QUANTUM FIELD THEORY & CONDENSED MATTER

(book by Shankar)

Feb 11, 2021

(I) THERMO - STAT MECH REVIEW

(a) Energy & Entropy in Thermo

Internal energy U: → state variable, i.e. sth which has a unique value associated w/ every state.

returns to same value if state is looped

→ 2 ways to change U: → $dU = -PdV$ (cons of energy)
$\qquad = \delta Q$. (heat)

→ get 1st law of thermo:

$$\boxed{dU = \delta Q - PdV}.$$

Q is not a state variable, so we write δQ.

2nd law introduces entropy $\boxed{dS = \dfrac{\delta Q}{T}}$ → heat added reversibly

S is a state variable since $\oint dS = 0$ for a quasi-static cyclic process.

If Q is added reversibly,

→ $dU = TdS - PdV$ → $U = U(S,V)$
$T = \partial U/\partial S|_V$
$-P = \partial U/\partial V|_S$

$\rightarrow$ get

$$dS = \frac{1}{T} dU + \frac{P}{T} dV$$

$$S = S(U, V)$$

$$\frac{1}{T} = \left.\frac{\partial S}{\partial U}\right|_V , \quad \frac{P}{T} = \left.\frac{\partial S}{\partial V}\right|_U$$

$U(S,V) \rightarrow$ "fundamental relation".

For ideal gas ...

$$P = -\left.\frac{\partial U}{\partial V}\right|_S = \frac{2}{3}\frac{U}{V}$$

$$T = \left.\frac{\partial U}{\partial S}\right|_V = \frac{2}{3nR} U$$

since $U(S,V) = C \left[\frac{e^{S/nR}}{V} \right]^{2/3}$

From these we find $\boxed{PV = nRT}$.

(b) Equilibrium as Max S

$\hookrightarrow$ S is max @ equilibrium

(c) Free Energy in Thermo

Suppose $V = \text{const}$, so $U = U(S) \Rightarrow T = \frac{\partial U}{\partial S} = \frac{dU}{dS}$

$\Rightarrow T = T(S) \Rightarrow S = S(T)$

$$\boxed{\text{Free energy: } F(T) = U(S(T)) - T \cdot S(T)}$$

we find that $\quad U - ST$

$$\boxed{\frac{dF}{dT} = \ldots = -S(T)}$$

Now, bring back V to get

$\rightarrow$

$$F = F(T,V) \qquad -S = \left.\frac{\partial F}{\partial T}\right|_V$$

$$-P = \left.\frac{\partial F}{\partial V}\right|_T \qquad dF = -S\,dT - P\,dV$$

(d) <u>Equilibrium as Min of F</u>

$F(V,T)$ has the same info as $U(V,S)$

(e) <u>Microcanonical Distribution</u>

<u>Microstate</u>: states described by maximal detail.

→ A system can statistically be in ~~any~~ many microstates, each with probability p_i, with which observable O has value $O(i)$

$$\rightarrow \boxed{\langle O \rangle = \sum_i p_i\, O(i)}$$

Fluctuation (variance) is $\boxed{(\Delta O)^2 = \langle O^2 \rangle - \langle O \rangle^2}$

Fundamental postulate of statistical mech...

A macroscopic isolated system in thermal equilibrium is equally likely to be found in any of its accessible microstates

→ "equal weight probability"

↳ (Microcanonical distribution)

→ Boltzmann: Entropy of isolated system:

$$\boxed{S = k \ln \Omega}$$

k: $1.38 \cdot 10^{-23}\ JK^{-1}$ Ω: # microstates

ideal gas constant

$$\boxed{R = N_A k}$$

N_A: Avogadro's number

2 independent systems → $\left\{ \begin{array}{l} \Omega_{tot} = \Omega_1 \times \Omega_2 \\ S_{tot} = S_1 + S_2 \end{array} \right\}$.

(f) <u>Gibbs's Approach: Canonical Distribution</u>

Boltzmann gave description of system for a definite U.

Gibbs did the same, but for a definite T

→ Relative probability of system in state i of energy E_i is $e^{-\beta E(i)}$ where

$$\beta = \frac{1}{kT}.$$

$$\rightarrow \boxed{p(i) = \frac{e^{-\beta E(i)}}{\sum_i e^{-\beta E(i)}} = \frac{e^{-\beta E(i)}}{Z}}$$

where Z is the "partition function"

$$Z = \sum_i e^{-\beta E(i)}$$

Ex $E(x,p) = \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 x^2$

$$\rightarrow Z(\beta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dp\; e^{-\beta\left(p^2/2m + \frac{1}{2} m\omega_0^2 x^2\right)}$$

In QM, we have

$$Z = \sum_i e^{-\beta E(i)} = \mathrm{Tr}\left(e^{-\beta H}\right)$$

$$= \int_{-\infty}^{\infty} dx\, \langle x| e^{-\beta H(x,p)} |x\rangle$$

this leads to the path integral

Ex $\langle E\rangle = \frac{\sum_i E_i e^{-\beta E_i}}{Z} = -\frac{\partial \ln Z}{\partial \beta} = U$

(g) Grand Canonical Distribution

↳ one that can exchange heat & particles.

In this case, $E(i)$ depends on chemical potential μ.

$$\boxed{\begin{aligned} &\rightarrow p(E(i), N) = \frac{e^{-\beta(E(i)-\mu N)}}{Z} \\ &\text{where } Z = \sum_N \sum_{i(N)} e^{-\beta(E(i)-\mu N)} \end{aligned}}$$

where

$$\boxed{\begin{aligned} &\langle N \rangle = \frac{1}{\beta} \frac{\partial \ln Z}{\partial \mu}\bigg|_\beta \\ &-\frac{d \ln Z}{d\beta} = \langle U \rangle - \mu \langle N \rangle \end{aligned}}$$

Ex QM:

$$\boxed{\langle N \rangle = n_{F/B} = \frac{1}{e^{\beta(\varepsilon-\mu)} \pm 1}}$$

where $(+) \rightarrow$ fermions

$(-) \rightarrow$ bosons.

For more info on ~~stat~~ mech & thermo

→ Review ~~P~~H 332 notes (on website)

(2) THE ISING MODEL in $d=0$ & $d=1$

(a) Ising model in $d=0$

- System has 2 d.o.f.: $\boxed{s_1, s_2 \, . \quad s_i = \pm 1.}$

  Config of the system is given by (s_1, s_2).

- Energy: $\boxed{E(s) = -J s_1 s_2 - B(s_1+s_2)}$

  models a system made of magnetic moments.

$J>0 \Rightarrow$ ferromagnetic (E favors aligned spins)
$J<0 \Rightarrow$ antiferromagnetic (- - antiparallel - -)

$B>0$ always. Spins like to align w/ $\vec{B} = B\hat{z}$.

With these - -

$$Z = \sum_{s_1, s_2} e^{K s_1 s_2 + h(s_1+s_2)}$$

where $K = \beta J$, $h = \beta B$.

$$Z = Z(K,h) = 2\cosh(2h) \cdot e^{K} + 2e^{-K}$$

high B / low T $\Rightarrow$ State of min E dominate
$\rightarrow$ spins align w/ B.

$\beta \to 0$ / high T $\Rightarrow$ equal weights $\rightarrow$ spins fluctuate!

To check this, look at "average magnetization"

$$M = \frac{s_1 + s_2}{2}$$

$$\hookrightarrow \langle M \rangle = \frac{1}{Z} \left\{ \sum_{s_1, s_2} \frac{1}{2}(s_1 + s_2) e^{K s_1 s_2 + h(s_1 + s_2)} \right\}$$

$$= \frac{1}{2Z} \frac{\partial Z(K,h)}{\partial h}$$

$$\rightarrow \langle M \rangle = \frac{1}{2} \frac{\partial \ln Z(K,h)}{\partial h}$$

Recall that free energy F is related to Z via

$$Z = e^{-\beta F}$$

$$\Rightarrow \langle M \rangle = \frac{1}{2} \frac{\partial[-\beta F(K,h)]}{\partial h}$$

With this, $-\beta F = \ln\left[2\cosh(2h) \cdot e^{K} + 2e^{-K}\right]$

so

$$\langle M \rangle = \frac{\sinh(2h)}{\cosh(2h) + e^{-2K}}$$

note that $\langle M \rangle \to 1$ as $h, K \to \infty$

$\langle M \rangle \to h$ as $K \to 0$

(as expected)

Suppose we want thermal average of a particular spin s_1. $\rightarrow$ need to add "source term" $\underline{h_1 s_1 + s_2 h_2}$ rather than $h(s_1+s_2)$.

$$\hookrightarrow \quad Z = \sum_{s_1, s_2} e^{K s_1 s_2 + h_1 s_1 + s_2 h_2} = e^{-\beta F(K, h_1, h_2)}$$

Can check that $\langle s_i \rangle = \frac{1}{Z}\frac{\partial Z}{\partial h_i} = \frac{\partial \ln Z}{\partial h_i} = \frac{\partial}{\partial h_i}\frac{(-\beta F(K, h_1, h_2))}{1}$

So, $$\frac{\partial^2}{\partial h_1 \partial h_2}\left\{-\beta F(K, h_1, h_2)\right\} = \frac{\partial}{\partial h_1}\left(\frac{1}{Z}\frac{\partial Z}{\partial h_2}\right)$$

$$= \frac{1}{Z}\frac{\partial^2 Z}{\partial h_1 \partial h_2} - \frac{1}{Z^2}\frac{\partial Z}{\partial h_1}\frac{\partial Z}{\partial h_2}$$

$$= \langle s_1 s_2 \rangle - \langle s_1 \rangle \langle s_2 \rangle$$

Let $$\boxed{\langle s_1 s_2 \rangle_c = \langle s_1 s_2 \rangle - \langle s_1 \rangle \langle s_2 \rangle}$$

correlation function

connected correlation function

What if h is uniform or zero? We would evaluate the derivative, then set $h_i = h$ or 0 $\forall i$.

Ex for uniform h...

$$\langle s_1 s_2 \rangle_c = \langle s_1 s_2 \rangle - \langle s_1 \rangle \langle s_2 \rangle = \left.\frac{\partial^2 \left[-\beta F(K, h_1, h_2)\right]}{\partial h_1 \partial h_2}\right|_{h_i = h \, \forall i}$$

//

High derivatives of $[-\beta F]$ gives fluctuations. Ex,

Magnetic susceptibility (χ) is the rate of change of the average magnetization with the applied field.

$$\rightarrow \boxed{\chi = \frac{1}{2}\frac{\partial \langle M \rangle}{dh} = \frac{1}{4}\frac{\partial^2 [-\beta F]}{dh^2} = \langle M^2 \rangle - \langle M \rangle^2}$$

"2" for 2 spins, which is the situation we're considering.

For N spins, we have (which we'll use later).

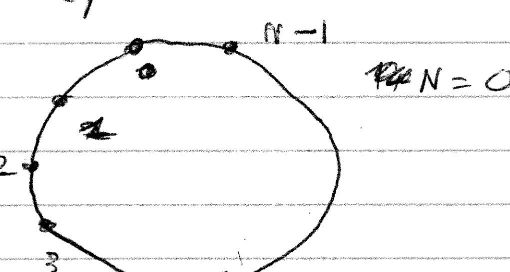
$$\boxed{\chi = \frac{1}{N}\frac{\partial \langle M \rangle}{dh} = \frac{1}{N^2}\frac{\partial^2 [-\beta F]}{dh^2} = \langle M^2 \rangle - \langle M \rangle^2}$$

(b) Ising model with $d = 1$

$\rightarrow$ 2 types of boundary conditions $<$ open / periodic.

(open)

(periodic)



We eventually want to take the $\boxed{\text{thermodynamic limit } N \to \infty}$

The model is of "nearest neighbor type"

$$E = -J \sum_{i=0}^{N-1} s_i s_{i+1}$$ where we have set $B = 0$ ($= h$)

From this, we get Z:

$$Z = \sum_{s_i = \pm 1} \exp \left\{ \sum_{i=0}^{N-1} K (s_i s_{i+1} - 1) \right\}$$

where $K = \beta J > 0$. We note that this term, which is spin-independent $(-K)$ is added to every site for convenience. This just shifts βF by NK.

Now, fix s_0 & define a relative spin variable.

$$t_i = s_i s_{i+1}.$$

With s_0 & $\{t_i\}$, we can construct the entire system.

$$Z = \sum_{t_i = \pm 1} \exp \left\{ \sum_{i=0}^{N-1} K (t_i - 1) \right\} = \sum_{t_i = \pm} \prod_{i=0}^{N-1} e^{K(t_i - 1)}$$

with which we find $\boxed{Z = \underbrace{2}_{\text{2 choices of } s_0} (1 + e^{-2K})^N}$

(!) (the math takes some time to work out...)

Now, we want to look at the free energy per site in the thermodynamic limit $N \to \infty$.

$$f(K) = - \lim_{N \to \infty} \frac{1}{N} \ln Z$$ (recall $Z = e^{-\beta F}$)

we see that $f(K) \sim -\ln(1+e^{-2K})$ upon drop pi[ng] the $\ln 2/N$ in $N\to\infty$

#

<u>What about correlation function?</u> (sites i & $j > i$)

$$\langle s_j s_i \rangle = \frac{1}{Z}\sum_{s_k}\left\{\exp\left[\sum_k K(s_k s_{k+1}-1)\right]\cdot s_j s_i\right\}$$

→ measure how likely s_i, s_j are on average point in the same direction.

well...

$$s_j s_i = s_i s_j = s_i \overbrace{s_{i+1} s_{i+1}}^{1} \cdots s_{j-1} s_j$$

$$= t_i t_{i+1} \cdots t_{j-1}, \text{ by the fact that } s_i^2 = 1$$

So,

$$\boxed{\langle s_j s_i \rangle = \langle t_i \rangle \langle t_{i+1} \rangle \cdots \langle t_{j-1} \rangle}$$

why?

~~$\langle s_j s_i \rangle = \frac{1}{Z}\left\{\sum_k t_i t_{i+1} \cdots t_{j-1} \exp\left[\sum_k K(t_k - 1)\right]\right\}$~~

~~$\langle t_i \rangle = \frac{1}{Z}\left[\sum_{t_k=\pm 1} t_i \exp\left\{\sum_k K(t_k - 1)\right\}\right]$~~

~~So note that there are $(j-i)$ terms in $\langle t_i \rangle \cdots \langle t_{j-1} \rangle$~~

<u>Why?</u>

Note that with this we have ...

$$\langle s_j s_i \rangle = \frac{1}{Z} \sum_{s_k} s_j s_i \exp\left\{ \Sigma_k K(s_k s_{k+1} - 1) \right\}$$

$$= \frac{1}{Z} \sum_{t_k} t_i \dots t_{j-1} \exp\left\{ \Sigma_k K(t_k - 1) \right\}$$

This sum is completely factored. The sums ~~over~~ that don't have t_n multiplying the exponential cancel the cosh in $Z = 2(1+e^{-2K})^N$. So basically after carefully working this out we get

$$\langle s_j s_i \rangle = \underbrace{\langle t_i \rangle \langle t_{i+1} \rangle \dots \langle t_{j-1} \rangle}_{} \quad j-i \text{ terms}$$

Now, the average for each t is easy:

$$\langle t \rangle = \frac{1}{Z} \sum_{t=-1}^{+1} t \exp\left(\Sigma_k K(t_k - 1) \right) \qquad t = -1, +1$$

$$= \frac{1 e^{-0 \cdot K} - 1 e^{-2K}}{e^{0 \cdot K} + e^{-2K}} = \tanh K.$$

So $\langle s_j s_i \rangle = (\tanh K)^{j-i} = \exp\left[(j-i) \ln \tanh K \right]$

Now, choose $i > j$, we see that in general,

$$\langle s_j s_i \rangle = (\tanh K)^{|j-i|} = \exp\left\{ |j-i| \ln \tanh K \right\}$$

At finite K, since $\tanh K < 1$

$$\boxed{\langle s_j s_i \rangle \to 0 \text{ as } |j-i| \to \infty \text{ exponentially}}$$

Note: the $s_k s_{k+1} = t_k$ transform is called the "duality transformation"

Note that $\langle s_j s_i \rangle$ only depends on the difference $|j-i|$

$\Rightarrow$ this is called $\boxed{\text{translational invariance}}$

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(c) The Monte Carlo method

When $d > 1$, we can't do these calculations exactly

To do this, we need Monte Carlo...

$$\langle s_i s_j \rangle = \frac{\Sigma_c \, s_i s_j \, e^{-E(c)/kT}}{\Sigma_c \, e^{-E(c)/kT}} = \Sigma_c \, s_i s_j(c) \, P(c)$$

configuration energy probability

we won't worry about details here, but the point is that we make the computer generate a bunch of configurations c, and carry out the calculation.

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(3) FROM STATISTICAL MECH TO QUANTUM MECH

Feb 12, 2021 We'll go through the same material, but through a different approach...

(a) Real-Time QM (as opposed to Euclidean QM)
(imaginary time)

SE : $\boxed{i\hbar \frac{d}{dt}|\psi\rangle = H|\psi(t)\rangle}$

propagator.

if H time independent then $|\psi(t)\rangle = \mathcal{U}(t)|\psi(0)\rangle$

$$\rightarrow \boxed{i\hbar \frac{d}{dt} U(t) = H U(t)}$$

Formal solution for $\mathcal{U}(t)$ is $U(t) = e^{-iHt/\hbar}$

H is self-adjoint $\Rightarrow$ $\mathcal{U}$ is unitary : $\underline{\mathcal{U}^\dagger \mathcal{U} = I}$.

Spectral decomp: $\mathcal{U}(t) = \sum_n |n\rangle\langle n| e^{-iE_n t/\hbar}$

where

$$H|n\rangle = E_n|n\rangle$$

matrix elements: $\mathcal{U}(x', x, t)$ in $|x\rangle$ basis...

$$\mathcal{U}(x', x, t) = \langle x'|\mathcal{U}(t)|x\rangle$$

$$= \sum_n \psi_n(x')\psi_n^*(x) e^{-iE_n t/\hbar}$$

where $\psi_n(x) = \langle x|n\rangle$

Free particle fcn stuff... I won't worry about this since already did in PH431.

Now, what we just looked at is the Schrödinger picture

↳ Can also look at Heisenberg picture
→ operators have time dependence.

A Heisenberg operator $\Omega(t)$ is related to Schrö op. Ω by

$$\boxed{\Omega(t) = \mathcal{U}^{\dagger}(t)\,\Omega\,\mathcal{U}(t)}$$

from which we define the <u>time-ordered Green's fn</u>

$$\boxed{iG(t) = \langle 0|\,\mathcal{T}(\Omega(t)\Omega(0))\,|0\rangle}$$

ground state of H

time-order symbol

$\mathcal{T}$ is defined like this...

$$\mathcal{T}(\Omega(t_1)\Omega(t_2)) = \theta(t_2-t_1)\,\Omega(t_2)\,\Omega(t_1) + \theta(t_1-t_2)\,\Omega(t_1)\,\Omega(t_2)$$

Heaviside step fn.

Now, in condensed matter, we want a generalization of the above ground state expectation value ~~to~~ to one at finite temp β when the system can be in any state of energy E_n with Boltzman weight:

$$\boxed{\begin{aligned} iG(t,\beta) &= \frac{\sum_n e^{-\beta E_n} \langle n | \mathcal{T}(\Omega(t)\Omega(0)) | n \rangle}{\sum_n e^{-\beta E_n}} \\ &= \frac{\mathrm{Tr}\left(e^{-\beta H}\mathcal{T}(\Omega(t)\Omega(0))\right)}{\mathrm{Tr}(e^{-\beta H})} \end{aligned}}$$

Sometimes we need to compute the "retarded Green's fn" to calculate responses to external probes...

$$\boxed{iG_R(t,\beta) = \frac{\mathrm{Tr}\left[e^{-\beta H}\theta(t)[\Omega(t),\Omega(0)]\right]}{\mathrm{Tr}(e^{-\beta H})}}$$

We won't worry to much about this.

(b) Imaginary - time QM

let $t = -i\tau$

then we have SE in img time:

$$\boxed{-\hbar \frac{d}{d\tau}|\Psi(\tau)\rangle = H|\Psi(\tau)\rangle}$$

The propagator $\boxed{\mathcal{U}(t) = e^{-iHt/\hbar} \Rightarrow e^{-H/\hbar\,\tau} = \mathcal{U}(\tau)}$

now $\mathcal{U}(\tau)$ is Hermitian, not unitary.

and in energy eigenbasis ..

$$\boxed{\mathcal{U}(\tau) = \sum_n |n\rangle\langle n|\, e^{-E_n\tau/\hbar}} \quad \text{w/ } H|n\rangle = E_n|n\rangle$$

Now, since $\mathcal{U}(\tau)$ is Hermitian & not unitary (oscillatory), $\mathcal{U}(\tau)$ kills all but its ground state projector...

$$\rightarrow \lim_{\tau\to\infty} \mathcal{U}(\tau)\,|\Psi(0)\rangle \longrightarrow |0\rangle\langle 0|\Psi(0)\rangle\, e^{-E_0\tau/\hbar}$$

☒ what about Heisenberg operators?

Recall that $\Omega(t) = \mathcal{U}^\dagger(t)\,\Omega\,\mathcal{U}(t)$.

