Math 262: Line and Surface Integrals

Scalar Line Integral $\int_C f\,ds$ (special case: arc length $\int_C ds$)

- If f is constant, then $\int_C f\,ds = f(\text{arc length of } C)$
- If f is odd in a variable and C is symmetric in that variable, then $\int_C f\,ds = 0$
- Otherwise, choose a parameterization of $\vec{s}(t)$ of C, with $a \le t \le b$ and evaluate

$$\boxed{\int_C f\,ds = \int_a^b f\frac{ds}{dt}\,dt}$$

Vector Line Integral $\int_C \vec{F}\cdot d\vec{s} = \int_C Mdx + Ndy + Pdz$

- If $\vec{F}$ is **conservative,** with a potential f, then $\boxed{\int_{C:A\to B} \vec{F}\cdot d\vec{s} = f(B) - f(A)}$
- If C is **closed,** we are in $\mathbb{R}^2$, and $\vec{F}$ is defined throughout the region D enclosed by C, then we can use **Green's theorem**:

Green:

$$\oint_{C=\partial D} \vec{F}\cdot d\vec{s} = \iint_D \operatorname{scurl}\vec{F}\,dA$$

or

$$\int_{C=\partial D} Mdx + Ndy = \iint_D (N_x - M_y)dA$$

- If C is **closed,** we are in $\mathbb{R}^3$, and $\vec{F}$ is defined throughout a surface S that "caps" C, then we can use **Stokes's theorem** (with S oriented according to the Right Hand Rule),

Stokes:

$$\oint_{C=\partial S} \vec{F}\cdot d\vec{s} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_S (P_y - N_z, M_z - P_x, N_x - M_y)\cdot d\vec{S}$$

- Otherwise, choose a parameterization of $\vec{s}(t)$ of C, with $a \le t \le b$ and evaluate

$$\int_C \vec{F}\cdot d\vec{s} = \int_a^b \vec{F}\cdot\frac{d\vec{s}}{dt}\,dt$$

or

$$\int_C Mdx + Ndy + Pdz = \int_a^b \left(M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt}\right)dt$$

Math 262: Line and Surface Integrals

Scalar Surface Integral $\iint_S f\, dS$ (special case: Surface area $\iint_S dS$)

- If f is **constant** on S, then $\iint_S f\, dS = f(\text{area of } S)$
- If f is **odd in one of the variables** and S is symmetric in that variable, then $\iint_S f\, dS = 0$.
- Otherwise, consider a **parameterization** $\vec{S}(s,t)$ of S on D, find the standard normal vector $\vec{N} = \frac{\partial \vec{S}}{\partial s} \times \frac{\partial \vec{S}}{\partial t}$ and let

$$\boxed{\iint_S f\, dS = \iint_D f \left\|\vec{N}\right\| dA}$$

Flux, $\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS$, where $\vec{n}$ is the orientation

- If $\vec{F} \cdot \vec{n}$ is **constant**, then $\iint_S \vec{F} \cdot \vec{n}\, dS = \left(\vec{F} \cdot \vec{n}\right)(\text{area of } S)$.
- If $\vec{F} \cdot \vec{n}$ is **odd in a variable** and S is symmetric in that variable, then $\iint_S \vec{F} \cdot d\vec{S} = 0$.
- If S is a **closed surface**, oriented outward, and if $\vec{F}$ is defined throughout the solid region W enclosed by S, then we can use **Gauss' Theorem**:

$$\boxed{\textbf{Gauss:} \quad \oiint_{S=\partial W} \vec{F} \cdot d\vec{S} = \iiint_W \text{div}\vec{F}\, dV = \iiint_W \left(M_x + N_y + P_z\right) dV}$$

- Otherwise, consider a parameterization $\vec{S}(s,t)$ of S (refer to the handout on parameterizations), find the standard normal vector $\vec{N} \doteq \frac{\partial \vec{S}}{\partial s} \times \frac{\partial \vec{S}}{\partial t}$ and let

$$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS = \pm \iint_D \vec{F} \cdot \vec{N}\, dt\, ds}$$

Choose the plus sign if $\vec{N}$ and $\vec{n}$ are parallel; choose the minus sign if $\vec{N}$ and $\vec{n}$ are anti-parallel.

VECTOR CALCULUS
MA262 - Otto Bretscher

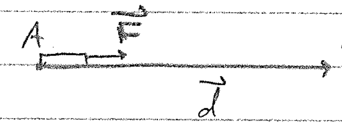
Sept 5, 2018

Two introductory examples

① Work (mechanical work)

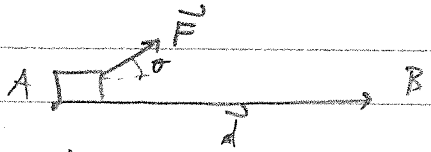
$\vec{F}$ is constant, parallel to $\vec{d}$

Case 1:



$$\text{Work} = W = \|\vec{F}\| \cdot \|\vec{d}\|$$

Case 2



$\vec{F}$ is constant, θ acute

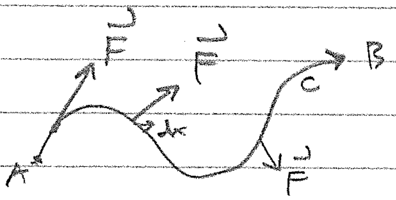
$$\text{Work} = W = \|\vec{F}\| \|\vec{d}\| \cos\theta$$

Only $\vec{F}_{\parallel}$ contributes to work...

so $\boxed{W = \|\vec{F}_{\parallel}\| \cdot \|\vec{d}\|} = \|\vec{F}\| \cos\theta \, \|\vec{d}\|$

$\boxed{W = \vec{F} \cdot \vec{d}}$ (dot product)

General case



$W = ?$

Use differentials

$dW = \vec{F} \cdot d\vec{x}$

"Add up":

$$W = \int dW = \int_C \vec{F} \cdot d\vec{x}$$

"To do" → talk about curves

→ what kind of function is $F(x, y)$?

{ Domain: points on curve C (vector fields)

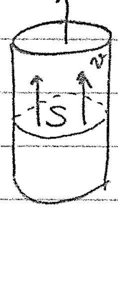
{ Target Space: vectors

→ VECTOR FIELDS

→ line integrals,

② Flux (rate of flow)

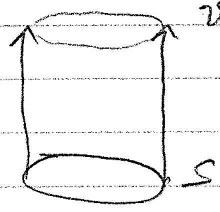
Case 1



Assume velocity is constant

pipe line over space & time

What is the flux? (in m^3/s)



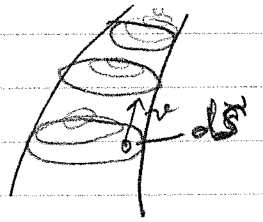
What happens in time t?

$f = \frac{V}{t} = \frac{S\,vt}{t} =$

$V = (\text{area of } S) \cdot \|\vec{v}\| t$

$\text{flux} = \frac{V}{t} = (\text{area of } S) \cdot \|\vec{v}\|$

<u>General case</u>



Find flux of $\vec{v}$ thru S

$$\iint_S \vec{v} \cdot d\vec{S}$$ (surface integral)

<u>"To do" (cont)</u> → surfaces, surface integrals, spatial vector fields

Sep 7, 2018

Notations & Terminology

$\mathbb{R}$: the set of real numbers

<u>Examples</u> $3, 0, -4, \frac{7}{8}, \cancel{i}, e, \cancel{\infty}$

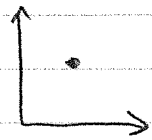
$\mathbb{R}^2$: the set of all pairs of real no. $= \{(a,b),\ a,b \in \mathbb{R}\}$

$\mathbb{R}^3$: the set of all triples of real... $\{(a,b,c),\ a,b,c \in \mathbb{R}\}$

$\mathbb{R}^n$: the set of all n-tuple of real no... $\{(a_1, a_2, \dots a_n),\ a_1 \dots a_n \in \mathbb{R}\}$

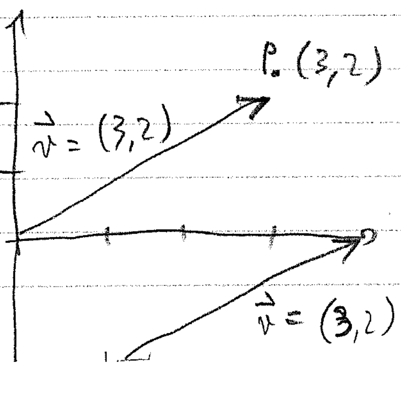
Geometrical interpretation of $\mathbb{R}^n$

<u>Ex</u> $(3,2) \in \mathbb{R}^2$ as point in Cartesian plane



<u>OR</u>

as ~~x~~ vectors (i.e classes of arrows with the same direction & magnitude)



Difference between points & vector...

↳ we can add/multiply by scalar vectors.

We can do algebra with vectors, but not with points

<u>Alternative notation</u> $\hat{i} = (1,0)$ $\hat{j} = (0,1)$ So $\vec{v} = 3\hat{i} + 2\hat{j}$

<u>What kind of functions do we study in Calculus?</u>

<u>121</u> $y = f(x) = x^2$ $f: \mathbb{R} \longmapsto \mathbb{R}$
(domain) (target space)

<u>122</u> $z = g(x,y) = x^2 + y^2$ $g: \mathbb{R}^2 \longmapsto \mathbb{R}$

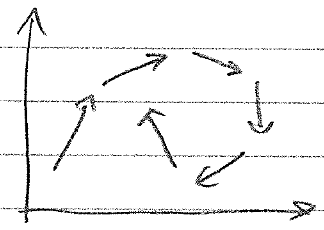
$w = h(x,y,z) = x^2 + y^2 + z^2$ $h: \mathbb{R}^3 \longmapsto \mathbb{R}$

domain = poi

$\Rightarrow$ <u>Scalar-valued functions</u>
, g & h are called <u>scalar fields</u>, meaning that the domain is $\mathbb{R}^m$ where $m > 1$

$\rightarrow$ Scalar fields : inputs <u>points</u> in $\mathbb{R}^m$ $(m > 1)$, output <u>= scalars</u>...

<u>262</u> (1) Velocity of water on the surface of the ocean...



$\vec{v}: \mathbb{R}^2 \longmapsto \mathbb{R}^2$
(points) (vectors)

$\boxed{\text{vector field on } \mathbb{R}^2}$

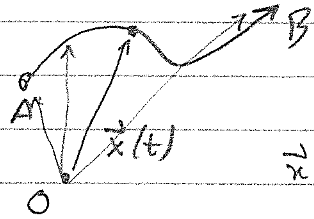
m, n have to be the same to be considered vector field

form
$: \mathbb{R}^n \mapsto \mathbb{R}$
(pts) (scalars)

$: \mathbb{R}^2 \mapsto \mathbb{R}^2$
(pts) (vectors)

A vector field in $\mathbb{R}^3$ is defined analogously.

(2) <u>Path</u>



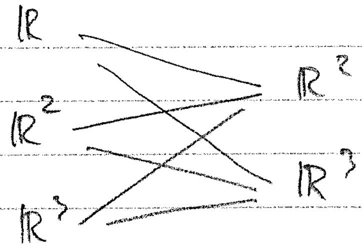
For a given time $t \in [a, b]$

$\vec{x}(t)$: position vector @ time t

$\vec{x}(t) : \mathbb{R} \mapsto \mathbb{R}^2$ (vector)

$\boxed{\underline{\text{Path}} : f: \mathbb{R} \mapsto \mathbb{R}^n}$

<u>Vector-valued functions</u>



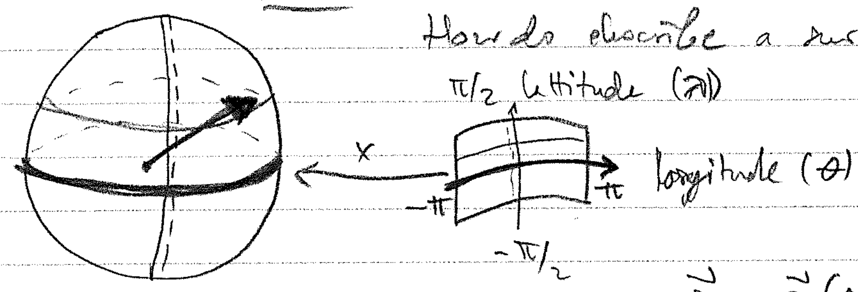
But we care abt

$\mathbb{R} \longmapsto \mathbb{R}^2$
$\mathbb{R} \longmapsto \mathbb{R}^3$
$\mathbb{R}^2 \longmapsto \mathbb{R}^2$
$\mathbb{R}^3 \longmapsto \mathbb{R}^3$

SURFACE

<u>Earth</u>

How do describe a surface? $\rightarrow$ Make a map



$\vec{x} = \vec{x}(\theta, \lambda)$: $\mathbb{R}^2 \longmapsto \mathbb{R}^3$
(pts) (vectors)

<u>so</u> $\boxed{\vec{x} = \vec{x}(\theta, \lambda)}$

Convention

When studying a function $\vec{f} : \mathbb{R}^m \longmapsto \mathbb{R}^n$, where $m, n \geq 1$ we will interpret the elements of <u>$\mathbb{R}^m$ as points</u> and elements of <u>$\mathbb{R}^n$ as vectors</u>, unless stated otherwise.

Vector calculus is the study of vector-valued functions, meaning that the <u>target space</u> is <u>$\mathbb{R}^n$</u>, where <u>$n > 1$</u>.

II. Review of Multivariable Calculus

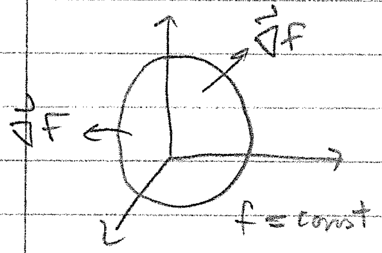
Want to be able to describe curved spacetime in general coord. systems + to be able to transform between them.

Gradient $\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

→ vector that points $\perp$ to the surfaces of $f = \text{const}$

e.g. $f(x,y,z) = x^2 + y^2 + z^2$ ← can't plot in 4D

instead make contour maps, $f = \text{const}$



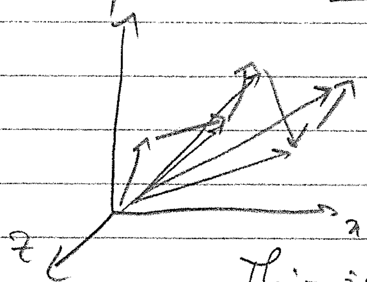
$\vec{\nabla} f = (2x, 2y, 2z) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

But $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ (radial)

so $\vec{\nabla} f$ is $\perp$ to $f = \text{constant}$

Curves in 3D space ⇒ parameterized by some param t, σ, τ

e.g. $\boxed{\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}}$



mapping $\mathbb{R}^1 \mapsto \mathbb{R}^3$

If $t = \text{time}$, then $\vec{v} = \frac{d\vec{r}}{dt}$ → tangent to the curve

This is always true for any parameterized curve

Let $\vec{r}(\sigma)$ be curve, then $\frac{d\vec{r}}{d\sigma} = \dot{\vec{r}}$ → tangent to $\vec{r}$

10, 2018

Review Vector Calculus is the study of smooth vector-valued functions

Smooth function (or C^∞) ~~we can~~

Definition → A function $f : \mathbb{R} \mapsto \mathbb{R}$ is said to be smooth if it has derivatives of all orders: $f', f'', \dots f^{(n)}$ exist for all n positive integer

