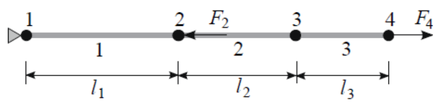
Direct Stiffness Method

huangyf15

1 Problem 1

For a three-element structure shown below determine displacements of all nodal points, compute values of stress σ_x for all elements, check local/global equilibrium. Data of the problem are as follows.

	Element number		
	1	2	3
E [GPa]	200	200	400
A [m^2]	10^{-4}	10^{-4}	10^{-4}



1.1 Stress calculation

The system has been subdivided into finite elements as shown in the figure, as well as the order of the nodes. And we could write the element equation as follows($\boldsymbol{e}$ is the label of each element, from 1 to 4):

$$\boldsymbol{F}^e = \boldsymbol{K}^e \boldsymbol{d}^e$$

where the term $\boldsymbol{K}^e$ is equal to $k^e \boldsymbol{K}_0$ in which $\boldsymbol{K}_0$ (which I called the "kernel") is defined as

$$\boldsymbol{K}_0 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

and k^e is the element stiffness

$$k^e = \frac{E^e A^e}{l^e}.$$

Now, as long as we get the global stiffness matrix, the equilibrium equations of the system could be easily written down. With the given boundary conditions, we will find the final answer. According to the corresponding rules between the local labels and the global ones, the generator matrix of label-transition matrix could be written as follows:

$$\boldsymbol{L} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$$

From the first row, we could make the label-transition matrix for the first element, which is a quasi-permutation matrix:

$$\boldsymbol{L}^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

and the other ones are the like. Then, the global stiffness matrix could be written as

$$\boldsymbol{K} = \sum_{i=1}^{3} \boldsymbol{L}^{i^T} \boldsymbol{K} \boldsymbol{L}^i = \begin{pmatrix} k^1 & -k^1 & & \\ -k^1 & k^1+k^2 & -k^2 & \\ & -k^2 & k^2+k^3 & -k^3 \\ & & -k^3 & k^3 \end{pmatrix}$$

So now we could write down the equilibrium equation of the system

$$\boldsymbol{K}\boldsymbol{d} = \boldsymbol{f} + \boldsymbol{r},$$

in which the term $\boldsymbol{d}$ is the nodal displacement, $\boldsymbol{f} = \begin{pmatrix} 0 & -F_2 & 0 & F_4 \end{pmatrix}^T$ is the eternal force, and $\boldsymbol{r}$ is the constraint reaction force.Using the reduction technique, we could easily get the nodal displacement as well as the unknown reaction force

$$\begin{aligned} \boldsymbol{d}_2 &= -\tfrac{\boldsymbol{F}_2 - \boldsymbol{F}_4}{k^1}, \\ \boldsymbol{d}_3 &= \tfrac{\boldsymbol{F}_4}{k^2} - \tfrac{\boldsymbol{F}_2 - \boldsymbol{F}_4}{k^1}, \\ \boldsymbol{d}_4 &= \tfrac{\boldsymbol{F}_4}{k^3} + \tfrac{\boldsymbol{F}_4}{k^2} - \tfrac{\boldsymbol{F}_2 - \boldsymbol{F}_4}{k^1}, \\ \boldsymbol{r} &= \boldsymbol{F}_2 - \boldsymbol{F}_4. \end{aligned}$$

Using the definition equation of the stress, that is $\sigma^e = \frac{E^e \Delta d^e}{l^e}$, we could finally find the stress in each element

$$\begin{aligned} \sigma^1 &= \tfrac{\boldsymbol{F}_4 - \boldsymbol{F}_2}{A} \\ \sigma^2 &= \tfrac{\boldsymbol{F}_4}{A} \qquad , \\ \sigma^3 &= \tfrac{\boldsymbol{F}_4}{A} \end{aligned}$$

where $A = A_1 = A_2 = A_3$ is the cross-sectional area of the elements.

1.2 Local/global equilibrium checking

The local equilibrium is shown from the following equations

$$\begin{aligned} &node\ 1: \sigma_1 A + r = 0;\\ &node\ 2: \sigma_1 A + F_2 = \sigma_2; A\\ &node\ 3: \sigma_2 A = \sigma_3 A;\\ &node\ 4: \sigma_3 A = F_4. \end{aligned}$$

As for the global one, we have

$$r - F_2 + F_4 = 0.$$

2 Problem 2

Problem 2.1 [F]:p37
For the spring system given in Figure 2.16,
a. Number the elements and nodes.
b. Assemble the global stiffness and force matrix.
c. Partition the system and solve for the nodal displacements.
d. Compute the reaction forces.

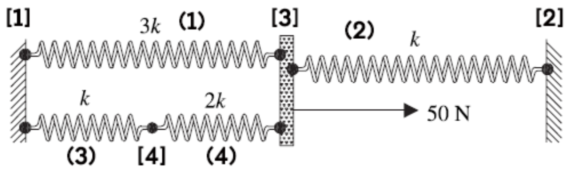


Figure 2.16 Data for Problem 2.1.

2.1 Number the elements and nodes

Just as shown in the figure.

2.2 Assemble the global stiffness and force matrix

Just like the procedures presented in the problem 1, we could easily write down the generator matrix of label-transition matrix

$$\boldsymbol{L} = \begin{pmatrix} 1 & 3 & 1 & 4 \\ 3 & 2 & 4 & 3 \end{pmatrix}.$$

And the global stiffness matrix could be written as follows

$$\boldsymbol{K} = \begin{pmatrix} 4k & & -3k & -k \\ & k & -k & \\ -3k & -k & 6k & -2k \\ -k & & -2k & 3k \end{pmatrix}$$

The force matrix $\boldsymbol{f} = \begin{pmatrix} 0 & 0 & 50 & 0 \end{pmatrix}^T$.

2.3 Solve for the nodal displacements and compute the reaction forces

Using the same method called "reduction technique" as in the problem 1, we finally find that

$$\boldsymbol{d}_3 = \tfrac{75}{7k},$$

$$\boldsymbol{d}_4 = \tfrac{50}{7k},$$

$$\boldsymbol{r}_1 = -\tfrac{275}{7}(N),$$

$$\boldsymbol{r}_2 = -\tfrac{75}{7}(N).$$

A Source Code

```
function [K,u,r] = FEM01_1D(m,n,p)
------Variables descriptions----
   m   the number of elements
   n   the number of nodes
   p   the number of constraints
--------------end---------------
syms k(t) F(t) u(t) r(t);
// stiffness, external/constraint force, displacement, constraint force
syms a;
k = [3*a,a,a,2*a];
F = [0;0;50;0];

K0 = [ 1 -1
      -1  1];
L0 = zeros(2,m);
L  = zeros(2,n);
K  = zeros(n);
for j = 1:m
    for i = 1:2
        L0(i,j) = input('L0');
    end
end

for j = 1:m
    for i = 1:2
        L(i, L0(i,j)) = 1;
    end
    K1 = k(j) * K0;
    K = L'* K1 * L + K;
    L = zeros(2,n);
end

p = p+1;
u = sym(K(p:n,p:n))\sym(F(p:n));
p = p-1;
r = sym(K(1:p,(p+1):n))*sym(u);
end
```