In τ, we have...

$$\boxed{\Omega(\tau) = e^{H/\hbar\,\tau}\,\Omega\,e^{-H\tau/\hbar}}$$

now note that in t: $\Omega^\dagger(t)$ & $\Omega(t)$ are adjoints

i.e. $(\Omega^\dagger)(t) = [\Omega(t)]^\dagger$

__But__

$$\Omega^\dagger(\tau) = e^{H/\hbar\,\tau}\,\Omega^\dagger\,e^{-H\tau/\hbar} \neq [\Omega(\tau)]^\dagger = e^{-H\tau/\hbar}\,\Omega^\dagger\,e^{H\tau/\hbar}$$

Now, time-ordered product...

$$\mathcal{T}[\Omega(\tau_2)\Omega(\tau_1)] = \theta(\tau_2-\tau_1)\Omega(\tau_2)\Omega(\tau_1) + \theta(\tau_1-\tau_2)\Omega(\tau_1)\Omega(\tau_2)$$

Same def as before, with exp value

$$\boxed{G(\tau_2-\tau_1) = -\langle 0|\mathcal{T}(\Omega(\tau_2)\Omega(\tau_1))|0\rangle \quad \text{in gnd state.}}$$

In all state @ Boltzmann weight:

$$\boxed{G(\tau_2-\tau_1,\beta) = -\frac{\mathrm{Tr}\{e^{-\beta H}\mathcal{T}(\Omega(\tau_2)\Omega(\tau_1))\}}{\mathrm{Tr}(e^{-\beta H})}}.$$

(c) The TRANSFER MATRIX

Now, let us come back to the $d=1$ Ising model and look at what QM comes out of it.

Recall the partition fn (without the duality trs fn)

$$\boxed{Z = \sum_{s_i}\prod_i e^{K(s_is_{i+1}-1)}}$$

now look at each

$$e^{K(s_is_{i+1}-1)} \longrightarrow \text{2 indices, 2 values of } s_i \rightarrow \text{4 total.}$$

Define $T_{s's} = \exp[K(s's-1)]$ $\longrightarrow$ elements of the 2×2 transfer matrix.

where
$$\boxed{\begin{array}{l} T_{++} = T_{--} = 1 \\ T_{+-} = T_{-+} = e^{-2k} \end{array} \Rightarrow T = \begin{pmatrix} 1 & e^{-2k} \\ e^{-2k} & 1 \end{pmatrix} = I + e^{-2k}\sigma_1}$$

σ_x

note that T is Real - Hermitian.

With this can rewrite Z:

$$Z = \sum_{s_i,\, i=1,\dots,N-1}' T_{s_N s_{N-1}} T_{s_{N-1} s_{N-2}} \cdots T_{s_2 s_1} T_{s_1 s_0}$$

But note that following thing... when multiplying matrices A & B,

$$(AB)_{ij} = \sum_{\cancel{k}} A_{ij} \cancel{B}_{jk}$$

$$\Rightarrow Z = \sum_{s_i,\, i=1,\dots,N-1} T_{s_N s_{N-1}} \cdots T_{s_2 s_1} T_{s_1 s_0}$$

$$= \sum_{s_1=-1}^{1} \sum_{s_2=-1}^{1} \cdots \sum_{s_{N-1}=-1}^{1} T_{s_N s_{N-1}} \cdots T_{s_2 s_1} T_{s_1 s_0}$$

Isolate $\displaystyle\sum_{s_2=-1}^{1} T_{s_3 s_2} T_{s_2 s_1} = (T^2)_{s_1 s_3} \dots$

If we keep "collapsing" like this, we get

$$\boxed{Z = \langle s_N | T^N | s_0 \rangle}$$ (for fixed boundary condition)

If we sum over the first & last spins to get free boundary conditions, we get

$$Z = \sum_{s_0} \sum_{s_N} \langle s_N | T^N | s_0 \rangle$$

Now, if we're looking at periodic BC, $s_N = s_0$, and so

$$\boxed{Z = \sum_{s_0} \langle s_0 | T^N | s_0 \rangle = \mathrm{Tr}(T^N)}$$

$K = \beta J$

where recall that $T = I + e^{-2K}\sigma_x$

With this, we can show that the free energy per site is insensitive to boundary conditions as $N \to \infty$.

Recall the formula: $f(K) = -\lim_{N\to\infty} \frac{1}{N} \ln Z$.

Let us write $T = \lambda_0 |0\rangle\langle 0| + \lambda_1 |1\rangle\langle 1|$, by spectral decomp.

$$\Longrightarrow \quad T^N = \lambda_0^N |0\rangle\langle 0| + \lambda_1^N |1\rangle\langle 1|$$

Now, use thm "Perron - Frobenius" associated

[A sqr matrix with positive entries will have a non-degenerate largest eigv & eigvector with strictly positive components]

$\Rightarrow$ Assume that T is such a matrix, for finite K.

$\therefore$ that $\lambda_0 > \lambda_1$. Then we have

$$\lim_{N\to\infty} T^N \simeq \lambda_0^N \left[|0\rangle\langle 0| + \mathcal{O}\left(\frac{\lambda_1}{\lambda_0}\right)^N \right]$$

vanishes since $\lambda_1 < \lambda_0$.

and so

$$Z = \langle s_N | T^N | s_0 \rangle \simeq \langle s_N | 0 \rangle \langle 0 | s_0 \rangle \lambda_0^N \left(1 + \cancelto{0}{\mathcal{O}\left(\lambda_1/\lambda_0\right)^N}\right)$$

and so

$$-f = \lim_{N\to\infty} \frac{1}{N} \ln Z$$

$$= \lim_{N\to\infty} \left[\frac{1}{N} \ln \left\{ \langle s_N|0\rangle\langle 0|s_0\rangle \lambda_0^N \left(1 + \mathcal{O}(\dots)^N\right) \right\} \right]$$

$$\simeq \lim_{N\to\infty} \left[\ln \lambda_0 + \frac{1}{N} \ln \left\{ \langle s_N|0\rangle\langle 0|s_0\rangle \right\} + \dots \right]$$

$$\to \ln \lambda_0 \quad \text{so} \quad \boxed{f \to -\ln \lambda_0} \quad (\text{OBC})$$

which is independent of the bd spins so long as

$\langle 0|s_0\rangle, \langle s_N|0\rangle \neq 0$., which is assured by the Perron-Frob thm which states that all components of the dominant eigvector are positive.

This can be done for periodic BC as well.

If $Z = \mathrm{Tr}(T^N) \Rightarrow f \to -\ln \lambda_0$ as before

(PBC)

Next, we want to compute the correlation func using this formulation. How do we do this?

Feb 15, 2021

To do this, we need to work out some identities

First, look at $T = I + e^{-2K}\sigma_1$ — σ_x

and

$$e^{K^*\sigma_1} = (\cosh K^*)I + (\sinh K^*)\sigma_1$$

$$= \cosh K^*(I + \tanh K^*)\sigma_1$$

$\tanh K = e^{-2K}$

now choose K^* such that $\boxed{\tanh K^* = e^{-2K}}$ then

$K^*(K)$

$$T = I + e^{-2K}\sigma_1 =$$

$$= I + (\tanh K^*)\sigma_1 = \boxed{\frac{e^{K^*\sigma_1}}{\cosh K^*} = T}$$

now, $\cosh K^*$ will often get dropped out of averages... and note that $K^* = K^*(K)$.

↳ $\boxed{K^* \text{ is the "dual" of } K}$

- Next, look at eig(T)...

$$\lambda_0 = e^{K^*} ; \quad \lambda_1 = e^{-K^*}, \quad \frac{\lambda_1}{\lambda_0} = e^{-2K^*}$$

and

$$|0\rangle, |1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$$

Now, consider $\langle s_j s_i \rangle$ where $j > i$. Assume that s_0, s_N are fixed. What is $\langle s_j s_i \rangle$?

→ Claim $\boxed{\langle s_j s_i \rangle = \frac{\langle s_N | T^{N-j} \sigma_3 T^{j-i} \sigma_3 T^i | s_0 \rangle}{\langle s_N | T^N | s_0 \rangle}}$

Pf: don't care about the denominator, b/c it's just Z.

→ look at numerator, which looks like...

$\langle s_N | T^N s_j s_i | s_0 \rangle$

now, insert

$I = (|+\rangle\langle+| + |-\rangle\langle-|)$

identity

eigenbasis of σ_3

(since we call $\{|0\rangle, |1\rangle\}$ basis for σ_1)

With this we get

$\langle s_N | T\, I\, T \ldots I\, T\, s_j s_i | s_0 \rangle$, which $= \langle s_N | T^{N-j} \sigma_3 T^{j-i} \sigma_3 T^i | s_0 \rangle$

Why? ~~Now~~, Reading from right to left, we just get sum over Boltzmann weight until we get to site i

$$\left\{ \begin{array}{l} \sigma_3 |s_i\rangle = s_i |s_i\rangle \\ T|s_0\rangle = |s_1\rangle, \text{ and so on,} \end{array} \right.$$

$\Rightarrow \sigma_3 T |s_{i-1}\rangle = \sigma_3 |s_i\rangle = s_i |s_i\rangle$

⤳ pull out a factor of s_i ✓.

And keep going for site j as well

$\rightarrow$ get $\langle s_N | T^{N-j} \sigma_3 T^{j-i} \sigma_3 T^i | s_0 \rangle$
$= \langle s_N | T^N s_j s_i | s_0 \rangle$. ✓

So,

$$\langle s_j s_i \rangle = \frac{\langle s_N | T^{N-j} \sigma_3 T^{j-i} \sigma_3 T^i | s_0 \rangle}{\langle s_N | T^N | s_0 \rangle}$$

But we can write this differently...

Define the "Heisenberg Operators" by

$$\boxed{\sigma_3(n) = T^{-n} \sigma_3 T^n}$$

{ the site index n plays the role of discrete integer-value time, and T is the time-evolution operator / propagator for one unit of Euclidean (imag) time

With this,

$$\boxed{\langle s_j s_i \rangle = \frac{\langle s_N | T^N \sigma_3(j) \sigma_3(i) | s_0 \rangle}{\langle s_N | T^N | s_0 \rangle}}$$

note that this kinda makes sense.

$\rightarrow$ Now, consider $N \to \infty$ & look at sites $j > i$ where j, i are far from the end points (0 & N).

Then we can approximate $T^\alpha \simeq \lambda_0^\alpha |0\rangle\langle 0|$, $\left(\alpha = N, N-j, i \right)$

• In this limit, we have (from the defn)

$$\langle s_j s_i \rangle = \frac{\langle s_N | T^{N-j} \sigma_3 T^{j-i} \sigma_3 T^{i} | s_0 \rangle}{\langle s_N | T^N | s_0 \rangle}$$

$$= \frac{\langle s_N | \lambda_0^{N-j} |0\rangle\langle 0| \sigma_3 T^{j-i} \sigma_3 \lambda_0^{i} |0\rangle\langle 0| s_0 \rangle}{\langle s_N | \underset{\substack{\downarrow \\ \lambda_0^N |0\rangle\langle 0|}}{T^N} | s_0 \rangle}$$

projector to $|0\rangle$

projector to $|0\rangle$

$$\text{(formally)} \cong \frac{\langle s_N | \lambda_0^N |0\rangle\langle 0| T^{-j} \sigma_3 T^{j} T^{-i} \sigma_3 T^{i} \underbrace{\cancel{|0\rangle\langle 0|}}_{|0\rangle\langle 0|} s_0 \rangle}{\langle s_N | \lambda_0^N |0\rangle\langle 0| s_0 \rangle}$$

$$= \langle 0 | \sigma_3(j) \sigma_3(i) | 0 \rangle$$

So, $\boxed{\langle s_j s_i \rangle = \langle 0 | \sigma_3(j) \sigma_3(i) | 0 \rangle}$

and dependence on the boundary dropped out.

Now, if $i > j$, we'll get $\sigma_3(i)\sigma_3(j)$, so, more precisely ...

$$\boxed{\langle s_j s_i \rangle = \langle 0 | T \{ \sigma_3(j) \sigma_3(i) \} | 0 \rangle}$$

as before:

$$T[\sigma_3(j)\sigma_3(i)] = \theta(j-i)\,\sigma_3(j)\sigma_3(i) + \theta(i-j)\,\sigma_3(i)\sigma_3(j)$$

Note: we're associating i & j with time! Take notice of this...

→ Now, let us evaluate the expression

$$\langle s_j s_i \rangle = \langle 0 | \sigma_3(j) \sigma_3(i) | 0 \rangle$$

eigv of σ_1
$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

in terms of the eigenvectors of T.

- Before this, let's look at the <u>mean magnetization</u>

$$\langle s_i \rangle = \langle \cancel{s_j} s_i \rangle = \langle 0 | I \cdot \sigma_3(i) | 0 \rangle$$

$$= \langle 0 | \sigma_3 | 0 \rangle \quad \text{(check this!)}$$

by ↑ approx T

$$\langle s \rangle = \langle s \rangle$$

⇒ $\boxed{\langle s_i \rangle = \langle 0 | \sigma_3 | 0 \rangle}$ ⇒ independent of i so long as i is <u>far from the ends</u>.

(long $i \ll N$, $i \gg 0$)

→ This of course fails @ 0 temp ($\Rightarrow K^* = 0$, at which $T = I$ → Perron-Frobenius thm no longer holds.

#

→ Now, back to evaluating $\langle s_j s_i \rangle = \langle 0 | \sigma_3(j) \sigma_3(i) | 0 \rangle$.

We note that even though it appears that $\langle s_j s_i \rangle$ only depends on the gnd state, we in fact have to know the full state even in the $N = \infty$ limit.

In any case, look at $j > i$ and insert the complete set of eigv of T (which is also $\{|0\rangle, |1\rangle\}$ between $\sigma_3(j)$ & $\sigma_3(i)$.

...

$$\langle s_j s_i \rangle = \langle 0 | \sigma_3(j) \{ |0\rangle\langle 0| + |1\rangle\langle 1| \} \sigma_3(i) |0\rangle$$

$$= \underbrace{\langle 0 | \sigma_3(j) | 0 \rangle}\underbrace{\langle 0 | \sigma_3(i) | 0 \rangle}$$

$\langle 0|\sigma_3|0\rangle = 0$

$i = j = 0$

$$+ \langle 0 | \sigma_3(j) | 1 \rangle \langle 1 | \sigma_3(i) | 0 \rangle$$

This is actually zero but we'll keep it here

$$= \langle s \rangle^2 + \langle 0 | \sigma_3(j) | 1 \rangle \langle 1 | \sigma_3(i) | 0 \rangle$$

$$= \langle s \rangle^2 + \langle 0 | T^{-j} \sigma_3(0) T^{j-i} | 1 \rangle \langle 1 | \sigma_3(0) T^{i} | 0 \rangle$$

$$= \langle s \rangle^2 + \left(\frac{\lambda_1}{\lambda_0}\right)^{j-i} | \langle 0 | \sigma_3 | 1 \rangle |^2$$

$$\Rightarrow \langle s_j s_i \rangle_c = \langle s_j s_i \rangle - \langle s \rangle^2$$

connected corr. fn.

$$= \left(\frac{\lambda_1}{\lambda_0}\right)^{j-i} | \langle 0 | \sigma_3 | 1 \rangle |^2$$

$$= e^{-2K^*(j-i)} | \langle 0 | \sigma_3 | 1 \rangle |^2$$

$i < j$ or $j < i$ → we'll find same thing

$$\Rightarrow \boxed{\langle s_j s_i \rangle_c = [\tanh K]^{|j-i|} \, | \langle 0 | \sigma_3 | 1 \rangle |^2}$$

replace $j-i$ by $|j-i|$, without loss of generality

So, in general, $\langle s_j s_i \rangle_c = [\tanh K]^{|j-i|} | \langle 0 | \sigma_3 | 1 \rangle |^2$

$$\rightarrow \boxed{\text{as } |j-i| \to \infty, \quad \langle s_j s_i \rangle \to \langle s \rangle^2}$$

With this, we can define the $\boxed{\text{correlation length}}$ (ξ)

For general models, not necessarily of Ising spins or $d=1$, we'll find that

$$\lim_{|j-i|\to\infty} \langle s_j s_i \rangle_c = \frac{e^{-|j-i|/\xi}}{|j-i|^{d-2+\eta}}$$

correlation length

some number

distance between 2 spins

The exponential decay dominates the power law asymptotically

$\rightarrow$ can extract ξ from the expression above to find

$$\xi^{-1} = \lim_{|j-i|\to\infty}\left[\frac{-\ln\langle s_j s_i\rangle_c}{|j-i|}\right]$$

Can check this:

$$\ln\langle s_j s_i\rangle_c = -\frac{|j-i|}{\xi} - \ln\left[|j-i|^{d-2+\eta}\right]$$

As $|j-i|\to\infty \rightarrow |j-i| \gg \ln|j-i|$

So we have

$$-\xi = \lim_{|j-i|\to\infty}\left\{\frac{-\ln\langle s_j s_i\rangle_c}{|j-i|}\right\}$$

Now, $\langle s \rangle = \langle 0 | \sigma_3 | 0 \rangle = \langle 0 | 1 \rangle = 0$

So, $\langle s_j s_i \rangle_c = \langle s_j s_i \rangle = \exp\{ |j-i| \tanh K \}$

So,

$$\xi^{-1} = \lim_{|j-i| \to \infty} \left\{ \frac{-\ln \langle s_j s_i \rangle_c}{|j-i|} \right\}$$

$$= -\ln \tanh K = \boxed{2K^* = \xi^{-1}}$$

→ Why is the correlation length ξ defined in terms of $\langle s_j s_i \rangle_c$?

↳ To see this, we turn on the magnetic field → $h > 0$ ⇒ $\langle s \rangle \neq 0$ (mean magnetization no longer zero).

I would still have $\lambda_1 > \lambda_0$, and the same argument would still show that ξ is determined by $\langle s_j s_i \rangle_c$.

→ So, what does $\langle s_j s_i \rangle$ tell us?

Well... $\langle s_j s_i \rangle$ is the Exp Val of product of 2 spins. If they fluctuate independently, then $\langle s_j s_i \rangle$ is as likely to be -1 or 1 ⇒ $\langle s_j s_i \rangle = 0$.

→ But they aren't independent! Why?

① $h > 0$ ⇒ spins tend to align. (even @ $K = 0$)
Even when $K = 0$, at which the probability distr for s_i, s_j factorizes into 2 indep dist
→ $\langle s_j s_i \rangle = \langle s \rangle^2$, independent of $|j-i|$

(2) Now with $K>0$, the aligned spins will generate via the K term additional internal field parallel to h, which enhances $\langle s \rangle$.

⇒ We are looking for additional correlations in the "fluctuations on top of the average $\langle s \rangle$.

→ Can define new correlation fn.

$$\langle s_j s_i \rangle_{new} = \Big\langle \big(s_j - \langle s \rangle \big) \big(s_i - \langle s \rangle \big) \Big\rangle$$

$$= \langle s_j s_i \rangle - \cancel{\langle s_j \rangle \langle s \rangle - \langle s_j \rangle \langle s \rangle + \langle s \rangle^2}$$

$$= \langle s_j s_i \rangle_c \quad \checkmark$$

which is what we defined earlier. Note too that we're using translational invariance in the cancellation.

⇒ In general, we will have $\boxed{\text{"clustering"}}$, of which the Ising model is an example.

↳ [Clustering]: when the connected correlation between 2 vars A & B die asymptotically.

i.e.

$$\lim_{|j-i|\to\infty} \langle A_j B_i \rangle = \langle A_j \rangle \langle B_i \rangle$$

→ joint pdf must asymptotically factorize.

→ This will come up again when we talk about phase transitions.

(i) The Hamiltonian

T : the transfer matrix, plays the role of the time-evolution operator (in img time), when we define the Heisenberg ops.

$$\sigma_3(j) = T^{-j}\sigma_3 T^{j}$$

Now... if we identify T ... with ... $\mathcal{U}(\tau) = e^{-H\tau}$
then what is τ? $\rightarrow$ One step in the discrete time lattice.
Well... now, we introduce a Hamiltonian eg

$$\boxed{T = e^{-H}}$$

real-Hermitian

This H is dim-less because unit of time is unity.

Note that T is real, Hermitian $\Rightarrow$ symmetric

$\rightarrow$ H also real, Herm $\Rightarrow$ symmetric.

$\Rightarrow$ T, H ~~are~~ share eigenbasis. $\{|0\rangle, |1\rangle\}$

We have ...

$$T = e^{K^*\sigma_1} \qquad H = -K^*\sigma_1$$

In summary,

$|0\rangle$: $T|0\rangle = \lambda_0|0\rangle = e^{K^*}|0\rangle$
$H|0\rangle = E_0|0\rangle = -K^*|0\rangle$

$f = -\ln\lambda_0$
$= -K^* = E_0$

$|1\rangle$: $T|1\rangle = \lambda_1|1\rangle = e^{-K^*}|1\rangle$
$H|1\rangle = E_1|1\rangle = K^*|1\rangle$

A few remarks on generality!

1. $\langle s_j s_i \rangle_c$ depends only on the ratio λ_1/λ_0, which are of eigv of T, and falls exp-ly with distance with coef $2K^{*}$.