$|x|,\ x|x|,\ x^6|x|, \dots$

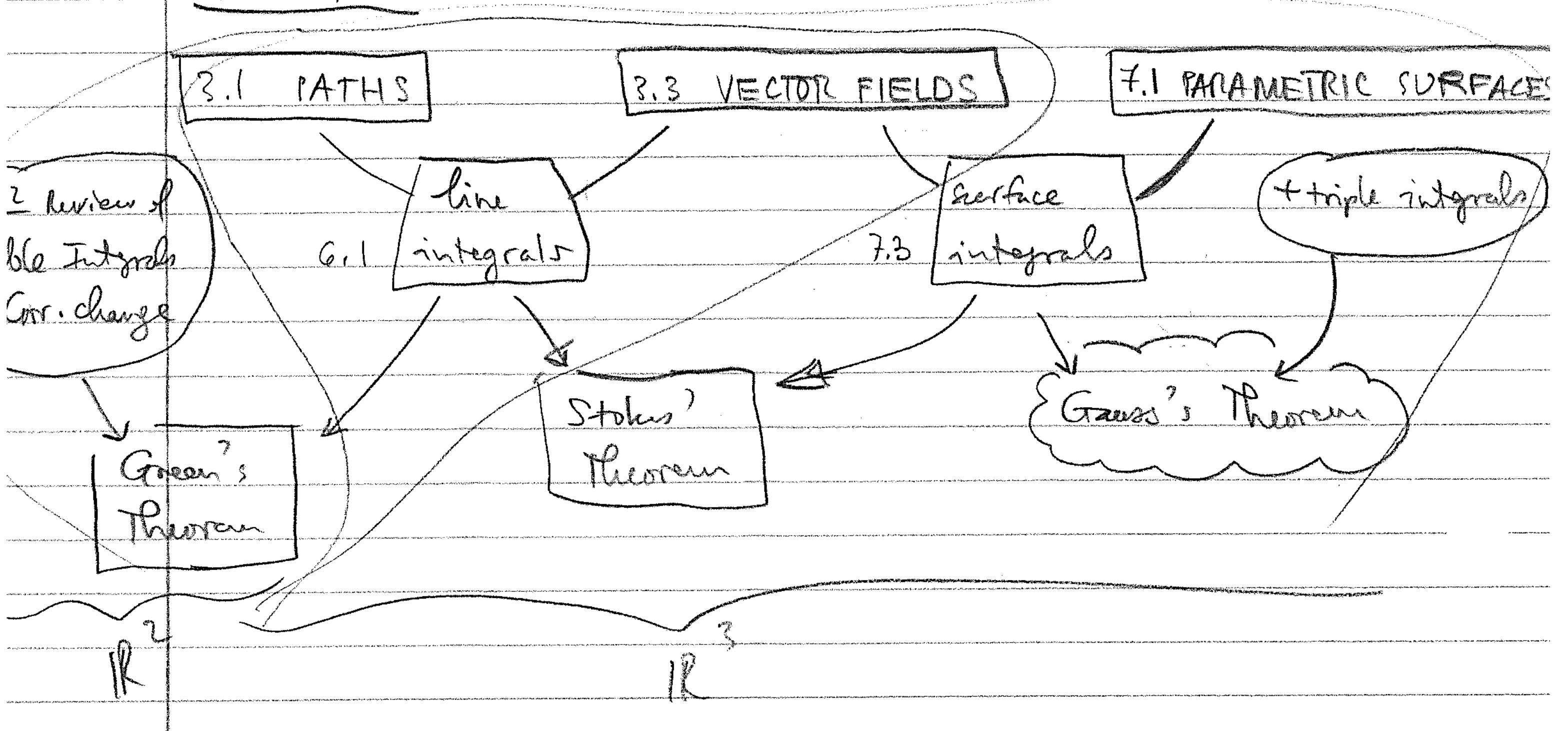
Definition → A function $f : \mathbb{R}^2 \mapsto \mathbb{R}$ is said to be smooth if all its partial derivatives of all orders exist: $f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy}, \dots$

The definition for a function $f : \mathbb{R}^3 \mapsto \mathbb{R}$ is analogous

Equality of Mixed Partials

$\boxed{\text{If } f(x,y) \text{ is smooth then } f_{xy} = f_{yx}}$

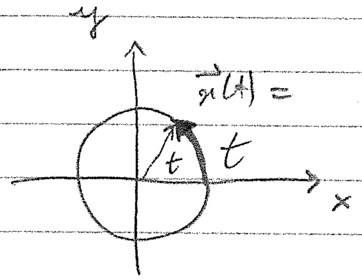
Course Plan



3.1 PATHS / TRAJECTORY / PARAMETRIC CURVE

Example 1 A small bug is crawling around the unit circle once, starting at (1, 0) in the counter clockwise direction, with speed of 1 unit per minute

Find the position of the bug after t minutes ($\vec{x}(t)$)



arclength $= r.\theta = t = t$

$$\boxed{\vec{x}(t) = (\cos t, \sin t)} \quad (0 \le t \le 2\pi$$

So $\vec{x} : [0, 2\pi] \mapsto \mathbb{R}^2$

Im (range) of $\vec{x}$ is the unit circle $x^2 + y^2 = 1$, called the underlying curve of the path

Definition

?

A path in $\mathbb{R}^2$ is a function $\vec{x} : I \mapsto \mathbb{R}^2$ where I is an interval on $\mathbb{R}$ (no gaps) (a path in $\mathbb{R}^3$ or $\mathbb{R}^n$ are defined analogously)

Sep 12, 2018

path ≡ function
curve ≡ point set

We can write $\vec{x}(t) = (x(t), y(t))$. The functions $x(t), y(t)$ are called the component functions or parametric equations of the path.

The image (or range) of $\vec{x}$ is called the underlying curve of the path

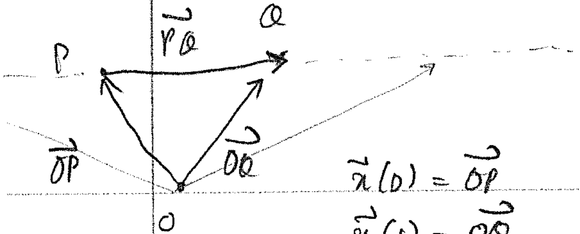
Definition

A path $\vec{x}(t) = (x(t), y(t))$ is said to be smooth if the component functions $x(t), y(t)$ are smooth

Ex Consider 2 distinct point P, Q in $\mathbb{R}^2$ (or $\mathbb{R}^3$). Define the path

$$\vec{x}(t) = \vec{OP} + t\vec{PQ}$$

Describe the underlying curve...



$\vec{x}(0) = \vec{OP}$ Curve ~~all lin~~ through P & Q

$\vec{x}(1) = \vec{OQ}$ ↳ the straight line

$\vec{x}(2) = \vec{OP} + 2\vec{PQ}$

$\vec{x}(-1) = \vec{OP} - \vec{PQ}$

Example 2.1

Parametrize (i.e. give / find parametric eqn for the line thru $P(1,2,3)$ & $Q(3,2,1)$)

Formula

$$\boxed{\vec{x}(t) = \vec{OP} + t\vec{PQ}} = (1,2,3) + t(\vec{OQ} - \vec{OP}) = (1,2,3) + t(2,0,-2) = (1+2t,\ 2,\ 3-2t)$$

So $\boxed{x = 1+2t,\ y = 2,\ z = 3-2t}$

Note ~~Linear~~ parametrization of a line can be non-linear (nonconstant velocity)

Example 3 Find the underlying curve of $\vec{x}(t) = (t, t^2)$. Do it algebraically

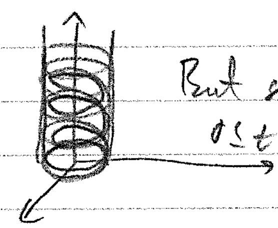
$x = t$

$y = t^2 = x^2$ So underlying curve is $\boxed{y = x^2}$ (parabola)

Example 3.1 We can parametrize the graph of $y = f(x)$ as $\begin{cases} x = t \\ y = f(t) \end{cases}$

Example 4 (non-planar curve) $\boxed{\vec{x} = (\cos t, \sin t, t) \quad 0 \le t \le 4\pi}$

Project into x-y plane $(\cos t, \sin t, 0)$



But since $0 \le t \le 4\pi$ → 2 times

$\vec{x}(0) = (\cos 0, \sin 0, 0)$ (helix)

$\vec{x}(4\pi) = (\cos 4\pi, \sin 4\pi, 4\pi)$

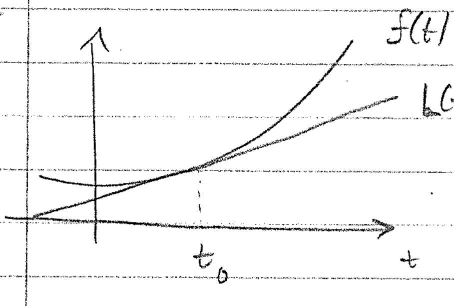
Example 5 Parametrize the intersection of the planes

$$x + 2y + 3z = 4$$
and
$$5x + 6y + 7z = 8$$

} express x, y in terms of z. Then $z = t$

Equation of Tangent

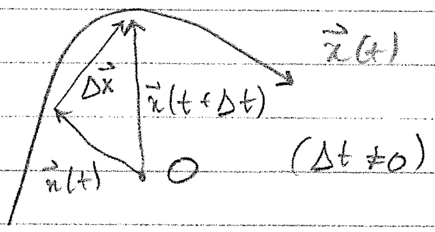
Sep 14, 2018



$$\boxed{L(t) = f(t_0) + (t-t_0) f'(t_0)}$$

3.1 Calculus of Paths (kinematics)

Consider a smooth path $\vec{x}(t) = (x(t), y(t))$



($\Delta t \neq 0$)

<u>Displacement</u>: $\vec{x}(t+\Delta t) - \vec{x}(t) = \Delta\vec{x}$

<u>Average velocity</u> $\dfrac{\text{displacement}}{\text{time elapsed}} = \dfrac{\Delta\vec{x}}{\Delta t}$ (difference quotient.)

In components

$\Delta\vec{x} = (\Delta x, \Delta y)$

$$\frac{\Delta\vec{x}}{\Delta t} = \left(\frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t}\right)$$

Instantaneous velocity

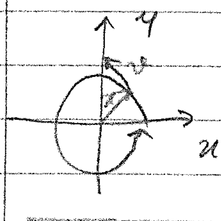
$$\vec{v}(t) = \frac{d\vec{x}}{dt} = \lim_{\Delta t\to 0} \frac{\Delta\vec{x}}{\Delta t} = \lim_{\Delta t \to 0}\left(\frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t}\right) := \left(\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t}, \lim_{\Delta t\to 0}\frac{\Delta y}{\Delta t}\right)$$

$$= \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \vec{x}'(t) = (x'(t), y'(t))$$

<u>Speed</u> $v(t) = \|\vec{v}(t)\|$

<u>Acceleration</u> $\vec{a}(t) = \dfrac{d\vec{v}}{dt} = \dfrac{d^2\vec{x}}{dt^2}$

<u>Example</u> $\vec{x}(t) = (\cos t, \sin t)$



$\vec{v}(t) = (-\sin t, \cos t)$

<u>So</u> $\vec{v} \perp \vec{x}$

$\|\vec{v}\| = 1$ $\quad \vec{a}(t) = (-\cos t, -\sin t) = -\vec{x}(t)$

Rules for derivatives of paths

Sum rule: $(\vec{p}+\vec{q})' = \vec{p}' + \vec{q}'$

Product rule:

$$(f(t)\,\vec{x}(t))' = f'(t)\,\vec{x}(t) + f(t)\,x'(t)$$

$$(\vec{p}(t)\cdot\vec{q}(t))' = \vec{p}\,'\cdot\vec{q} + \vec{p}\cdot\vec{q}\,'$$

Chain rules

$$\vec{x}(f(t))' = \vec{x}\,'(f(t))\,f'(t) = \boxed{f'(t)\;\vec{x}\,'(f(t))}$$

Proof of chain rule

$$\begin{aligned}(\vec{x}(f(t)))' &= (x(f(t)),\, y(f(t)))' \\ &= ((x(f(t)))',\, (y(f(t)))') \\ &= (x'f',\, y'f') \\ &= f'(t)\,(x'(f(t)),\, y'(f(t))) = f'(t)\,\vec{x}\,'(f(t))\end{aligned}$$

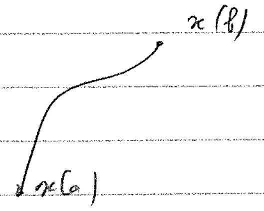
Example 0 Linear motion

Find the velocity, speed, and acceleration of a linear path

$$\begin{aligned}\vec{x}(t) &= \vec{b} + t\vec{c} \\ \vec{v}(t) &= \vec{c} \\ \|\vec{v}(t)\| &= \|\vec{c}\| \\ \vec{a}(t) &= \vec{0}\end{aligned}$$

Distance travelled?

$a \le t \le b$, in $\mathbb{R}^n$ $\quad \vec{x}(t)$ is path



Define Distance travelled (arc length) as

$$\boxed{L = \int_a^b \underbrace{v(t)}_{\text{speed}}\,dt}$$

More generally, we can define the distance $s(t)$ travelled between $t = a$ and an arbitrary $a \le t \le b$

$$\boxed{s(t) = \int_a^t v(\tau)\,d\tau}$$

Example find length of $\vec{x}(t) = (\cos t, \sin t, t)$ $\quad 0 \le t \le 2\pi$

$$\vec{x}\,'(t) = (-\sin t, \cos t, 1) \qquad L = \int_0^{2\pi} \sqrt{2}\,dt = 2\sqrt{2}\pi$$

$$v(t) = \sqrt{2}$$

<u>Example 2</u>, Find $s(t)$ ~~based~~ at $a=0$, $\vec{x}(t) = \left(t, \frac{\sqrt{2}}{2}t^2, \frac{1}{3}t^3\right)$

$$\vec{v}(t) = \left(1, \sqrt{2}t, t^2\right)$$

$$v(t) = \sqrt{1+2t^2+t^4} = (t^2+1)$$

$$s(t) = \int_0^t (\tau^2+1)\,d\tau = \frac{1}{3}t^3 + t$$

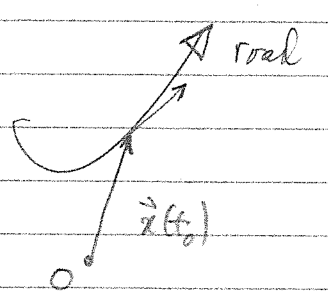
$\boxed{\text{When does } \vec{v}(t) \perp \vec{x}(t)?}$ For which paths $\vec{x}(t)$ is $\vec{x}(t) \perp \vec{v}(t)$

i.e. $\vec{x} \cdot \frac{d\vec{x}}{dt} = 0$

<u>Claim</u> $\boxed{\vec{x}(t) \perp \frac{d\vec{x}}{dt} \text{ iff } \|\vec{x}\| \text{ is constant.}}$

$$\left\{ \begin{aligned} \|\vec{x}(t)\| \text{ constant} &\Leftrightarrow \|\vec{x}(t)\|\,\|\vec{x}(t)\| \text{ constant} \Leftrightarrow \vec{x}(t)\cdot\vec{x}(t) \text{ constant} \\ &\Leftrightarrow \frac{d}{d\vec{x}}\left(\vec{x}(t)\cdot\vec{x}(t)\right) = 0 \Leftrightarrow 2\vec{x}(t)\cdot\frac{d\vec{x}}{dt} = 0 \\ &\Leftrightarrow \vec{x}(t)\cdot\frac{d\vec{x}}{dt} = 0 \end{aligned} \right\}$$

$\boxed{\text{Example 4}}$



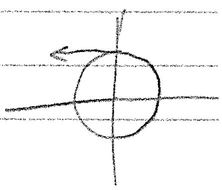
@ time $t = t_0$, the car is sliding off the road and proceeding with constant velocity $\vec{x}'(t_0)$ for $t \geq t_0$

Find $L(t_0)$, called the tangent trajectory or linearization of $\vec{x}(t)$)

$$\vec{L}(t) = \vec{x}(t_0) + \cancel{t}\left(\vec{v}(t_0)\right)(t - t_0)$$

$\boxed{\text{Example 4.15}}$ Find tangent trajectory for $\vec{x} = (\cos t, \sin t)$ @ $t = \frac{\pi}{2}$

$$\vec{L}(t) = (0,1) + \left(t - \frac{\pi}{2}\right)(-1,0) = \left(\frac{\pi}{2} - t, 1\right)$$



? 18, 2012 **Example** We wish to drive from Diamond to Walmart.
GPS: "after 1.6 mi ... do something..."

→ arc length parametrization... $\vec{x}(s)$. The position is given in terms of the arc length $\vec{x}(s)$, called the "arc length parametriza[tion]" of this oriented curve.

Note 2 arc length param (since we can start from either end)

If $\vec{x}(t)$ is our path from Diamond to Walmart, what is the relationship between $\frac{d\vec{x}}{dt}$ and $\frac{d\vec{x}}{ds}$?

Chain rule

$t \longrightarrow s \longmapsto x$

$$\boxed{\frac{d\vec{x}}{dt} = \frac{d\vec{x}}{ds}\frac{ds}{dt} \rightarrow \vec{v} = \frac{d\vec{x}}{ds}v}$$

So $\boxed{\frac{d\vec{x}}{ds} = \frac{\vec{v}}{v}}$ → tangent unit vector to the oriented curve denoted by $\vec{T}$

$\vec{T} = \vec{v}/v$

We can interpret arc length param as the unit speed param, whereby we travel with speed of 1.

Example Find arc length param for $\vec{x}(t) = (8\cos t,\ 8\sin t,\ \cancel{6t})$
$0 \le t \le 2\pi$

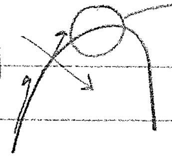
Strategy Find $\vec{v}(t)$, $v(t)$, $s(t)$, then find $\cancel{t}(s)$ and plug in

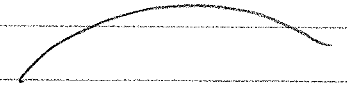
$\vec{v}(t) = (-8\sin t,\ 8\cos t,\ 6) \qquad v = 10$

So $s = 10(2\pi) = 20\pi \rightarrow t(s) = \frac{1}{10}s$

So $\vec{x}(s) = \left(8\cos\frac{s}{10},\ 8\sin\frac{s}{10},\ \frac{6}{10}s\right)$. $\qquad 0 \le s \le 20\pi$

$\boxed{\text{Curvature}}$





How do we quantify curvature?

