Chapter 1

Symmetric rank-k update: $C := A^T A + C$

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1.1 Operation

Consider the operation

$$C := A^T A + C$$

where A is a $m \times m$ lower triangular matrix and C is a $m \times m$ matrix. This is a special case of triangular matrix-matrix multiplication, with a matrix being multiplied by its transpose on the left. We will refer to this operation as SYRK_LT where the LT indicates the two matrices are transposed and the matrix being updated is stored in the lower triangular part.

1.2 Precondition and postcondition

In the precondition

$$C = \widehat{C}$$

$\widehat{C}$ denotes the original contents of C. This allows us to express the state upon completion, the postcondition, as

$$C = A^T A + \widehat{C}.$$

It is implicitly assumed that C is a symmetric matrix.

1.3 Partitioned Matrix Expressions and loop invariants

There is one PME for this operation.

1.3.1 PME

To derive the PME, partition

$$C \rightarrow \left(\begin{array}{c|c} C_{TL} & C_{BL}^T \\ \hline C_{BL} & C_{BR} \end{array}\right), \quad \text{and} \quad A \rightarrow \left(\begin{array}{c|c} A_L & A_R \end{array}\right)$$

Substituting these into the postcondition yields

$$\left(\begin{array}{c|c} C_{TL} & C_{BL}^T \\ \hline C_{BL} & C_{BR} \end{array}\right) = \left(\begin{array}{c} A_L^T \\ \hline A_R^T \end{array}\right) \left(\begin{array}{c|c} A_L & A_R \end{array}\right) + \left(\begin{array}{c|c} \widehat{C}_{TL} & C_{BL}^T \\ \hline \widehat{C}_{BL} & \widehat{C}_{BR} \end{array}\right)$$

or, equivalently,

$$\left(\begin{array}{c|c} C_{TL} & C_{BL}^T \\ \hline C_{BL} & C_{BR} \end{array}\right) = \left(\begin{array}{c|c} A_L^T A_L + \widehat{C}_{TL} & A_L^T A_R + C_{BL}^T \\ \hline A_R^T A_L + \widehat{C}_{BL} & A_R^T A_R + \widehat{C}_{BR} \end{array}\right)$$

From this, we can choose four loop invariants:

Invariant 1: $\left(\begin{array}{c|c} C_{TL} & C_{BL}^T \\ \hline C_{BL} & C_{BR} \end{array}\right) = \left(\begin{array}{c|c} A_L^T A_L + \widehat{C}_{TL} & C_{BL}^T \\ \hline \widehat{C}_{BL} & \widehat{C}_{BR} \end{array}\right).$

(The top left part has been left alone and the rest have been partially computed).

Invariant 2: $\left(\begin{array}{c|c} C_{TL} & C_{BL}^T \\ \hline C_{BL} & C_{BR} \end{array}\right) = \left(\begin{array}{c|c} \widehat{C}_{TL} & C_{BL}^T \\ \hline \widehat{C}_{BL} & A_R^T A_R + \widehat{C}_{BR} \end{array}\right).$

(The bottom right part has been left alone and the rest have been partially computed).

Invariant 3: $\left(\begin{array}{c|c} C_{TL} & C_{BL}^T \\ \hline C_{BL} & C_{BR} \end{array}\right) = \left(\begin{array}{c|c} A_L^T A_L + \widehat{C}_{TL} & C_{BL}^T \\ \hline A_R^T A_L + \widehat{C}_{BL} & \widehat{C}_{BR} \end{array}\right).$

(The left part has been left alone and the right part has been partially computed).

Invariant 4: $\left(\begin{array}{c|c} C_{TL} & C_{BL}^T \\ \hline C_{BL} & C_{BR} \end{array}\right) = \left(\begin{array}{c|c} \widehat{C}_{TL} & C_{BL}^T \\ \hline A_R^T A_L + \widehat{C}_{BL} & A_R^T A_R + \widehat{C}_{BR} \end{array}\right).$

(The bottom part has been left alone and the top part has been partially computed).

1.3.2 Notes

How do I decide to partition the matrices in the postcondition?