   Now, $2K^* = K^* - (-K^*)$, which is the gap to the 1st excited state of H where $T = e^{-H}$.

   $$\Rightarrow \boxed{\xi^{-1} = E_1 - E_0 = m} \text{ is very general}$$

   "mass gap"

2. $\langle s_j s_i \rangle_c$ also depends on $|\langle 0 | \sigma_3 | 1 \rangle|^2$. $(\lambda_0 > \lambda_1)$

   gnd state

   next exc state.

   $\Rightarrow$ This is also a general feature.
   If $\langle 0 | \sigma_3 | 1 \rangle = 0$, then we must go up till we find a state that is connected to the gnd state by σ_3.

   Here, $\langle 0 | \sigma_3 | 1 \rangle = \langle 0 | 0 \rangle = 1 \neq 0$.

If T is bigger than 2×2, the sum over states will have more than 2 terms,

$\rightarrow$ $\langle s_j s_i \rangle_c$ will be Σ of decaying exps, and a unique ξ will emerge only asymptotically when the small mass gap dominates!

(ii) Turning on h

Now, add the magnetic field back into the Hamiltonian

-1 ~~$H = \Sigma$~~ $\to$ add term $h \sum_i s_i$.

What is the transfer matrix now?

$\to$ Suppose we define $T = e^{K^* \sigma_1} e^{h\sigma_3} \equiv T_K T_h$

then T reproduces Boltzmann weight, but is not symmetric (Hermitian).

$\to$ To fix, write

$$h \sum_i s_i = \frac{1}{2} h \sum_i (s_i + s_{i+1})$$

to get
$$\boxed{T = T_h^{1/2}\, T_K\, T_h^{1/2}}$$

which is symmetric. (Recall that T's defined via its elements $T_{\pm\pm}$, $T_{\mp\pm}$)

Define $T = e^{-H}$. Before, when $h=0$, $H = -K^* \sigma_1$. But here, ~~now~~ we can't find an explicit formula for H...

Ex The eigenvalues of T are: $T = \begin{pmatrix} e^{h} & e^{-2K} \\ e^{-2K} & e^{h} \end{pmatrix}$

$$\text{eigv}(T) = \frac{1}{2} e^{-h-2K} \left(e^{2K} + e^{2h+2K} \mp \sqrt{4e^{2h} + e^{4K} - 2e^{2h+4K} + e^{4h+4K}} \right)$$

Calculate magnetisation: recall that mean magnetisation $\equiv$ magnetisation per site, which is related to the free energy...

Recall that $M = \frac{-\partial F}{\partial h} \Rightarrow \langle s \rangle = \frac{M}{N} = \frac{-1}{N}\frac{\partial F}{\partial h}$

$= \begin{pmatrix} e^{h} & e^{-2k} \\ e^{-2k} & e^{-h} \end{pmatrix}$

where $\frac{F}{N} = f = -\ln \lambda_0 \longrightarrow$ larger eigv.

$$= -\ln \left\{ \frac{1}{2} e^{-2h+2K}(-1+e^{2h}) \times \left[e^{2K}(-1+e^{2h}) + \sqrt{4e^{2h} + e^{4K}(-1+e^{2h})^2} \right] \right\}$$

$$\Rightarrow M = \langle s \rangle = \frac{-1}{N}\frac{\partial F}{\partial h} = -\frac{\partial f}{\partial h} = \frac{e^{2K}(-1+e^{2h})}{\sqrt{4e^{2h} + e^{4K}(-1+e^{2h})^2}}$$

$=$ (simplification)

$$\boxed{\langle s \rangle = \frac{\sinh(h)}{\sqrt{e^{-4K} + \sinh^2(h)}}}$$

note that when $h = 0$, $\langle s_i \rangle = 0$ as expected

$\boxed{?}$ Can we evaluate $\langle s \rangle \overset{?}{=} \langle s_j \rangle$ by definition?

NO, because we no longer have translational invariance! $\langle s \rangle \neq \langle s_j \rangle$

$\hookrightarrow$ We have to use $\frac{\partial f}{\partial h}$ instead. (IMPORTANT)

(d) Classical to Quantum Mapping: Dictionary

(1) Schrödinger operators are σ_3 | Ω

Heisenberg ops are $\sigma_3(j)$ | $\Omega(t)$

in stat mech | before, in QM

(2) Transfer matrix T | Propagator in imaginary time ($\Delta\tau$)

$$T \Longleftrightarrow \mathcal{U}(\Delta\tau)$$

Units: $[\Delta\tau] = 1$.

With this,

$$Z(s_i, s_f) = \langle s_N = s_f | T^N | s_0 = s_i \rangle \Longleftrightarrow \langle s_f | \mathcal{U}(\Delta\tau) | s_i \rangle$$

matrix elements corresponding to propagator $\mathcal{U}$.

(3) Heisenberg - Schrö:

$$\sigma_3(j) = T^{-j}\sigma_3 T^j \Longleftrightarrow \mathcal{U}^{-1}(j\Delta\tau)\,\sigma_3\,\mathcal{U}(j\Delta\tau)$$

$$= \sigma_3(\tau = j\Delta\tau)$$

(4) Hamiltonian: $T = e^{-H} = e^{-H\Delta\tau}$, from which we have that the dominant eigv of T is the gnd state of H. $(|0\rangle)$

So it's important to remember that

$$\boxed{\begin{aligned}|0\rangle &= \text{"excited state" of } T \text{ (dominant eigvect)} \\ &= \text{ground state of the Hamiltonian } H\end{aligned}}$$ (transfer matrix)

(5) Correlation function: in $N \to \infty$ (thermo limit)

= gnd state expectation value of the time-ordered product of the corresponding Heisenberg op:

$$\boxed{\langle s_j s_i \rangle \Longleftrightarrow \langle 0 | \mathcal{T} \{ \sigma_3(j) \sigma_3(i) \} | 0 \rangle}$$

gnd state of H

excited state of T

—4—

IV. QUANTUM TO STATISTICAL MECHANICS

Here we'll go from Imaginary time QM to classical SM.
by retracing the path that led to the transfer matrix

$\Rightarrow$ and map the problem of Quantum Stat Mech.

Central object is the partition fn:

$$\boxed{Z_Q = \mathrm{Tr}\, e^{-\beta H} = \mathrm{Tr}\, e^{-\beta \hbar H/\hbar} = \mathrm{Tr}\, \mathcal{U}(\tau = \beta\hbar)}$$

H : Quantum Hamiltonian

$\hookrightarrow$ Z_Q is just the trace of the imaginary-time evolution operator for a time $\beta\hbar$.

$\hookrightarrow$ We want to study a more general object:

$$\mathcal{U}(f, i; \tau) = \langle f | \mathcal{U}(\tau) | i \rangle$$

(a) From $\mathcal{U}$ to Z

Feb 16, 2021

Begin with the matrix element of the propagator

$$\mathcal{U}(f, i; \tau) = \langle f | \mathcal{U}(\tau) | i \rangle$$

We will set $\hbar \equiv 1$ for now.

(b) Example from Spin 1/2

Consider Hamiltonian

$$\boxed{\mathcal{H} = -B_1 \sigma_1 - B_3 \sigma_3}$$

which describes a spin-$\frac{1}{2}$ in a magnetic field

$$\vec{B} = \hat{i} B_1 + 0\hat{j} + B_3 \hat{k} \quad (\text{in } x\text{-}z \text{ plane})$$

Now factor...

$$\rightarrow \quad \mathcal{U}(\tau) = \left\{ \mathcal{U}(\tau/N) \right\}^N$$

where

$$\mathcal{U}(\tau/N) \equiv T_\varepsilon = e^{-\varepsilon H}, \text{ with } \varepsilon = \tau/N$$

Put in $N-1$ intermediate ~~states~~ sums over σ_3 eigenstates & renaming $|i\rangle$, $|f\rangle$ to $|s_0\rangle$, $|s_N\rangle$, resp., we get

$$\boxed{\mathcal{U}(s_N, s_0) = \sum_{s_i} \prod_{i=0}^{N-1} \langle s_{i+1} | e^{-\varepsilon H} | s_i \rangle}$$

$$\langle s_N | \mathcal{U}(\tau) | s_0 \rangle = \langle s_N | (e^{-\varepsilon H})^N | s_0 \rangle \quad (\text{like before})$$

$$= \sum_{s_i} \prod_{i=0}^{N-1} \langle s_{i+1} | e^{-\varepsilon H} | s_i \rangle$$

So, $\mathcal{U}(s_N, s_0)$ is just the Z that we've seen before (for $N+1$ spins)

$\Rightarrow$ The transfer matrix is defined by the elements:

$$\boxed{T_{s_{i+1} s_i} = \langle s_{i+1} | e^{-\varepsilon H} | s_i \rangle}$$

Now, what are the classical $\{s_i\}$ parameters?

$\Rightarrow$ to find them, we equate the general expression for the Boltzmann weight of the $\{s_i\}$ problem to the matrix elements of the QM problem --

$$e^{K(s's-1) \pm \frac{h}{2}(s'+s)+c} = \langle s'| e^{-\epsilon H} |s\rangle$$

$$= \langle s'| e^{+\epsilon(B_1\sigma_1 + B_3\sigma_3)} |s\rangle$$

Boltzmann weight

QM

$\boxed{?}$ why does the exponent on the LHS have such a form?

$\hookrightarrow$ Pf Let $T_{s's} = e^{R(s',s)}$.

Now, let $R(s',s) = \sum_{i,j} (s')^i (s)^j B_{ij}$

Since $\underline{s^2 = (s')^2 = 1}$ and that $T_{s's}$, $R(s',s)$

must both be symmetric in $s \leftrightarrow s'$,

$R(s',s)$ must have the form $Ass' + B(s'+s) + C$

$\#$

Now, we want to solve for K, h, c in terms of B_1, B_3 & c.

$\Rightarrow$ Choose $s = s' = \pm 1$ & $s = -s'$. Then we can show that

$$\begin{cases} \tanh(h) = \dfrac{B_3}{B} \tanh(\varepsilon B) \\ e^{-2k+c} = \dfrac{B_1}{B} \sinh(\varepsilon B) \\ e^{2c}(1-e^{-4K}) = 1 \end{cases}$$

$\det(e^{\varepsilon \vec{\sigma}\cdot\vec{B}}) = 1$

using the identity $\boxed{e^{\varepsilon\vec{\sigma}\cdot\vec{B}} = \cosh(\varepsilon B) + \frac{\vec{B}\cdot\vec{\sigma}}{B}\sinh(\varepsilon B)}$

Note that the classical parameters K, h, c depend on ε, the time slice ($\varepsilon = \tau/N$)

→ Changing ε will change the params K, h, c, but Z will remain constant.

How do we fix the problem of discrete time?
→ We have to take $N \to \infty$ & $\varepsilon \to 0$.

In Euclidean time QM, the lattice in time is an artifact that must be banished at the end by taking $N \to \infty$ or $\varepsilon = \tau/N \to 0$

More concretely... consider a typical situation where we want to evaluate...

$$G(\tau_2 - \tau_1) = \frac{\mathrm{Tr}\{e^{-H\tau}[\mathcal{T}(\sigma_3(\tau_2)\sigma_3(\tau_1))]\}}{\mathrm{Tr}[e^{-H\tau}]}$$

If we map this to the PBC Ising model, we'll just get

$$\boxed{G(\tau_2-\tau_1) = \langle s_{i_2} s_{i_1} \rangle}$$

where ~~$i_2 i_1$~~ $\tau_1 = i_1 \varepsilon$; $\tau_2 = i_2 \varepsilon$

How do we work in the limit $\varepsilon \to 0$?

↳ In the infinitesimal limit of ε, ($T_\varepsilon = e^{-\varepsilon H}$)

$$\boxed{\mathcal{U}(\tau/N) = \mathcal{U}(\varepsilon) = T_\varepsilon \longrightarrow I - \varepsilon H + O(\varepsilon^2)}$$

So the matrix elements of T_ε are basically those of εH.

With $H = -B_1\sigma_1 - B_3\sigma_3$, we have

$$e^{K(s's-1) + h/2 (s'+s) + c}$$

$$= \langle s' | e^{-\varepsilon H} | s \rangle$$

$$= \langle s' | I + \varepsilon B_1 \sigma_1 + \varepsilon B_3 \sigma_3 | s \rangle \quad (\varepsilon \to 0)$$

And thus, as $\varepsilon \to 0$

$$\boxed{\begin{aligned} T_{\pm\pm} &= e^{\pm h + c} = 1 \pm \varepsilon B_3 \\ T_{+-} &= e^{-2K + c} = \varepsilon B_1 \end{aligned}}$$

Also, $T_{++} \cdot T_{--} = e^{2c} \simeq 1 \Rightarrow c = 0$. So the parameters are

$$\boxed{h = \varepsilon B_3 \; ; \; e^{-2K} = \varepsilon B_1}$$

which agree with ~~these~~ the exact results earlier, in the $\varepsilon \to 0$ limit.

↳ (with sinh, tanh, etc)

(c) $\boxed{\text{The } T\text{-continuum limit of Fradkin-Susskind}}$

We won't worry too much about this, except noting that

- when $\varepsilon \to 0$, the QM problem maps to Classical, where the parameters h, e^{-2k} are $\propto \varepsilon$, i.e. infinitesimal.

This corresponds with $T \simeq I$,

(d) $\boxed{\text{Two } N \to \infty \text{ limits, 2 Two Temperatures}}$

$N \to \infty$ has different meanings depending on what we hold constant.

- If T is fixed then $N \to \infty$ means taking the thermodynamic limit. In this limit, $T^N \simeq \lambda_0^N |0\rangle\langle 0|$ where λ_0 is the dominating eigv.
  If we let $T = e^{-H}$ then the system is dominated by the gnd state of H, which is $|0\rangle$.

8 TZ

- If we choose $T_\varepsilon = \mathcal{U}(\varepsilon)$ then the parameters in T_ε vary, but $Z = \mathrm{Tr}\, T^N$ or $\langle s_N | T^N | s_0\rangle$ is constant.
  ↳ The system has finite extent in I, but the pts in that time interval become dense as $N \to \infty$
  ↳ Physics is NOT dom. by $\max(\mathrm{eigv}(T))$. This holds only if $I \to \infty$.

↳

- If we want a quantum system @ $T=0$, i.e. $\beta \to \infty$
  - $\hookrightarrow$ must let $\boxed{\bar{\beta} = \beta\hbar \to \infty \Rightarrow \text{gnd state dominates}}$
    - $\hookrightarrow$ This can be a problem for Monte Carlo method b/c the # of pts increases really fast...

There are actually 2 temperatures when we map the quantum ~~Hb~~ partition fn to a classical Ising model

Temperatures

We actually have
- $\beta \Rightarrow$ controls length in τ (spatial extent of Ising problem)
- $K = J/kT$
  - $\hookrightarrow$ temperature of the Ising model, which ravis the parameters of the quantum problem.

Now, we go to the Feynman Path Integral!

V : THE FEYNMAN PATH INTEGRAL

(i) The Feynman Path Integral in Real Time

Consider time-indep Hamiltonian:

$$\boxed{\mathcal{H} = \frac{p^2}{2m} + V(x)}$$

From which we have:

$$\exp\left\{\frac{-i\varepsilon}{\hbar} H\right\} = \exp\left\{\frac{-i\varepsilon}{\hbar}\left[\frac{p^2}{2m} + V(x)\right]\right\}$$

$$\simeq \exp\left\{\frac{-i\varepsilon}{\hbar}\cdot\frac{p^2}{2m}\right\} \exp\left\{\frac{-i\varepsilon}{\hbar} V(x)\right\}$$

because the commutators in

$$e^A e^B = \exp\left\{(A+B) + \frac{1}{2}[A,B] + \dots\right\}$$

Baker Campbell Hausdorff formula

are proportional to higher pwr of ε. i.e.

$$\frac{1}{2}[A,B], \dots = O(\varepsilon^2)$$

$$\text{and} \quad \varepsilon \to 0 \to [A,B]^n \to 0.$$

Next, we can split $V(x)$ into $\frac{V(x)}{2} + \frac{V(x)}{2}$ and write

$$H = \frac{V(x)}{2} + p^2/2m + V(x)/2$$

which mimicks writing the magnetic field term as $T_h^{\gamma/2}\, T_K\, T_h^{\gamma/2}$.

However, we won't do that here b/c there's no gain.

- Anyway... we have to compute the following...

$$\langle x' | \left(e^{-i\epsilon H/\hbar}\right)^N | x\rangle \cong \langle x' | \underbrace{\underbrace{\left(e^{\frac{-i\epsilon p^2}{2m\hbar}}\, e^{\frac{-i\epsilon V(x)}{\hbar}}\right)}\, e^{\frac{-i\epsilon V(x)}{\hbar}}}_{N \text{ times}} | x\rangle \quad (\epsilon \to 0)$$

time-~~evolution~~ operator

→ now insert $I = \int_{-\infty}^{\infty} |x\rangle\langle x|\, dx$ between each of the

factor $\mathcal{U}(t/N) = \exp\left\{\frac{-i\epsilon p^2}{2m\hbar}\right\} \exp\left\{\frac{-i\epsilon V(x)}{\hbar}\right\}$ ~~gives~~.

→ what does this give? Consider $N=3$...

$$\mathcal{U}(x_3, x_0; t) = \langle x_3 | e^{\frac{-i\epsilon p^2}{2m\hbar}}\, e^{\frac{-i\epsilon V(x)}{\hbar}} \int |x_2\rangle\langle x_2|\, dx_2\; e^{\frac{-i\epsilon p^2}{2m\hbar}}\, e^{\frac{-i\epsilon V(x)}{\hbar}}$$

$$\cdot \int |x_1\rangle\langle x_1|\, dx_1\; e^{\frac{-i\epsilon p^2}{2m\hbar}}\, e^{\frac{-i\epsilon V(x)}{\hbar}} |x_0\rangle$$

$$= \int \prod_{n=1}^{2} dx_n\, \langle x_3 | e^{\frac{-i\epsilon p^2}{2m\hbar}}\, e^{\frac{-i\epsilon V}{\hbar}} | x_2\rangle$$

$$\cdot \langle x_2 | e^{\frac{-i\epsilon p^2}{2m\hbar}}\, e^{\frac{-i\epsilon V}{\hbar}} | x_1\rangle \langle x_1 | e^{\frac{-i\epsilon p^2}{2m\hbar}}\, e^{\frac{-i\epsilon V}{\hbar}} | x_0\rangle$$

Now, consider the matrix element $\langle x_n | e^{\frac{-i\epsilon p^2}{2m\hbar}}\, e^{\frac{-i\epsilon V(x)}{\hbar}} | x_{n-1}\rangle$

note that $|x_{n-1}\rangle$ is an eigenvector of $V(X)$, so

$$e^{-i\varepsilon V(X)/\hbar}|x_{n-1}\rangle = |x_{n-1}\rangle \cdot e^{-i\varepsilon V(x_{n-1})/\hbar}$$

So,

$$\langle x_n|\exp\left\{\frac{-i\varepsilon P^2}{2m\hbar}\right\}\exp\left\{\frac{-i\varepsilon V(X)}{\hbar}\right\}|x_{n-1}\rangle$$

$$= \underbrace{\langle x_n|\exp\left\{\frac{-i\varepsilon P^2}{2m\hbar}\right\}|x_{n-1}\rangle}\exp\left\{\frac{-i\varepsilon V(x_{n-1})}{\hbar}\right\}.$$

this is now just the free particle propagator from x_{n-1} to x_n in time ε.

<u>We know that</u>

$$\rightarrow \boxed{\begin{aligned}\mathcal{U}_{free}(x_n, x_{n-1}; \varepsilon) &= \langle x_n|\exp\left\{\frac{-i\varepsilon P^2}{2m}\right\}|x_{n-1}\rangle \\ &= \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{1/2}\cdot\exp\left\{\frac{im(x_n-x_{n-1})^2}{2\hbar\varepsilon}\right\}\end{aligned}}$$

<u>Why?</u> Recall that $I = \int_{-\infty} \frac{dp}{2\pi\hbar}|p\rangle\langle p|$, and so

$$\langle x_n|\exp\left\{\frac{-i\varepsilon P^2}{2m\hbar}\right\}|x_{n-1}\rangle = \int_{\infty}\frac{dp}{2\pi\hbar}\langle x_n|p\rangle\langle p|\exp\left\{\frac{-i\varepsilon P^2}{2m\hbar}\right\}|x_{n-1}\rangle$$

$$= \frac{1}{2\pi\hbar}\int\frac{dp}{1}e^{ipx_n/\hbar}\cdot\langle p|\exp\left\{\frac{-i\varepsilon P^2}{2m\hbar}\right\}|x_{n-1}\rangle$$

$$= \frac{1}{2\pi\hbar}\int dp\; e^{ipx_n/\hbar}\exp\left\{\frac{+i\varepsilon p^2}{2m\hbar}\right\}\langle p|x_{n+1}\rangle$$

$$= \frac{1}{2\pi\hbar} \int dp \; e^{ip(x_n - x_{n-1})/\hbar} \exp\left\{ \frac{-i\varepsilon p^2}{2m\hbar} \right\}$$

= (Mathematica) Feynman Gaussian Integral trick ...)

$$= \left(\frac{m}{2\pi i \hbar \varepsilon} \right)^{1/2} \exp\left\{ \frac{im(x_n - x_{n-1})^2}{2\hbar\varepsilon} \right\}$$

$\square$

<u>Remark</u>: note that we didn't expand $e^{-i\varepsilon p^2/2m\hbar}$ out to order ε because $\hat{p}^2$ has ~~a~~ singular matrix elements between $\langle x_n|$ & $|x_{n-1}\rangle$.

