2D Elastic Problem

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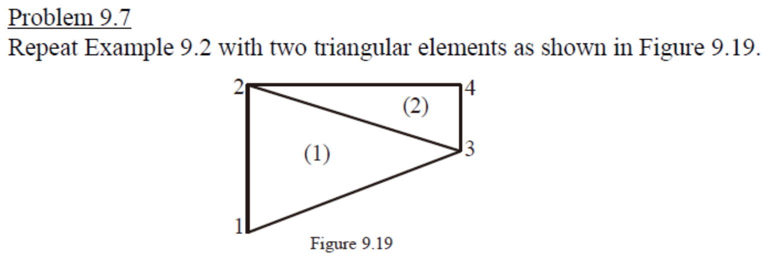
1 Problem 9.7



Problem 9.7

Repeat Example 9.2 with two triangular elements as shown in Figure 9.19.

Figure 1: Problem 9.7

1.1 Preprocessing, Formation & Assembly

首先写出材料的本构矩阵

$$\boldsymbol{D} = \frac{E}{1-\nu^2}\begin{pmatrix} 1 & \nu & \\ \nu & 1 & \\ & & \frac{1-\nu}{2} \end{pmatrix}$$

其中，$E = 3 \times 10^7 Pa, \nu = 0.3$，单元节点标号顺序分别为 $(1) 2 \to 1 \to 3$ 与 $(2) 2 \to 3 \to 4$，于是单元坐标矩阵可写为

$$\begin{pmatrix} \boldsymbol{x}^{(1)} & \boldsymbol{y}^{(1)} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 2 & 0.5 \end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{x}^{(2)} & \boldsymbol{y}^{(2)} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 0.5 \\ 2 & 1 \end{pmatrix}$$

由此容易求得单元应变矩阵与 $Jacobi$ 矩阵

$$\boldsymbol{B}^{(1)} = \frac{1}{2}\begin{pmatrix} -0.5 & 0 & -0.5 & 0 & 1 & 0 \\ 0 & 2 & 0 & -2 & 0 & 0 \\ 2 & -0.5 & -2 & -0.5 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{B}^{(2)} = \begin{pmatrix} -0.5 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \\ 0 & -0.5 & -2 & 0 & 2 & 0.5 \end{pmatrix}$$

$$\boldsymbol{J}^{(1)} = \begin{pmatrix} -2 & 0.5 \\ -2 & -0.5 \end{pmatrix}$$

$$\boldsymbol{J}^{(2)} = \begin{pmatrix} -2 & 0 \\ 0 & -0.5 \end{pmatrix}$$

为了计算单元刚度阵，注意到单元内部为常应变状态且 $Jacobi$ 矩阵为常数矩阵，因此有

$$\boldsymbol{K}^e = \frac{|\boldsymbol{J}^e|}{2}\boldsymbol{B}^{eT}\boldsymbol{D}\boldsymbol{B}^e$$

单元刚度阵的计算结果为

$$\boldsymbol{K}^{(1)} = 10^7 \times \begin{pmatrix} 1.5399 & -0.5357 & -0.9478 & -0.0412 & -0.4121 & 0.5769 \\ -0.5357 & 3.3688 & 0.0412 & -3.2246 & 0.4945 & -0.1442 \\ -0.9478 & 0.0412 & 1.3599 & 0.5357 & -0.4121 & -0.5769 \\ -0.0412 & -3.2246 & 0.5357 & 3.3688 & -0.4945 & -0.1442 \\ -0.4121 & 0.4945 & -0.4121 & -0.4945 & 0.8242 & 0 \\ 0.5769 & -0.1442 & -0.5769 & -0.1442 & 0 & 0.2885 \end{pmatrix}$$

$$\boldsymbol{K}^{(2)} = 10^7 \times \begin{pmatrix} 0.4121 & 0 & 0 & 0.4945 & -0.4121 & -0.4945 \\ 0 & 0.1442 & 0.5769 & 0 & -0.5769 & -0.1442 \\ 0 & 0.5769 & 2.3077 & 0 & -2.3077 & -0.5769 \\ 0.4945 & 0 & 0 & 6.5934 & -0.4945 & -6.5934 \\ -0.4121 & -0.5769 & -2.3077 & -0.4945 & 2.7198 & 1.0714 \\ -0.4945 & -0.1442 & -0.5769 & -6.5934 & 1.0714 & 6.7376 \end{pmatrix}$$