__Note__ "tight" $\to$ $\vec{T}$ changes quickly (w.r.t arc length)

$\boxed{\text{Define curvature}}$

kappa

$$\boxed{\begin{aligned} \kappa &= \left\| \frac{d\vec{T}}{ds} \right\| = \left\| \frac{d^2\vec{x}}{ds^2} \right\| \\ &= \left\| \frac{d\vec{T}}{dt} \frac{dt}{ds} \right\| = \left\| \frac{1}{v} \frac{d\vec{T}}{dt} \right\| = \frac{1}{v} \left\| \frac{d\vec{T}}{dt} \right\| \end{aligned}}$$

__Note__ $\frac{d\vec{T}}{ds} \perp$ curve

$\boxed{\text{Curvature of circle of radius } R}$

$\vec{x}(t) = r(\cos t, \sin t)$ $\qquad v = r$

$\vec{x}'(t) = r(-\sin t, \cos t)$ $\qquad \vec{T} = \frac{\vec{v}}{v} = (-\sin t, \cos t)$

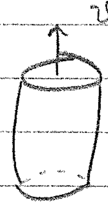
$\left\| \frac{d\vec{T}}{dt} \right\| = 1$ $\qquad \kappa = \frac{1}{v} \left\| \frac{d\vec{T}}{dt} \right\| = \frac{1}{r}$

__Sep 21, 2018__

$\boxed{\text{Vector fields}}$

__Example__

$\vec{v}(x, y, z)$: velocity of water at point $P(x, y, z)$



__Def__ A vector field on $\mathbb{R}^n$ ($n = 2$ or 3) is a function

$$F : X \subset \mathbb{R}^n \mapsto \mathbb{R}^n,$$
(points) (vectors)

where X is a subset of $\mathbb{R}^n$

__Examples__

$\hookrightarrow$ velocity fields

displacement fields

force fields.

__Ex__ $\vec{F}(x, y) = (2, 1)$ $\qquad \vec{F}(x, y) = (-x, -y)$ $\qquad \vec{F}(x, y) = (-y, x)$

$\vec{F}(x, y) = (-y, x) \perp (x, y)$ since $(-y, x) \cdot (x, y) = 0$

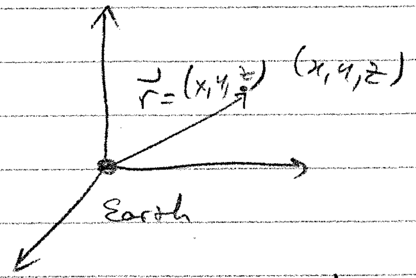
$\|F(x, y)\| = \|(-y, x)\|$

Ex $\vec{F}(x,y) = (x+2y,\ 3x+4y)$

A vector field $\vec{F}(x,y)$ on $\mathbb{R}^2$ can be written as $\vec{F}(x,y) = (M(x,y), N(x,y))$ where the component functions are scalar fields on $\mathbb{R}^2$.

A vector field f is said to be smooth if its component functions are smooth...

Ex



Find the gravitational force on m

$$\|\vec{F}\| = \frac{GMm}{r^2}$$

(inverse square field) $\quad \vec{F} = \frac{-GMm}{\|\vec{r}\|^3}\vec{r} = \frac{c}{\|\vec{r}\|^3}\vec{r}$

Example

$f(x,y) = x^2+y^2$

$\nabla f = (2x, 2y)$ (vector field on $\mathbb{R}^2$)

↳ perpendicular to level curve thru (p).

Not all smooth vector fields have smooth ~~as~~ scalar field

Question for which smooth vector field $\vec{F}(x,y)$ on $\mathbb{R}^2$ does there exist a scalar field $f(x,y)$ on $\mathbb{R}^2$ s.t.

$$\vec{F}(x,y) = \nabla f(x,y)$$

If so, f is called a <u>potential</u> of $\vec{F}$ and $\vec{F}$ is called a <u>conservative</u> vector field.

Ex if $\vec{F} = (M,N)$ conservative, what is the relationship between M_y and N_x?

$\vec{F}(M,N) = \nabla f(x,y) \rightarrow M = f_x,\ N = f_y \ \Rightarrow\ M_y = f_{xy},\ N_x = f_{yx} \Big\}\ M_y = N_x$

(equality of mixed partials (smooth

Example $\vec{F}(x,y) = (2x+3y,\ 4x+5y)$, $\quad M_y \neq N_x$

Sep 24, 2018

Aside Two differential equations...

① Find all smooth functions $\vec{x}(t)$ such that $\frac{dx}{dt} = 4t$ and $x(0) = 3$

(i) "Integrate" $\boxed{x(t) = 2t^2 + 3}$

$\frac{dx}{dt} = 4x$

(ii) Place x and dx on one side & dt on the other... $(x \neq 0)$

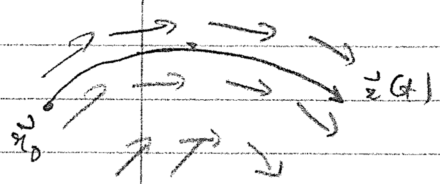
$$\int \frac{1}{x} dx = \int 4 dt = \ln x = 4t + C$$

$$x(t) = Ce^{4t}. \quad x(0) = 3 \quad \text{to} \quad \boxed{x(t) = 3e^{4t}}$$

Fact $\boxed{\text{Solution of } \frac{dx}{dt} = kx, \ x(0) = C \text{ is } x(t) = Ce^{kt}}$

3.3. Flow lines of a vector field. Let $\vec{v}(x,y)$ be velocity field of water on the surface of a river

If we place a rubber duck at position $\vec{x}_0$ at time $t_0 = 0$. What path $\vec{x}(t)$ will the duck trace out as it goes with the flow...



$\vec{x}(t)$ is called a flow line of a vector field $\vec{v}(x,y)$

Definition Flow line of a vector field $\vec{v}(\vec{x}(t))$ in $\mathbb{R}^2$ is a smooth path $\vec{x}(t)$ such that

$$\frac{d\vec{x}}{dt} = \vec{v}(x(t))$$

Example 1

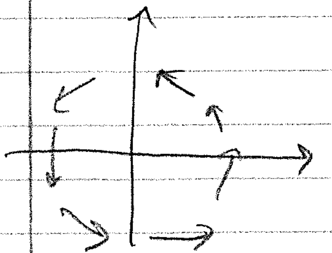
Find the flow line $\vec{x}(t)$ of $\vec{v}(x,y) = (1, y)$, with $x(0) = (0, 1)$

Solve $\frac{d\vec{x}}{dt} = \vec{v}(x,y) = \left(\frac{dx(t)}{dt}, \frac{dy(t)}{dt}\right) = (1, y) \rightarrow \begin{cases} \frac{dx}{dt} = 1 \\ \frac{dy}{dt} = y \end{cases}$

So $x(t) = t$, $y(t) = e^t$ Underlying curve : $y = e^x$

Example 2 Same... $\vec{v} = (-y, x)$ $\vec{x}(0) = (1,0)$

$$\boxed{\begin{aligned} \frac{dx}{dt} &= -y & \quad x(0) &= 1 \\ \frac{dy}{dt} &= x & \quad y(0) &= 0 \end{aligned}}$$



Since $\vec{v}(x,y) \perp (x,y)$ → the underlying curve of the flow line will be on the unit circle...

The speed will be $\dot{x} = \|\vec{v}\| = \sqrt{y^2+x^2} = 1$

Thus

$x = \cos t$

$y = \sin t$

//

Back to potentials

↳ Find $f(x,y)$ of $\vec{F} = (2x+3y, 3x+4y)$, if you can.

Want $\nabla f = \vec{F}$

$$\frac{\partial f}{\partial x} = 2x + 3y$$

$$\frac{\partial f}{\partial y} = 3x + 4y$$

$f(x,y) = x^2 + 3xy + g(y) = 3xy + 2y^2 + h(x)$

So $f(x,y) = x^2 + 3xy + 2y^2 + C$

Sept 26, 2018

Aside Notation $C^\infty(\mathbb{R}^m, \mathbb{R}^n)$ set of all smooth functions from $\mathbb{R}^m$ to $\mathbb{R}^n$ ~~of all smooth~~

Examples paths $m=1$
$n=2,3$

Scalar fields : $m=2,3$
$n=1$

Vector fields $m=n=2,3$

Example Find domain & target space of the gradient on $\mathbb{R}^2$

Domain $C^\infty(\mathbb{R}^2, \mathbb{R})$ "$\text{grad}(\underbrace{C^\infty(\mathbb{R}^2,\mathbb{R})}_{\text{scalar field}}) = \underbrace{C^\infty(\mathbb{R}^2, \mathbb{R}^2)}_{\text{vector field}}$"

Target space $C^\infty(\mathbb{R}^2, \mathbb{R}^2)$

$$\boxed{\text{grad} : C^\infty(\mathbb{R}^2, \mathbb{R}) \longmapsto C^\infty(\mathbb{R}^2, \mathbb{R}^2)}$$

$$\boxed{\text{Im}(\text{grad}) = \text{conservative vector fields}}$$

grad fails to be one-to-one since $\text{grad}(1) = \text{grad}(2) = \vec{0}$

One definition

An operator is a function whose domain & target space consist of functions

We will consider first-order differential operators (FODO's), meaning that the components of the output are expressed as linear combination of derivatives (partial) of the components of the input

Examples of FODO

(1) $D : C^\infty \to C^\infty$ as $D(f(t)) = f'(t) = \frac{\partial}{\partial t}(f(t))$ OR $\boxed{D \equiv \frac{\partial}{\partial t}}$

(2) $P : C^\infty(\mathbb{R}^2, \mathbb{R}) \longrightarrow C^\infty(\mathbb{R}^2, \mathbb{R})$

$$P(f(x,y)) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) \quad \underline{\text{OR}} \quad P \equiv \frac{\partial}{\partial x}$$

(3)

$$\operatorname{grad}(f(x,y)) = \left(\frac{\partial}{\partial x} f(x,y), \frac{\partial}{\partial y} f(x,y)\right) =: \underbrace{\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)}_{\vec{\nabla}} f$$

"del" or "nabla" symbol

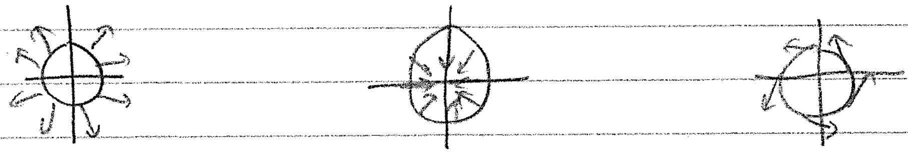
So $\boxed{\operatorname{grad} f = \vec{\nabla} f}$

(4) $D : C^\infty(\mathbb{R}^2, \mathbb{R}^2) \longrightarrow C^\infty(\mathbb{R}^2, \mathbb{R})$

$$D(\vec{F}) = \vec{\nabla} \cdot \vec{F}$$

in components. $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \cdot (M, N) = \frac{\partial N}{\partial x} + \frac{\partial N}{\partial y} = M_x + N_y$

$\underline{\text{Ex}}$ $\vec{F}(x,y) = (x,y)$, $\vec{G}(x,y) = (-x,-y)$, $\vec{H}(x,y) = (-y, x)$



$\vec{\nabla} \cdot \vec{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} = 2$ $\quad \vec{\nabla} \cdot \vec{G} = -1 - 1 = -2$ $\quad \vec{\nabla} \cdot \vec{H} = 0$

$\boxed{\text{This is called the "divergence"}}$

$$\begin{cases} \operatorname{div}(F) = \vec{\nabla} \cdot \vec{F} \\ \operatorname{grad}(f) = \vec{\nabla} f \end{cases}$$

Sept 28, 20[illegible]

Del symbol $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

$$\boxed{\begin{aligned} \text{grad}(f) &= \vec{\nabla} f \\ \text{div}(\vec{F}) &= \vec{\nabla}\cdot\vec{F} \end{aligned}} \qquad \boxed{\text{curl}(\vec{F}) = \vec{\nabla}\cdot\vec{F}}$$

Example $C: C^{\infty}(\mathbb{R}^3, \mathbb{R}^3) \longrightarrow C^{\infty}(\mathbb{R}^3, \mathbb{R}^3)$

(27th dimensional)

$$C(\vec{F}) = \vec{\nabla}\times\vec{F} = \det\begin{pmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{pmatrix}$$

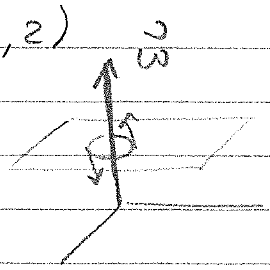
$$= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z},\ -\frac{\partial P}{\partial x} + \frac{\partial M}{\partial z},\ \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

Example $\vec{F} = (x, y, z)$

$$C(\vec{F}) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\times(x, y, z) = (0, 0, 0)$$

Example $\vec{F} = (-y, x, 0)$

$$C(\vec{F}) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\times(-y, x, 0) = (0, 0, 2) = 2\hat{k} = 2\vec{\omega}$$



Angular speed $\omega = \frac{v}{r} = 1$

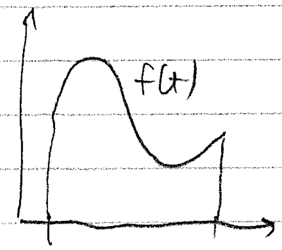
angular velocity $\vec{\omega} = (0, 0, 1) = \hat{k}$

$C(\vec{F})$ is called the curl of $\vec{F}$

$$\boxed{\text{curl}(\vec{F}) = \vec{\nabla}\times\vec{F}}$$

$\boxed{\text{If } \vec{F} \text{ is a velocity field, then curl} \vec{F} = 2\vec{\omega}, \text{ the angular velocity}}$

$\boxed{5.1 - 5.4}$ Review on Integration



$a = t_0$, $b = t_n$

"Find area A under the curve"

Set up Riemann Integral

Find over estimates of A. — Divide interval $[a,b]$ with points (equally spaced) $a = t_0 < \dots < t_n = b$

$$\Delta t = \frac{b-a}{n}$$

Let $f(\tau_k)$ be maximal value of $f(t)$ on the interval $[t_{k-1}, t_k]$

$$\sum_{k=1}^{n} f(\tau_k)\Delta t \geq A$$

By part 4, 10,

$$\boxed{\lim_{n\to\infty} \sum_{k=1}^{n} f(\tau_k)\Delta t \cancel{=A} =: \int_a^b f(t)\,dt}$$

defined as definite integral of $f(t)$ on $[a,b]$

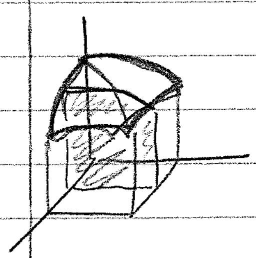
$\boxed{\text{Leibniz' intuition behind the notation integral } \int_a^b f(t)\,dt}$

dt : infinitesimal, so small that $f(t)$ has no chance to change

↳ $f(t)dt$ = area of rectangle. (thin slice)

$\int f(t)\,dt$ = "sum" of all areas...

