

Lecture 2: More review; the Bernoulli process

Outline:

- Expectations
- Indicator random variables
- Multiple random variables
- IID random variables
- Laws of large numbers in pictures
- The Bernoulli process
- Central limit theorem for Bernoulli

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Expectations

The distribution function of a rv X often contains more detail than necessary. The expectation $\bar{X} = E[X]$ is sometimes all that is needed.

$$E[X] = \sum_i a_i p_X(a_i) \quad \text{for discrete } X$$

$$E[X] = \int x f_X(x) dx \quad \text{for continuous } X$$

$$E[X] = \int F_X^c(x) dx \quad \text{for arbitrary nonneg } X$$

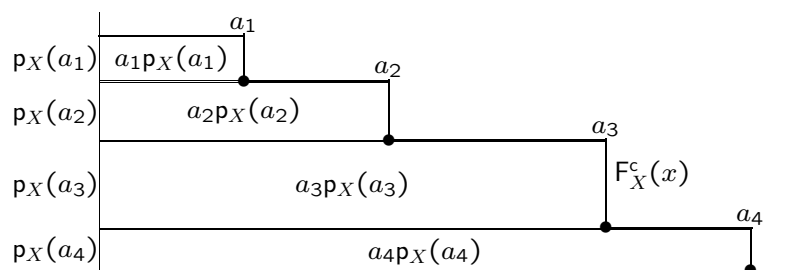
$$E[X] = \int_{-\infty}^0 F_X(x) dx + \int_0^{\infty} F_X^c(x) dx \quad \text{for arbitrary } X.$$

Almost as important is the standard deviation,

$$\sigma_X = \sqrt{E[(X - \bar{X})^2]}$$

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Why is $E[X] = \int F_X^c(x) dx$ for arbitrary nonneg X ?
 Look at discrete case. Then $\int F_X^c(x) dx = \sum_i a_i p_X(a_i)$.



If X has a density, the same argument applies to every Riemann sum for $\int_x x f_X(s) dx$ and thus to the limit.

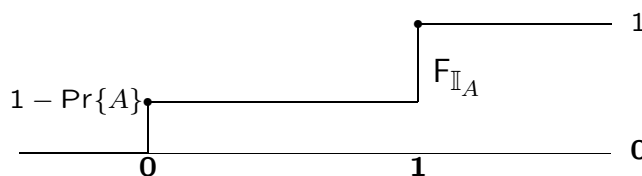
It is simpler and more fundamental to take $\int F_X^c(x) dx$ as the general definition of $E[X]$. This is also useful in solving problems

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Indicator random variables

For every event A in a probability model, an indicator rv \mathbb{I}_A is defined where $\mathbb{I}_A(\omega) = 1$ for $\omega \in A$ and $\mathbb{I}_A(\omega) = 0$ otherwise. Note that \mathbb{I}_A is a binary rv.

$$p_{\mathbb{I}_A}(0) = 1 - \Pr\{A\}; \quad p_{\mathbb{I}_A}(1) = \Pr\{A\}.$$



$$E[\mathbb{I}_A] = \Pr\{A\} \quad \sigma_{\mathbb{I}_A} = \sqrt{\Pr\{A\} (1 - \Pr\{A\})}$$

Theorems about rv's can thus be applied to events.

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Multiple random variables

Is a random variable (rv) X specified by its distribution function $F_X(x)$?

No, the relationship between rv's is important.

$$F_{XY}(x, y) = \Pr\{\{\omega : X(\omega) \leq x\} \cap \{\omega : Y(\omega) \leq y\}\}$$

The rv's X_1, \dots, X_n are **independent** if

$$F_{\vec{X}}(x_1, \dots, x_n) = \prod_{m=1}^n F_{X_m}(x_m) \quad \text{for all } x_1, \dots, x_n$$

This product form carries over for PMF's and PDF's.

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For discrete rv's, independence is more intuitive when stated in terms of **conditional probabilities**.

$$p_{X|Y}(x|y) = \frac{p_{XY}(x, y)}{p_Y(y)}$$

Then X and Y are **independent** if $p_{X|Y}(x|y) = p_X(x)$ for all sample points x and y . This essentially works for densities, but then $\Pr\{Y = y\} = 0$ (see notes). This is not very useful for distribution functions.

NitPick: If X_1, \dots, X_n are independent, then all subsets of X_1, \dots, X_n are independent. (This isn't always true for independent events).