Note that $\langle x_n | \exp\left\{ \frac{-i\varepsilon p^2}{2m\hbar} \right\} | x_{n-1} \rangle$ has not a series expansion in ε.

With this, we have the following ...

$$\boxed{\begin{aligned} &\langle x_n | \exp\left\{ \frac{-i\varepsilon p^2}{2m\hbar} \right\} \exp\left\{ \frac{-i\varepsilon V(x)}{\hbar} \right\} | x_{n-1} \rangle \\ &\qquad = \left(\frac{m}{2\pi i \hbar \varepsilon} \right)^{1/2} \cdot \exp\left\{ \frac{im(x_n - x_{n-1})^2}{2\hbar\varepsilon} \right\} \cdot \exp\left\{ \frac{-i\varepsilon}{\hbar} V(x_{n-1}) \right\} \end{aligned}}$$

So, by collecting all these factors, we have for general N,

$$\boxed{\begin{aligned} U(x_N, x_0; t) &= \left(\frac{m}{2\pi i \hbar \varepsilon} \right)^{1/2} \left\{ \int \prod_{n=1}^{N-1} \left(\frac{m}{2\pi i \hbar \varepsilon} \right)^{1/2} dx_n \right\} \\ &\quad \times \exp\left\{ \sum_{n=1}^{N} \frac{im(x_n - x_{n-1})^2}{2\hbar\varepsilon} - \frac{i\varepsilon}{\hbar} V(x_{n-1}) \right\} \end{aligned}}$$

Now, rewrite

$$\exp\left(\sum_{i=1}^{N}\left\{\frac{im(x_n-x_{n-1})^2}{2\hbar\epsilon}-\frac{i\epsilon}{\hbar}V(x_{n-1})\right\}\right)$$

$$=\exp\left(\frac{i\epsilon}{\hbar}\sum_{n=1}^{N}\left\{\frac{m(x_n-x_{n-1})^2}{2\epsilon^2}-V(x_{n-1})\right\}\right)$$

kinetic − potential = $\mathcal{L}$

so that we can obtain the continuum version as $\underline{\epsilon\to 0}$:

$$\mathcal{U}(x,x';t)=\int[\mathcal{D}x]\exp\left\{\frac{i}{\hbar}\int_0^t\mathcal{L}(x,\dot{x})dt\right\}$$

where

$$\int[\mathcal{D}x]=\lim_{N\to\infty}\left(\frac{m}{2\pi\hbar i\epsilon}\right)^{1/2}\int\left[\prod_{i=1}^{N-1}\left(\frac{m}{2\hbar i\epsilon\pi}\right)^{1/2}dx_n\right]$$

and the Lagrangian $\mathcal{L}(x,\dot{x})$ is the fn from classical mech

$$\mathcal{L}(x,\dot{x})=\underbrace{\frac{1}{2}m\dot{x}^2}_{\text{kinetic}}-\underbrace{V(x)}_{\text{potential}}=K-V$$

The quantity $S[x(t)]=\int_{t_1}^{t_2}\mathcal{L}\,dt$ is called the <u>action</u>

functional

Schematically, we write

$$\boxed{\mathcal{U}(x',x;t)=\int[\mathcal{D}x]\,e^{\frac{i}{\hbar}S}}$$

<u>action</u>

Stationary points of S contribute the most to the propagator amplitude $\Rightarrow$ obey the Euler - Lagrange Eqn:

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}}$$

principle of least action.

As $\hbar \to 0$, we may forget all but the classical path and its coherent neighbors & write

$$\boxed{\mathcal{U}(x', x; t) \simeq A e^{i/\hbar S_c}}$$

where S_c is the action evaluated @ classical path.
A is the prefactor that reflects the sum over other paths.

↳ we can have saddle-point approximations to help evaluate these (just like method of stationary phase in osc. integrals)

↳ We won't worry abt this for now.

We'll also skip some of the calculations & examples here.

(b) $\boxed{\text{The Feynman Phase Space Path Integral}}$

To do this, we look back at

$$\underbrace{\langle x_N| \left(e^{-i\epsilon p^2/2m\hbar}\, e^{-i\epsilon V(x)/\hbar}\right) \cdots \quad e^{-i\epsilon V(x)/\hbar}}_{N\text{-times } e^{-i\epsilon H/\hbar}} |x_0\rangle$$

Introduce resolution of identities:

$$I = \int_{\infty} |x\rangle\langle x|\, dx = \int_{\infty} \frac{dp}{2\pi\hbar} |p\rangle\langle p|.$$

where ~~$|x\rangle\langle p|$~~ $\langle x|p\rangle = e^{ip\cdot x/\hbar}$

- Let $N = 3$. Then we insert 3 resolutions of identity in p and 2 in terms of x alternatingly...

This gives

$$\boxed{\begin{aligned} U(x_3, x_0; t) = \int [\mathcal{D}p\, \mathcal{D}x]\, & \langle x_3| e^{-i\varepsilon p^2/2m\hbar} |p_3\rangle\langle p_3| e^{-i\varepsilon V(x)/\hbar} |x_2\rangle \\ \times & \langle x_2| e^{-i\varepsilon p^2/2m\hbar} |p_2\rangle\langle p_2| e^{-i\varepsilon V(x)/\hbar} |x_1\rangle \\ \times & \langle x_1| e^{-i\varepsilon p^2/2m\hbar} |p_1\rangle\langle p_1| e^{-i\varepsilon V(x)/\hbar} |x_0\rangle \end{aligned}}$$

where

$$\int [\mathcal{D}p\, \mathcal{D}x] = \underbrace{\int_{\infty} \cdots \int_{\infty}}_{2N-1 \text{ times}} \prod_{n=1}^{N} \frac{dp_n}{2\pi\hbar} \prod_{n=1}^{N-1} dx_n$$

- Observe that evaluating each matrix element is very easy b/c we're only working with eigvalues & eigvector

~~→ Gather all terms and take the $\varepsilon \to 0$ limit we get~~

Gather all terms to get the following...

$$\mathcal{U}(x,x';t) = \int [\mathcal{D}p\,\mathcal{D}x] \exp\left\{ \sum_{i=1}^{N} \left(\frac{-i\varepsilon}{2m\hbar} p_n^2 + \overbrace{\frac{i}{\hbar} p_n (x_n - x_{n-1})}^{\sim p\dot{x}} - \frac{i\varepsilon}{\hbar} V(x_{n-1}) \right) \right\}$$

- As $N \to \infty$ i.e. $\varepsilon \to 0$ we can write in continuous time (after manipulating ε like before ---) we get

$$\sum_{i=1}^{N} \left(\frac{-i\varepsilon p_n^2}{2m\hbar} + \frac{i}{\hbar} p_n (x_n - x_{n-1}) - \frac{i\varepsilon}{\hbar} V(x_{n-1}) \right)$$

$$= \frac{i\varepsilon}{\hbar} \sum_{i=1}^{N} \left[\left(\frac{-p_n^2}{2m} - V(x_{n-1}) \right) + \frac{p_n (x_n - x_{n-1})}{\varepsilon} \right]$$

- ε acts like time, so, as $N \to \infty$; i.e. $\varepsilon \to 0$,

$$\begin{cases} 1^{st} \text{ term} \Rightarrow \text{becomes } \mathcal{H}(x,p) \\ 2^{nd} \text{ term} \Rightarrow \text{becomes } p\dot{x} \end{cases}$$

So, in the continuum limit,

$$\boxed{\mathcal{U}(x,x',t) = \int [\mathcal{D}p\,\mathcal{D}x] \exp\left\{ \frac{i}{\hbar} \int_0^t [p\dot{x} - \mathcal{H}(x,p)]\,dt \right\}}$$

$x = x(t)$ $\qquad$ $p = p(t)$

"phase space path integral"

Note that since p is quadratic in the exponent, we can actually integrate it out, and get the usual "configuration space" path as expected ...

(c) $\boxed{\text{The Feynman Path Integral for Imag. Time}}$

Repeat the same process, but with imaginary time, we get

$$\mathcal{U}(x, x'; \tau) = \langle x | \mathcal{U}(\tau) | x' \rangle$$

$$= \underbrace{\int [\mathcal{D}x]}_{\lim\limits_{N\to\infty} \left(\frac{m}{2\hbar\pi\varepsilon}\right)^{1/2} \prod\limits_{i=0}^{N-1} \left(\frac{m}{2\pi\hbar\varepsilon}\right)^{1/2} dx_i} \exp\left[-\frac{1}{\hbar}\int_0^\tau \mathcal{L}_E(x, \dot{x})\, d\tau\right]$$

$$\boxed{\mathcal{L}_E = \frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x)}$$

$\mathcal{L}_E$ is the Euclidean Lagrangian.

= Sum of KE & real-time potential.

⊛ ⟹ particle actually sees the potential flipped upside down

Similarly, we can define the Euclidean action...

$$S_E = \int \mathcal{L}_E \, d\tau$$

Principle of least action still applies...

$e^{-S_E/\hbar}$ is biggest when S_E smallest...

(d) Classical – Quantum Connection

- Path integral from $x_0 \to x_N$ is identical in form to a classical partition function of a system with $N+1$ coordinates x_n, with BC: x_0, x_N fixed.

  ↳ x_n: intermediate state labels of the quantum problem = classical variables summed over in the partition function.

- S_E (Euclidean action) $\equiv$ Energy in the partition fn.

- The role of $\hbar$ is played by T.
  As $\hbar$ (or T) $\to 0$, the sum / integral over all configs is dominated by the min of S_E, or energy & fluctuations are suppressed.

- $K_1 = m/2\hbar\varepsilon$; $K_2 = m\omega^2\varepsilon/2\hbar$

- $\varepsilon \to 0$ in the QM problem, so classical params must also approach limits $K_1 \to \infty$, $K_2 \to 0$.

- The single QM degree of freedom is traded for a 1-D array of classical d.o.f (spin ± 1).

  ↳ Dimensionality goes up by one as we go from QM $\to$ CM.

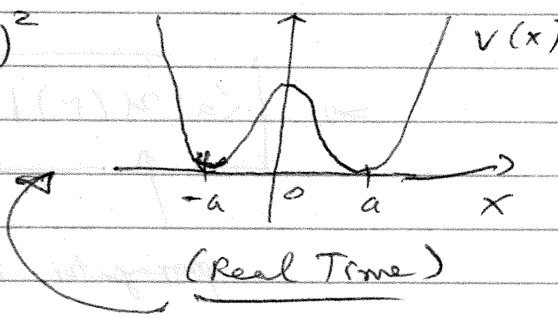
(e) Tunneling by Euclidean Path Integrals

Recall that we can approximate path integrals by looking @ contribution from the classical path as $\hbar \to 0$.

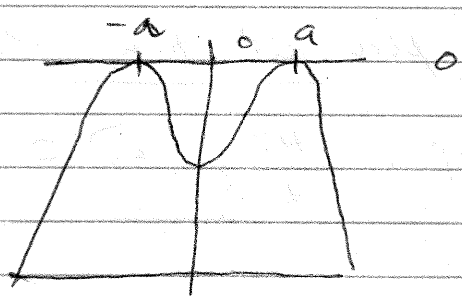
→ But if ∃ Barrier ⇒ we get no classical path.
→ can't find tunneling amplitudes

→ However in Euclidean dynamics, the potential is turned upside down...
↳ Tunneling can be found by this formalism (^up)

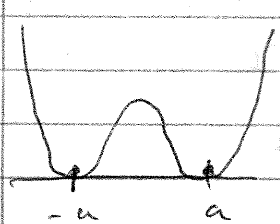
$\boxed{Ex}$ let $V(x) = A^2(x^2 - a^2)^2$



In img. time:



In real time, we have a degeneracy in gnd state



→ call them $\boxed{|\pm a\rangle}$.

$$\mathcal{H} = \begin{pmatrix} E_0 & 0 \\ 0 & E_0 \end{pmatrix}$$

→ Let's shift reference so that $\underline{E_0 = 0}$.

then the energy levels will be split into

$$E = \pm H_{+-} = \pm H_{-+} = \pm \langle a|H|-a\rangle .$$

and the eigenstates will be $|S\rangle$ & $|A\rangle$

symmetric comb of $|\pm a\rangle$ — antisymmetric comb of $|\pm a\rangle$.

$\Rightarrow$ There's no classical path between $-a$ & $+a$ in real time, but there is one in img time (so we invert the potential).

$\Rightarrow$ Consider ...

$$\Rightarrow \boxed{\langle a|\mathcal{U}(\tau)|-a\rangle = \langle a|\exp\left\{\frac{-H\tau}{\hbar}\right\}|-a\rangle}$$

propagator from $-\tau/2 \rightarrow \tau/2$

Now ... pick out term linear in τ ...

$$\boxed{\langle a|\exp\left\{\frac{-H\tau}{\hbar}\right\}|-a\rangle \simeq 0 - \frac{1}{\hbar}\tau\langle a|H|-a\rangle + O(\tau^2)}$$

same H as in real time.

Now, from the semiclassical approximation we also know that - up to the term linear in τ ...

$$\boxed{\langle a|e^{-H\tau/\hbar}|-a\rangle \simeq \tau e^{-S_{cl}/\hbar}}$$

Why? See p. 56 of the book for ideas ...

So, we infer that

$$\boxed{\langle a|H|-a\rangle \simeq e^{-S_{cl}/\hbar}}$$

where

$$S_{cl} = \int (T+V)\,d\tau \quad \text{because } E_0 = \frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 - V = T - V \quad \text{conserved}$$

$$= \int 2T\,d\tau \longleftarrow \text{want 0-energy solution} \rightarrow T = V$$

$$= \int m\dot{x}\dot{x}\,d\tau$$

$$= \int_{-a}^{a} p(x)\,dx = \int_{-a}^{a} \sqrt{2mV(x)}\,dx$$

→ Tunneling amplitude is $e^{-\frac{1}{\hbar}\int_{-a}^{a}\sqrt{2mV(x)}\,dx}$ which agrees with Schrödinger's approach.

Note that S_{cl} does not depend on τ_0 → every path has the same action.

Sum over everything gives...

$$\langle a|\mathcal{U}(\tau)|-a\rangle \simeq \tau e^{-S_{cl}/\hbar}$$

compare this to $\langle a|e^{-\frac{1}{\hbar}\tau H}|-a\rangle \simeq 0 - \frac{\tau}{\hbar}\langle a|H|-a\rangle + O(\tau^2)$

$$\rightarrow \boxed{H_{+-} \simeq e^{-\frac{1}{\hbar}S_{cl}}} \quad \text{From here we have...}$$

$$|S\rangle = \frac{1}{\sqrt{2}}\left(|a\rangle + |-a\rangle\right) \quad ; \quad |A\rangle = \frac{1}{\sqrt{2}}\left(|a\rangle - |-a\rangle\right)$$

$$E_S = -e^{-\frac{1}{\hbar}S_{cl}} \quad ; \quad E_A = +e^{-\frac{1}{\hbar}S_{cl}}$$

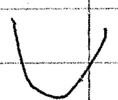
(f) Spontaneous Symmetry Breaking

Feb 21, 2021

Consider a Hamiltonian which has a symmetry, say under parity.

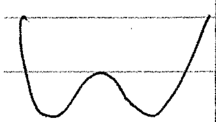
If the lowest-energy state of the problem is itself not invariant under the symmetry, we say symmetry is spontaneously broken

Ex • Single-well oscillator. Hamiltonian is invariant under parity.



Gnd state is particle sitting at the bottom of the well

→ This state respects the symmetry.



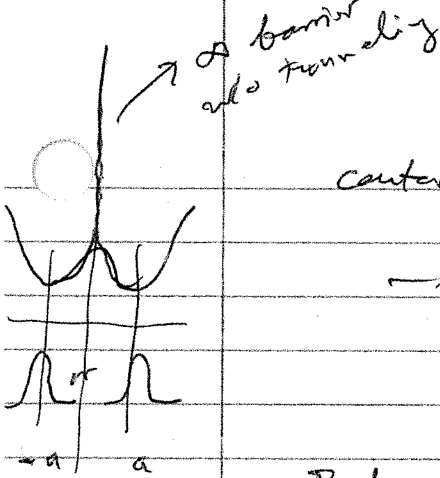
• Now consider the double well with minima at $x = \pm a$. → 2 lowest energy configurations available → does not obey parity symmetry.

"Spontaneous" in that particle has to make a choice

→ ∃ more than one gnd state. States are not invariant under sym. Rather, one gnd state gets mapped to the other ...

• Consider the quantum case, but with an infinite barrier between the wells. (so that no tunneling occurs --- so the barrier is not δ-function).

→ particle has 2 choices: being Gaussian-like functions



centered at either one of the 2 troughs: $|\pm a\rangle$.

→ this has feature of symmetry breaking (degenerate 2 non-invariant under transf by parity i.e. $\langle x \rangle \neq 0$).

<u>But</u> since this is QM we can have a superposition...

$$|S/A\rangle = \frac{[\,|+a\rangle \pm |-a\rangle\,]}{\sqrt{2}}$$

and the parity transformation goes like

$$\Pi\, |S/A\rangle = \pm\, |S/A\rangle$$

So $|S/A\rangle$ are eigenstates of parity Π; which implies that

$$\boxed{[\Pi, H] = 0}$$

which (circularly) implies that $|S/A\rangle$ <u>can</u> be formed

But <u>should</u> they be formed? $\boxed{\text{No}}$ b/c of the ∞ barrier

→ symmetry <u>is</u> spontaneously broken.

Ex Now, back to the finite barrier problem → $|S/A\rangle$ are possible and are not degenerate.

Now note that $|S\rangle$, in general, will be <u>the unique gnd</u> state. The <u>instanton</u> calculation (tunneling) tells us that the double well, despite having 2 classical states that break symmetry, has in QM a unique, symmetric grnd state.

→ The symmetry of the Hamiltonian is the symmetry of the grnd state

Sym. breaking does not take place in the double-well pottam (finite)

→ We have that tunneling restores symmetry.

Ex Consider periodic potential $V(x) = 1 - \cos 2\pi x$
→ minima @ $x = n \in \mathbb{Z}$.

Symmetry is $x \to x+1$.

The approx states $|n\rangle$ which are Gaussian centered @ ~~each~~ a classical minima n breaks the symmetry & are converted to each other by the translation opertr T:

$$T|n\rangle = |n+1\rangle$$

Now, due to tunneling, we can form the symmetric ground state: (assuming that size = N)

$$|s\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} |n\rangle$$

[Ex] Assume that

$$\mathcal{H} = \sum_{1}^{N} E_0 |n\rangle\langle n| - t\left(|n\rangle\langle n+1| + |n+1\rangle\langle n|\right)$$

describes the low-energy Hamiltonian of a particle in a periodic potential w/ minima @ $n \in \mathbb{Z}$. $n \in \{1, \dots, N\}$

on a ring.

→ Problem has sym under translation by 1 site

- First term of $\mathcal{H}$ represents the energy of the Gaussian state centered @ n.

- Second term represents the tunneling to adjacent minima with tunneling amplitude t.

→ Consider $\boxed{|\theta\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{in\theta} |n\rangle}$.

Show that $|\theta\rangle$ is an eigstate of T & eigvalue. Use $T^N = 1$ to restrict the allowed values of θ & make sure we still have N states

Show that $|\theta\rangle$ is eigstate of $\mathcal{H}$ w/ eigv
$$\boxed{E(\theta) = E_0 - 2t\cos\theta}$$

Let, $N = 2$ & regain the double-well result

~~Solution:~~ $\cancel{T|\theta\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{i\theta n} T|n\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{i\theta n} |n+1\rangle}$

~~[illegible]~~ $\cancel{= \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{i\theta} \dots |n\rangle = |\theta\rangle}$

Now, $\frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{in\theta} |n+1\rangle$

$$= \frac{1}{\sqrt{N}} e^{-i\theta} \sum_{n=1}^{N} e^{i(n+1)\theta} |n+1\rangle$$

$$= \frac{1}{\sqrt{N}} e^{-i\theta} \left\{ e^{i2\theta} |2\rangle + \dots + e^{i(N+1)\theta} |N+1\rangle \right\}$$

$e^{i\theta} |1\rangle$ since $T^N = 1$

$\to e_{jr} = e^{-i\theta}$. , $T^N = 1 \to e^{-iN\theta} = 1 \to \boxed{\cos(N\theta) = 1}$

$\to \theta = 2\pi k, \; k \in \mathbb{Z}$ (?)

$$H(A) = \sum_{n=1}^{N} E_0 |n\rangle\langle n| - t\left(|n\rangle\langle n+1| + |n+1\rangle\langle n|\right)$$

$$\times \left\{ \sum_{m=1}^{N} e^{im\theta} |m\rangle \right\}$$

$$= \sum_{n=1}^{N} \sum_{m=1}^{N} E_0 |n\rangle\langle n|m\rangle e^{im\theta}$$

$$- t\left(|n\rangle\langle n+1|m\rangle + |n+1\rangle\langle n|m\rangle\right) e^{im\theta}$$

$$= \sum_{n=1}^{N} E_0 |n\rangle e^{in\theta}$$

$$- t\left(|n+1\rangle e^{in\theta} + |n\rangle e^{i(n+1)\theta}\right)$$

$$= \sum_{n=1}^{N} E_0 |n\rangle e^{in\theta} - t \sum_{n=1}^{N} |n\rangle e^{i(n+1)\theta}$$

$$- t \sum_{n=1}^{N} e^{in\theta} |n+1\rangle$$

$$= d + u + (-t) \sum$$

→ this ~~problem~~ is phrased a bit weirdly, but we can expect the answer to be the symmetric wfn

$$|s\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{in\theta} |n\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} |n\rangle .$$

Since θ can only be $2\pi k$, $k \in \mathbb{Z}$.

