Contents

Formulas

This is a cheat sheet for USAAAO

Orbital Dynamics

Newton Gravity Law

$$F_g = \frac{GMm}{r^2}$$

This is known as the *Inversed-Squared* law of gravity. G is the Gravitational Constant.

Gravity Potential Energy: Point Mass Assuming the potential energy at infinity is *zero*, by integrating the gravity law, we have the potential energy at R:

$$U = \int_{\infty}^{R} \frac{GMm}{r^2} dr = -\frac{GMm}{R}$$

Therefore the *Sun* can be visualized as a *Gravitational Well* , in which the deeper you get, the less energy you have.

Gravity Potential Energy: Uniform Ball A ball with mass M and radius R, assuming uniform density:

$$\rho = \frac{M}{\frac{4}{3}\pi R^3}$$

The potential energy is:

$$\begin{aligned} U = \int_0^R dU &= \int_0^R -\frac{GM(r)dm}{r} \\ &= \int_0^R -\frac{G \cdot \frac{4}{3}\pi r^3 \rho \cdot 4\pi r^2 dr \rho}{r} \\ &= -\frac{3GM^2}{R^6} \int_0^R r^4 dr \\ &= -\frac{3}{5}\frac{GM^2}{R} \end{aligned}$$

Together with *viral theorem*: $\langle K \rangle = -\frac{1}{2}\langle U \rangle$, one can link the observational properties (velocities->kinetic energy) to its mass

Conservation of Momentum

Examples:

Without the effects of *force*, the momentum of the system is conserved:

$$\vec{P} = \sum_m \vec{p} = \sum_m m\vec{v} = const$$

Conservation of Angular Momentum

Examples:

Without the effects of *torque*, the angular momentum of the system (referenced at a give point) is conserved

$$\vec{L} = \sum_m \vec{l} = \sum_m \vec{r} \times m\vec{v} = const$$

Conservation of Energy

The total energy: Kinetic+Potential is **conserved** for planets:

$$E = K + U = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

For a elipse orbit with semi-major axis a: (Derivation: Conservation of energy at aphelia and perihelia)

$$E = -\frac{GMm}{2a} = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

Orbital Energy $E = -\frac{GMm}{2a}$ is known as the *orbital energy*. One immediately notices three properties:

- E is *negative* for elipse ($a > 0$), *zero* for parabola ($a = \infty$), *positive* for hyperbola ($a < 0$)
- Increase in *orbital energy* will increase a until it becomes a parabola, or even hyperbola
- A meteorite is *trapped* when $E < 0$, it *escapes* when $E \geq 0$.

Vis-Viva Equation Due to the conservation of orbital energy, one calculate velocity based on distance r:

$$v = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)}$$

This is known as the *vis-viva* equation. The *escape* velocity is:

$$v_{excape} = \sqrt{\frac{2GM}{r}}$$

Examples: 2021-Q15

Viral Theorem In statistical mechanics, people are often interested in the averaged behavior of an ensemble of particles, one of the most important results is the *viral theorem*:

$$\langle K \rangle = -\frac{1}{2}\langle U \rangle$$

And therefore the total energy:

$$\langle E \rangle = \langle K \rangle + \langle U \rangle = -\langle K \rangle = \frac{1}{2}\langle U \rangle$$

Examples: 2021-Q9

Kepler's Law

Examples: 2023-Q21

First Law *The orbit of a planet is an ellipse with the Sun at one of the two foci.*

Second Law *line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.*

This is effectively the *Conservation of Angular Momentum*, because:

- For a small object orbiting a central star, the *Gravity force* is point towards the star, therefore the change in augular momentum:

$$d\vec{L} = \vec{r} \times \vec{F}_g = 0$$

And the angular momentum:

$$\vec{L} = \vec{r} \times m\vec{v}$$

is conserved.

- Constant $\vec{L}$ is identical to * sweeps out equal areas during equal intervals of time*

$$\vec{r} \times m\vec{v}dt \propto \vec{r} \times \vec{v}dt$$

is the small change in the area

Third Law *The square of a planet's orbital period is proportional to the cube of the length of the semi-major axis of its orbit.*

Simple derivation can be inferred from circular orbit, where the **centrifugal force balances the gravity force**:

$$\frac{GMm}{a^2} = m\omega^2 a = m\left(\frac{2\pi}{T}\right)^2 a \frac{GM}{4\pi^2} = \frac{a^3}{T^2}$$

Sidereal Day and Solar Day

Examples: 2020-Q12;

We denote sidereal day as t_{sid} and Solar Day as t_{sol}, we have:

$$t_{sid} = \frac{2\pi}{\omega_0}$$

$$t_{sol} = \frac{2\pi}{\omega_0 - \omega_1 \cos\theta}$$

where ω_0 is the angular velocity of the planet's rotation, and ω_1 is the angular velocity of orbital revolution, θ is the tilt angle.