1, 2012 $\boxed{\text{Double Integral}}$ $f(x,y) = 20 - x^2 - y^2$, $0 \le x \le 2$, $0 \le y \le 3$



Find the volume of this region

Let $A(x)$ be the area of the cross section for a fixed x, $0 \le x \le 2$

$$A(x) = \int_{y=0}^{3} f(x,y)\,dy = \int_{y=0}^{3} 20 - x^2 - y^2\,dy = (20-x^2)y - \frac{1}{3}y^3\Big|_0^3$$

$$= 60 - 3x^2 - 9$$

$$= (51 - 3x^2)$$

Next, $dV = A(x)dx \to V = \int_0^2 A(x)dx = \int_0^2 51 - 3x^2\,dx$

$$= 102 - 8 = 94$$

So $V = \int_{x=0}^{2}\int_{y=0}^{3} 20 - x^2 - y^2\,dy\,dx$: double integral of $f(x,y)$ over domain

$D = [0,2]\times[0,3]$

$:= \{(x,y) \in \mathbb{R}^2,\ 0 \le x \le 2,\ 0 \le y \le 3\}$

This integral is denoted by

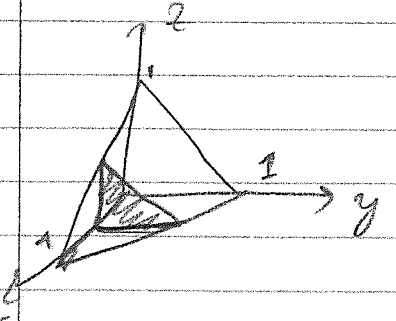
$$\iint_D f(x,y)\,dy\,dx = \iint_D f(x,y)\,dA = \iint_D f\,dA$$

Alternatively, we could reverse the order of integration and write

$$\iint_D f\,dA = \int_{y=0}^{3}\int_{x=0}^{2} f(x,y)\,dx\,dy$$

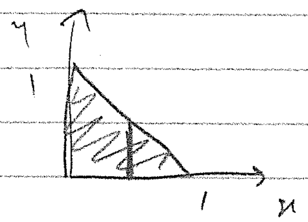
(The 2 iterated integrals are the same by Fubini's Theorem)

Example 2 V of solid region under the surface $z = f(x,y) = 1 - x - y$ above the triangle D with vertices at $(0,0,0)$, $(1,0,0)$ and $(1,0,0)$



$$V = \frac{\text{area}(D)\cdot\text{height}}{3} = \frac{1}{6}$$

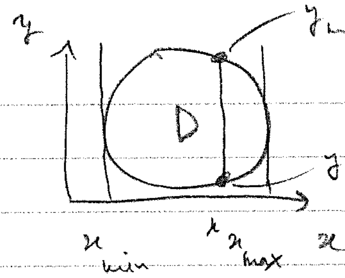
$$A(x) = \int_{y=0}^{(1-x)} f(x,y)\,dy \quad \text{so} \quad V = \int_{x=0}^{x=1} A(x)\,dx$$



$$V = \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} 1 - x - y\,dy\,dx = \frac{1}{6}$$

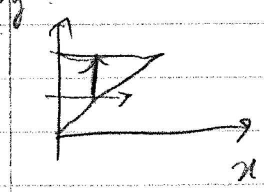
skip computation

In general



$$\int_{x_{min}}^{x_{max}} \int_{y=y_{min}(x)}^{y=y_{max}(x)} f(x,y)\,dy\,dx$$

Ex 3 $\iint_D \frac{\sin y}{y}\,dA$ D: triangle with vertices $(0,0)$, $(0,\pi)$, (π,π)



$$\int_{x=0}^{\pi}\int_{y=x}^{y=\pi} \frac{\sin y}{y}\,dy\,dx \longrightarrow \text{no elementary antiderivative}$$

Idea → reverse order of int $\displaystyle\int_{y=0}^{\pi}\underbrace{\int_{x=0}^{x=y}\frac{\sin y}{y}\,dx}_{\sin y}\,dy = 2$

2/20

TRIPLE INTEGRAL

Density average density $= \frac{\text{mass } m}{\text{volume } V}$

$f(P) =$ local density at point P $= \lim_{r\to 0^+} \frac{\text{mass of } B_r(P)}{\text{volume of } B_r(P)}$

How to recover mass from density f?

↳ Consider box W. For a given positive integer n, we chop box up into n equal pieces in each direction, thus creating n^3 equal subboxes.

Choose a point (x_i, y_j, z_k) $1 \le i,j,k \le n$

$$m = \lim_{n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} f(x_i, y_j, z_k)\,\Delta x\,\Delta y\,\Delta z$$

show $\lim_{n\to 0}(U_n - L_n) = 0$. Use MVT with f' replaced by ∇f

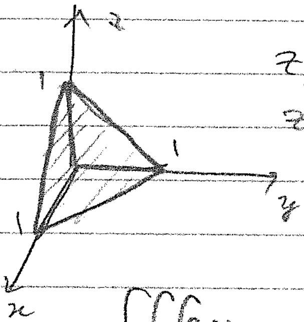
This limit is defined as $\iiint_W f(x,y,z)\,dx\,dy\,dz$

$\overset{m}{\uparrow}$ $\overset{m}{\uparrow}$

$L_n \le R_n \le U_n \longrightarrow R_n \to m$

<u>More generally</u>, $\iiint_W f\,dV$

<u>Ex</u> W is the region in the first octant bounded by $x+y+z=1$



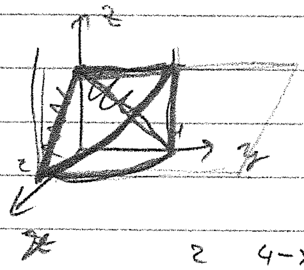
$z_{max} = 1-x-y$ $\quad y_{max} = 1-x$ $\quad x_{max} = 1$

$z_{min} = 0$ $\quad y_{min} = 0$ $\quad x_{min} = 0$

$z_{max} = z_{min}$ $\quad y_{max} = y_{min}$

$$\iiint_W f\,dV = \int_{x\,min}^{x\,max}\int_{y\,min(x)}^{y\,max(x)}\int_{z\,min(x,y)}^{z\,max(y,x)} f\,dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} f\,dz\,dy\,dz$$

$W \rightarrow$ region in the 1st octant bounded by $x+z=2$ & $y=4-x^2$

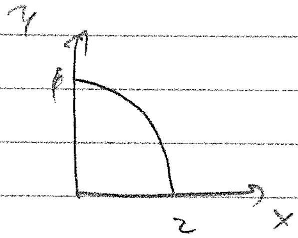


$z\ min = 0$

$z\ max = 2-x$

$z_{min} = z_{max}$

$\rightarrow x = 2$



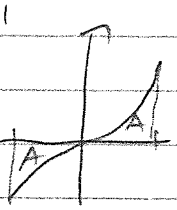
$$\int_0^2\int_0^{4-x^2}\int_0^{2-x} f\,dz\,dy\,dx$$

$x \quad y \quad z$

Oct 5, 2011

$\boxed{\text{Some more integrals}}$

① $\int_{-1}^{1} \sin^3(x)\,dx = 0$ since $\sin^3(x)$ odd, over symmetric interval



$\int_{-1}^{1} \sin^3(x)\,dx = -A + A = 0$

<u>Proof</u> Mid point sums are 0

another proof $\int_{-1}^{1} \sin^9(x^3)\,dx = +\int_{1}^{-1} \sin^9(u^3)\,du = -\int_{-1}^{1} \sin(u^3)\,du$

let $u = -x$ so $I = -I$

In general $\boxed{\int_{-a}^{a} f(x)\,dx = 0 \text{ if } f(x) \text{ odd}}$

(2) $$\iiint_{x^2+y^2+z^2\le 1} \sin(z^3)\,dV = \int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \underbrace{\int_{z=-\sqrt{1-y^2-x^2}}^{z=\sqrt{1-y^2-z^2}} \sin(z^3)\,dz}_{0}\,dy\,dx = 0$$

In general

if $(x,y,z)\in W$ then $(x,y,-z)\in W$

$$\boxed{\iiint_W f(x,y,z)\,dV = 0 \text{ if } W \text{ is "symmetric in } z\text{" and } f(x,y,z) \text{ odd in } z. \ (f(x,y,-z) = -f(x,y,z))}$$

Analogously for x and y

(3) $$\iiint_{x^2+y^2+z^2\le 1} e^{xy^2z^3}\,dV = \iiint_{x^2+y^2+z^2\le 1} e^{x^3yz^2}\,dV$$

swap x, z

(x, y, z are "dummy variables". We can permute them.)

$\boxed{\text{Linear function}}$

($f: \mathbb{R}^2 \mapsto \mathbb{R}$ is said to be a linear function if $f(x,y) = ax + by + c$ for some $a, b, c \in \mathbb{R}$.)

($g: \mathbb{R}^2 \to \mathbb{R}$ is said to be a linear map if $g(x,y) = ax + by$ for some $a, b \in \mathbb{R}$.)

($h: \mathbb{R}^m \to \mathbb{R}$ is said to be a linear map if $h(x_1, x_2, \ldots x_m) = \sum_{i=1}^{m} a_i x_i$ for some constants $a_1 \ldots a_m$)

($\vec{F}: \mathbb{R}^m \mapsto \mathbb{R}^n$ is said to be a linear function (map) if its component functions are linear functions or (maps).)

Example $\vec{F}(x,y,z) = (x+2y,\ 3+4z,\ x+y+z) \to$ linear fn, not ma

Matrices An $n \times m$ matrix A is a rectangular array of real numbers arranged in n rows and m columns.

The set of all $n \times m$ matrices is defined by $\mathbb{R}^{n \times m}$

An $1 \times m$ matrix is called a row vector, and an $n \times 1$ matrix is a (column) vectors. We will identify $\mathbb{R}^n$ with $\mathbb{R}^{n \times 1}$

$\boxed{\text{addition of matrices}}$ $\to$ entry-wise

$\boxed{\text{Scalar multiplication}}$ $\to$ entry-wise

$\boxed{\text{Dot product}}$ If $\vec{v}$ & $\vec{w}$ are row/column vectors with n components, we define

$$\vec{v} \cdot \vec{w} = \sum_{i=1}^{n} v_i w_i$$

Def If A is an $n \times p$ matrix and B is a $p \times m$ matrix, then

AB is the $n \times m$ matrix whose ij^{th} entry is the dot product of the i^{th} row of A with the j^{th} column of B.

Theorem

8, 2018

> Consider a vector field $\vec{F}(x,y,z)$ defined on all of $\mathbb{R}^3$
> (a) $\vec{F}$ conservative $\Leftrightarrow$ $\operatorname{curl}(\vec{F}) = \vec{0}$
> (b) $\vec{F}$ has a vector potential $\Leftrightarrow$ $\operatorname{div}(\vec{F}) = 0$

Ex $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ define $\vec{L}(\vec{x}) = A\vec{x}$ from $\mathbb{R}^2$ to $\mathbb{R}^2$

$$\vec{L}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ 3x+4y \end{pmatrix}$$, a linear map

A : coef. matrix of $\vec{L}$

Fact if A is an $n \times m$ matrix, then the function $\vec{L}(x) = A\vec{x}$ from $\mathbb{R}^m$ to $\mathbb{R}^n$, ~~then~~ is a linear map,
and all linear maps $\mathbb{R}^m \to \mathbb{R}^n$ are of this form

Linear functions $\vec{L}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y+3 \\ 4x+5y+6 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ 6 \end{pmatrix} = A\vec{x} + \vec{b}$

$\boxed{\vec{L}(\vec{x}) = A\vec{x} + \vec{b}}$

Goal find derivative of a function $\vec{F} : \mathbb{R}^m \mapsto \mathbb{R}^n$

Ex $\vec{F}(x,y) = (M(x,y), N(x,y))$ 4 partial derivatives

$$\begin{bmatrix} M_x & M_y \\ N_x & N_y \end{bmatrix} = \begin{bmatrix} \operatorname{grad}(M)(x,y) \\ \operatorname{grad}(N)(x,y) \end{bmatrix}$$

Convention we will write grad f as a row vector unless stated otherwise

Def

> The derivative of a function $\vec{F} : \mathbb{R}^m \mapsto \mathbb{R}^n$ is the $n \times m$ matrix whose rows are the gradients of n component functions ~~$f_1, \ldots, f_n$~~ $F_1 \ldots F_n$
>
> $$\begin{bmatrix} \operatorname{grad} F_1 \\ \vdots \\ \operatorname{grad} F_n \end{bmatrix}$$, denoted by $(D\vec{F})(\vec{x})$ or $(D\vec{F})$
>
> (other names: derivative matrix, matrix of partials, Jacobian matrix)

Ex $\vec{F}(x,y) = \begin{pmatrix} xy^2 \\ x^3y^4 \end{pmatrix} \longrightarrow D\vec{F} = \begin{pmatrix} y & 2xy \\ 3x^2y^4 & 4x^3y^3 \end{pmatrix}$

Ex $\vec{L}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad D\vec{F} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

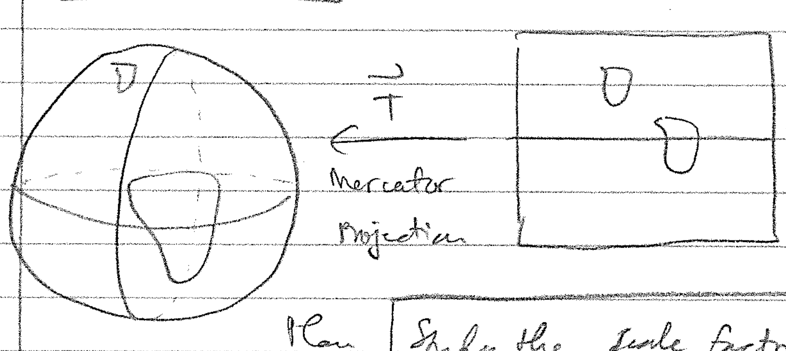
The derivative of a linear function is its coeff matrix

Linearization | $f: \mathbb{R} \mapsto \mathbb{R}$. The linearization $L(x)$ of $f(x)$ at $x = a$ is the linear function such that $L(a) = f(a)$ & $L'(a) = f'(a)$

$L(x) = f'(a)(x-a) + f(a)$

Generalize $\boxed{\vec{L}(\vec{x}) = D\vec{F}(\vec{a})(\vec{x}-\vec{a}) + \vec{F}(\vec{a})}$ ($\vec{F}(\vec{x})$ smooth $\mathbb{R}^n \mapsto \mathbb{R}^d$)

Oct 17, 2012 | **Scale Factors** | Motivation



Areas get distorted

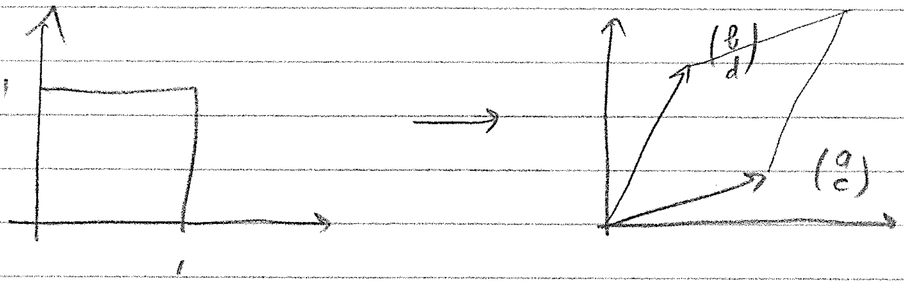
$\mathbb{R}^2 \mapsto \mathbb{R}^3$

Plan | Study the scale factor of functions $\vec{T}: \mathbb{R}^n \mapsto \mathbb{R}^n$, $\dfrac{\text{area of } T(D^*)}{\text{area of } D^*}$

Study $\vec{T}: \mathbb{R}^2 \mapsto \mathbb{R}^2$

The linear case... $\vec{L}\begin{pmatrix} x \\ y \end{pmatrix} = \overbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}^{A}\begin{pmatrix} x \\ y \end{pmatrix}$ $\qquad \vec{L}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$

$\vec{L}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$



Embed $\mathbb{R}^2$ into $\mathbb{R}^3$

area of $D = \left\| \begin{pmatrix} a \\ c \\ 0 \end{pmatrix} \times \begin{pmatrix} b \\ d \\ 0 \end{pmatrix} \right\|$

$= \| ad - bc \| = |de$

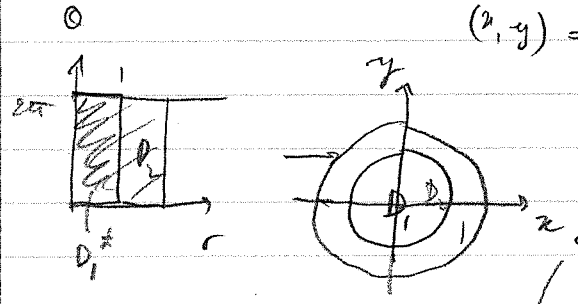
Scale factor $\dfrac{\text{area of } D}{\text{area of } D^*} = |\det A|$ $\boxed{L(\vec{x}) = A\vec{x}}$

Theorem $\boxed{\text{(Homework): The scale factor of a linear map } \mathbb{R}^2 \to \mathbb{R}^2 \text{ on any coordinate rectangle is } |\det(A)| = |\det D\vec{L}|}$

The scale factor of a linear function $L(\vec{x}) = A\vec{x} + \vec{b}$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ is $|\det A| = |\det D\vec{L}|$.

$\boxed{\text{The nonlinear case}}$ Example Polar → Cartesian $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$(x,y) = T(r,\theta) = (r\cos\theta, r\sin\theta)$



$\underline{\text{Scale factor on } D_1} = \dfrac{A(T(D_1))}{A(D_1^*)} = \dfrac{\pi}{2\pi} = \dfrac{1}{2}$

$\underline{\text{Scale fact. on } D_2} = \dfrac{3\pi}{2\pi} = \dfrac{3}{2}$

→ scale factor isn't the same!