#

→ Will the gnd state always be invariant under the symmetry operation that commutes with $\mathcal{H}$?

↳ Yes, as long as the barrier is finite. (or as long as tunneling can occur)

#

Instantons in the $d=1$ Ising model

Recall that the Hamiltonian is (in the σ_z basis)

$$\mathcal{H} = -\begin{pmatrix} 0 & K^* \\ K^* & 0 \end{pmatrix}$$

Suppose that $K^* = e^{-2k} = T \to 0$. We can do a "semi-classical" approximation in which the sum over paths is dominated by the one of least action.

↳ The "instanton" that connects the $|\uparrow\rangle$ & $|\downarrow\rangle$ gnd state is a configuration in which the spin is up until some time when it flips to down & stay down. This tunneling restores symmetry @ $T \neq 0$.

(9) $\boxed{\text{The Classical Limit of Quantum Stat Mech}}$

Consider particle of mass m in potential V, then

$$Z(\beta) = \int dx \int_x^x [\mathcal{D}x] \exp\left\{\frac{-1}{\hbar}\int_0^{\beta\hbar}\left[\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V[x(\tau)]\right] d\tau\right\}$$

in imaginary time, and we're considering paths starting & ending on the same point x.

Consider $\beta\hbar \to 0$ (i.e. $\hbar \to 0$ (classical limit) or high temp $\beta \to 0 \leftrightarrow \frac{1}{kT} \to \infty$)

Look at any x. Need to look @ particle @ x, go somewhere in time $\beta\hbar$, and return to x.

If the particle goes a distance Δx, the KE is like

$$\simeq m\left(\frac{\Delta x}{\beta\hbar}\right)^2$$

$\rightarrow$ Boltzmann factor for this is

$$\sim e^{-1/\hbar\, m(\Delta x/\beta\hbar)^2 \beta\hbar}$$

So we have

$$\Delta x \simeq \sqrt{\frac{\beta}{m}}\,\hbar$$

so that $e^{-1/\hbar\, m\, 1/\beta m \cdot \beta\hbar} = e \sim O(1)$.

If the potential does not vary much over such length scale, then we can pull it out ...

$$Z(\beta) \simeq \int dx\, e^{-\beta V(x)} \int_x^x [\mathcal{D}x] \exp\left\{ \frac{-1}{\hbar} \int_0^{\beta\hbar} \left[\frac{m}{2} \left(\frac{dx}{d\tau} \right)^2 \right] d\tau \right\}$$

$$= \int dx\, e^{-\beta V(x)} \sqrt{\frac{m}{2\pi\hbar\beta\hbar}}$$

where the last step is obtained by viewing that integral as the amplitude for a free particle to go from x to x in time $\beta\hbar$ (view previous pages in the book for more details)

→ How does this compare with classical soln? Recall that if we replace the sum over states by an integral over phase space...

$$Z = A \int dx \int dp \exp\left\{ -\beta \left(\frac{p^2}{2m} + V(x) \right) \right\}$$

→ do the p-integral & compare the result to the classical limit of the path integral we see that QM fixes

$$A = \frac{1}{2\pi\hbar} \quad \text{~~in accordance~~}$$

What is A? ⟹ A reflects one's freedom to multiply Z by a constant w/o changing anything physical

→ A corresponds to the fact that the # of classical states in the region $(dx\,dp)$ of phase space is <u>not uniquely defined</u>.

⟹ if $A = \frac{1}{2\pi\hbar}$ then we see that it is in accordance with the uncertainty principle.

COHERENT STATE PATH INTEGRALS for SPINS, BOSONS, and FERMIONS

a) Spin Coherent State Path Integral

Consider spin - S particle. Hilbert space is $(2S+1)$-dim. Let S_z eigenstates be basis -

Resolution of Id: $I = \sum_{-S}^{S} |S_z\rangle\langle S_z|$

→ Consider the spin-coherent state:

$$\boxed{|\Omega\rangle \equiv |\theta, \varphi\rangle = \mathcal{U}[R(\Omega)]\,|SS\rangle}$$

around x-axis; around z-axis; rotation; fully polarized state

Given that $\langle SS|\vec{S}|SS\rangle = \hat{k}S$, we have

$$\langle\Omega|\vec{S}|\Omega\rangle = \langle SS|\,\mathcal{U}^{+}(R(\Omega))\,\vec{S}\,\mathcal{U}(R(\Omega))\,|SS\rangle$$

$$= S\left(\hat{i}\sin\theta\cos\varphi + \hat{j}\sin\theta\sin\varphi + \hat{k}\cos\theta\right)$$

If $S = 1$, then $\sigma(S_z) = \{0, \pm 1\}$.

The coherent state ≡ one in which the spin operator has a nice exp value = equal classical spin of length S, but point in the direction of Ω.

(not an eigvec though)

Now, look at the following eqn...

$$\langle \Omega_2 | \Omega_1 \rangle = \left(\cos\frac{\theta_2}{2} \cos\frac{\theta_1}{2} + e^{i(\phi_1 - \phi_2)} \sin\frac{\theta_2}{2} \sin\frac{\theta_1}{2} \right)^{2S}$$

How is this true?

~~can check this pretty easily~~

If $S = \frac{1}{2}$, ~~then~~ the up spinor along (θ, ϕ) is

$$|\Omega\rangle = |\theta\phi\rangle = \cos\frac{\theta}{2}\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle + e^{i\phi}\sin\frac{\theta}{2}\left|\tfrac{1}{2}\,\tfrac{-1}{2}\right\rangle$$

~~Then~~ Suppose $2S$ spin-$\frac{1}{2}$ particles join to form a spin-S state $\rightarrow$ there's only one product state with $S_z = S$, which is where all spin-$\frac{1}{2}$ are $\uparrow$.

$$\rightarrow \quad |SS\rangle = \left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle \otimes \left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle \otimes \cdots \otimes \left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle$$

If we rotate this, it becomes a tensor product of rotated states, and ~~we~~ when we form the inner product on the LHS of ______, we obtain the RHS of ______

The resolution of the identity in terms of these states is

$$I = \frac{2S+1}{4\pi}\int d\Omega\, |\Omega\rangle\langle\Omega| \qquad \text{where } d\Omega = d\cos\theta\, d\phi$$

Try for $S = \frac{1}{2}$, then we have

$$|\Omega\rangle\langle\Omega| = \left(\cos\frac{\theta}{2}\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle + e^{i\phi}\sin\frac{\theta}{2}\left|\tfrac{1}{2}\,\tfrac{-1}{2}\right\rangle\right)$$

Idea

$$\left(\cos\frac{\theta}{2}\left\langle\tfrac{1}{2}\,\tfrac{1}{2}\right| + e^{-i\phi}\sin\frac{\theta}{2}\left\langle\tfrac{1}{2}\,\tfrac{-1}{2}\right|\right)$$

$$= \cos^2\frac{\theta}{2} \left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\left\langle\tfrac{1}{2}\ \tfrac{1}{2}\right| + \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\phi}\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\left\langle\tfrac{1}{2}\ \tfrac{-1}{2}\right|$$

$$+ \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\phi}\left|\tfrac{1}{2}\ \tfrac{-1}{2}\right\rangle\left\langle\tfrac{1}{2}\ \tfrac{1}{2}\right| + \sin^2\frac{\theta}{2}\left|\tfrac{1}{2}\ \tfrac{-1}{2}\right\rangle\left\langle\tfrac{1}{2}\ \tfrac{-1}{2}\right|$$

$$\int \cos^2\frac{\theta}{2}\, d\cos\theta = -\int \cos^2\frac{\theta}{2}\sin\theta\, d\theta = 1$$

$$\int \sin^2\frac{\theta}{2}\, d\cos\theta = -\int \sin^2\frac{\theta}{2}\sin\theta\, d\theta = 1.$$

other terms are zero.

So $\displaystyle\int d\Omega\, |\Omega\rangle\langle\Omega| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot 2\pi$

$$\rightarrow \frac{2S+1}{4\pi}\int d\Omega\, |\Omega\rangle\langle\Omega| = \frac{2\cdot\frac{1}{2}+1}{4\pi}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot 2\pi$$

$$= I \checkmark$$

(b) $\boxed{\text{Real-Time Path Integral for Spin}}$

Feb 26 2021

Approximately, the path integral looks like (as terms)

$$\ldots \langle \Omega(t+\varepsilon)|\, I - \frac{i\varepsilon}{\hbar}H(\vec{S})\, |\Omega(t)\rangle.$$

where $|\Omega\rangle$ is a coherent state. Now there are 2 parts to this...

$$\langle \Omega(t+\varepsilon)|\Omega(t)\rangle \text{ and } \langle \Omega(\varepsilon+t)|\, \frac{-i\varepsilon}{\hbar}H\, |\Omega(t)\rangle.$$

- $\langle \Omega(t+\varepsilon) | -\frac{i\varepsilon}{\hbar} \mathcal{H}(\vec{S}) | \Omega(t) \rangle$ (up to order ε)

$\simeq -\frac{i\varepsilon}{\hbar} \langle \Omega(t) | \mathcal{H}(\vec{S}) | \Omega(t) \rangle$

$\equiv -i\varepsilon\, \mathcal{H}(\Omega)$ ← definition

- $\langle \Omega(t+\varepsilon) | \Omega(t) \rangle \simeq 1 - i\varepsilon S(1-\cos\theta)\dot{\phi}$

$\simeq e^{iS(\cos\theta - 1)\dot{\phi}\varepsilon}$

we got this by expanding the expression (which we derived before)

$$\left\{ \langle \Omega_2 | \Omega_1 \rangle = \left(\cos\frac{\theta_2}{2}\cos\frac{\theta_1}{2} + e^{i(\phi_1-\phi_2)} \sin\frac{\theta_2}{2}\sin\frac{\theta_1}{2} \right)^{2S} \right.$$

to first order in $\Delta\theta$ & $\Delta\phi$.

(we can check this quite easily, but I won't do it out here to save time...)

From there, we can calculate the "representation" of the propagator $\mathcal{U}(t)$ in the continuum limit...

$$\boxed{\langle \Omega_f | \mathcal{U}(t) | \Omega_i \rangle = \int [\mathcal{D}\Omega] \exp\left\{ i\int_{t_1}^{t_2} \left(S\cos\theta\,\dot{\phi} - \mathcal{H}(\Omega) \right) dt \right\}}$$

How?

We'll come back to the rest of the details on spins later... Since we're mostly interested in bosonic/fermionic systems, we will ~~do the~~ look at these topics first...

(c) $\boxed{\text{Bosonic Coherent States}}$

Recall harmonic oscillator...

$$\begin{cases} \mathcal{H} = \left(a^\dagger a + \frac{1}{2}\right)\hbar\omega \\ [a^\dagger, a] = \mathbb{1} \\ \mathcal{H}|n\rangle = E_n|n\rangle \qquad n = 0, 1, 2, \ldots \\ E_n = \left(n + \frac{1}{2}\right)\hbar\omega . \end{cases}$$

Rather than saying the HO is in nth state, we could ignore $n=0$ and say that there is one state of energy $\hbar\omega$ - n quanta in it.

→ This is how photons / phonons / etc are viewed

→ Any level $\hbar\omega$ can be occupied by any number of quanta n. ⇒ BOSONS.

→ Bosonic coherent state is defined as

$$\boxed{|z\rangle = e^{a^\dagger z}|0\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle}$$

We can check that $\boxed{a|z\rangle = z|z\rangle}$

and $\boxed{\langle z|a^\dagger = \langle z|z^*}$

$$a|z\rangle = a\sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n-1\rangle\sqrt{n} \qquad n=0 \rightarrow 0$$

$$= z\sum_{n=1}^{\infty} \frac{z^{n-1}}{\sqrt{(n-1)!}}|n-1\rangle$$

$$= z|z\rangle \checkmark$$

Next, using the identity $\underline{e^A e^B = e^B e^A e^{[A,B]}}$

where $[A,B]$ is a c-number, we can show that

$$\langle z_2|z_1\rangle = e^{z_2^* z_1}$$

$$\boxed{\langle z_2|z_1\rangle} = \langle 0|e^{z_2^* a} e^{a^\dagger z_1}|0\rangle$$

$$= \langle 0|e^{a^\dagger z_1} e^{z_2^* a} e^{[z_2^* a, a^\dagger z_1]}|0\rangle$$

$$= \langle 0|e^{a^\dagger z_1} e^{z_2^* a} e^{z_2^* z_1 \underbrace{(aa^\dagger - a^\dagger a)}_{\mathbb{1}}}|0\rangle$$

$$= e^{z_2^* z_1}\langle 0|e^{a^\dagger z_1} \underbrace{e^{z_2^* a}|0\rangle}_{|0\rangle}$$

$$= e^{z_2^* z_1}\langle 0|e^{a^\dagger z_1}|0\rangle = \boxed{e^{z_2^* z_1}}$$

March 1 2021

Resolution of Identity:

$$I = \int \frac{dz\,dz^*}{2\pi i} e^{-z^* z} |z\rangle\langle z|$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dx\,dy}{\pi} e^{-z^* z} |z\rangle\langle z|$$

$$= \int_0^{\infty}\int_0^{2\pi} \frac{r\,dr\,d\theta}{\pi} e^{-z^* z} |z\rangle\langle z|$$

verification is left as an exercise, but the idea is to ~~rere~~ write $|z\rangle$ as $|n\rangle$ and $\langle z|$ as $\langle m|$, then to see angular ~~part~~ & radial parts to show that I reduces to $\sum_n |n\rangle\langle n|$.

Need to recall the defn of the Γ function.

With this, we can write down the path integral.

$$\boxed{\langle z_f| \exp\left\{\frac{-i}{\hbar} : \mathcal{H}(a^\dagger, a) : t\right\} |z_i\rangle}$$

where $:\mathcal{H}(a^\dagger, a):$ $\rightarrow$ normal-order Hamiltonian

all creation ops: left
all annihilation ops: right

$\rightarrow$ chop t into N pieces ... $\varepsilon = t/N$ & repeatedly insert resolution of ID we get we'll get a string with factors of the form...

$$\Big\{ \ldots |z_{n+1}\rangle\langle z_{n+1}| \left\{ 1 - \frac{i\varepsilon}{\hbar} :\mathcal{H}(a^\dagger, a): |z_n\rangle\langle z_n| \frac{dz_n\, dz_n^*\, e^{-z_n^* z_n}}{2\pi i} \right\}$$

$$\times \left\{ 1 - \frac{i\varepsilon}{\hbar} \ldots \right\} \ldots$$

$$= \ldots |z_{n+1}\rangle \frac{dz_n\, dz_n^*}{2\pi i} \exp\left[(z_{n+1}^* - z_n^*) z_n - \frac{i\varepsilon}{\hbar} :\mathcal{H}(z_n^*, z_n): \right] \langle z_n|$$

where we've used the facts that...

→ a, $a^\dagger$ in $:\mathcal{H}:$ are acting on their eigenstates on the left & right, and with $\varepsilon \ll 1$, we may set $z_{n+1} = z_n$ inside it...

So how have we got the path integral in the continuum limit

$$\boxed{\begin{aligned} &\langle z_f | \exp\left\{ \frac{-i}{\hbar} :\mathcal{H}(a^\dagger, a): t \right\} |z_i\rangle \\ &= \int_{z_i}^{z_f^*} [\mathcal{D}z^* \mathcal{D}z] \exp\left\{ \frac{i}{\hbar} \int_0^t \left(i\hbar z^* \frac{dz}{dt} - :\mathcal{H}(z^*, z): \right) dt \right\} \end{aligned}}$$

(There's some sloppiness in the math, but we won't worry too much about it for now...)

(d) THE FERMION PROBLEM

Unlike bosons, fermions have "fermionic oscillator", which has only one level.

↳ which can contain of only one or zero quanta due to Pauli exclusion principle.

There are a few things we need to sort out before getting to the path integral for fermions...

1. Fermionic oscillator: spectrum & thermodynamics
2. Resolution of Id.
3. Non-interacting fermions only... we won't worry about interacting fermions...

(e) FERMIONIC OSCILLATOR: SPECTRUM & THERMODY.

↳ Hamiltonian: $H_0 = \Omega_0 \Psi^\dagger \Psi$

↳ Fermions obey the anti-comm relations:

$$\boxed{\begin{aligned} \{\Psi^\dagger, \Psi\} &= \Psi^\dagger\Psi + \Psi\Psi^\dagger = \mathbb{1} \\ \{\Psi, \Psi\} &= \{\Psi^\dagger, \Psi^\dagger\} = 0 \end{aligned}}$$

↳ $\Psi\Psi = \Psi^2 = (\Psi^\dagger)^2 = 0.$

↳ we will use this all the time...

This is Pauli exclusion principle...

Also have the number operator... $\boxed{N = \psi^\dagger \psi}$

which is idempotent. $\boxed{N^2 = N}$

$$N^2 = \psi^\dagger \psi \psi^\dagger \psi = \psi^\dagger (1 - \psi^\dagger \psi) \psi = \psi^\dagger \psi = N.$$

So eigs of N can only be 0 or 1.

$$\boxed{N|0\rangle = 0|0\rangle \quad ; \quad N|1\rangle = 1|1\rangle}$$

So

$$\psi^\dagger |0\rangle = \psi^\dagger (1 - \psi^\dagger \psi)|0\rangle = \psi^\dagger \psi \psi^\dagger |0\rangle = N \psi^\dagger |0\rangle$$

$\Rightarrow$ $\boxed{\psi^\dagger |0\rangle = |1\rangle}$ $\to$ with unity norm.

Similarly, can show that $\boxed{\psi |1\rangle = |0\rangle}$

$\to$ there are no other vectors in the Hilbert space.

since $\psi^2 = \psi^{\dagger 2} = 0$.

$\to$ States are either singly occupied or empty.

And so the Fermionic oscillator

$$\boxed{\mathcal{H}_0 = \Omega_0 \Psi^\dagger \Psi} \qquad \sigma(\mathcal{H}_0) = \{0, \Omega_0\}.$$

has eigvalues $\boxed{0 \,,\, \Omega_0}$

But we won't work with $\mathcal{H}_0$. Rather, we work with

$$\boxed{\mathcal{H} = \mathcal{H}_0 - \underbrace{\mu}_{\text{chemical potential}} N = (\Omega_0 - \mu) \psi^\dagger \psi}$$

Grand partition function

$$\boxed{Z = \mathrm{Tr}\left(e^{-\beta(H_0 - \mu N)}\right)}$$

trace over any complete set of eigenstates.

in the N basis, this is easy...

$$Z = \mathrm{Tr}\left(e^{-\beta(H_0-\mu N)}\right) = \underbrace{1}_{N=0} + \underbrace{e^{-\beta(\Omega_0-\mu)}}_{N=1}$$

From here, we can find the rest...

Mean occupation number...

$$\langle N\rangle = \frac{1}{\beta Z}\frac{\partial Z}{\partial \mu} = \frac{1}{\beta}\frac{\partial \ln Z}{\partial \beta} = \frac{1}{1+e^{\beta(\Omega_0-\mu)}} \equiv n_F$$

this is Fermi-Dirac statistics -

At $T=0$ (zero temperature...)

$$\boxed{\langle N\rangle = \theta(\mu-\Omega_0) \quad @\ T=0}$$

Heaviside step fn, which says that the fermion is present if its energy $\Omega_0 < \mu$ & absent if $\Omega_0 > \mu$.

(f) COHERENT STATE FOR FERMIONS

Mar 8, 2021

Coherent states are eigenstates of the annihilation op.

- for fermions, we have

$$\Psi|\psi\rangle = \psi|\psi\rangle$$

and more importantly $\psi^2 = 0$ since $\boxed{\Psi^2 = 0}$

$\Rightarrow$ But ψ wouldn't be interesting if itself is 0.

$\Rightarrow$ $\boxed{\psi \text{ is a Grassman Variable}}$

↳ Defn variable that anti-commute with each other & w/ all fermionic annihilation – creation operators.

(and therefore commute with a string containing an even number of such operators).

→ What, then, is the coherent state?

Well... $\boxed{|\psi\rangle = |0\rangle - \psi|1\rangle}$ → Fermionic coherent state.

grassmann number.

Note that we can show $\Psi|\psi\rangle = \psi|\psi\rangle$ as follows

$$\Psi|\psi\rangle = \Psi(|0\rangle - \psi|1\rangle)$$

(–) → (+) by anti-commutativity

$$= 0 + \psi\Psi|1\rangle$$

$$= \psi|0\rangle = \psi(|0\rangle - \psi|1\rangle) = \psi|\psi\rangle \checkmark$$

We can similarly verify that $\langle\bar{\Psi}|\Psi^\dagger = \langle\bar{\Psi}|\bar{\Psi}$

when $\langle\bar{\Psi}| = \langle 0| - \langle 1|\bar{\Psi} = \langle 0| + \bar{\Psi}\langle 1|$

anti-commutativity.