Celestial Coordinates and Time

Examples: 2023-Q1|17|18; 2022-Q24|25|28|30

Trigonometry

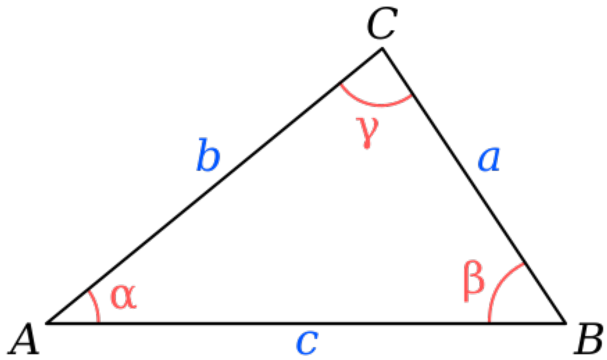

Figure 1: Trigonometry

Law of Cosine

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos\alpha \\ b^2 &= a^2 + c^2 - 2ac\cos\beta \\ c^2 &= a^2 + b^2 - 2bc\cos\gamma \end{aligned}$$

Derivation:

$$\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$$

$$\begin{aligned} |\overrightarrow{BC}|^2 &= |\overrightarrow{AC} - \overrightarrow{AB}|^2 \\ &= |\overrightarrow{AC}|^2 + |\overrightarrow{AB}|^2 - 2\overrightarrow{AC}\cdot\overrightarrow{AB} \\ a^2 &= b^2 + c^2 - 2bc\cos\alpha \end{aligned}$$

Law of Sine

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

Spherical Trigonometry

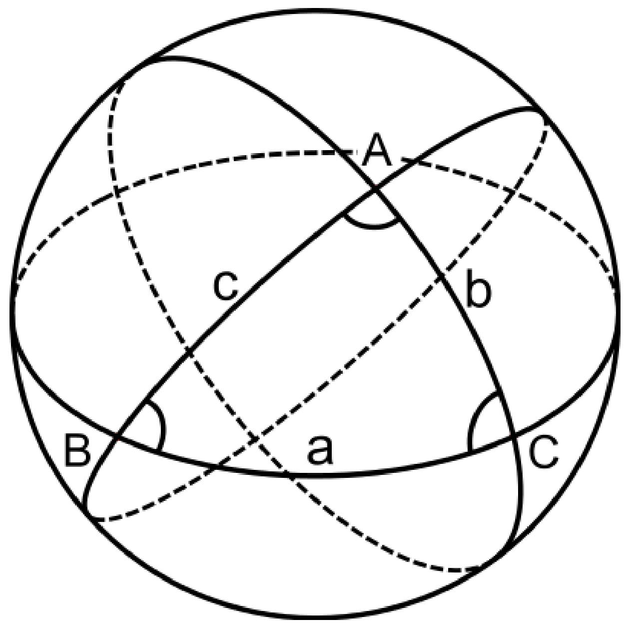

Figure 2: Spherical Trinometry

Spherical Law of Cosine

$$\begin{aligned} \cos a &= \cos b\cos c + \sin b\sin c\cos A \\ \cos b &= \cos c\cos a + \sin c\sin a\cos B \\ \cos c &= \cos a\cos b + \sin a\sin b\cos C \end{aligned}$$