组装系统刚度阵时只需要将 $\boldsymbol{K}^{(1)}$ 和 $\boldsymbol{K}^{(2)}$ 添加至 $\boldsymbol{K}(1:6, 1:6)$ 与 $\boldsymbol{K}(3:8, 3:8)$

处即可，具体为

$$
\boldsymbol{K} = 10^7 \times \begin{pmatrix}
1.3599 & 0.5357 & -0.9478 & 0.0412 & -0.4121 & -0.5769 & 0 & 0 \\
0.5357 & 3.3688 & -0.0412 & -3.2246 & -0.4945 & -0.1442 & 0 & 0 \\
-0.9478 & -0.0412 & 1.7720 & -0.5357 & -0.4121 & 1.0714 & -0.4121 & -0.4945 \\
0.0412 & -3.2246 & -0.5357 & 3.5130 & 1.0714 & -0.1442 & -0.5769 & -0.1442 \\
-0.4121 & -0.4945 & -0.4121 & 1.0714 & 3.1319 & 0 & -2.3077 & -0.5769 \\
-0.5769 & -0.1442 & 1.0714 & -0.1442 & 0 & 6.8819 & -0.4945 & -6.5934 \\
0 & 0 & -0.4121 & -0.5769 & -2.3077 & -0.4945 & 2.7198 & 1.0714 \\
0 & 0 & -0.4945 & -0.1442 & -0.5769 & -6.5934 & 1.0714 & 6.7376
\end{pmatrix}
$$

下面组装右端力向量，将单元 (2) 的 $1-4$ 边上的分布力等效为节点力，有

$$
\begin{aligned}
t_{x2} &= t_{x4} = 0\,(N), \\
t_{y2} &= t_{y4} = -20\,(N).
\end{aligned}
$$

这样就有

$$
\boldsymbol{f} = \begin{pmatrix} 0 & 0 & 0 & -20 & 0 & 0 & 0 & -20 \end{pmatrix}^T.
$$

于是就可写系统刚度方程

$$
\boldsymbol{K}\boldsymbol{d} = \boldsymbol{f} + \boldsymbol{r}. \tag{1.1}
$$

其中，$\boldsymbol{K}, \boldsymbol{f}$ 的具体数值前已给出，$\boldsymbol{d}, \boldsymbol{r}$ 的形式如下

$$
\begin{aligned}
\boldsymbol{d} &= \begin{pmatrix} 0 & 0 & 0 & 0 & d_{x3} & d_{y3} & d_{x4} & d_{y4} \end{pmatrix}^T, \\
\boldsymbol{r} &= \begin{pmatrix} r_{x1} & r_{y1} & r_{x2} & r_{y2} & 0 & 0 & 0 & 0 \end{pmatrix}^T.
\end{aligned}
$$

1.2 Solution & Postprocessing

这时可直接应用分块降阶法，可依次求得节点位移与约束反力向量如下

$$
\boldsymbol{d} = 10^{-5} \times \begin{pmatrix} 0 & 0 & 0 & 0 & -0.0387 & -0.6657 & 0.1235 & -0.7041 \end{pmatrix}^T, \tag{1.2}
$$

$$
\boldsymbol{r} = \begin{pmatrix} 40.0000 & 11.5154 & -40.0000 & 28.4846 & 0 & 0 & 0 & 0 \end{pmatrix}^T (N). \tag{1.3}
$$

在与精确解比照后可以知道，与 4Q 单元相比，3T 单元的精度要差很多。

2 Problem 9.8

Problem 9.8
Develop MATLAB finite element program for the three-node constant strain triangular element. Note that in this case, element matrices can be computed without numerical integration. Test the code in one of the following two ways:
(a) against manual calculations for a two-element problem (see Problem 9.7) or
(b) against the MATLAB code for a quadrilateral element provided in chapter 12.
In the latter case, consider very fine meshes (at least 64-element mesh for the problem in Figure 9.14 for quadrilaterals)

Figure 2: Problem 9.8

节点位移与约束反力的计算结果与 **Problem 9.7** 中的结果是一致的。其中，MATLAB 程序改动之处包括

- Preprocessor:
  - `plotmesh` 函数增加 `nen=3` 的判断；
- Formation:
  - `BmatElast2D` 函数增加计算 3T 单元应变矩阵与 *Jacobi* 行列式的分支；
  - `elast2Delem` 函数增加组装 3T 单元刚度阵及体积力向量的分支；
- Postprocessor:
  - `displacements` 函数增加 `nen=3` 的判断；
  - `get_stress` 与 `nodal_stress` 函数增加 3T 常应变单元的应力恢复分支。