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IID random variables

The random variables X_1, \dots, X_n are **independent** and **identically distributed** (IID) if for all x_1, \dots, x_n

$$F_{\vec{X}}(x_1, \dots, x_n) = \prod_{k=1}^n F_X(x_k)$$

This product form works for PMF's and PDF's also.

Consider a probability model in which \mathbb{R} is the sample space and X is a rv.

We can always create an extended model in which \mathbb{R}^n is the sample space and X_1, X_2, \dots, X_n are IID rv's. We can further visualize $n \rightarrow \infty$ where X_1, X_2, \dots is a stochastic process of IID variables.

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We study the sample average, $S_n/n = (X_1 + \dots + X_n)/n$. The laws of large numbers say that S_n/n **'essentially becomes deterministic'** as $n \rightarrow \infty$.

If the extended model corresponds to repeated experiments in the real world, then S_n/n corresponds to the **arithmetic average** in the real world.

If X is the indicator rv for event A , then the sample average is the relative frequency of A .

Models can have two types of difficulties. In one, a sequence of real-world experiments are not sufficiently similar and **isolated** to correspond to the IID extended model. In the other, the **IID extension** is OK but the basic model is not.

We learn about these problems here through study of the models.

Science, symmetry, analogies, earlier models, etc. are all used to model real-world situations.

Trivial example: Roll a white die and a red die. There are 36 sample outcomes, $(i, j), 1 \leq i, j \leq 6$, taken as equiprobable by symmetry.

Roll 2 indistinguishable white dice. The white and red outcomes (i, j) and (j, i) for $i \neq j$ are now indistinguishable. There are now 21 'finest grain' outcomes, but no sane person would use these as sample points.

The appropriate sample space is the 'white/red' sample space with an 'off-line' recognition of what is distinguishable.



Neither the axioms nor experimentation motivate this model, i.e., modeling requires judgement and common sense.

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Comparing models for similar situations and analyzing limited and defective models helps in clarifying fuzziness in a situation of interest.

Ultimately, as in all of science, some experimentation is needed.

The outcome of an experiment is a sample point, not a probability.

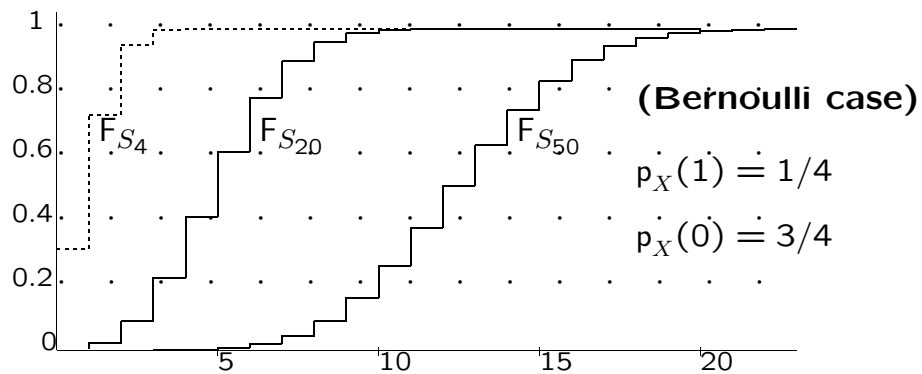
Experimentation with probability requires multiple trials. The outcome is modeled as a sample point in an extended version of the original model.

Experimental tests of an original model come from the laws of large numbers in the context of an extended model.

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Laws of large numbers in pictures

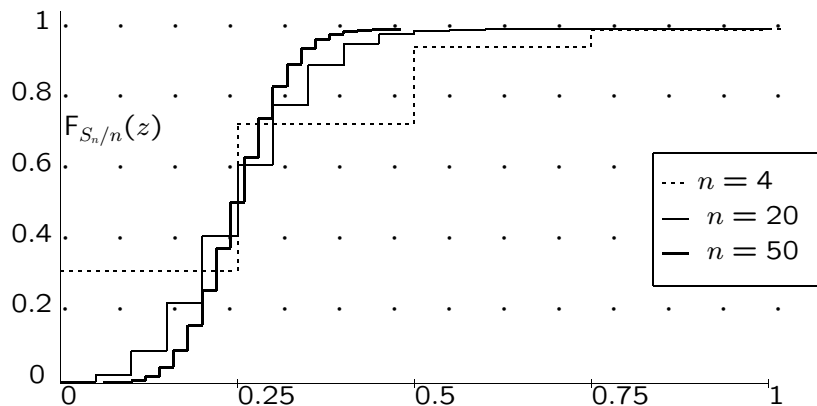
Let X_1, X_2, \dots, X_n be IID rv's with mean \bar{X} , variance σ^2 . Let $S_n = X_1 + \dots + X_n$. Then $\sigma_{S_n}^2 = n\sigma^2$.