- Note that fermionic coherent states are not in the complex vector space since its coefs have grassman numbers.
- $\bar{\Psi}$ is NOT the complex conjugate of Ψ, Ψ^*
- $\langle\bar{\Psi}|$ is NOT the adjoint of $|\Psi\rangle$, $\langle\Psi|$

↳ Inner product of 2 coherent states...

$$\langle\bar{\Psi}|\Psi\rangle = \left(\langle 0| - \langle 1|\bar{\Psi}\right)\left(|0\rangle - \Psi|1\rangle\right)$$

$$= \langle 0|0\rangle + \langle 1|\bar{\Psi}\Psi|1\rangle$$

(−) (−)

$$= 1 + \bar{\Psi}\Psi$$

$$\Rightarrow \quad \langle\bar{\Psi}|\Psi\rangle = e^{\bar{\Psi}\Psi}$$

→ This is formally similar to $\langle z_2|z_1\rangle = e^{z_2^* z_1}$ for bosons.

We remark that any function of a Grassmann variable can be expanded as follows.

$F(\Psi) = F_0 + F_1\Psi$, there being no higher power possible.

(g) $\boxed{\text{Integration over Grassmann Numbers}}$

→ we need to know how to do this before learning the path integral for fermions...

→ have to define integrals over Grassmann numbers

↳ these have no geometrical interpretations and we only define them formally.

<u>But</u> the nice thing is that we only need to define these integrals over 1 & ψ, since any function $F(\psi)$ can be written as

$$F(\psi) = F_0 + F_1 \psi .$$

↳ There are only 2 integrals we need to know:

$$\boxed{\begin{aligned} \int \psi \, d\psi &= 1 \\ \int 1 \, d\psi &= 0 \end{aligned}}$$

<u>and that's it.</u>

The integral is <u>postulated</u> to be translationally invariant under a ~~sh~~ shift by another Grassmann number η:

$$\boxed{\int F(\psi + \eta) \, d\psi = \int F(\psi) \, d\psi}$$

Note that since $F(\psi) = F_0 + F_1 \psi$, we must have that

$$\boxed{\int \eta \, d\psi = 0} \quad \forall \ \eta \in \text{Grassmann}$$

In general, for a collection of Grassmann #
$(\psi_1, \psi_2, \ldots, \psi_N)$, we postulate that

$$\boxed{\int \psi_i \, d\psi_j = \delta_{ij}}$$

→ A few things to note here:

- There are no limits on the integrals
- $d\psi_j$ is also a Grassmann number so

$$\int d\psi_j \, \psi_i = -\delta_{ij}$$

- Linear: the integral is linear ...

- What about change of variables? i.e. Jacobians?

↳ Start with $\boxed{\chi = a\phi}$

χ: Grassmann, a: ordinary number

Since χ is Grassmann,

$$1 = \int \chi \, d\chi = \int a\phi \frac{d\chi}{d\phi} d\phi = a \frac{d\chi}{d\phi} \int \phi \, d\phi = a \frac{d\chi}{d\phi}$$

and so $\boxed{\dfrac{d\chi}{d\phi} = \dfrac{1}{a}}$, rather than $\dfrac{d\chi}{d\phi} = a$ as you would expect.

More generally, for a linear transformation...

$$\boxed{\phi_i = \sum_j M_{ij} \chi_j \Rightarrow d\phi_i = \sum_j d\chi_j M^{-1}_{ji}}$$

which ensures that new variables have the same integral value as the old, i.e.

$$\int \phi_i d\phi_j = \delta_{ij} = \int \chi_i d\chi_j$$

We will often use the following result:

$$\int \bar{\psi}\psi\, \overline{d\psi d\psi} = 1$$

and

$$\int \bar{\psi}\psi\, \overline{d\psi d\psi} = -1$$

With these, we can do Gaussian integrals...

$$\boxed{\begin{aligned} \int e^{-a\bar{\psi}\psi} d\bar{\psi} d\psi &= a \\ \int e^{-\bar{\psi}M\psi} [d\bar{\psi} d\psi] &= \det M \end{aligned}}$$

{ To check this, simply expand the exponentials and do the integrals.

We also will need the "averages" over the Gaussian integrals, i.e. higher moments of the Gaussian integrals...

$$\boxed{\langle \bar{\psi}\psi \rangle = \frac{\int \bar{\psi}\psi e^{-a\bar{\psi}\psi} d\bar{\psi} d\psi}{\int e^{-a\bar{\psi}\psi} d\bar{\psi} d\psi} = \frac{1}{a} = -\langle \psi\bar{\psi} \rangle}$$

The proof of this is actually quite straightforward.

Now, let's consider a more general problem with 2 sets of Grassmann numbers..

$$\begin{cases} \psi = [\psi_1 \ldots \psi_N] \\ \bar{\psi} = [\bar{\psi}_1 \ldots \bar{\psi}_N] \end{cases}$$

and a Gaussian action. $S = -\bar{\psi} M \psi = -\sum_{ij} \bar{\psi}_i M_{ij} \psi_j$.

Assume that M is Hermitian, we will show that

$$\boxed{\int e^{-\bar{\psi} M \psi} [\mathcal{D}\bar{\psi}\mathcal{D}\psi] = \det M} \quad (\star)$$

where $[\mathcal{D}\bar{\psi}\mathcal{D}\psi] = \prod_1^N d\bar{\psi}_i\, d\psi_i$

Proof. To do this, we ortho-diag M, so that we have

$$\left\{ \begin{array}{l} M V_n = \lambda_n V_n \quad , \quad V_n^\dagger M = V_n^\dagger \lambda_n \\ V_n^\dagger V_m = \delta_{mn} \end{array} \right\}.$$

and write $\psi = \sum \chi_n V_n$; $\bar{\psi} = \sum_n \bar{\chi}_n V_n^\dagger$

Grassmann numbers

Then,

$$Z = \exp\left\{ -\sum_n \lambda_n \bar{\chi}_n \chi_n \right\} [d\bar{\chi}\, d\chi] = \prod_n \lambda_n = \det(M).$$

where we use $\int e^{-a\bar{\psi}\psi} d\bar{\psi} d\psi = a$ for a single pair & the fact that under a unitary transform $[V_{ij}]$ of Grassmann numbers the Jacobian is unity.

(we won't go into details here since the proofs are actually not too hard and probably not entirely instructive).

(b) $\boxed{\text{Resolution of Identity \& Trace}}$

Recall that the resolution of identity in the bosonic land is

$$I = \int |z\rangle\langle z| e^{-z^* z} \frac{dz^* dz}{2\pi i}$$

Here, we claim that $\boxed{I = \int |\psi\rangle\langle\bar{\psi}| e^{-\bar{\psi}\psi} d\bar{\psi} d\psi}$

Proof.

$$\int |\psi\rangle\langle\bar{\psi}| e^{-\bar{\psi}\psi} d\bar{\psi} d\psi$$

$$= \int |\psi\rangle\langle\bar{\psi}| (1-\bar{\psi}\psi) d\bar{\psi} d\psi$$

$$= \int (|0\rangle - \psi|1\rangle)(\langle 0| - \langle 1|\bar{\psi}) (1-\bar{\psi}\psi) d\bar{\psi} d\psi$$

$$= \int (|0\rangle\langle 0| + \psi|1\rangle\langle 1|\bar{\psi}) (1-\bar{\psi}\psi) d\bar{\psi} d\psi$$

$$= |0\rangle\langle 0| \int (-\bar{\psi}\psi) d\bar{\psi} d\psi + |1\rangle\langle 1| \int \psi\bar{\psi} d\bar{\psi} d\psi$$

$$= |0\rangle\langle 0| + |1\rangle\langle 1| = \mathbb{1}.$$

The trace of any <u>bosonic operator Ω</u> (an operator made up of an even # of Fermi operators)...

$$\boxed{\mathrm{Tr}\,\Omega = \int \langle -\bar{\psi}|\Omega|\psi\rangle e^{-\bar{\psi}\psi}\, d\bar{\psi}\, d\psi}$$

The proof then similar to that for the resolution of id

Lastly, we want the δ-function for Grassmann #:

↳ Suppose ψ, χ, η are Grassmann #, then

$$\boxed{\int e^{(\eta-\chi)\psi} F(\eta)\, d\psi\, d\eta = F(\chi)}$$

so that

$$\boxed{\int e^{(\eta-\chi)\psi}\, d\psi = \delta(\eta-\chi)}$$

To see that, we write $F(\eta) = F_0 + F_1\eta$, and expand everything, keeping only terms that survive integration over η & ψ.

$$\text{e.g.}\ \begin{cases} e^{(\eta-\chi)\psi} = 1 + (\eta-\chi)\psi. \\ F(\eta) = F_0 + F_1\eta. \end{cases}$$

(This is straight fwd, so we'll skip over the details).

(i) Thermodynamics of a Fermi Oscillator

→ consider partition fn of a single oscillator...

$$Z = \mathrm{Tr}\, \exp\{-\beta(\Omega_0-\mu)\Psi^\dagger\Psi\}$$

(by defn) $$= \int \langle -\bar\psi | \exp\{-\beta(\Omega_0-\mu)\Psi^\dagger\Psi\}|\psi\rangle e^{-\bar\psi\psi}\, d\bar\psi\, d\psi$$

- But notice that we can't expand this... b/c there are infinitely many terms + Grassmann variables getting mixed up...

→ have to rewrite the operator to support using Grassmann numbers...

$$\boxed{\exp\{-\beta(\Omega_0-\mu)\Psi^\dagger\Psi\} = 1 + (e^{-\beta(\Omega_0-\mu)}-1)\Psi^\dagger\Psi .}$$

note that RHS = LHS if $\Psi^\dagger\Psi = 0$ or $\mathbb{1}$, which is good...

With this, we can expand the integral...

$$Z = \int \langle -\bar\psi | 1 + \{e^{-\beta(\Omega_0-\mu)}-1\}\Psi^\dagger\Psi|\psi\rangle e^{-\bar\psi\psi}\, d\bar\psi\, d\psi$$

$$= \int \underbrace{\langle -\bar\psi|\psi\rangle}\left[1 + (e^{-\beta(\Omega_0-\mu)}-1)(-\bar\psi\psi)\right] e^{-\bar\psi\psi}\, d\bar\psi\, d\psi$$

$$= \int \{1 - (\underbrace{e^{-\beta(\Omega_0-\mu)}-1}_{A})(\bar\psi\psi)\} e^{-2\bar\psi\psi}\, d\bar\psi\, d\psi$$

$$= 2 - A\int \bar\psi\psi\, e^{-2\bar\psi\psi}\, d\bar\psi\, d\psi$$

as expected ✓

$$= 2 - A\left(\frac{-1}{2}\right)\cdot 2 = 2 + A = \boxed{1 + e^{-\beta(\Omega_0-\mu)}}$$

✓

(j) Fermionic Path Integral

We're now ready to map the quantum problem of fermions to a path "integral".

→ Begin with

$$\boxed{Z = e^{-\beta H}}$$

where H is the normal-ordered operator $H(\Psi^{\dagger}, \Psi)$

→ Next, write the exponent in discrete time:

$$e^{-\beta H} = \lim_{N\to\infty} \left(e^{(-\beta H/N)}\right)^N$$

$$= \underbrace{(1-\varepsilon H)\cdots(1-\varepsilon H)}_{N \text{ times} \to \infty}, \quad \varepsilon = \beta/N$$

we have, by the defn of the trace & inserting res of Id. $N-1$ times

$$Z = \mathrm{Tr}\left(e^{-\beta H}\right) = \mathrm{Tr}\left[(1-\varepsilon H)\cdots(1-\varepsilon H)\right]$$

$$= \int \langle -\bar{\psi}_0|(1-\varepsilon H)|\psi_{N-1}\rangle e^{-\bar{\psi}_{N-1}\psi_{N-1}} \langle \bar{\psi}_{N-1}|(1-\varepsilon H)$$

$$|\psi_{N-2}\rangle e^{-\bar{\psi}_{N-2}\psi_{N-2}} \langle \psi_{N-2}|\cdots|\psi_1\rangle e^{-\bar{\psi}_1\psi_1}$$

$$\langle \bar{\psi}_1|(1-\varepsilon H)|\psi_0\rangle e^{-\bar{\psi}_0\psi_0} \prod_{i=0}^{N-1} d\bar{\psi}_i\, d\psi_i$$

Now,

$$\langle \bar{\psi}_{i+1}|(1-\varepsilon H)|\psi_i\rangle = \langle \bar{\psi}_{i+1}|1-\varepsilon H(\Psi^{\dagger},\Psi)|\psi_i\rangle$$

$$= \langle \bar{\psi}_{i+1}|1-\varepsilon H(\bar{\psi}_{i+1},\psi_i)|\psi_i\rangle \sim e^{\bar{\psi}_{i+1}\psi_i} e^{-\varepsilon H(\bar{\psi}_{i+1},\psi)}$$

→ see Fetis, expand everything out...

where in the last step we've anticipating the limit $\varepsilon \to 0$

Now, let us define an additional pair of vars...

$$\boxed{\bar{\psi}_N = -\bar{\psi}_0 \quad \& \quad \psi_N = -\psi_0}$$

With these, we can replace $\langle -\bar{\psi}_0 | \to \langle \bar{\psi}_N |$,
Putting everything together, we find...

$$Z = \int \prod_{i=0}^{N-1} e^{\bar{\psi}_{i+1}\psi_i} e^{-\varepsilon H(\bar{\psi}_{i+1}, \psi_i)} e^{-\bar{\psi}_i \psi_i} d\bar{\psi}_i d\psi_i$$

$$= \int \prod_{i=0}^{N-1} \exp\left\{ \varepsilon \left[\frac{\bar{\psi}_{i+1} - \bar{\psi}_i}{\varepsilon} \psi_i - H(\bar{\psi}_{i+1}, \psi_i) \right] \right\} d\bar{\psi}_i \psi_i$$

$$\Rightarrow \boxed{Z \simeq \int e^{S(\bar{\psi}, \psi)} \left[\mathcal{D}\bar{\psi} \mathcal{D}\psi \right]}, \text{ where}$$

$$S = \int_0^\beta \Big[\underbrace{\bar{\psi}(\tau) \left(\frac{-\partial}{\partial \tau} \right) \psi(\tau)}_{p\dot{q}} - H(\bar{\psi}(\tau), \psi(\tau)) \Big] d\tau .$$

$$\underbrace{p\dot{q} - H}_{\mathcal{L}}$$

From here, we find that, for $\tau_1 > \tau_2$

$$\boxed{\begin{aligned} \langle \psi(\tau_1) \bar{\psi}(\tau_2) \rangle &= \frac{\mathrm{Tr}\left[e^{-H(\beta - \tau_1)} \Psi e^{-H(\tau_1 - \tau_2)} \Psi^\dagger e^{-H\tau_2} \right]}{\mathrm{Tr}\left[e^{-\beta H} \right]} \\ &= \frac{\mathrm{Tr}\left[e^{-\beta H} \Psi(\tau_1) \Psi^\dagger(\tau_2) \right]}{\mathrm{Tr}\, e^{-\beta H}} \end{aligned}}$$

upon invoking $\Psi(\tau_i) = e^{H\tau_i} \Psi e^{-H\tau_i}$, likewise for $\Psi^\dagger$

If $\tau_1 < \tau_2$, have to reorder the Grassmann numbers...

$$\psi(\tau_1)\bar{\psi}(\tau_2) = -\bar{\psi}(\tau_2)\psi(\tau_1)$$

And in general we write:

$$\langle \psi(\tau_1)\bar{\psi}(\tau_2)\rangle = \frac{\mathrm{Tr}\left(e^{-\beta H}\mathcal{T}[\Psi(\tau_1)\Psi^{+}(\tau_2)]\right)}{\mathrm{Tr}\, e^{-\beta H}}$$

with:

$$\boxed{\begin{aligned}\mathcal{T}(\psi(\tau_1)\psi^{+}(\tau_2)) = {} & \theta(\tau_1-\tau_2)\,\Psi(\tau_1)\Psi^{+}(\tau_2) \\ & -\theta(\tau_2-\tau_1)\,\Psi^{+}(\tau_2)\Psi(\tau_1)\end{aligned}}$$

« remember this »

⟨j1⟩ Finite - Temperature Green's function:

The Green's function is directly related to $\langle \Psi(\tau_1)\Psi^{+}(\tau_2)\rangle$

$$G(\tau = \tau_1 - \tau_2) = -\frac{\mathrm{Tr}\left[e^{-\beta H}\mathcal{T}\{\Psi(\tau_1)\Psi^{+}(\tau_2)\}\right]}{\mathrm{Tr}\, e^{-\beta H}}$$

$$= -\langle \psi(\tau_1)\bar{\psi}(\tau_2)\rangle = +\langle \bar{\psi}(\tau_2)\psi(\tau_1)\rangle$$

the (−) sign is just a convention.

Due to anti-commutativity, we have that G is anti-periodic, i.e.

$$\boxed{G(\tau-\beta) = -G(\tau) \quad , \quad 0 \le \tau \le \beta}$$

<u>Proof.</u> Choose $\tau_1 = \beta$, $\tau_2 > 0$ with $\beta > \tau_2$.

$$\begin{aligned}
Z\,G(\beta,\tau_2) &= -\mathrm{Tr}\left[e^{-\beta H}\left(e^{-\beta H}\Psi e^{-\beta H}\right)\left(e^{H\tau_2}\Psi^{+}e^{-H\tau_2}\right)\right] \\
&= -\mathrm{Tr}\left[e^{-\beta H}\Psi^{+}(\tau_2)\Psi(0)\right] \\
&= -\mathrm{Tr}\left[e^{-\beta H}\,T\left\{\Psi^{+}(\tau_2)\Psi(0)\right\}\right] \\
&= +\mathrm{Tr}\left[e^{-\beta H}\,T\left\{\Psi(0)\Psi^{+}(\tau_2)\right\}\right] \\
&= -Z\,G(0,\tau_2)
\end{aligned}$$

Since a sign change by β flips sign of G, it is periodic on $[-\beta,\beta] \Rightarrow$ take FT:

$$\begin{cases} G(\tau) = \displaystyle\sum_{m=-\infty}^{\infty} e^{-i\omega_m\tau}\,G(\omega_m) \quad \text{where } \omega_m = \frac{2\pi m}{2\beta} = \frac{m\pi}{\beta} \\ \text{and} \quad G(\omega_m) = \displaystyle\frac{1}{2\beta}\int_{-\beta}^{\beta} G(\tau)\,e^{i\omega_m\tau}\,d\tau \end{cases}$$

But we can actually split $G(\omega_m)$ in half b/c of the anti-periodicity of G...

Let $\bar{\tau} = \tau + \beta$, we may write ...

$$\frac{1}{2\beta}\int_{-\beta}^{0} G(\tau)\, e^{i\omega_m \tau}\, d\tau = \frac{1}{2\beta}\int_{0}^{\beta} G(\bar{\tau}-\beta)\, e^{-i\omega_m \beta} e^{i\omega_m \bar{\tau}}\, d\bar{\tau}$$

$$= (-1)\, \frac{1}{2\beta}\, e^{-i\omega_m \beta} \int_0^{\beta} G(\bar{\tau})\, e^{i\omega_m \bar{\tau}}\, d\bar{\tau}$$

{ anti-periodicity }

Note further that $\omega_m = \frac{m\pi}{\beta} \Rightarrow e^{i\omega_m \beta} = (-1)^m$.

$\Rightarrow$ If m even, then the integral $\equiv 0$ when in $[0, \beta]$
If $m = (2n+1)$, the integrals are equal in both ranges

$$\Rightarrow \boxed{\begin{aligned} & G(\omega_n) = \frac{1}{\beta}\int_0^{\beta} G(\tau)\, e^{i\omega_n \tau}\, d\tau \\ & \text{where } \omega_n = \frac{(2n+1)\pi}{\beta} \\ & G(\tau) = \sum_n e^{-i\omega_n \tau}\, G(\omega_n) \\ & \int_0^{\beta} e^{i\omega_n \tau} e^{-i\omega_m \tau} = \beta\, \delta_{mn} \end{aligned}}$$

$\rightarrow$ we will use these extensively later.

{ Matsubara frequency }

(Fermionic frequency)

#

(j2) $G(\tau)$ for a free-fermion

Consider a free-fermion for which

$$\boxed{\mathcal{H} = (\Omega_0 - \mu)\, \Psi^\dagger \Psi}.$$

chemical potential

In Euclidean time, the EOM for the Heisenberg op $\Psi(\tau)$ is

$$\boxed{\frac{d}{d\tau}\Psi(\tau) = [\mathcal{H}, \Psi(\tau)] = -(\Omega_0 - \mu)\Psi(\tau)}$$

with solution

$$\begin{cases} \Psi(\tau) = e^{-(\Omega_0-\mu)\tau}\,\Psi . \\ \Psi^\dagger(\tau) = e^{(\Omega_0-\mu)\tau}\,\Psi^\dagger \end{cases}$$

→ Choose $\tau_1 = \tau$, $\tau_2 = 0$, WLOG we find

$$\boxed{\begin{aligned} G(\tau) &= -\theta(\tau)\,\frac{\mathrm{Tr}\{e^{-\beta H}\,\Psi(\tau)\Psi^\dagger(0)\}}{Z} + \theta(-\tau)\,\frac{\mathrm{Tr}\{e^{-\beta H}\,\Psi^\dagger(0)\Psi(\tau)\}}{Z} \\ &= -\theta(\tau)\, e^{-(\Omega_0-\mu)\tau}\left(1 - n_F(\Omega_0-\mu)\right) \\ &\quad + \theta(-\tau)\, e^{-(\Omega_0-\mu)\tau}\, n_F(\Omega_0-\mu) \end{aligned}}$$

where $n_F(\Omega_0-\mu) = \dfrac{\mathrm{Tr}\{e^{-\beta H}\,\Psi^\dagger\Psi\}}{Z} = \dfrac{1}{e^{\beta(\Omega_0-\mu)}+1}$

thermally averaged occupation number.