But locally, it is! Idea: Define scale factor locally @ point of domain

Def: For a function $\vec{T}: \mathbb{R}^2 \mapsto \mathbb{R}^2$, $\vec{T}(u,v) = (x,y)$ we define the (local) scale factor at the point (u,v) in $\mathbb{R}^2$ to be the scale factor of the linearization $\vec{L}(u,v)$ at that point, which is the $|\det(D\vec{L}_{(u,v)})| = |\det D\vec{T}(u,v)|$ (by definition)

Terminology: $\det D\vec{T}(u,v)$ is called the Jacobian of $\vec{T}$, denoted $\boxed{\dfrac{\partial(x,y)}{\partial(u,v)}}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det(D\vec{T}) = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

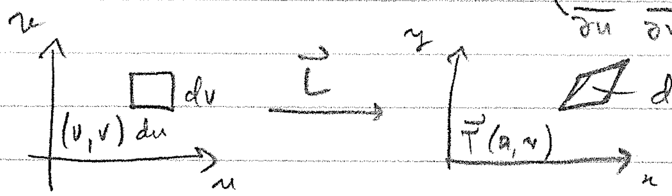
So Scaling factor is $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|$

Oct 19, 2018

Change of Variables $\quad \vec{T}: \mathbb{R}^2 \mapsto \mathbb{R}^2 \quad (x,y) = \vec{T}(u,v)$

$$\text{scale factor} = |\underline{\text{Jacobian}}| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = |\det D\vec{T}|$$

$$\text{Jacobian} = \det D\vec{T} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \qquad \text{Scale factor} = |\text{Jacobian}|$$



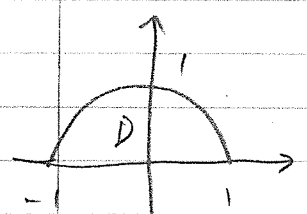
$$\boxed{dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv}$$

Ex Polar $\quad (x,y) = (r\cos\theta,\ r\sin\theta)$

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \left|\det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}\right| = r$$

So $\boxed{dA = r\,d\theta\,dr}$

Example Find $\iint_D y\,dA$ where D given by $x^2+y^2 \le 1$, $y \ge 0$

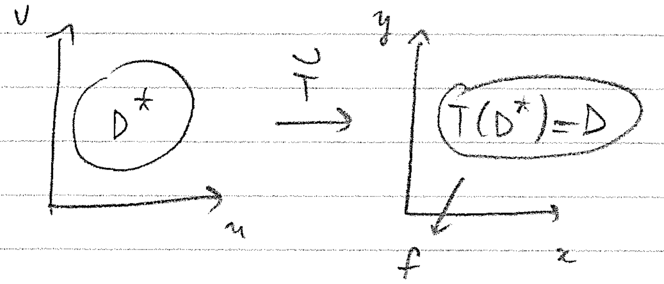


Use polar coords. Change of vars

$D: 0 \le r \le 1,\ 0 \le \theta \le \pi,\ y = r\sin\theta$

$$\hookrightarrow \iint_D y\,dA = \int_0^{\pi}\int_0^1 r\sin\theta\, r\,dr\,d\theta = \int_0^1 r^2\,dr \int_0^{\pi} \sin\theta\,d\theta = \frac{2}{3}$$

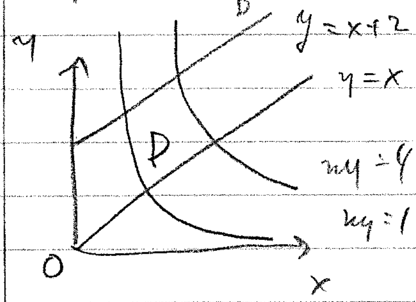
Polar coordinates are usually indicated when D is bounded by circles centered at the origin and rays from the origin



$$\boxed{\iint_D f(x,y)\,dA = \iint_{D^*} f(T(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right|}$$

Example 4 $\iint (y^2-x^2)\, e^{xy}\, dA$ D: region in 1st quadrant bounded by

$xy=1,\ xy=4,\ y=x,\ y=x+2$



Let $u = xy$, $v = y-x$ $\left\{\begin{matrix} 1 \le u \le 4 \\ 0 \le v \le 2 \end{matrix}\right\}$

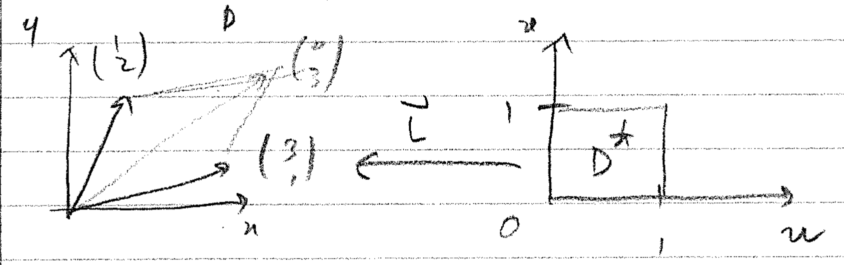
$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \left|\det\begin{pmatrix} y & x \\ -1 & 1 \end{pmatrix}\right| = |(x+y)| = x+y$$

$$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \frac{1}{(x+y)}\,du\,dv$$

$$\rightarrow \iint_{D^*} (y-x)\cancel{(y+x)}\frac{1}{\cancel{x+y}}\,du\,dv$$

$$\Rightarrow \int_0^2\int_1^4 e^u v\,du\,dv = \int_1^4 e^u\,du \int_0^2 v\,dv$$

Example 3 $\iint_D (3y-x)\,dA$ D with vertices $\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}4\\3\end{pmatrix}$



$$\vec{L}(\vec{x}) = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}\vec{x}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \qquad |\det(A)| = 5$$

$$\iint_D (3y-x)\,dA = \iint_{D^*} [3(u+2v) - (3u+v)]\cdot(5\,du\,dv)$$

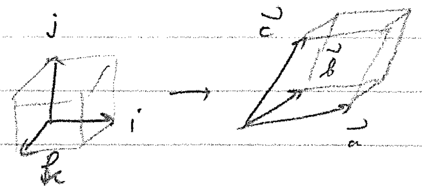
$$= \int_0^1\int_0^1 25v\,du\,dv = \frac{25}{2}$$

t 22, 2012

Aside: 3×3 Determinant Geometrical Meaning: volume of parallelepiped spanned by the three column vectors

The linear map $T(\vec{x}) = A(\vec{x})$ $\mathbb{R}^3 \mapsto \mathbb{R}^3$

Scales volume by exactly $|\det A|$

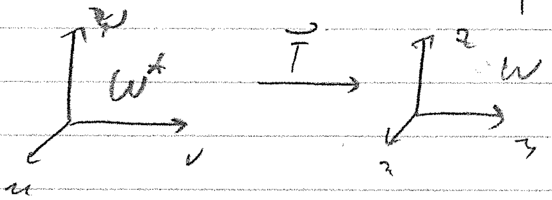


$\boxed{\text{Change of variables in 3D}}$ $(x,y,z) = \vec{T}(u,v,w)$ $\mathbb{R}^3 \mapsto \mathbb{R}^3$

Jacobian $\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \left|\det\begin{pmatrix} \frac{\partial x}{\partial u} & \cdots & \frac{\partial x}{\partial w} \\ \vdots & & \frac{\partial z}{\partial w}\end{pmatrix}\right| = |\det D\vec{T}|$ is the scale factor for volume

Volume Element

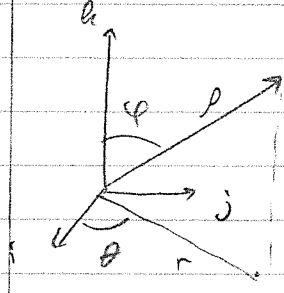
$$dV = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$



$W = T(W^*)$

$$\iiint_W f(x,y,z)\,dV = \iiint_{W^*} f(T(u,v,w))\,du\ldots \cdot \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| \cdot$$

$\boxed{\text{Spherical coords}}$ Location of point on sphere.



$0 \leq \theta \leq 2\pi$ φ: colatitude

$0 \leq \varphi \leq \pi$ θ: longitude.

$0 \leq \rho \leq R$ (arbitrary)

$x = \rho\sin\varphi\cos\theta$ volume element

$y = \rho\sin\varphi\sin\theta$

$z = \rho\cos\varphi$

$$\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} = \rho^2\sin\varphi$$

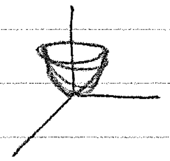
$$\boxed{dV = \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta}$$

Ex $\iiint_W z\,dV$ W: 1st octant of unit sphere

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho^3 \cos\varphi\sin\varphi\,d\rho\,d\varphi\,d\theta = \frac{1}{4}\left(\frac{\pi}{2}\right)\left[-\frac{1}{2}\cos(2\varphi)\right]_0^{\pi/2} = \frac{1}{4}\cdot\frac{-\pi}{4}\cdot[-1-1] = \boxed{\frac{\pi}{8}}$$

Spherical coords indicated when W bounded by

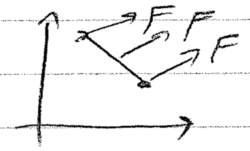
Ex $\iiint_W (x^2+y^2)\,dV$ W: $2x^2+2y^2-6 \le z \le 12$



→ polar for x, y, leave z alone

→ cylindrical coords...

$\boxed{\text{6.1 Line integral}}$ $\boxed{\text{Force, Work, Power}}$

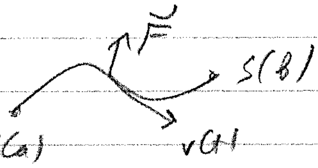


$s(t) = \cancel{\quad} \ \vec{b} + t\vec{v}$ $0 \le t \le 1$

Work $W(t) = \vec{F}\cdot[\vec{s}(t) - \vec{s}(0)] = \vec{F}\cdot\vec{v}(t)\cdot t$

Power: "rate at which work is being done" $\boxed{P = \dfrac{dW}{dt}} = \vec{F}\cdot\vec{v}$

"general case"



Define "power" as $P = \vec{F}\cdot\vec{v}$ $W = \int_a^b P\,dt = \int_a^b \vec{F}\cdot\vec{v}\,dt$

So $W = \int_a^b \vec{F}\cdot\frac{d\vec{s}}{dt}\,dt$

$= \int_{s(a)}^{s(b)} \vec{F}\cdot d\vec{s} = \int_s \vec{F}\cdot d\vec{s}$ is called the

(vector) line integral of $\vec{F}$ over the path $\vec{s}(t)$, $a \le t \le b$

Ex **Find** $\int_s \vec{F}\cdot d\vec{s}$ $\vec{F} = (-y, x)$ & $\vec{s}(t) = (\cos t, \sin t)$ $0 \le t \le 2\pi$

$= (-\sin t, \cos t)$ on $\vec{s}$

$\vec{v} = \frac{d\vec{s}}{dt} = (-\sin t, \cos t) = \vec{F}$

$\int \vec{F}\cdot\vec{v}\,dt = \int_0^{2\pi} dt = (2\pi)$

For closed path $\vec{s}(t)$, with $\vec{s}(a) = \vec{s}(b)$

> $\int \vec{F}\cdot d\vec{s}$ is called the circulation of $\vec{F}$ around the curve $\vec{s}(t)$, denoted by
>
> $\oint_s \vec{F}\cdot d\vec{s}$

{Written in components} $\vec{F} = (M, N, P)$

$$d\vec{s} = \frac{d\vec{s}}{dt}dt = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) dt = (dx, dy, dz)$$

$$\boxed{\int \vec{F}\cdot d\vec{s} = \int (M,N,P)\cdot(dx\, dy\, dz) = \int_{s} \underbrace{M dx + N dy + P dz}_{\text{1-form}}}$$

Find

$$\int_{\vec{s}} z\,dx + y\,dy + x\,dz \quad \text{for} \quad \begin{array}{l} x = t \\ y = t^2 \\ z = t^3 \end{array} \left| \begin{array}{l} dx = dt \\ dy = 2t\,dt \\ dz = 3t^2\,dt \end{array}\right.$$

$$= \int_{\vec{s}} t^3 dt + 2t^3 dt + 3t^3 dt$$

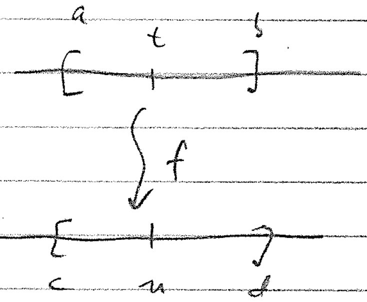
[Line int intrinsic?]

Oct 26, 2018

Theorem Consider an oriented curve C from $A \to B$ in $\mathbb{R}^n$ and a smooth vector $\vec{F}$ defined on C. If $\vec{r}(t)$, $a \le t \le b$, and $\vec{s}(u)$, $c \le u \le d$ are parameterization of C (with $\vec{r}(a) = \vec{s}(c) = \vec{OA}$ and $\vec{r}(b) = \vec{s}(d) = \vec{OB}$), then

$$\int_{\vec{r}} \vec{F}\cdot d\vec{r} = \int_{\vec{s}} \vec{F}\cdot d\vec{s} = \int_C \vec{F}\cdot d\vec{s}.$$

We say that the vector line integral is intrinsic for oriented curves



(reparameterization)

$f(a) = c$

$f(b) = d$

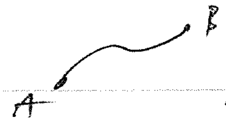
Assume that $\vec{r}$ & $\vec{s}$ are 1-to-1 ~~and~~

where $f = \vec{s}^{-1} \circ \vec{r}$ so that $\vec{s} \circ f = \vec{r}$

So $\vec{s}(f(t)) = \vec{r}(t)$ $\forall\, a \le t \le b$

$$\int_{\vec{r}} \vec{F} d\vec{r} = \int_a^b \vec{F}\, \vec{r}'(t)\, dt = \int_a^b \vec{F}\cdot \vec{s}'(f(t)) \underbrace{f'(t)\, dt}_{du} = \int_c^d \vec{F}\, \vec{s}'(u)\, du = \int_{s} \vec{F}\cdot d\vec{s}$$

$\boxed{\text{Scalar line integrals}}$



param: $\vec{s}(t)$, $a \le t \le b$

Introduction Let C be a curve between A & B, representing a string. Consider density $\lambda = \frac{dm}{dl}$

Total mass $\quad m = \int_C \lambda\, ds = \int_C \lambda \frac{ds}{dt}\, dt = \int_a^b \lambda\, v(t)\, dt = \int_a^b \lambda \left\| \frac{d\vec{s}}{dt} \right\| dt$

$$\left\{ \int_C f\, ds = \int_a^b f \left\| \frac{d\vec{s}}{dt} \right\| dt \right.$$ is the scalar line integral of fn $f(x,y)$ on C over a path $\vec{s}(t)$, $a \le t \le b$ $\}$

Ex Int. $f(x) = 3x^2 + 3y^2$ over the straight line segment C between $P(4,3)$, $Q(1,7)$

Find $\int_C f\, ds$. Choose param:

$$\vec{s}(t) = \vec{OP} + t\vec{PQ} = (4,3) + t(-3,4) = \begin{pmatrix} 4-3t \\ 3+4t \end{pmatrix}$$

$$\frac{d\vec{s}}{dt} = (-3,4) \quad . \quad \frac{ds}{dt} = 5$$

$$\int_C f\, ds = \int_0^1 f.(5dt) = \int_0^1 (4-3t)^2 + (3+4t)^2\, dt \,.\, (15) = 500$$

Oct 29, 2018

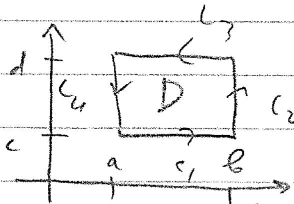
$\boxed{\text{Fundamental Theorem of Calculus}}$

$$\int_a^b g'(x)\,dx = g(b)-g(a)$$

$$\int_a^b f_x(x,c)\,dx = f(b,c)-f(a,c)$$

$\boxed{\text{Green's Theorem}}$ (Found in 1828 by Green & Ostrogradsky)

Example



Let c be the boundary of D, with counterclockwise orientation, denoted by ∂D.