The center of the distribution varies with n and the spread (σ_{S_n}) varies with \sqrt{n} .

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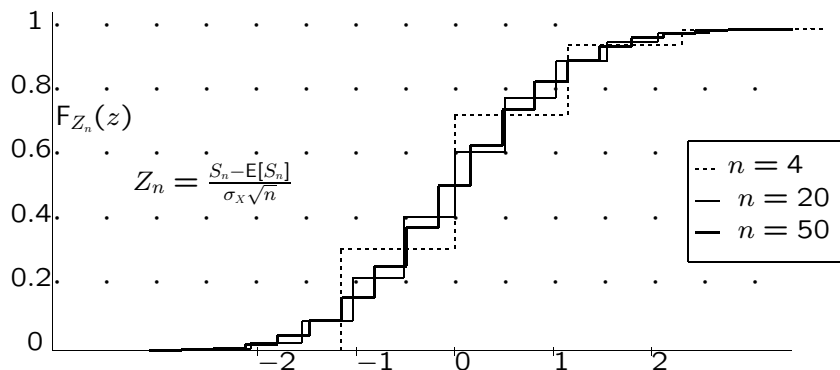
The sample average is S_n/n , which is a rv of mean \bar{X} and variance σ^2/n .



The center of the distribution is \bar{X} and the spread decreases with $1/\sqrt{n}$.

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Note that $S_n - n\bar{X}$ is a zero mean rv with variance $n\sigma^2$. Thus $\frac{S_n - n\bar{X}}{\sqrt{n}\sigma}$ is zero mean, unit variance.



Central limit theorem:

$$\lim_{n \rightarrow \infty} \left[\Pr \left\{ \frac{S_n - n\bar{X}}{\sqrt{n}\sigma} \leq y \right\} \right] = \int_{-\infty}^y \frac{1}{\sqrt{2\pi}} \exp \left(\frac{-x^2}{2} \right) dx.$$

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The Bernoulli process

$$S_n = Y_1 + \cdots + Y_n \quad p_Y(1) = p > 0, \quad p_Y(0) = 1 - p = q > 0$$

The n -tuple of k 1's followed by $n - k$ 0's has probability $p^k q^{n-k}$.

Each n tuple with k ones has this same probability. For $p < 1/2$, $p^k q^{n-k}$ is largest at $k = 0$ and decreasing in k to $k = n$.

There are $\binom{n}{k}$ n -tuples with k 1's. This is increasing in k for $k < n/2$ and then decreasing. Altogether,

$$p_{S_n}(k) = \binom{n}{k} p^k q^{n-k}$$

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$$p_{S_n}(k) = \binom{n}{k} p^k q^{n-k}$$

To understand how this varies with k , consider

$$\begin{aligned} \frac{p_{S_n}(k+1)}{p_{S_n}(k)} &= \frac{n!}{(k+1)!(n-k-1)!} \frac{k!(n-k)!}{n!} \frac{p^{k+1}q^{n-k-1}}{p^k q^{n-k}} \\ &= \frac{n-k}{k+1} \frac{p}{q} \end{aligned}$$

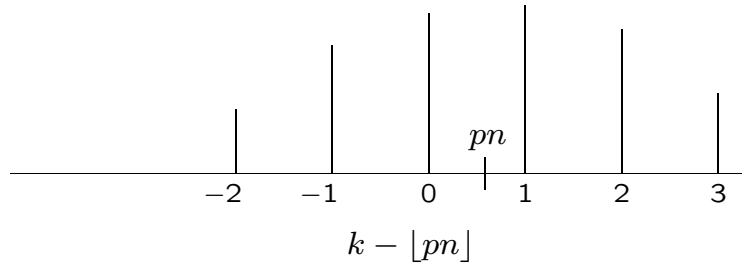
This is strictly decreasing in k . It also satisfies

$$\frac{p_{S_n}(k+1)}{p_{S_n}(k)} \begin{cases} < 1 & \text{for } k \geq pn \\ \approx 1 & \text{for } k < pn < k+1 \\ > 1 & \text{for } k+1 \leq pn \end{cases}$$