From here, we calculate $G(\omega_n)$ by defn:

$$G(w_n) = \frac{1}{\beta}\int_0^\beta e^{iw_n\tau} e^{-(\Omega_0-\mu)\tau}\left(1-n_F(\Omega_0-\mu)\right) d\tau$$

$$\boxed{G(w_n) = \frac{1}{\beta}\frac{1}{iw_n-(\Omega_0-\mu)}}$$

<u>As $T \to 0$, we have</u>

$$G(\tau) = -\theta(\tau)\, e^{-(\Omega_0-\mu)\tau} \quad (\mu < \Omega_0)$$

$$= +\theta(\tau)\, e^{-(\Omega_0-\mu)\tau} \quad (\mu > \Omega_0)$$

(j3) <u>Fermion Path Integral in frequency space</u>

Recall the path integral for fermions:

$$Z = \int e^{S(\bar{\Psi},\Psi)} [\mathcal{D}\bar{\Psi}\, \mathcal{D}\Psi] \text{ where}$$

$$S = \int_0^\beta \left[\bar{\Psi}(\tau)\left(\frac{-\partial}{\partial\tau}\right)\Psi(\tau) - H(\bar{\Psi}(\tau),\Psi(\tau))\right] d\tau$$

with $\mathcal{H} = (\Omega_0-\mu)\,\Psi^\dagger\Psi$, we have...

$$S = \int_0^\beta \bar{\Psi}(\tau)\left(\frac{-\partial}{\partial\tau} - w_0 + \mu\right)\Psi(\tau)\, d\tau.$$

Here, we want to express S in frequency space.

To do this, need to write $\Psi(\tau)$, $\bar{\Psi}(\tau)$ in terms of Matsubara freqs...

$$w_n = \frac{(2n+1)\pi}{\beta}..$$

$$\begin{cases} \bar{\psi}(\tau) = \sum_n e^{i\omega_n \tau} \bar{\psi}(\omega_n) \\ \psi(\tau) = \sum_n e^{-i\omega_n \tau} \psi(\omega_n) \end{cases}$$

these are NOT conjugates!

We then get inversions ...

$$\begin{cases} \psi(\omega_n) = \frac{1}{\beta} \int_0^\beta \psi(\tau) e^{i\omega_n \tau} d\tau \\ \bar{\psi}(\omega_n) = \frac{1}{\beta} \int_0^\beta \bar{\psi}(\tau) e^{-i\omega_n \tau} d\tau \end{cases}$$

where we use the orthogonality property:

$$\frac{1}{\beta} \int_0^\beta e^{i(\omega_n - \omega_m)\tau} d\tau = \delta_{nm}$$

if $\beta \to \infty$ then become $\omega_n = \frac{(2n+1)\pi}{\beta}$ implying that when $n' = n+1$, ω_n changes by $2\pi/\beta$, so that

$$\frac{1}{\beta} \sum_n \xrightarrow{\text{continuum}} \int \frac{d\omega}{2\pi}$$

With this, we can see how the action of the Fermi oscillator transforms under FT:

$$\boxed{\begin{aligned} S &= \int_0^\beta \bar{\psi}(\tau) \left(\frac{-\partial}{\partial \tau} - (\Omega_0 - \mu) \right) \psi(\tau) \, d\tau \\ &= \beta \sum_n \bar{\psi}(\omega_n) \left[i\omega_n - (\Omega_0 - \mu) \right] \psi(\omega_n) \end{aligned}}$$

In the $\beta \to \infty$ limit, $\omega_n \to$ continuous ω.

If we introduce rescaled Grassmann vars

$$\bar{\Psi}(\omega) = \beta\bar{\Psi}(\omega_n) \quad , \quad \Psi(\omega) = \beta\Psi(\omega_n)$$

and use $\frac{1}{\beta}\sum_n \to \int \frac{d\omega}{2\pi}$, we find that the action when $\beta \to \infty$ is

$$S = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \bar{\Psi}(\omega)\left[i\omega - \Omega_0 + \mu\right]\Psi(\omega)$$

Because the Jacobian for $(\bar{\Psi}(\tau), \Psi(\tau)) \to (\bar{\Psi}(\omega), \Psi(\omega))$ is $\underline{1}$, the path integral is

$$\boxed{Z = \int \exp\left\{\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \bar{\Psi}(\omega)\left[i\omega - \Omega_0 + \mu\right]\Psi(\omega)\right\}\left[\mathcal{D}\bar{\Psi}(\omega)\,\mathcal{D}\Psi(\omega)\right]}$$

What about correlation function?

$$\langle \bar{\Psi}(\omega_1)\Psi(\omega_2)\rangle = \frac{1}{Z}\int \bar{\Psi}(\omega_1)\Psi(\omega_2)\exp\left[\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\bar{\Psi}(\omega)\left(i\omega - \Omega_0 + \mu\right)\Psi(\omega)\right]\left[\mathcal{D}\bar{\Psi}\mathcal{D}\Psi\right]$$

$$\Rightarrow \quad \boxed{\langle \bar{\Psi}(\omega_1)\Psi(\omega_2)\rangle = \frac{2\pi\,\delta(\omega_1 - \omega_2)}{i\omega_1 - \Omega_0 + \mu}}$$

because $G(\omega) = \beta^2 G(\omega_n)$ due to the rescaling

And, in particular...

$$\boxed{G(\omega) = \langle \bar{\Psi}(\omega)\Psi(\omega)\rangle = \frac{2\pi\,\delta(0)}{i\omega - \Omega_0 + \mu} = \frac{\beta}{i\omega - \Omega_0 + \mu}}$$

With this, we can calculate the mean occupation #:

$$\langle N \rangle = \frac{1}{\beta \mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial \mu}$$

single-particle system, so outer integral vanishes.

$$= \frac{1}{\beta} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \langle \bar{\Psi}(\omega) \Psi(\omega) \rangle$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega 0^+}}{i\omega - \Omega_0 + \mu}$$

allows us to close contour in upper half plane. (to make integral converge).

$$= \Theta(\mu - \Omega_0)$$

which is what we found in the operator approach.

(k) Generating Functions $\mathcal{Z}(J)$ & $W(J)$

→ generating functions allow us to generate "moments" i.e. all correlation functions.

We will discuss generating fns for
- Ising spins
- real bosonic vars
- complex bosonic vars
- Grassmann vars.

(k1) Ising Correlators

Recall spins in a magnetic field... $h = (h_1, \ldots, h_N)$

$$\boxed{\mathcal{Z}(h_1, \ldots, h_N) \equiv \mathcal{Z}(h) = \sum_{s_i} \exp\left\{ -E(K, s_i) + \sum_i h_i s_i \right\}}$$

where $E(K, s_i)$ ~> energy as fn of spins, other params, ...

From here, we have ...

$$\begin{cases} \langle s_i \rangle = \dfrac{1}{Z}\dfrac{\partial Z}{\partial h_i} \\ \\ \langle s_i s_j \rangle = \dfrac{1}{Z}\dfrac{\partial^2 Z}{\partial h_i \partial h_j} \\ \quad \vdots \end{cases}$$

Just like in probability where we also have the MGF, we have Z.

We can also define a fn $W(h)$ by

$$\boxed{Z(h) = e^{-W(h)} = e^{-\beta F}}$$

We can see that derivatives of $W(h)$ give the <u>connected</u> correlation fns...

$$\frac{-\partial^2 W}{\partial h_i \partial h_j} = \langle s_i s_j \rangle_c = \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle$$

$$\begin{aligned} \frac{-\partial^4 W}{\partial h_i \partial h_j \partial h_k \partial h_l} &= \langle s_i s_j s_k s_l \rangle_c \\ &= \langle s_i s_j s_k s_l \rangle_c - \Big[\langle s_i s_j \rangle \langle s_k s_l \rangle + \langle s_i s_k \rangle \langle s_j s_l \rangle \\ &\qquad + \langle s_i s_l \rangle \langle s_j s_k \rangle \Big] \\ &\ \ \vdots \end{aligned}$$

(k2) Real Scalar Vars

This is introductory QFT, so we won't go into much detail here...

The idea is that with $|x) = \{x_1 \ldots x_N\}$, $x_i \in \mathbb{R}$, we have

$$Z(J) = \prod_{i=1}^{N} \int_{-\infty}^{\infty} dx_i \, e^{-S_0(x) + \langle J|x\rangle} = e^{-W(J)}$$

$S_0(x)$: don't involve J

$\langle J|x\rangle$: source (generic)

$$\langle J|x\rangle = \sum J_i x_i$$

Correspondence w/ Ising correlators: $\begin{cases} s_i \to x_i \\ h_i \to J_i \end{cases}$

$$-\left.\frac{\partial^2 W}{\partial J_i \partial J_j}\right|_{J=0} = (x_i x_j)_c = \langle x_i x_j\rangle$$

and so on...

$\boxed{\text{If}}$ $S_0(x)$ is quadratic then everything is nice.

i.e.

$$\boxed{\text{all connected correlators} = 0 \text{ except } \langle x_i x_j\rangle}$$

(Wick's Thm in QFT)

Recall that $$\int_{-\infty}^{\infty} e^{-\frac{1}{2}mx^2 + Jx} = \sqrt{\frac{2\pi}{m}} \exp\left[\frac{J^2}{2m}\right]$$

Consider $S(J) = -\frac{1}{2}\langle x|M|x\rangle + \langle J|x\rangle$

where M symmetric real. Then we can show (look at old notes...) that

$$\boxed{Z(J) = e^{\frac{1}{2}\langle J|M^{-1}|J\rangle} \cdot \frac{(2\pi)^{N/2}}{\sqrt{\det M}}}$$

and

$$W(J) = \underbrace{-\frac{1}{2}\langle J|M^{-1}|J\rangle + \frac{1}{2}\ln(\det M)) + \text{etc.}}$$

$$\hookrightarrow \quad \langle x_i x_j \rangle = \frac{-\partial^2 W}{\partial J_i \partial J_j}\bigg|_{J=0} = (M^{-1})_{ij} \equiv G_{ij}$$

(there are results about $\langle x_i x_j x_k x_\ell \rangle$ which we won't worry about here...)

If S is not quadratic, then we may expand the exponential in pwr series ~~then~~ and ~~do~~ calc. things perturbatively...

↳ There are also results about replacing $|X\rangle$ with the field $\phi(x)$, but that's QFT.

↳ We will pull results for QFT to here at some point, but perhaps not now.

(we won't worry about it now...

#

(h3) Complex Scalar Variables

↳ look at partition function

$$Z = \int_{-\infty}^{\infty} \frac{dz\,dz^*}{2\pi i} e^{-m z^* z}.$$

use $\dfrac{dz\,dz^*}{2\pi i} \longrightarrow \dfrac{dx\,dy}{\pi} = \dfrac{r\,dr\,d\theta}{\pi}$ — Real Jeez

and get $Z = \displaystyle\int_{-\infty}^{\infty} \frac{dx\,dy}{\pi} e^{-m(x^2+y^2)} = \frac{1}{m}$

↳ & that

$$\langle z^* z \rangle = \frac{1}{Z} \int_{-\infty}^{\infty} z^* z \, e^{-a z^* z} \, dz\,dz^*/2\pi i = \frac{1}{a}.$$

and $\langle z z \rangle = \langle z^* z^* \rangle = 0.$

- In general, if $z \in \mathbb{C}^d$, then we have (for Hermitian M)

$$\boxed{Z = \int [\mathcal{D}_z^* \mathcal{D}_z^\dagger] \, e^{-z^\dagger M z}}$$

where $[\mathcal{D}_z \mathcal{D}^\dagger z] = \displaystyle\prod_{i=1}^{N} \int dz_i\,dz_i^* / 2\pi i$

eigvs of M.

$z = (z_1 \ldots z_N)$, $z^\dagger = (z_1^* \ldots z_N^*)^T$

and

$$\boxed{Z = \prod_i \frac{1}{m_i} = [\det(M)]^{-1}}$$

We can check that $\boxed{\langle z_i^\dagger z_j \rangle = \frac{\delta_{ij}}{m_i} \equiv \langle \bar{i} j \rangle}$

Wick's Thm: $\langle z_i^\dagger z_j z_k^\dagger z_l \rangle = \langle \bar{i} j \rangle \langle \bar{k} l \rangle + \langle \bar{i} l \rangle \langle \bar{k} j \rangle$

- Now, add the source $\vec{J}$: 2 sources: $\vec{J}$, $\vec{J}^\dagger$

$$\mathcal{Z} = \mathcal{Z}(J, J^\dagger) = e^{-W(J,J^\dagger)}$$
$$= \int [\mathcal{D}z \mathcal{D}^\dagger z]\, e^{-z^\dagger M z + J^\dagger z + z^\dagger J}$$

We can of course move to complex fields ϕ in 4-d. But we won't worry about that now...

(since there (standard) results can be found in any QFT text book...)

(L4) Grassmann Variables

All the preceding machinery can be applied to Grassmann integrals...

→ We just have to recall the appropriate defns first...

$$\boxed{\begin{aligned} &Z(J,\bar{J}) = \int \exp\left[S(\bar{\Psi},\Psi) + \bar{J}\Psi + \bar{\Psi}J\right][\mathcal{D}\bar{\Psi}\mathcal{D}\Psi] \\ &\bar{J}\Psi = \sum_i \bar{J}_i \Psi_i \qquad \text{where } (\bar{J},J) \text{ are also grassmann variables.} \\ &\bar{\Psi}J = \sum_i \bar{\Psi}_i J_i \\ &[\mathcal{D}\bar{\Psi}\mathcal{D}\Psi] = \prod_i^N d\bar{\Psi}_i\, d\Psi_i \end{aligned}}$$

We can readily check that

$$\langle \bar{\Psi}_\beta \Psi_\alpha \rangle = \frac{-1}{Z}\frac{\partial^2 Z}{\partial J_\beta \partial \bar{J}_\alpha}$$

by following these steps

- expand source term to linear ord in exp
- take derivatives, w/ $\partial/\partial J$, $\partial/\partial\bar{J}$ commute w/ even terms like $\bar{\Psi}J$, $\bar{J}\Psi$
- remember that $\langle \bar{\Psi}_\beta \Psi_a\rangle = -\langle \Psi_a \bar{\Psi}_\beta\rangle$

→ Next, can define $W(\bar{J},J)$ by

$$\boxed{Z(J,\bar{J}) = e^{-W(\bar{J},J)}}$$

from which we get the 2-pt connected correlator:

$$\left.\frac{\partial^2 W}{\partial J_\beta \partial \bar{J}_\alpha}\right|_{J,\bar{J}=0} = \langle \bar{\Psi}_\beta \Psi_\alpha\rangle_c = \langle \bar{\Psi}_\beta \Psi_\alpha\rangle \equiv \langle \bar{\beta}\alpha\rangle$$

where we have used $\langle \bar{\Psi}_\beta\rangle = \langle \Psi_a\rangle = 0$ when $(J,\bar{J}) = 0$

The connected 4-pt correlation fn is:

$$\left.\frac{-\partial^4 W}{\partial J_\alpha \partial J_\beta \partial \bar{J}_\gamma \partial \bar{J}_\delta}\right|_{(\bar{J},J)=0} = \langle \bar{\alpha}\bar{\beta}\gamma\delta\rangle_c$$

$$= \langle \bar{\alpha}\bar{\beta}\gamma\delta\rangle - \left[\langle\bar{\alpha}\delta\rangle\langle\bar{\beta}\gamma\rangle - \langle\bar{\alpha}\gamma\rangle\langle\bar{\beta}\delta\rangle\right]$$

The central result, when Ψ becomes a tuple, is

$$\boxed{\int e^{-\bar{\Psi}M\Psi + \bar{J}\Psi + \bar{\Psi}J}\,[d\bar{\Psi}d\Psi] = \det(M)\, e^{\bar{J}M^{-1}J}}$$

Proof. making the following translation:

$$\Psi \to \Psi + M^{-1}J$$
$$\bar{\Psi} \to \bar{\Psi} + \bar{J}M^{-1}$$

And so, for a Gaussian action,

$$\boxed{W(J,\bar{J}) = -\ln\det(M) - \bar{J}M^{-1}J}$$

from which we find

$$\boxed{\langle\bar{\Psi}_\beta\Psi_\alpha\rangle_c = \frac{\partial^2 W}{\partial J_\beta\,\partial\bar{J}_\alpha} = -M^{-1}_{\alpha\beta}}$$

and

$$\boxed{\langle\bar{\alpha}\bar{\beta}\gamma\delta\rangle_c = 0}$$

since we can differentiate W only twice...

→ ~~the~~ Only the connected 2-pt corr fn is non zero in a fermionic Gaussian theory, as in bosonic theories

7: THE 2D ISING MODEL

Mar 19 2021

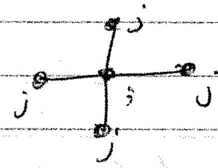
→ This is the easiest example of a solvable system.

ⓐ Intro → Now, we'll focus on the magnetic transition in the Ising model.

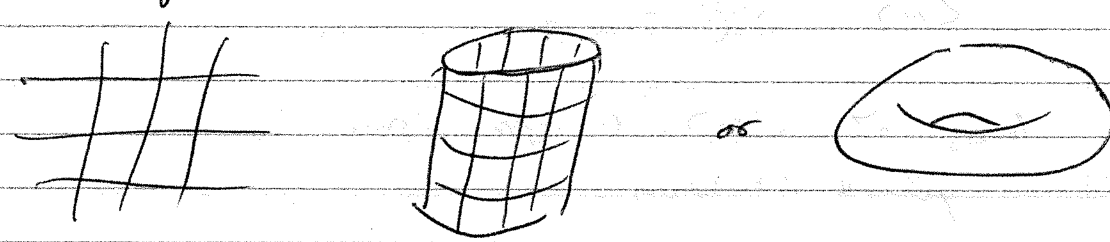
Model: Consider model with N cols × M rows

$$Z = \sum_{s_i} \exp\left\{ K \sum_{\langle i,j \rangle} s_i s_j \right\}$$

where $K = J/kT$, $\langle i,j \rangle$ means nearest neighbor



3 types of boundary conditions:



open — cylinder (periodic in 1 direction) — torus (periodic in 2 directions)

{ Assume total = $\mathcal{N} = M \cdot N \ggg 1$

Assume that we're not near the "end" of the model

→ Total # bonds = $2\mathcal{N}$.

<u>Extremal limits</u>:

$K = J/_{kT} \to \infty$ i.e. $T \to 0$ $\Rightarrow$ all spin up or down

→ system is magnetized
→ $\boxed{\langle M \rangle = \pm 1}$ → can pick $\langle M \rangle = +1$
↓
average
spin per site

$K = J/_{kT} \to 0$ i.e. $T \to \infty$ $\boxed{\langle M \rangle = 0}$ since Boltzmann weights will be 1 for all configurations

→ How does $\langle M \rangle$ look like as a fn of T?

$$\begin{cases} \langle M \rangle = 1 \text{ @ } T = 0 \\ \langle M \rangle = 0 \text{ @ } T = \infty \end{cases}$$

Now, if $\langle M \rangle = 0$ after some T_c then we have a phase transition.

If this is true then there must be a singularity @ T_c ~~fine~~ (since nice analytic functions don't do this)

→ To see this singularity, we must go to the thermodynamic limit.

{ → Particularly want to look @ "free energy per site"

In 1D → no phase transition @ finite T

In 2D however, there is finite-T phase transition.

(b) High-temp expansion (MIT ocw)

Basic idea: $K=0 \Leftrightarrow T=\infty \rightsquigarrow$ all Boltzmann weights are 1.

$\rightarrow$ For each bond we have (since $(\sigma_i\sigma_j)^2=1$)

$$\boxed{e^{K\sigma_i\sigma_j} = \frac{e^K+e^{-K}}{2} + \frac{e^K-e^{-K}}{2}\sigma_i\sigma_j}$$

(check this (easy))

so we can write

$$e^{K\sigma_i\sigma_j} = \cosh K\left(1+\tanh K\,\sigma_i\sigma_j\right)$$

Applying this to find Z.

$$\boxed{Z = \sum_{\{s_i\}} e^{K\sum_{\langle i,j\rangle} s_i s_j} = (\cosh K)^{\#\text{bonds}} \sum_{s_i}\prod_{\langle i,j\rangle}(1+\tanh K\, s_i s_j)}$$

For N_b bonds or N bonds in the lattice, the product generates 2^N terms, which can be represented diagrammatically by drawing a line connecting sites i,j for each factor of $\tanh K(s_i s_j)$

$\rightarrow$ There can be at most 1 line for each bond
$\rightarrow$ either empty or occupied.

$\rightarrow$ each site obtains a factor of $s_i^{p_i}$ where p_i is # occupied bonds emanating from i.