Consider the vector field $f(x,y) = (0, N(x,y))$

assume that $N(x,y) > 0$, $N_x(x,y) > 0$

$$\int_{c_1} \vec F \cdot d\vec s = \int_{c_3} \vec F \cdot d\vec s = 0, \quad \int_{c_2} \vec F \cdot d\vec s > 0, \quad \int_{c_4} \vec F \cdot d\vec s < 0$$

$$\oint_{c=\partial D} \vec F \cdot d\vec s = \oint_c (0,N)\cdot(dx,dy) = \oint_c N\,dy = \int_{c_2} N\,dy - \int_{c_4^{opp}} N\,dy$$

c_2: $x=b$, $y=t$ $(c \le t \le d)$

c_4^{opp}: $x=a$, $y=t$ $(c \le t \le d)$

$$\rightarrow \oint_{\partial D} \vec F \cdot d\vec s = \int_c^d N(b,t)\,dt - \int_c^d N(a,t)\,dt$$

$$= \int_c^d N(b,t) - N(a,t)\,dt$$

$$= \int_c^d \int_a^b N_x(x,t)\,dx\,dt$$

$$= \int_c^d \int_a^b N_x(x,y)\,dx\,dy = \iint_D N_x\,dA$$

So $$\boxed{\oint_{c=\partial D} N\,dy = \iint_D N_x\,dA}$$

Analogously, consider a vector field $\vec{F}(x,y) = (M(x,y), 0)$

<u>Assume</u>, $M(x,y) > 0$
$M_y(x,y) > 0$ $\rightarrow$ $\boxed{\oint_c M\,dx = -\iint_D \underbrace{M_y}_{>0}\, dA}$

<u>So</u>

$$\oint_C N\,dy + M\,dx = \iint_D (N_x - M_y)\, dA$$

<u>Green's Theorem</u> (preliminary)

$\boxed{\text{if } D \text{ is a compact ~~[illegible]~~, convex region in } \mathbb{R}^2 \text{ and } \vec{F} \text{ is a smooth vector field whose domain includes } D, \text{ then}}$

circulation $\longrightarrow$ $$\oint_{c=\partial D} M\,dx + N\,dy = \oint_{c=\partial D} \vec{F}\cdot d\vec{s} = \iint_D -M_y + N_x\, dA$$

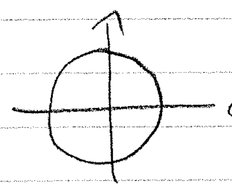
<u>Example</u> Evaluate $\oint_c (x-y)\,dx + (x+y)\,dy$ where C is the border of $(0,0),(0,1),(1,0$

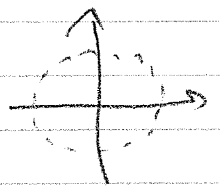
$$\oint_c \underbrace{(x-y)}_{M}\,dx + \underbrace{(x+y)}_{N}\,dy = \iint_D 1-(-1)\,dA = 2\iint_D dA = 2\cdot\left(\frac{1}{2}\right) = \boxed{1}$$

31, 2018

<u>Aside</u>: $D \subseteq \mathbb{R}^2$ is said to be bounded if it fits inside some circle

<u>Compact</u> if it is bounded <u>and</u> contains all of its limit points

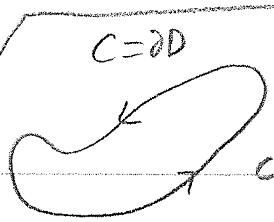

compact


bounded, not compact.

"We integrate over compact sets"

$\boxed{\text{Last time}}$ <u>Green's Theorem</u>

Green's Theorem



compact region D without holes

$\vec{F} = (M,N)$ defined on all of D.

$$\oint_{C=\partial D} M\,dx + N\,dy = \iint_D \left(-M_y + N_x\right) dA$$

(I) Vector Form of Green's Theorem

$$\operatorname{Curl}(M,N,0) = \det\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & 0 \end{vmatrix} = (N_x - M_y)\,\hat{k} = \begin{pmatrix} 0 \\ 0 \\ N_x - M_y \end{pmatrix}$$

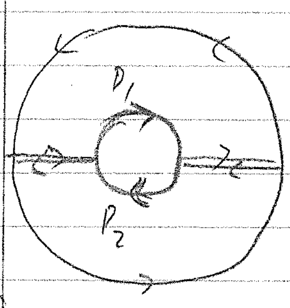
For a vector field $\vec{F} = (M,N)$ on $\mathbb{R}^2$, we define the scalar curl

$\boxed{\operatorname{SCurl}(\vec{F}) = N_x - M_y}$

Green's Theorem

$$\boxed{\oint_{C=\partial D} \vec{F}\cdot d\vec{s} = \iint_D \operatorname{scurl}(\vec{F})\,dA}$$

(II) Holes in domain of $\vec{F}$



Use Green's Theorem for D_1 & D_2

$$\oint_{\partial D_1} \vec{F}\cdot d\vec{s} = \iint_{D_1} (\operatorname{scurl}\vec{F})\,dA$$

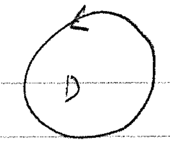
$$\oint_{\partial D_2} \vec{F}\cdot d\vec{s} = \iint_{D_2} (\operatorname{scurl}\vec{F})\,dA$$

inside boundary oriented clockwise

$$\boxed{\oint_{\partial D} \vec{F}\cdot d\vec{s} = \iint_D (\operatorname{scurl}\vec{F})\,dA}$$

boundary of hole is oriented clockwise.

III Motivation



$c = \partial D$

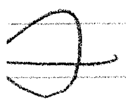
$$\oint_{\partial D} x\,dy = \iint_D M_x\,dA = \text{area of } D = \iint_D 1\,dA$$

Fact $\boxed{\oint_{\partial D} x\,dy = \oint_{\partial D} -y\,dx = \text{area of } D}$

$-1 \le t \le 1$

Example

Find the area of region D enclosed by path $\vec{s}(t) = (1-t^2, 15t-15t^3)$



$$\text{Area} = \oint_{\partial D} x\,dy = \int_{-1}^{1} (1-t^2)(15-45t^2)\,dt = 8$$

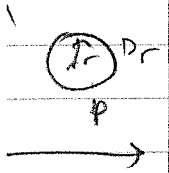
IV Interpretation of scurl$(\vec{F})$

Assume scurl$(\vec{F})$ = constant $\rightarrow \oint_{C=\partial D} \vec{F}\cdot d\vec{s} = \text{scurl}(\vec{F}) \iint_D dA$

So $\boxed{(\text{scurl}\,\vec{F}) = \frac{1}{\text{area of } D} \cdot \oint_{\partial D} \vec{F}\cdot d\vec{s}} \Longleftrightarrow \boxed{\text{circulation density}}$

$\Rightarrow$ (scurl $\vec{F}$) represents circulation density

If scurl $\vec{F}$ fails to be constant



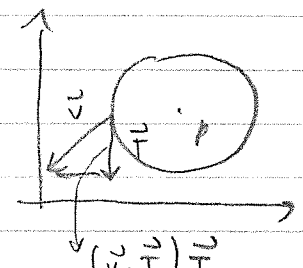
$$\boxed{(\text{scurl}\,\vec{F})(P) = \lim_{r\to 0} \frac{\oint_{\partial D_r} \vec{F}\,d\vec{s}}{\pi r^2}}$$

Nov 2, 2018

$\boxed{\text{Scurl \& angular velocity}}$

Let $\vec{v}(x,y)$ be the velocity at the surface of a body of water
Assume that scurl of v = constant

Place a lifesaver into the water, centered at p, anchored, touching the surface in a circle.

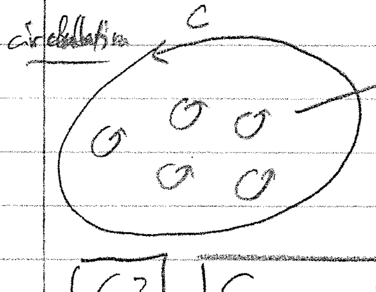


$$\omega = \frac{\text{average of } \vec{v}\cdot\vec{T} \text{ over } C}{r} = \frac{\oint_C (\vec{v}\cdot\vec{T})\,ds}{(2\pi r)\,r}$$

$$= \frac{\oint_C \vec{v}\,d\vec{s}}{2\pi r^2} = \cancel{\text{scurl}(\vec{v})}$$

$$= \frac{1}{2\pi r^2}\iint_D \text{scurl}(\vec{v})\,dA = \text{scurl}(\vec{v})\cdot\frac{\pi}{2\pi}$$

So $\boxed{\omega = \frac{1}{2}\,\text{scurl}(\vec{v})}$ $\boxed{\text{Vorticity } \cancel{\Longleftrightarrow} \text{ scurl}}$



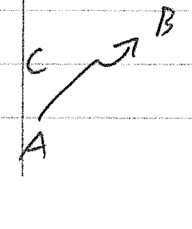
circulation → accumulation of scurl ...

$\boxed{6.3}$ $\boxed{\text{Conservative Vector Fields}}$ (in $\mathbb{R}^2$ & $\mathbb{R}^3$)

A vector field $\vec{F}$ on $\mathbb{R}^n$ is said to be conservative if $\vec{F} = \text{grad} f$ for some scalar field f, called a potential of $\vec{F}$.

if $\vec{F}$ on $\mathbb{R}^2$ is conservative, then $\text{scurl}(\vec{F}) = 0 = \text{scurl}(\text{grad} f)$
$= f_{xy} - f_{yx} = 0$

Example Consider oriented curve C from A to B and a conservative $\vec{F} = \text{grad} f$ defined on C



$$\int_C \vec{F}\,d\vec{s} = \ ? \quad \text{in terms of } f$$

Choose parame[trization] $\vec{s}(t)$ of C, with $a \le t \le b$
$\vec{s}(a) = A$, $\vec{s}(b) = B$

$$\int_C \vec{F} \cdot d\vec{s} = \int_a^b (\text{grad} f)_{(\vec{s}(t))} \frac{d\vec{s}}{dt}\, dt = \int_a^b \frac{df}{dt}\Big|_{\vec{s}(t)} dt \overset{FTC}{=} f(\vec{s}(b)) - f(\vec{s}(a)) = f(B) - f(A)$$

FTC for gradient $\rightarrow$ if C is an oriented curve in $\mathbb{R}^2$ or $\mathbb{R}^3$, $\vec{F} = \text{grad} f$ is a conservative vector field defined on C then,

$$\int_C (\text{grad} f) \cdot d\vec{s} = f(B) - f(A)$$ where C runs from A to B

Ex $\vec{F}(\vec{r}) = \dfrac{\vec{r}}{r^3}$ $\quad (r = \|\vec{r}\|)$ along any curve from $(5,0,0) \to (2,6,-3)$

Potential is $f(x,y,z) = \dfrac{-1}{\sqrt{x^2+y^2+z^2}}$

$$\left.\begin{aligned} \int_C \vec{F} \cdot d\vec{s} &= f(2,6,-3) - f(5,0,0) \\ &= -\frac{1}{7} + \frac{1}{5} = \frac{2}{35} \end{aligned}\right\} \text{path independence}$$

- Circulation of conservative vector field is 0

Nov 5, 2018

$\boxed{\text{FTC of single variable - calculus}}$

$$\begin{cases} \int_a^b f'(t)\,dt = f(b) - f(a) \\ \frac{d}{dt}\int_a^x g(t)\,dt = g(x) \end{cases}$$

$\boxed{\text{Last time}}$ FTC of gradient

If C is a curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ and $\vec{F} = \vec{\nabla} f$ is a vector field defined on C, then

$$\int_C \text{grad}\, f\, d\vec{s} = \int_C \vec{\nabla} f\, d\vec{s} = f(B) - f(A)$$

- Circulation is 0 $\oint_C \vec{\nabla} f\, d\vec{s} = 0$
- $\int_{C_1} \vec{\nabla} f\, d\vec{s} = \int_{C_2} \vec{\nabla} f\, d\vec{s}$ if $C_1(t_0) = C_2(t_0)$, $C_1(t_f) = C_2(t_f)$

$= f(B) - f(A)$

Theorem: For a vector field $\vec{F}$ on $\mathbb{R}^2$ or $\mathbb{R}^3$. The following are equivalent:

(1) $\vec{F}$ is conservative ($\exists f$ s.t. $\vec{\nabla} f = \vec{F}$)

(2) $\oint_C \vec{F}\, d\vec{s} = \oint_C \vec{\nabla} f\, d\vec{s} = 0$ $\forall C$ in domain of $\vec{F}$

(3) Path indep. line-integrals $\int_{C_1} \vec{F}\, d\vec{s} = \int_{C_2} \vec{F}\cdot d\vec{s}$ if $C_1(t_0) = C_2(t_0)$, $C_2(t_f) = C_2(t_f)$

↳ depends only on the end points.

$\boxed{\text{Proof}}$ $\boxed{(\mathrm{I}) \xrightarrow{\text{FTC}} (\mathrm{II}) \xrightarrow{?} (\mathrm{III}) \to (\mathrm{I})}$ ✓ ✓

$(\mathrm{I}) \Rightarrow (\mathrm{II}) \Downarrow (\mathrm{III}) \Rightarrow (\mathrm{I})$

(?) Let C be the concatenation of C_1, C_2 opp

$$\Rightarrow 0 = \oint_C \vec{F}\, d\vec{s} = \int_{C_1} \vec{F}\, d\vec{s} + \int_{C_2\text{opp}} \vec{F}\cdot d\vec{s}$$

$$\Rightarrow \int_{C_1} \vec{F}\, d\vec{s} = \int_{C_2} \vec{F}\, d\vec{s} \quad \text{(path-indep.)}$$

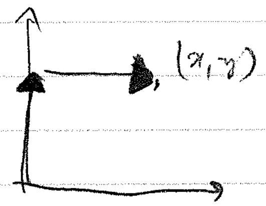
$\boxed{III \to I}$ Assume that $\vec{F}$ is defined throughout $\mathbb{R}^2$ (or $\mathbb{R}^3$) and path-indep (III)

Define $f(x,y) = \int_C \vec{F}\,d\vec{s}$ along any curve from $(0,0)$ to (x,y)

Can denote
$$\int_C \vec{F}\,d\vec{s} = \int_A^B \vec{F}\,d\vec{s}$$

<u>To show</u> $\operatorname{grad} f = \vec{F} = (M,N)$

or $\dfrac{\partial f}{\partial x} = M$ & $\dfrac{\partial f}{\partial y} = N$



$$f_x = \underbrace{\frac{\partial}{\partial x}\int_{(0,0)}^{(0,y)} \vec{F}\,d\vec{s}}_{0} + \frac{\partial}{\partial x}\int_{(0,y)}^{(x,y)} \vec{F}\,d\vec{s}$$

$$= 0 + \frac{\partial}{\partial x}\int_{(0,y)}^{(x,y)} M\,du + \cancel{N\,dv}^{\,0}$$

$$= \frac{\partial}{\partial x}\int_{0}^{x} M(t,y)\,dt = \boxed{M(x,y)}$$

So $\dfrac{\partial}{\partial y}\displaystyle\int = N(x,y)$ by analogy

//

What about $\boxed{\vec{F} \text{ being conservative} \Leftrightarrow \operatorname{scurl}(\vec{F}) = \vec{0}}$?

$$\left\{\begin{array}{l} \vec{F} \text{ conservative} \Rightarrow \operatorname{scurl}(\vec{F}) = \vec{0} \\ \operatorname{scurl}(\vec{F}) \not\Rightarrow \vec{F} \text{ conservative} \end{array}\right\}$$

↳ <u>Counter exp</u> : $\vec{F} = \left(\dfrac{-y}{x^2+y^2}, \dfrac{x}{y^2+x^2}\right)$

why? because $\oint_C \vec{F}\cdot d\vec{s} \neq 0$

<u>why?</u> $\vec{F}$ undefined @ $(0,0)$

Def A region D in $\mathbb{R}^n$ is said to be <u>connected</u> if $\exists$ a curve in D connecting any two points $A, B \in D$.

Nov 7, 2018

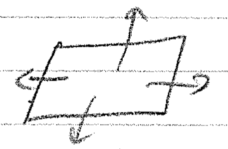
If $\vec{F}$ ~~on~~ $\mathbb{R}^2$ is defined on all of $\mathbb{R}^2$ and $\text{scurl}(\vec{F}) = 0$, then $\vec{F}$ is conservative.

<u>proof</u> show that for any simple closed curve, the circulation is 0
By Green's
$$\oint_C \vec{F}\, d\vec{s} = \iint_D \text{scurl}\,\vec{F}\, dA = 0$$

SURFACES What is a surface? "looks like $\mathbb{R}^2$ locally"

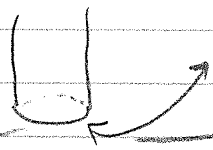
A small neighborhood of each point $P \in S$ looks like a distorted disk in $\mathbb{R}^2$, or like half a disk. The latter points are said to form the boundary of S, denoted ∂S.

For us, surfaces are subsets of $\mathbb{R}^3$. <u>Ex</u> sphere: $\partial S = \phi$
hemisphere ∂S = equator

<u>Ex</u> A plane  $\partial S = \phi$

cylinder : $\partial S = \phi$ side

Möbius strip $\to$ 1 connected boundary

 ∂S

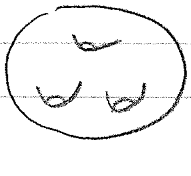
A surface is said to be <u>compact</u> if it is bounded & closed under limits

A surface is said to be closed if it is compact and $\partial S = \phi$ (has no boundary)

"boundedness" has nothing to do with "boundary"...