Spherical Law of Sine

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

Area of the Spherical Triangle

$$Area\ of\ triangle = A + B + C - \pi,$$

Celestial Coordinate

Circumpolar

$$90 - \delta < \lambda$$

λ is latitude, δ is declination

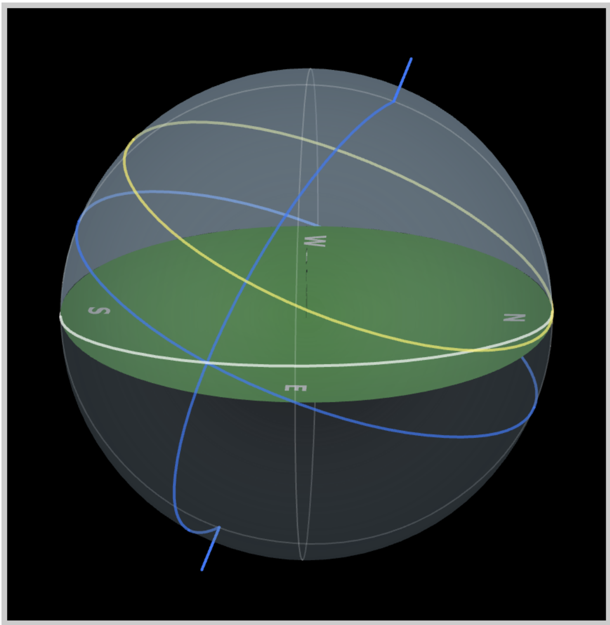

Figure 3: Circumpolar

Telescope & Star Magnitudes

Parallax

Examples:

One *parsec* $\approx 3.26\ ly$ is the parallax of the distant star from a triangle of 1AU and 1 arcsec

Some confusing notations:

- mac: micro-arcsec $= 10^{-3} arcsec$
- Mpc: Million-parsec $= 10^{6} pc$

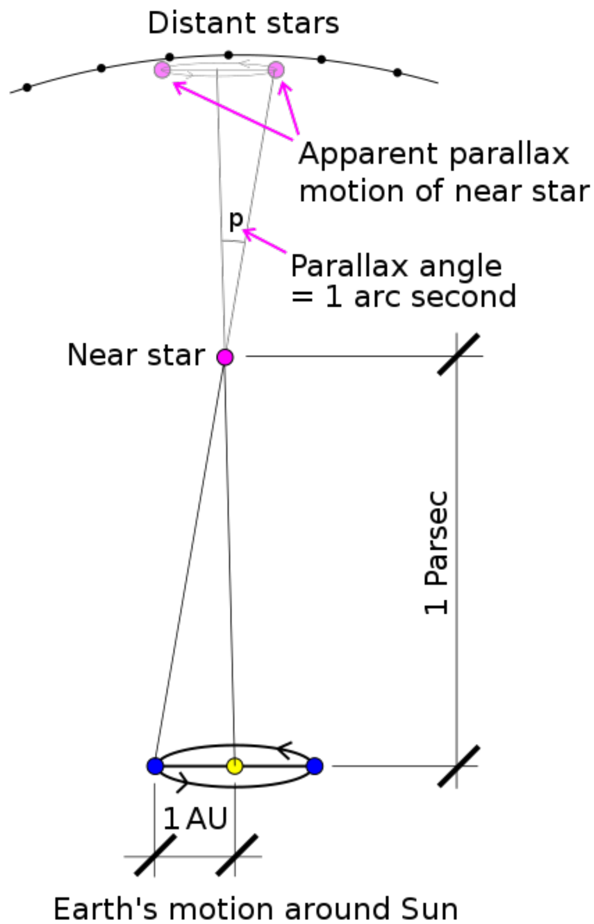

Figure 4: Parallax

The Airy Spot

Examples: 2022-Q6

Due to the diffraction of light, the *best-focused spot* of light has a limited angular size.

$$\sin\theta \approx \theta \approx 1.22\frac{\lambda}{d}$$

where λ is the light wavelength, d is the diameter of the lens. To **differentiate** two light source, they have to be θ away from each other.

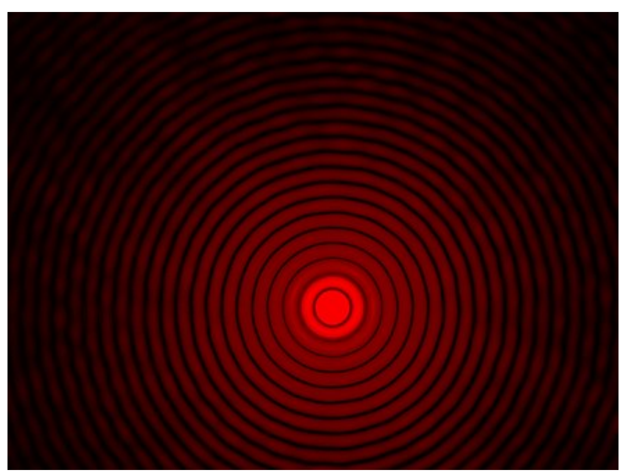

Figure 5: Airy Pattern

Telescope Parameters

Examples: 2021-Q13, 2023-Q2|Q24

f number (focal ratio) The focal ratio is the ratio between the focal length f and the diameter of the aperture d:

$$N = \frac{f}{d}$$

This number is usually denoted as f/N.