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$$\frac{p_{S_n}(k+1)}{p_{S_n}(k)} = \frac{n-k}{k+1} \frac{p}{q} \quad (1)$$

$$\frac{p_{S_n}(k+1)}{p_{S_n}(k)} \begin{cases} < 1 & \text{for } k \geq pn \\ \approx 1 & \text{for } k < pn < k+1 \\ > 1 & \text{for } k+1 \leq pn \end{cases}$$



In other words, $p_{S_n}(k)$, for fixed n , is increasing with k for $k < pn$ and decreasing for $k > pn$.

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CLT for Bernoulli process

$$\frac{p_{S_n}(k+1)}{p_{S_n}(k)} = \frac{n-k}{k+1} \frac{p}{q}$$

We now use this equation for large n where k is relatively close to pn . To simplify the algebra, assume pn is integer and look at $k = pn + i$ for relatively small i . Then

$$\begin{aligned} \frac{p_{S_n}(pn + i + 1)}{p_{S_n}(pn + i)} &= \frac{n - pn - i}{pn + i + 1} \frac{p}{q} = \frac{nq - i}{pn + i + 1} \frac{p}{q} \\ &= \frac{1 - \frac{i}{nq}}{1 + \frac{i+1}{np}} \\ \ln \left[\frac{p_{S_n}(pn + i + 1)}{p_{S_n}(pn + i)} \right] &= \ln \left[1 - \frac{i}{nq} \right] - \ln \left[1 + \frac{i+1}{np} \right] \end{aligned}$$

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Recall that $\ln(1 + x) \approx x - x^2/2 + \dots$ **for** $|x| \ll 1$.

$$\begin{aligned} \ln \left[\frac{p_{S_n}(pn + i + 1)}{p_{S_n}(pn + i)} \right] &= \ln \left[1 - \frac{i}{nq} \right] - \ln \left[1 + \frac{i+1}{np} \right] \\ &= -\frac{i}{nq} - \frac{i}{np} - \frac{1}{np} + \dots \\ &= -\frac{i}{npq} - \frac{1}{np} + \dots \end{aligned}$$

where we have used $1/p + 1/q = 1/pq$ and the neglected terms are of order i^2/n^2 .

This says that these log of unit increment terms are essentially linear in i . We now have to combine these unit incremental terms.

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$$\ln \left[\frac{p_{S_n}(pn + i + 1)}{p_{S_n}(pn + i)} \right] = -\frac{i}{npq} - \frac{1}{np} + \dots$$

Expressing an increment of j terms as a telescoping sum of j unit increments,

$$\begin{aligned} \ln \left[\frac{p_{S_n}(pn + j)}{p_{S_n}(pn)} \right] &= \sum_{i=0}^{j-1} \ln \left[\frac{p_{S_n}(pn + i + 1)}{p_{S_n}(pn + i)} \right] \\ &= \sum_{i=0}^{j-1} -\frac{i}{npq} - \frac{1}{np} + \dots \\ &= -\frac{j(j-1)}{2npq} - \frac{j}{np} + \dots \approx \frac{-j^2}{2npq} \end{aligned}$$

where we have used the fact that $1 + 2 + \dots + j - 1 = j((j-1)/2)$. We have also ignored terms linear in j since they are of the same order as a unit increment in j .

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Finally,

$$\begin{aligned} \ln \left[\frac{p_{S_n}(pn + j)}{p_{S_n}(pn)} \right] &\approx \frac{-j^2}{2npq} \\ p_{S_n}(pn + j) &\approx p_{S_n}(pn) \exp \left[\frac{-j^2}{2npq} \right] \end{aligned}$$

This applies for j both positive and negative, and is a quantized version of a Gaussian distribution, with the unknown scaling constant $p_{S_n}(pn)$. Choosing this to get a PMF,

$$p_{S_n}(pn + j) \approx \frac{1}{\sqrt{2\pi npq}} \exp \left[\frac{-j^2}{2npq} \right],$$

which is the discrete PMF form of the central limit theorem. See Section 1.5.3 for a different approach.

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6.262 Discrete Stochastic Processes

Spring 2011

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