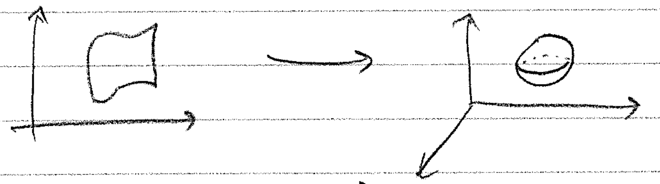
Types of Closed Surfaces

 Bagel

Pretzel  ... pretzel with n holes

 sphere

$\boxed{\text{Parameterized Surfaces}}$ The prototype is a topographical map



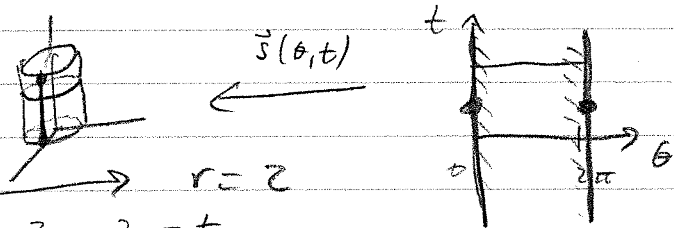
$\mathbb{R}^2 \longrightarrow \mathbb{R}^3$

Def A parameterized surface is a mapping $\mathbb{R}^2 \to \mathbb{R}^3$ (smooth) $\vec{S}$

$\vec{S} : D \to \mathbb{R}^3$ where D is a region in $\mathbb{R}^2$. The map is required to be one-to-one except possibly on the boundary

1 9, 2018

$\boxed{\text{Parametrized Surfaces}}$



(1) Param Cylinder $x^2+y^2=4 \longrightarrow r=2$

$x = 2\cos\theta, \quad y = 2\sin\theta, \quad z = z = t$

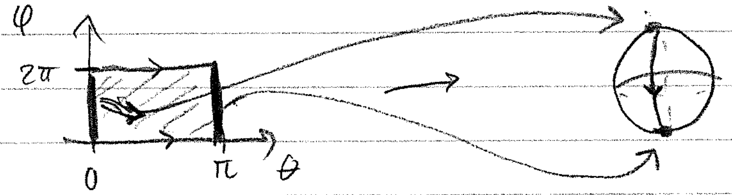
$\vec{s}(\theta,t) = (2\cos\theta, 2\sin\theta, t) \qquad 0 \le \theta \le 2\pi$

$\boxed{\text{Scaling factor } \sqrt{\det A^T A}}$ ↳ No Jacobian here

(2) Param sphere $x^2+y^2+z^2 = 9$

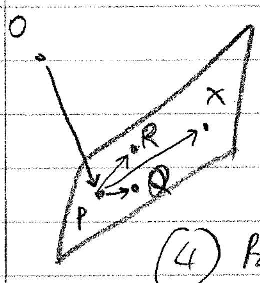
$x = 3\sin\theta\cos\varphi, \quad y = 3\sin\theta\sin\varphi, \quad z = 3\cos\theta \qquad 0 \le \varphi < 2\pi$

$\vec{s}(\theta,\varphi) = (3\sin\theta\cos\varphi, 3\sin\theta\sin\varphi, 3\cos\theta) \qquad 0 \le \theta \le \pi$



$\boxed{\text{Scaling factor} = 9\sin\theta}$

(3) Param plane thru 3 pts P, Q, R



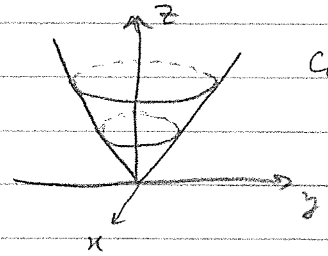
write $\vec{PX} = a\vec{PQ} + b\vec{PR} = s\vec{PQ} + t\vec{PR}$

$$\boxed{\vec{OX} = \vec{OP} + \vec{PX} = \vec{OP} + s\vec{PQ} + t\vec{PR}}$$

(4) Param graph x^2+y^2. Let $x = s, \; y = t, \; z = x^2+y^2 = s^2+t^2$

$\to \boxed{\vec{s}(s,t) = (s, t, s^2+t^2)}$

Ex Parametrize $x^2+y^2=z^2$ $z \geq 0$ $r^2=z^2 \to \boxed{r=z}$



$x=0$ Cone

$x=r\cos\varphi$

$y=r\sin\varphi$

$z=r$

in cylindrical coo

$0 \le \varphi \le 2\pi$, $r \ge 0$

$$\boxed{\vec{S}(r,\varphi)=(r\cos\varphi,\ r\sin\varphi,\ r)}$$

Vertical scaling factor $=\sqrt{2}$

Horizontal scaling factor: r $\Big\} \to r\sqrt{2}$

Nov 12, 2018

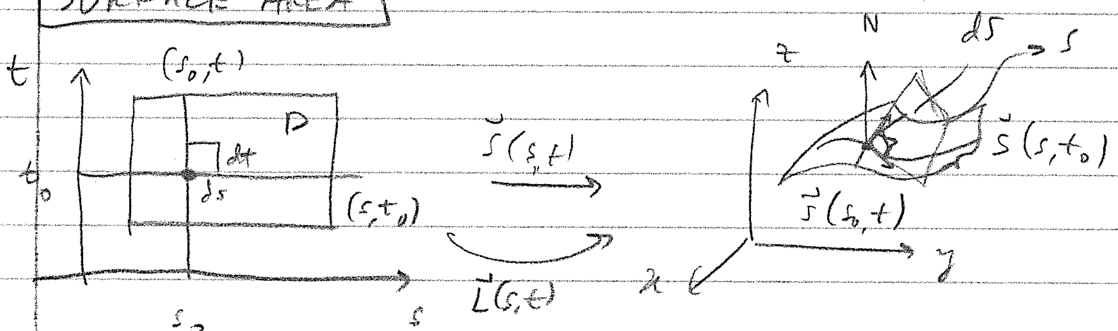
$\boxed{\text{Topics for Test}}$ (7) Scalar & Vector line integrals, conservative $\vec{F}$, potential, FTC for gradient, Green's theorem

(5) Change of variables, Jacobian.

Vector fields with curl $=0$.

(5) Surfaces, Surface Areas ...

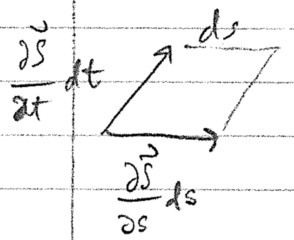
$\boxed{\text{SURFACE AREA}}$



Find a normal vector $\vec{N}$ to S at $\vec{S}(s_0,t_0)$ $\quad \dfrac{\partial \vec{S}}{\partial t}, \dfrac{\partial \vec{S}}{\partial s}$

$$\boxed{\vec{N}(s,t)=\frac{\partial \vec{S}}{\partial s}\times\frac{\partial \vec{S}}{\partial t}}$$ is normal to $\vec{S}$

Scaling factor Consider linearization $[ds \times dt] \to$ piece of tangent plane



$$dS=\left\|\frac{\partial \vec{S}}{\partial t}dt\times\frac{\partial \vec{S}}{\partial s}ds\right\|=\left\|\frac{\partial \vec{S}}{\partial t}\times\frac{\partial \vec{S}}{\partial s}\right\|dt\,ds$$

$$\boxed{dS=\left|\vec{N}(s,t)\right|dt\,ds} \to \text{area element ...}$$

$\to$ standard normal vector

$$\boxed{A = \iint_D \|\vec{N}\|\, dt\, ds = \iint_S dS}$$

Example Surface area of sphere of radius $\rho = a$

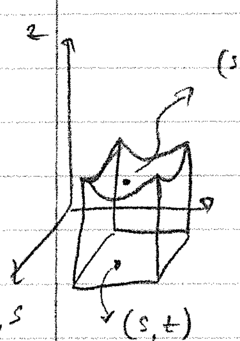
Scaling factor

$\hookrightarrow \rho^2 \sin\theta = a^2 \sin\varphi$

$$A = \int_0^{\pi}\int_0^{2\pi} a^2 \sin\varphi \, d\varphi\, d\theta = 2\pi a^2 \int_0^{\pi} \sin\varphi\, d\varphi = \boxed{4\pi a^2}$$

Scaling factor for a graph of $z = f(x,y)$

$\vec{S}(s,t) =$



$\vec{S}(s,t) = (s, t, f(s,t))$

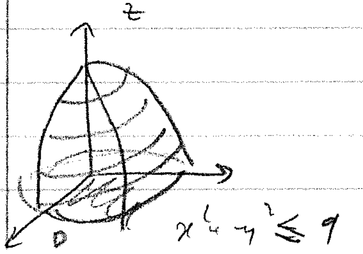
$\frac{\partial \vec{S}}{\partial s} = (1, 0, f_s) \quad \frac{\partial \vec{S}}{\partial t} = (0, 1, f_t)$

So $\boxed{\vec{N} = \frac{\partial \vec{S}}{\partial s} \times \frac{\partial \vec{S}}{\partial s} = (-f_s, -f_t, 1)}$

So $\boxed{\|\vec{N}\| = \sqrt{1 + f_s^2 + f_t^2}} \longrightarrow \geq 1$

or $\boxed{\|\vec{N}\| = \sqrt{1 + f_x^2 + f_y^2}}$

(3) Find area of portion of paraboloid $z = 9 - x^2 - y^2$ above x, y plane ($f(x,y) =$)



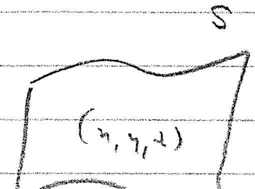
$$A = \iint_D \|N\|\, dx\, dy = \iint \sqrt{1 + 4x^2 + 4y^2}\, dA = \int_0^{2\pi}\int_0^{3} \sqrt{1 + 4r^2}\, r\, dr\, d\theta = 2\pi \int_0^{3} \sqrt{1+4r^2}\, r\, dr$$

$x =$

Nov 19, 2018

$$\boxed{\text{Surface element : } dS = \|\vec{N}\|\, dt\, ds \quad \text{where } \vec{N} = \frac{\partial \vec{S}}{\partial s} \times \frac{\partial \vec{S}}{\partial t}}$$

Scalar surface integral



a piece of paper. Let (x, y, z) be the area density (measured by kg/m^2)

mass of surface element

↳ $f(x,y)\,dS$

→ total mass $\iint_S f(x,y)\,dS$

Def. Consider surface S parameterized by $\vec{S}(s,t)$ on D and the smooth function $f(x,y,z)$ on S. The scalar surface int

$$\iint_{\vec{S}} f\, dS := \iint_D f\, \|\vec{N}\|\, dt\, ds$$

or

$$\iint_S f(x,y,z)\, dS = \iint_D f(\vec{S}(s,t))\, \|\vec{N}\|\, dt\, ds$$

Theorem The scalar surface int is intrinsic, depending on the underlying surface only (not the parameterization)

Outline of proof → $\vec{S}_2^{-1} \circ \vec{S}_1 = \vec{T} \rightarrow \vec{S}_2(T(s,t)) = \vec{S}_1(s,t)$

$$\|\vec{N}_{\vec{S}_1}\| \overset{?}{\iff} \|\vec{N}_{\vec{S}_2}\| \rightarrow \boxed{\|\vec{N}_{\vec{S}_1}\| = \|\vec{N}_{\vec{S}_2}\| \cdot \left|\frac{\partial \vec{T}(u,v)}{\partial(s,t)}\right|}$$

$$\iint_{\vec{S}_1} f\, dS = \iint_{D_1} f\, \|\vec{N}_{\vec{S}_1}\|\, dt\, ds = \iint_{D_1} f\, \|\vec{N}_{\vec{S}_2}\| \cdot \underbrace{\left|\frac{\partial(u,v)}{\partial(s,t)}\right| dt\, ds}_{du\,dv} = \iint_{D_2} f\, \|\vec{N}_2\|\, dv\, du = \iint_{\vec{S}_2} f\, dS$$

Ex $\iint_S (x^2+y^2)\,dS \longrightarrow$ sphere $x^2+y^2+z^2 \leq a^2$

$$\|N\| = a^2 \sin\theta \qquad 0 \leq \theta \leq \pi$$
$$0 \leq \varphi \leq 2\pi$$

$$\int_0^{2\pi}\int_0^{\pi} [a \sin\theta]^2\, a^2 \sin\theta \, d\theta\, d\varphi = \frac{8}{3}\pi a^4$$

v 26, 2018

$\boxed{\text{Flux}}$ (aka vector surface integrals)

Ea water flowing in pipeline, with $\vec{v}(x,y,z)$. At what rate, in m^3/s is the water flowing through a given cross section S, in a given direction

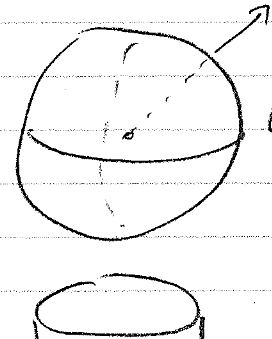
Aside $\boxed{\text{orientation of surfaces}}$. An orientation of surface S is a smooth vector field $\vec{n}(x,y,z)$ of unit normal vectors on S

Möbius strip doesn't have an orientation
Klein bottle doesn't have an orientation

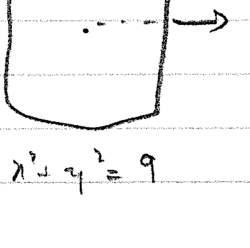
Ex sphere: has 2 orientations (outward / inward)

Will define flux through oriented surface $(S, \vec{n})$

Ex Find outward orientation of the sphere $x^2+y^2+z^2=9$



$$\boxed{\vec{n} = \frac{1}{3}(x,y,z)}$$



$$\boxed{\vec{n} = \frac{1}{3}(x,y,0)}$$

$\vec{\nabla}f =$ grad $f = (2x, 2y, 0)$
normal $(x,y,0)$

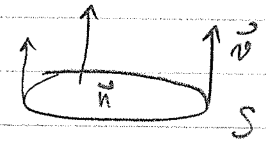
$x^2+y^2=9$

Parameterizations of some Surfaces

Surface	Parameterization $\vec{S}(s,t)$	Standard Normal $\vec{N}=\dfrac{\partial\vec{S}}{\partial s}\times\dfrac{\partial\vec{S}}{\partial t}$	Scaling Factor $\|\vec{N}$
Cylinder $x^2+y^2=a^2$	$\vec{S}(\theta,t)$ $x=a\cos\theta,\quad 0\le\theta\le 2\pi$ $y=a\sin\theta$ $z=t$	$\vec{N}=\left(a\cos\theta, a\sin\theta, 0\right)$	$\|\vec{N}\|=a$ $\boxed{\tilde{n}=\frac{\pm 1}{a}(x,y,0)}$
Sphere $x^2+y^2+z^2=a^2$	$\vec{S}(\theta,\phi)$ $x=a\sin\phi\cos\theta,\quad 0\le\theta\le 2\pi$ $y=a\sin\phi\sin\theta,\quad 0\le\phi\le\pi$ $z=a\cos\phi$	$\vec{N}=-a\sin\phi\,\vec{S}(\theta,\phi)$ $-a^2\sin\phi\left(\sin\phi\cos\theta,\sin\phi\sin\theta,\cos\phi\right)$	$\|\vec{N}\|=a^2\sin\phi$ $\boxed{\tilde{n}=\frac{\pm 1}{R}(x,y,z)}$
Plane though points P, Q, and R	$\vec{S}(s,t)=$ $OP+sPQ+tPR$	$\vec{N}=PQ\times PR$	$\|\vec{N}\|=\|PQ\times PR\|$
Graph of $z=f(x,y)$	$\vec{S}(s,t)$ $x=s$ $y=t$ $z=f(s,t)$	$\vec{N}=\left(-f_x,-f_y,1\right)$	$\|\vec{N}\|=\sqrt{f_x^2+f_y^2+1}$
Cone $x^2+y^2=z^2,\quad z\ge 0$	$\vec{S}(\theta,r)$ $x=r\cos\theta,\quad 0\le\theta\le 2\pi$ $y=r\sin\theta,\quad r\ge 0$ $z=r$	$\vec{N}=\left(r\cos\theta, r\sin\theta, -r\right)$	$\|\vec{N}\|=\sqrt{2}\,r$

Simple Cases of Flux

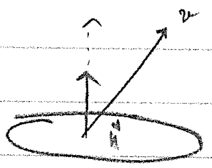
Case 1



Set $\vec v \uparrow\uparrow \vec n$ $\quad v = \|\vec v\| =$ constant

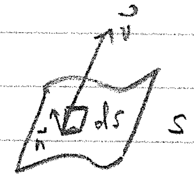
Flux $= \boxed{v\ (\text{area of } S)}$

Case 2



$(\vec v\cdot\vec n)\vec n$ $\quad \boxed{\text{Flux}\ \ (\vec v\cdot\vec n)(\text{area of } S)}$

Case 3 **General**



Flux through element $= \boxed{(\vec v\cdot\vec n)(dS)}$

Total flux $\boxed{\iint_S (\vec v\cdot\vec n)\,dS}$ → scalar surface integral

Def If $(S,\vec n)$ is an oriented compact surface in $\mathbb{R}^3$ and $\vec F$ is a ve[ctor] field on S, then the flux of $\vec F$ thru $(S,\vec n)$ is defined to be

$$\iint_S (\vec F\cdot\vec n)\,dS$$, a scalar surface integral

↳ intrinsic...

