) \tag{57}$$

In summary, the sole master equation Eq. (54) does not take into account the exclusion cases. In order to represent the exclusive setting, we assume Eq. (54) is valid for the whole space, and then apply an additional condition on this equation like Eq. (57). This condition is similar to the partial derivative of the PDF is the same at the collision boundary $x_1 = x_2$ in the continuous space case.

Finally, in the similar way we can derive the exclusion condition for the cases $N > 2$. However, it is not difficult to check that in cases of more than two particles, the three body collision or four body collision cases do not give out new conditions, just the linear combination of the two body collision conditions like Eq. (57). This is fundamentally a result of Yang-Baxter Equation is satisfied for the ASEP model.