- Pick a matrix (operand), any matrix.
- If that matrix has
  - a triangular structure (in storage), then you want to either partition is into four quadrants, or not at all. Symmetric matrices and triangular matrices have a triangular structure (in storage).

- no particular structure, then you partition it vertically (left-right), horizontally (top-bottom), or not at all.

- Next, partition the other matrices similarly, but conformally (meaning the resulting multiplications with the parts are legal).

Take our problem here: $C := A^T A + C$. Start by partitioning A horizontally:

$$\left(A_L \,\middle|\, A_R \right)$$

Then, the transpose of A will give us:

$$\left(\begin{array}{c} A_L^T \\ \hline A_R^T \end{array} \right)$$

which is easier to express as a matrix partitioned vertically. The postcondition will now look like:

$$C = \left(\begin{array}{c} A_L^T \\ \hline A_R^T \end{array} \right) \left(A_L \,\middle|\, A_R \right) + \widehat{C}$$

Now, the way partitioned matrix multiplication works, A would yield a matrix partitioned into quadrants:

$$C = \underbrace{\left(\begin{array}{c} A_L^T \\ \hline A_R^T \end{array} \right) \left(A_L \,\middle|\, A_R \right)}_{\left(\begin{array}{c|c} A_L^T A_L & A_L^T A_R \\ \hline A_R^T A_L & A_R^T A_R \end{array} \right)} + \widehat{C}$$

So, we need to also partition C into quadrants:

$$\left(\begin{array}{c|c} C_{TL} & C_{BL}^T \\ \hline C_{BL} & C_{BR} \end{array} \right) = \left(\begin{array}{c|c} A_L^T A_L & A_L^T A_R \\ \hline A_R^T A_L & A_R^T A_R \end{array} \right) + \left(\begin{array}{c|c} \widehat{C}_{TL} & C_{BL}^T \\ \hline \widehat{C}_{BL} & \widehat{C}_{BR} \end{array} \right)$$

1.4 Deriving all unblocked algorithms

The below table summarizes all loop invariants, with links to all files related to this operation.

The worksheet and code skeletons were genered using the ☛ Spark webpage.

	Invariant	Derivations	Implementations
1	$\left(\begin{array}{c\|c} C_{TL} & C_{BL}^T \\ \hline C_{BL} & C_{BR} \end{array}\right) = \left(\begin{array}{c\|c} A_L^T A_L + \widehat{C}_{TL} & C_{BL}^T \\ \hline \widehat{C}_{BL} & \widehat{C}_{BR} \end{array}\right)$	PDF	syrk_lt_unb_var1.mlx syrk_lt_unb_var1.c
2	$\left(\begin{array}{c\|c} C_{TL} & C_{BL}^T \\ \hline C_{BL} & C_{BR} \end{array}\right) = \left(\begin{array}{c\|c} \widehat{C}_{TL} & C_{BL}^T \\ \hline \widehat{C}_{BL} & A_R^T A_R + \widehat{C}_{BR} \end{array}\right)$	PDF	syrk_lt_unb_var2.mlx syrk_lt_unb_var2.c
3	$\left(\begin{array}{c\|c} C_{TL} & C_{BL}^T \\ \hline C_{BL} & C_{BR} \end{array}\right) = \left(\begin{array}{c\|c} A_L^T A_L + \widehat{C}_{TL} & C_{BL}^T \\ \hline A_R^T A_L + \widehat{C}_{BL} & \widehat{C}_{BR} \end{array}\right)$	PDF	syrk_lt_unb_var3.mlx syrk_lt_unb_var3.c
4	$\left(\begin{array}{c\|c} C_{TL} & C_{BL}^T \\ \hline C_{BL} & C_{BR} \end{array}\right) = \left(\begin{array}{c\|c} \widehat{C}_{TL} & C_{BL}^T \\ \hline A_R^T A_L + \widehat{C}_{BL} & A_R^T A_R + \widehat{C}_{BR} \end{array}\right)$	PDF	syrk_lt_unb_var4.mlx syrk_lt_unb_var4.c