This integral is also denoted by $\iint_S \vec F\cdot d\vec S$ → $\vec n\,dS$

Writing flux in terms of parametrization. $\iint_S \vec F\cdot d\vec S = \iint_S \vec F\cdot\vec n\,dS = \iint_S \cdot$

So $$\iint_S \vec F\cdot d\vec S = \iint_S \vec F\cdot\vec n\,dS = \iint_D (\vec F\cdot\vec n)\,\|\vec N\|\,dt\,ds$$

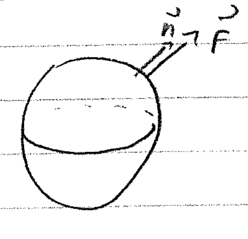
$$= \iint_D (\vec F\cdot\vec n)\|\vec N\|\,dt\,ds = \iint \vec F\cdot(\|\vec N\|\,\vec n)\,dt\,ds = \pm\iint_S \vec F\cdot\vec N\,dt\,ds$$

$\vec n = \pm\dfrac{\vec N}{\|\vec N\|}$

28, 2018 (+) if $\vec{N} \uparrow\uparrow \vec{n}$ (−) if $\vec{N} \downarrow\uparrow \vec{n}$

$$\boxed{\text{Use } \iint_S \vec{F}\cdot\vec{n}\, dS \text{ if you can, use } \pm\iint_D \vec{F}\cdot\vec{N}\, dt\, ds \text{ if you must}}$$

Ex Find outward flux of field $\vec{F} = (x,y,z)$ through sphere $S: x^2+y^2+z^2=1$



$\vec{F}\cdot\vec{n} = (x,y,z)\cdot(x,y,z) = 1$

$\rightarrow \iint_S \vec{F}\cdot\vec{n}\, dS = \iint_S dS = 4\pi$

Ex Find the upward flux of $\vec{F} = (z^2, x, -3z)$ through $S: z = 4-y^2$

$0 \le x \le 1$

$\boxed{\text{Parametrize as graph}}$ $f(x,y) = 4-y^2$ $0 \le y \le 2$

$D = [0,1]\times[0,2]$

$\vec{F} = \left((4-y^2)^2, x, -3(4-y^2)\right)$ $\vec{N} = \left(-f_x, -f_y, 1\right)$

$= (0, 2y, 1)$

$\vec{F}\cdot\vec{N} = 2xy - 12 + 3y^2$ "upward" if last component is >0

$\rightarrow (\vec{N} /\!/ \vec{n})$ $\rightarrow \vec{N}$ "upward", $\vec{n}$ also upward

$\rightarrow$ Flux $= \iint_S \vec{F}\cdot d\vec{S} = + \int_0^1\int_0^2 2xy - 12 + 3y^2\, dy\, dx = -14$

Ex Outward flux of vector field $\vec{F} = (x^2, 0, 0)$, $\vec{G} = (x,0,0)$, through

$x^2+y^2 = 1$, $z \in [0,3]$

(a) $\vec{n} = (x,y,0)$ $\rightarrow \vec{F}\cdot\vec{n} = (x^2,0,0)(x,y,0) = x^3$

Flux $= \iint_S \vec{F}\cdot\vec{n}\, dS = \iint_S x^3\, dS = 0$ by symmetry

x^3 odd in x, S symmetric

(b) $\vec{G}\cdot\vec{n} = (x,0,0)\cdot(x,y,0) = x^2 \rightarrow \iint_S x^2 dS = \iint_S y^2 dS = \frac{1}{2}\iint_S dS = \frac{3}{2}2\pi = 3\pi$

$\boxed{\text{Alternative way}}$ $\displaystyle\iint_S x^2\,dS = \iint_S \vec{G}_1\, d\vec{S} =$

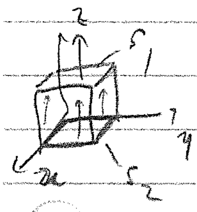
D: $0 \le \varphi \le 2\pi$, $0 \le z \le 3$ $\quad \vec{G} = (\cos\varphi, 0, 0)$ $\quad \vec{N} = (\cos\varphi, \sin\varphi, 0)$

So $\vec{G}\cdot\vec{N} = \cos^2\varphi$

$$\text{Flux} = \int_0^{2\pi}\int_0^3 \cos^2\varphi\, dz\, d\varphi = 3\int_0^{2\pi} \cos^2\varphi\, d\varphi = 3\pi$$

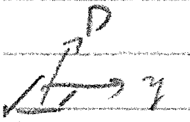
...30, 2018

$\boxed{\text{Gauss' Theorem}}$ – Divergence Theorem – Ostrogradsky Theorem



Example. Consider $W = [0,1]^3$ solid region

Let $\partial W = S$ be the boundary of W with outward orientation. Consider $P(x,y,z) > 0$, $P_z(x,y,z) > 0$



Note $\displaystyle\iint_{\text{sides}} \vec{F}\cdot d\vec{S} = 0$ so $\displaystyle\oiint_S \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\, d\vec{S} - \iint_{S_2\,\text{opp}} \vec{F}\, d\vec{S}$

So $$\oiint_S \vec{F}\, d\vec{S} = \iint_D (0,0,P(x,y,1))\cdot(0,0,1)\,dy\,dx$$
$$- \iint_D (0,0,P(x,y,0))\cdot(0,0,1)\,dy\,dx$$
$$= \iint_D P(x,y,1) - P(x,y,0)\,dy\,dx = \int_{x=0}^1\int_{y=0}^1\int_{z=0}^1 P_z(x,y,z)\,dz\,dy\,dx$$

So $$\oiint_{S=\partial W} \vec{F}\cdot d\vec{S} = \iiint_W P_z\, dV = \oiint_{S=\partial W} (0,0,P(x,y,z))\, d\vec{S}$$

Analogously, $$\oiint_{S=\partial W} (M,0,0)\, d\vec{S} = \iiint_W M_x\, dV$$

$$\oiint_{S=\partial W} (0,N,0)\, d\vec{S} = \iiint_W N_y\, dV$$

<u>Add</u> $\displaystyle\oiint_{S=\partial W} (M,N,P)\,d\vec{S} = \iiint_W M_x + N_y + P_z \, dV$

<u>&</u> $\displaystyle\oiint_{S=\partial W} \vec{F}\cdot d\vec{S} = \iiint_W \underbrace{\vec{\nabla}\cdot\vec{F}}_{\text{"flux density"}} \, dV$

Ostrogradsky's Theorem	If W is a compact solid region in $\mathbb{R}^3$ & $\vec{F}$ is a smooth vector field on W, then $\displaystyle\oiint_{S=\partial W} \vec{F}\cdot d\vec{S} = \iiint_W \vec{\nabla}\cdot\vec{F}\,dV$

where $S=\partial W$ is oriented <u>away</u> from the region W

Interpret $\vec{\nabla}\cdot\vec{F}$ if it is constant

$$\oiint_{S=\partial W} \vec{F}\cdot d\vec{S} = \iiint_W \vec{\nabla}\cdot\vec{F}\,dV = \vec{\nabla}\cdot\vec{F}\iiint_W dV$$
$$= \vec{\nabla}\cdot\vec{F}\cdot(\text{Volume } W)$$

<u>So</u> $\displaystyle\vec{\nabla}\cdot\vec{F} = \frac{1}{V_W}\oiint_{S=\partial W} \vec{F}\cdot d\vec{S}$, hence "flux density"

$\boxed{\text{"div } \vec{F} \text{ is flux density"}, \quad \text{"curl } \vec{F} \text{ is circulation density"}}$

<u>Ex</u> $\vec{F}=(z,z,x)$ thru unit sphere.

$\operatorname{div}\vec{F} = 1$ $\quad\displaystyle\oiint_{S=\partial B} \vec{F}\cdot d\vec{S} = \iiint_B \underbrace{\operatorname{div}\vec{F}}_{1}\,dV = \iiint_B dV = \boxed{\frac{4\pi}{3}}$

$\boxed{\text{Dec 3, 2018}}$ $\boxed{\text{Ex}}$ $\vec F = (x, y, z^2 - 1)$ $S = \partial W$ "closed" → n-hole lout

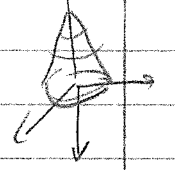
$W: x^2 + y^2 \le 4, \quad 0 \le z \le 1$

$$\Phi = \iiint_W 1 + 1 + 2z \cdot dV$$

$$= \iiint_W (2z+2)\, dV = \int_0^{2\pi}\int_0^{2}\int_0^{1} (2+2z)\, r\, dz\, dr\, d\theta$$

$$= 2\pi\,(2) \int_0^1 (2+2z)\,dz = (4\pi)\cdot 3 = 12\pi$$

$\boxed{\text{Ex}}$ $\vec F = (e^{y}\cos z, \sqrt{x^3+1}\,\sin z, 3)$ on $S = z = (1-x^2-y^2)\,e^{1-x^2-y^2}$, $x^2+y^2 \le 1$

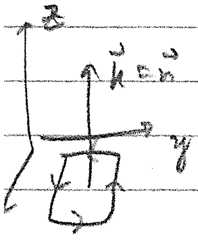


Close the surface up by adding S_1: $x^2+y^2 \le 1$, $z = 0$, oriented downward.

$$\Phi = \underbrace{\iiint_W \operatorname{div} \vec F\, dV}_{0} - \iint_{S_1} \vec F \cdot d\vec S = -\iint_{S_1} \vec F \cdot \vec n\, dS$$

$$= -\iint_{S_1} (-3)\, dS = 3\pi$$

$\boxed{\text{STOKES' THEOREM}}$ found by Lord Kelvin



"Move Green's theorem into $\mathbb{R}^3$"

Green's theorem → $\oint_{\partial D} \vec F \cdot d\vec s = \iint_D (\text{scurl}\, \vec F)\, dA$

$\vec F = (M, N, 0)$ $\quad \vec\nabla \times \vec F = (-\partial_z N, \partial_z M, \partial_x N - \partial_y M)$

$$= (0, 0, \text{scurl}\, \vec F)$$

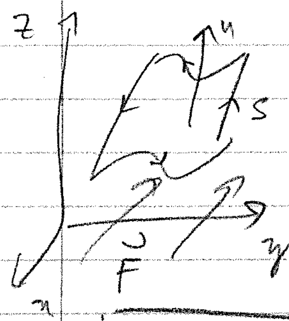
$\boxed{\text{scurl}\, \vec F = (\vec\nabla \times \vec F)\cdot \vec k}$ or $\vec n$

$\boxed{\text{Translate Green's Theorem}}$

$$\oint_{\partial S} \vec{F}\cdot d\vec{s} = \iint_S (\vec{\nabla}\times\vec{F})\,\vec{n}\,dS = \iint_S (\vec{\nabla}\times\vec{F})\cdot d\vec{S}$$

Stokes' Theorem

If $(S, \vec{n})$ is an oriented compact surface in $\mathbb{R}^3$ and $\vec{F}$ is a smooth vector field on S, then



$$\oint_{C=\partial S} \vec{F}\cdot d\vec{s} = \iint_S (\vec{\nabla}\times\vec{F})\cdot d\vec{S}$$ ∂S is oriented by the right hand rule.

$\boxed{\text{curl } \vec{v} = \vec{\nabla}\times\vec{v} = 2\vec{\omega}}$

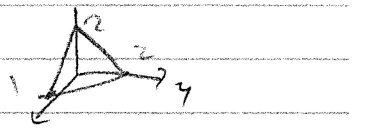
(except @ (0,0,0))

If $\vec{F}$ defined throughout $\mathbb{R}^3$, $\vec{\nabla}\times\vec{F} = \vec{0}$, then $\vec{F}$ conservative.

$\boxed{\text{TRUE}}$ $\rightarrow$ $\oint_{\partial S} \vec{F}\cdot d\vec{s} = \iint \underbrace{\vec{\nabla}\times\vec{F}}_{0}\, dS = 0$ (any surface)

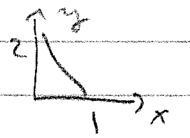
If entire z axis is missing, then not true

<u>Ex</u> $S: 2x+y+z = 2$ $\quad \vec{F} = (2xz,\ xy,\ -xz)$



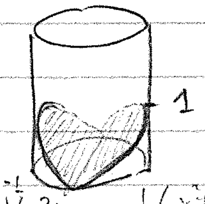
$x, y, z \geq 0$

$$\oint_{\partial S} \vec{F}\cdot d\vec{s} = \int_0^1\int_0^{2-x} \underbrace{(0,\ 2x+z,\ y)(2,1,1)}_{2}\, dy\,dx$$



$$= \iint (2x+z+y)\,dy\,dx = 2\cdot 1 = 2 \qquad (\vec{n} \parallel \vec{N})$$

<u>Ex</u> $x^2+y^2=1,\ z=y^2$ $\quad \vec{F} = (e^x,\ x+\sin y,\ \cos z)$



$S: z = y^2,\ x^2+y^2 \leq 1$ $\quad \vec{\nabla}\times\vec{F} = (0,0,1)$

$\vec{N} = (0, -2y, 1)$

$(\vec{\nabla}\times\vec{F})\,\vec{N} = 1 \Rightarrow \phi = \pi$ (area of D)

$(x^2+y^2)^{1/2} + \frac{1}{6}x(x^2+y^2)^{-\frac{1}{2}}\cdot 2x = \frac{1}{3}\left(\frac{x^2+y^2+x^2}{\sqrt{x^2+y^2}}\right) + \frac{1}{3}\ \frac{y^2+x^2+x^2}{} = \sqrt{x^2+y^2}$

$\boxed{\star}$ Converse true if defined on all of $\mathbb{R}^2, \mathbb{R}^3$

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$$SF(\mathbb{R}^2) \xrightarrow{\nabla} VF(\mathbb{R}^2) \xrightarrow{\text{scurl}} SF(\mathbb{R}^2)$$

$$f(p) \qquad \int_C \vec{F}\cdot d\vec{s} \qquad \iint_D f\,dA$$

$$f(B)-f(A) = \int_C \vec{\nabla} f \cdot d\vec{s} \qquad \int_{\partial D} \vec{F}\cdot d\vec{s} = \iint_D \text{scurl}(\vec{F})\,dA \quad (\text{Green's})$$

$$SF(\mathbb{R}^3) \xrightarrow{\nabla} VF(\mathbb{R}^3) \xrightarrow{\vec{\nabla}\times} VF(\mathbb{R}^3) \xrightarrow{\vec{\nabla}\cdot} SF(\mathbb{R}^3)$$

$$f(p) \qquad \int_C \vec{F}\cdot d\vec{s} \qquad \iint_S \vec{F}\cdot d\vec{S} \qquad \iiint_W f\,dV$$

$$f(B)-f(A) = \int_C \vec{\nabla} f\, d\vec{s} \qquad \cancel{\text{Stokes's}} \rightarrow \oint_{\partial S} \vec{F}\cdot d\vec{s} = \iint_S \vec{\nabla}\times\vec{F}\cdot d\vec{S} \qquad \iint_{S=\partial W} \vec{F}\cdot d\vec{S} = \iiint_W \underbrace{\text{div}\,\vec{F}}_{\vec{\nabla}\cdot\vec{F}}\,dV$$

(Gauss's)

<u>Stokes' Theorem</u> $\displaystyle\int_{\partial M} \omega = \int_M d\omega$

<u>Forms</u>

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

<u>Ex</u> $\displaystyle\oint_C (6y+x)\,dx + (y+2x)\,dy$, $(x-2)^2 + (y-3)^2 = 4$, ccw

$\vec{F} = (6y+x,\ y+2x)$ $\quad$ scurl $\vec{F} = N_x - M_y = 2-6 = -4 \rightarrow \oint = -4\int dA$

$= (-4)(2^2)\pi = \boxed{\ }$

<u>Ex</u> Int $(1,0,0) \rightarrow (2,-2,1)$ $\quad \vec{F}(\vec{r}) = \vec{r}/\|\vec{r}\|^3$

$r=1 \qquad r=3 \rightarrow \int \vec{F}\,d\vec{r} = \left.\frac{-1}{r}\right|_1^3 = -\frac{1}{3}+1 = \frac{2}{3}$

$f(r) = -1/r$

$\boxed{\text{For cheat sheet}}$

	$\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$	$\tilde{r}/\lVert\tilde{r}\rVert^3$
div	0	0
curl	$\tilde{0}$	$\tilde{0}$
scalar potential	Nope	$-1/r$
vector potential		
circulation		
flux thru sphere		

Ex Outward Φ of $\tilde{F} = (x,y,z)$ out of region given by $x^2+y^2 \le 1$, $x^2+z^2 \le 1$
Express answer in terms of V and A.

$$\operatorname{div} \tilde{F} = 3 \qquad \oint\!\!\oint_S \tilde{F}\cdot d\vec{s} = \iiint_W \operatorname{div} \tilde{F}\, dW = 3V = A$$

$$\iint \tilde{F}\cdot\tilde{n}\, dS = \iint_S (x,y,z)\cdot(x,y,0)\, dS = \iint_S 1\, dS = 1\iint_S dS = A$$

Ex C. $x^2+y^2+z^2=1$, $z = -y$. $\tilde{F} = (z, 2x, 3y)$

$$\oint_C \vec{F}\, d\vec{s} = \iint (\tilde{\nabla}\times\tilde{F})\, dS = \iint (3,1,2)\cdot(0,1,1)\, dS = 3\iint dS = 3\pi$$