For example, $f/2$ means $f = 2d$, the *larger the number, the worse the telescope.*

Magnification The magnification $m = f_o/f_e$ is the ratio between the focal length of *objective* and *eyepiece* lens.

The Apparent and Absolute Magnitude

Examples: 2023-Q2|13; 2022-Q8|18|20; 2021-Q22|23; 2020-Q13; 2019-Q13, 2018-Q13|21

Magnitude and Flux The ultimate physical carrier of light is the flux of photons (or electric-magnetic field), which follows the *inversed-squared law.* Magnitude is a *representation* of the *relative* amount of flux. The definition is that:

> Five unit of *magnitude* = 100 difference in *flux*

$$100^{\frac{m_1 - m_2}{5}} = \frac{F_2}{F_1}$$

This can be rewritten in terms of distance (for same type of star):

$$10^{\frac{m_1 - m_2}{5}} = \frac{d_1}{d_2}$$

And in log10 in terms of distance:

$$m_1 - m_2 = 5 \log_{10} d_1 - 5 \log_{10} d_2$$

Absolute Magnitude M The apparent magnitude of a star measured at $10pc$ ($\log_{10}(10pc) = 1$):

$$M = m - 5 \log_{10}(d_{pc}) + 5$$

Extinction Due to the existence of dust, the light can dim:

$$m - M = 5 * \log(d) - 5 + a_V * d$$

Where a_V is the interstellar extinction in the unit of mag/pc or mag/kpc

Examples: 2021-Q23, 2019-Q13

Special Relativity, Hubble's Law & Red Shift

Special Relativity

If the velocity is comparable to the speed of light c, the relativity effects can not be ignored.

Mass-Energy Equation Examples: 2023-Q11, 2022-Q12|15, 2021-Q20, 2020-Q15

The mass and energy is equivalent:

$$E = mc^2$$

The loss of mass is identical to the loss of energy. This is the ultimate source of energy in the universe: *Fusion in the stars.*

Lorentz Coefficient

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

For a moving body, the time flow is slower "*Time dilation*": (S' is the moving frame)

$$\Delta t' = \gamma \Delta t$$

The "*length contraction*":

$$\Delta x' = \frac{\Delta x}{\gamma}$$

Examples: 2023-Q15,

Hubble's Law & Red Shift

Examples: 2023-Q27; 2022-Q29;2021-Q8|12|26

The universe is constantly expanding with a coefficient $H_0 = 70km/s/Mpc$, the expanding speed is:

$$v = H_0 D$$

The resulting “red-shift velocity” is **defined** to be:

$$v_{rs} = cz$$

where z is the red shift. In low velocity case, this can be related to the real red-shift in observed wavelength using the *Fizeau-Doppler Formula*:

$$z = \frac{\lambda_\mathrm{o}}{\lambda_\mathrm{e}} - 1 = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} - 1 \approx \frac{v}{c}$$

where λ_o and λ_e is the observed and emitted wavelength. Since the speed of light is constant, this can also be used to calculate the change in frequency:

$$\frac{\nu_e}{\nu_o} = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

Critical Density of the Universe Replace the escape velocity with the speed of the light from Hubble’s expansion:

$$c = H_0 r = \sqrt{\frac{2GM}{r}}$$

We have:

$$\rho = \frac{M}{\frac{4}{3}\pi r^3} = \frac{3H_0^2}{8\pi G} \simeq 9.22 \times 10^{-27} kg \cdot m^{-3}$$

MISC

Signal to Noise Ratio

Proportional to $\sqrt{N}$, where N is the number of measurements or exposure time

Examples: 2022-Q27

Energy of E&M Wave

Poynting Flux:

$$\vec{S} = \vec{E} \times \vec{B}$$

is *independent* of frequency

Constants and Notations

Constants

1. The absolute magnitude of the Sun: 4.83
2. Age of the Universe: 13.3 Billion years
3. Visible wavelength: 310 nm (ultraviolet) - 1100 nm (infrared)

Notations

1. Length: pm/Å/nm/μm/mm/cm/km/Mm: $10^{-12}, 10^{-10}, 10^{-9}, 10^{-6}, 10^{-3}, 10^{-2}, 10^{3}, 10^{6} m$