Since $\sigma_i = \pm 1$, summing over gives a factor of

$$\begin{cases} 2 & \text{if } p_i \text{ even} \\ 0 & \text{if } p_i \text{ odd} \end{cases}$$

→ The only graphs that survive the sum have an <u>even</u> # of lines passing through each site

⇒ The resulting graphs are collections of closed paths and

(needs some explaining)

$$\boxed{Z = 2^N \times (\cosh K)^{N} \sum_{\substack{\text{all}\\ \text{closed}\\ \text{graphs}}} (\tanh K)^{\#\text{ bonds in graph}}}$$

For d-dimensional hypercubic lattice, the smallest closed graph is a square of 4 bonds which has $d(d-1)/2$ possible orientations, and so on...

$$\Rightarrow \boxed{Z = 2^N (\cosh K)^{dN} \left\{ 1 + \frac{d(d-1)N}{2} (\tanh K)^4 + d(d-1)(2d-3)(\tanh K)^6 + \dots \right\}}$$

→ when $d = 2$, we have

$$\boxed{Z = 2^N (\cosh K)^{2N} \left\{ 1 + N(\tanh K)^4 + 2N(\tanh K)^6 + (\tanh K)^8 \left(6N + \frac{1}{2} N(N-5)\right) + O(t^{10}) \right\}}$$

How do we get this expression, really?

→ look at $Z = \sum_{s_i} \prod_{\substack{\text{bond}\\ ij}} \underbrace{(\cosh K)}_{A} (1 + s_i s_j \underbrace{\tanh K}_{B})$

$$= \sum_{s_i} \prod_{\substack{\text{bond}\\ ij}} A(1 + s_i s_j \cancel{B})$$

- There are 2^N terms in the product over bonds (each site has $s_i = \pm 1$).
- Each bond contributes a (1) or a ($\tanh K$).

$$= (\cosh K)^{2N} \sum_{s_i} \prod_{\substack{\text{bond}\\ ij}} (1 + s_i s_j B)$$

Sum over all configs

product over links in the lattice. (2^N terms)

{ (1) is represented by an empty edge and ($\tanh K$) $s_i s_j$ by an occupied edge }

→ $\sum_{s_i} \prod_{\substack{\text{bond}\\ ij}} (1 + s_i s_j B) = \sum_{s_i} \boxed{?}$

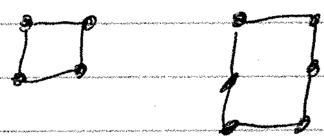
ij ← n.n only

This $\boxed{?}$ is $\displaystyle\sum_{G \subseteq \text{lattice}} (\tanh K)^{\#\text{edges of } G} \prod_{(ij)\in G} s_i s_j$

graphs on the lattice.

To get non zero contributions, every factor s_j should occur with even power, and sum over s_j gives a factor of 2.

↳ selects G which are collections of closed polygons



etc ...

Summing over spins gives...

$$Z = (\cosh K)^{2N} \, 2^{N} \sum_{G \in C} (\tanh K)^{\#\text{edges of } G}$$

set of closed graphs

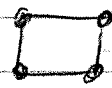
Now, what is

$$\sum_{G\in C} (\tanh K)^{\#\text{edges of } G} \; ?$$

Well... there is a factor of $2^N \cosh K^{2N}$ for every bond...

First nonzero contribution is $(\tanh K)^4$: picking 4

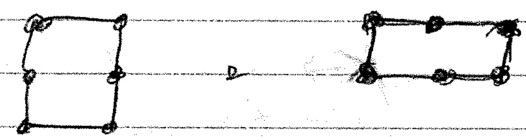
4 bonds to form a square. The spin @ each corner appears twice



$\hookrightarrow$ get $2^N (\cosh K)^{2N} \cdot N \tanh^4 K$

# of squares we can have (by labeling)

With 6 bonds, there are 2 possible orientations:



and so on for

So we have

$$\frac{Z(K)}{2^N (\cosh K)^{2N}} = 1 + N \tanh^4 K + 2N \tanh^6 K + \ldots$$

okay... so we have, in general...

$$\frac{Z(K)}{2^N (\cosh K)^{2N}} = \sum_{\text{closed loops}} C(L) \tanh^L(K)$$

loops of length L without covering any bond more than once

With this, can calculate free ~~energy~~ per site

$$-\frac{f}{kT} = \frac{1}{N} \ln Z = \ln\left[2\cosh^2 K\right] + \frac{1}{N} \ln\left(1 + N \tanh^4 K + \ldots\right)$$

$$\cong \ln\left[2\cosh^2 K\right] + \tanh^4 K + \ldots$$

This is not too important now...

(c) Low Temp Expansion

- So before, we did the expansion when $T >> 1$ in which case all Boltzmann weights are 1.

- When $T = 0$, $K \to \infty$. The spins tend to be aligned in one direction...

  ↳ Assume this direction is up.

Boltzmann weight now is e^K on each of the $2N$ bonds. In this case, the bonds are unbroken.

If, however, one spin is flipped down, 4 bonds linking to it will be reduced by e^{-8K}

(energy excitation)

↳ This can occur at any site, we have

$$Z = e^{2NK}(\underset{\substack{\uparrow \\ \text{no flip}}}{1} + \underset{\substack{\uparrow \\ \text{one flip}}}{N}e^{-8K} + \dots)$$

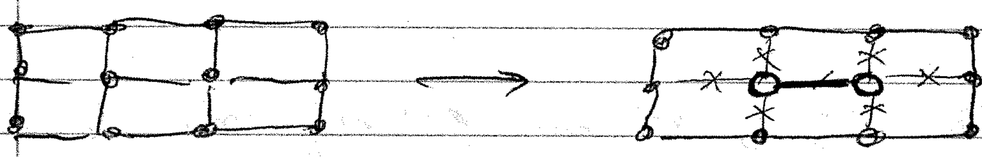
what if 2 spins flip? → we only care if the 2 are nearest neighbor... In this case, there are:

- N ways to choose the first spin.

- 2 ways to choose the 2^{nd} (not 4, since we'll double-count)

So we have: $$Z = e^{2NK}\left(1 + Ne^{-8K} + 2Ne^{-12K} + \cdots\right)$$

Why e^{-12K}: since the bond between the flipped spin isn't broken, but between the flipped spins & other (6) are.



→ things are reduced by e^{-12K}.

We can keep going to get

$$Z = e^{2NK}\left[1 + Ne^{-8K} + 2Ne^{-12K} + \frac{N(N-3)}{2}e^{-16K} + \cdots\right]$$

Can find the free energy per site by

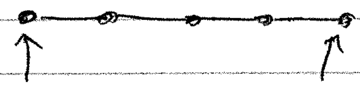
$$\left(-\frac{f}{kT} = \frac{1}{N}\ln Z\right).$$

See that if we create an island of spins pointing opposite to the majority, it costs an energy proportional to the perimeter of the island –

↳ reflected in the Boltzmann factor e^{-2KL}

(perimeter of island.)

- In the 1D model ... the cost of an island is always $\boxed{e^{-2KL}}$ regardless of size



perim = 1 + 1 = 2.

$\Downarrow$

the 1D model loses magnetization above $T = 0$.

- But in 2D, we can actually find T_c by asking when large islands will go unsuppressed.

↳ Imagine having a loop of length L ... at each stage we can go in 3 directions ... so

Energy cost is $\boxed{3^L e^{-2KL}}$ (roughly speaking)

↳ loops of arbitrary size no longer suppressed when we reach ...

$$e^{\underbrace{(-2K_c + \ln 3)}_{0} L} = 1 \Leftrightarrow \boxed{K_c \sim 0.5493}$$

(The exact result is $K_c = 0.4407$)

Can do similar analysis for high temp series to estimate K_c:

$$(\tanh K_c)^L \cdot 3^L \sim 1 \Rightarrow \boxed{K_c = 0.3466}$$

→ The correct answer:

$$0.3466 < 0.4407 < 0.5493$$

(d) Kramers - Wannier Duality

Kramers & Wannier discovered a hidden symmetry that relates properties of Ising model on the square lattice at low & high temps.

↳ Compare low & high temp expansion...

Low: $$Z = e^{2NK}\left[1 + Ne^{-4\times 2K} + 2Ne^{-6\times 2K} + \cdots\right]$$

$$= e^{2NK} \sum_{\substack{\text{Island}\\\text{of } (-)\\\text{spins}}} e^{-2K\times L} \leftarrow \text{perimeter of island}$$

High: $$Z = 2^N(\cosh K)^{2N}\left[1 + N\tanh K^4 + 2N\tanh K^6 + \cdots\right]$$

$$= 2^N(\cosh K)^{2N} \sum_{\substack{\text{Graphs}\\\text{w/ } 2/4 \text{ lines}\\\text{per site}}} (\tanh K)^{\text{Length of graph}}$$

As the boundary of any island of spins serves as an acceptable graph

↳ there is a 1-1 correspondence between 2 series...

↳ Define function g to indicate the log of the above series...

↳ Free energy $= \dfrac{\ln Z}{N} = 2K + g(e^{-2K})$

$= \ln(2) + 2\ln\cosh K + g(\tanh K)$

The argument of g in the eqn above are related by the duality condition

$$\boxed{e^{-2\tilde{K}} \longleftrightarrow \tanh K \Rightarrow \tilde{K} = D(K) \equiv -\frac{1}{2} \ln \tanh K}$$

g must have a special symmetry that relates its value for dual arguments. –

(1) Low temp mapped to high temp & vice versa

(2) The map connects pairs of points since

$$D(D(K)) = K \, :$$

$$\sinh 2K = 2 \sinh K \cosh K = 2 \tanh K \cosh^2 K$$

$$= \frac{2 \tanh K}{1 - \tanh^2 K} = \frac{2 e^{-2\tilde{K}}}{1 - e^{-4\tilde{K}}}$$

$$= \frac{2}{e^{2\tilde{K}} - e^{-2\tilde{K}}} = \frac{1}{\sinh 2\tilde{K}} .$$

$$\boxed{(\sinh 2K)(\sinh 2\tilde{K}) = 1}$$

how the dual interactions are symmetrically related by this relation.

If $g(K)$ is singular at a point K_c, it must also be singular at $\tilde{K}_c$.

$\Rightarrow$ Since the free energy is expected to be analytic everywhere except at the transition

$\hookrightarrow$ Critical model must be SELF-DUAL.

$\Rightarrow$ At self-dual point,

$$e^{-2K_c} = \tanh K_c = \frac{1 - e^{-2K_c}}{1 + e^{-2K_c}}.$$

$$\hookrightarrow e^{-4K_c} + 2e^{-K_c} - 1 = 0 \Rightarrow e^{-2K_c} = -1 \pm \sqrt{2}$$

$\Rightarrow$ Only positive solution is acceptable..

$$\boxed{K_c = -\frac{1}{2}\ln(\sqrt{2}-1) = \frac{1}{2}(\sqrt{2}+1) = 0.4407.}$$

If $K_c \sim 0.4407$, then we can work out what T_c is by the relation $K = J/kT$.

ⓔ <u>Correlation Function in the Tanh Expansion</u>

At high temp:

$$\langle s_i s_f \rangle = \frac{\sum_s s_i s_f \prod_{\text{bonds}} (1 + s_m s_n \tanh K)}{\sum_s \prod_{\text{bonds}} (1 + s_m s_n \tanh K)}$$

$$= (\tanh K)^{|i-f|} (1 + \ldots)$$

We won't worry about this for now

Mar 20, 2021

8: EXACT SOLUTION OF THE 2D ISING MODEL

(a) Transfer Matrix in Terms of Pauli Matrices

Consider model:

$$\boxed{Z = \sum_{s} \exp\left[\sum_{i} K_x s_i s_{i+x} + K_\tau (s_i s_{i+\tau} - 1)\right]}$$

where $i+x$ & $i+\tau$ are neighbors of site i in the x & τ directions, where we have subtracted the 1 from $s_i s_{i+\tau}$ so we can borrow 1D results.

↳ Get transfer matrix for a lattice with N columns is:

$$T = \underbrace{\frac{\exp\left[\sum_{n=1}^{N} K_\tau^* \sigma_1(n)\right]}{[\cosh K_\tau^*]^N}}_{V_1} \cdot \underbrace{\exp\left[\sum_{n=1}^{N} K_x \sigma_3(n)\sigma_3(n+1)\right]}_{V_3}$$

$$\boxed{T \equiv V_1 V_3}$$

where $\sigma_1(n)$, $\sigma_3(n)$ are Pauli matrices @ site n.

<u>Check</u> eigenvalues of $\sigma_3(n)$, $n = 1, 2, 3, \dots N$

$$\langle s_1' s_2' \dots s_N' | T | s_1 \; s_2 \dots s_n \rangle$$
$$\|$$
$$\langle s_1' s_2' \dots s_N' | V_1 V_3 | s_1 s_2 \dots s_n \rangle$$
$$\downarrow$$

$\sim \sigma_3(n)\sigma_3(n+1) \Rightarrow$ turn into s_n and so on

→ give Boltzmann weights associated w/ <u>horizontal bonds</u> of the row containing s_n's

For V_1 --- we can factorize, and look at contribution

$$\langle s_n' | \frac{e^{K_\tau^* \sigma_1(n)}}{\cosh K_\tau^*} | s_n \rangle = \langle s_n' | (1 + \sigma_1(n) \tanh K_\tau^*) | s_n \rangle$$

$$= (\delta_{s_n s_n'} + \tanh K_\tau^* \, \delta_{s_{n}, -s_n'})$$

$$= (\delta_{s_n s_n'} + e^{-2K_\tau} \delta_{s_{n}, -s_n'})$$

which is the Boltzmann weight due to vertical bond at site n

So we roughly see that $\begin{cases} V_3 \text{ transfer horizontally} \\ V_1 \text{ transfer vertically.} \end{cases}$

→ note that T not hermitian, so we can use the following version

$$\boxed{T = V_3^{1/2} V_1 V_3^{1/2}}$$

But we might ignore the hermiticity since it won't matter in the continuum limit.

(b) The Jordan-Wigner transformation & Majorana Fermions

It's hard to diagonalize the transfer matrix T.

↳ why? B/c Pauli matrices are neither Bosonic nor Fermionic, since they anti-commute at one site but commute at different sites.

Ex

So, we need to trade Pauli matrices for Majorana Fermions, which have canonical anti-comm relation
↳ even before & after FT.

- How to make Majorana Fermions?

↳ Start w/ Fermions (Dirac) Ψ which obey the usual anti-comm relation

$$\begin{cases} \{\Psi, \Psi^+\} = 1, \quad \{\Psi, \Psi\} = \{\Psi^+, \Psi^+\} = 0. \\ n_F = \Psi^+ \Psi = \{0, 1\}. \end{cases}$$

→ Make Majorana Fermions by ~~be~~ combining these

$$\boxed{\begin{aligned} \psi_1 &= \frac{1}{\sqrt{2}} (\Psi + \Psi^+) \\ \psi_2 &= \frac{1}{\sqrt{2}i} (\Psi - \Psi^+) \end{aligned}}$$

So that $\boxed{\{\psi_i, \psi_j\} = \delta_{ij}}$

The algebra Majorana fermions obey is called the Clifford Algebra.

<u>Inverse relation:</u>
$$\begin{cases} \Psi = \dfrac{\psi_1 + i\psi_2}{\sqrt{2}} \\ \Psi^+ = \dfrac{\psi_1 - i\psi_2}{\sqrt{2}}. \end{cases}$$

At site n, suppose we have a pair of Majorana Fermions

$\psi_1(n)$ & $\psi_2(n)$ $\Rightarrow$ live in 2D space.

→ If we have a lattice the fermions will need a Hilbert space of dimension 2^N.

↳ This is of course also the dimensionality of Pauli matrices, However, the relation between Pauli matrices & Majorana fermions is non-local

$$\boxed{\begin{aligned} \psi_1(n) &= \begin{cases} \frac{1}{\sqrt{2}}\left[\prod_{\ell=1}^{n-1}\sigma_1(\ell)\right]\sigma_2(n) & n>1 \\ \frac{1}{\sqrt{2}}\,\sigma_2(1) & n=1 \end{cases} \\ \psi_2(n) &= \begin{cases} \frac{1}{\sqrt{2}}\left[\prod_{\ell=1}^{n-1}\sigma_1(\ell)\right]\sigma_3(n) & n>1 \\ \frac{1}{\sqrt{2}}\,\sigma_3(1) & n=1 \end{cases} \end{aligned}}$$

This is the Jordan-Wigner Transformation

We can check that

$$\boxed{\{\psi_i(n),\ \psi_j(n')\} = \delta_{ij}\,\delta_{nn'}}$$

The role of the string (the product $\prod \sigma_{\ldots}$) is to ensure that ψ's @ different sites n & n' anti-commute. Example: $n' > n$ ⇒ then in the product of 2 fermions, the $\sigma_1(n)$ in $\psi_{1,2}(n')$ will anti-commute with both the $\sigma_2(n)$ & $\sigma_3(n)$ in $\psi_{1,2}(n)$.

Nice properties

$$\boxed{\begin{aligned} \sigma_1(n) &= -2i\,\psi_1(n)\,\psi_2(n) \\ \sigma_3(n)\,\sigma_3(n+1) &= 2i\,\psi_1(n)\,\psi_2(n+1) \\ \sigma_3(n) &= \sqrt{2}\left[\prod_{l=1}^{n-1}\left(-2i\,\psi_1(l)\,\psi_2(l)\right)\right]\psi_2(n) \end{aligned}}$$

This is why no one has been able to solve the 2D Ising in a magnetic field using fermions.

—//—

With these, we can return to the transfer matrix T (dropping the $(\cosh K_\tau^*)^{-N}$ factor...

After JW transform →

$$T = \exp\left[\sum_{n=1}^{N} -2i K_\tau^* \psi_1(n)\psi_2(n)\right]\exp\left[\sum_{n=1}^{N} 2i K_x \psi_1(n)\psi_2(n+1)\right]$$

$$\equiv V_1 V_3$$

→ See that V_1, V_3 both contain expressions quadratic in fermions

↳ which means we can diagonalize them in momentum space.

(!) But also see that we have 2 exponentials...

→ These issues will be resolved after we do a Fourier Transform (go to k-space).

(c) Boundary Conditions.

We will impose periodic boundary condition in order to analyze finite-chain Ising models.

We might expect that ↳ $1+N=1$

$$\sigma_3(N)\sigma_3(1) = 2i\,\psi_1(N)\,\psi_2(1)$$

But this is WRONG, since $\psi_2(1)$ has no string of σ_1's to cancel the string that $\psi_1(N)$ has.

(can check this)

The correct result is (we can check this, by defn)

$$\boxed{\begin{aligned}\sigma_3(N)\sigma_3(1) &= -\left[\prod_{l}^{N}\sigma_1(l)\right]2i\,\psi_1(N)\psi_2(1)\\ &\equiv (-1)\cdot \underbrace{\mathcal{P}}_{\left(\prod_{l}^{N}\sigma_1(l)\right)\ \text{(non-local)}}\cdot\, 2i\,\psi_1(N)\,\psi_2(1)\end{aligned}}$$

$\mathcal{P}$ commutes with the transfer matrix T

($\mathcal{P}$ is a symmetry of T, since it flips all spins)

$$\mathcal{P}^2 = 1 \Rightarrow \sigma(P) = \{1,-1\}$$

$\Rightarrow$ Simultaneous eigenstates of $\mathcal{P}$ & T can be divided into those where $\mathcal{P}=1$ (even sector) & $\mathcal{P}=-1$ (odd sector)

In the odd sector,

$$\sigma_3(N)\,\sigma_3(1) = 2i\,\psi_1(N)\,\psi_2(1)$$

$$= 2i\,\psi_1(N)\,\psi_2(1+N)$$

since $\psi_2(1+N) \equiv \psi_2(1)$

$\Rightarrow$ The periodic spin-spin interaction becomes

(odd) $$\boxed{\sum_{n=1}^{N} \sigma_3(n)\,\sigma_3(n+1) = 2i \sum_{n=1}^{N} \psi_1(n)\,\psi_2(n+1)}$$

In the even sector, we want $\psi_2(N+1) = -\psi_2(1)$

in order to obtain $\sum_{n=1}^{N} \sigma_3(n)\sigma_3(n+1) = 2i\sum_{n=1}^{N} \psi_1(n)\psi_2(n+1)$

$\Rightarrow$ This means that fermions return to minus itself when we go around the loop & come back to the same point.

$\hookrightarrow$ just like spinors do after a rotation by 2π.

$\longrightarrow$ Need (ANTI) PERIODIC Boundary conditions for the fermion in the (even) odd sector with $P = (-1)\cdot 1$

Now need to characterize the sectors in the fermionic language. (not Majorana)

At site n, we first form a Dirac fermion:

$$\Psi(n) = \frac{\psi_1(n) + i\psi_2(n)}{\sqrt{2}}, \quad \Psi^\dagger(n) = \frac{\psi_1(n) - i\psi_2(n)}{\sqrt{2}}$$

so $N_\Psi(n) = \Psi^\dagger(n)\Psi(n)$

$$= \frac{1 + 2i\psi_1(n)\psi_2(n)}{2} = \frac{1 - \sigma_1(n)}{2}.$$

If we now use

$$\sigma_1 = -ie^{i\pi/2\,\sigma_1} = e^{i\pi/2\,(\sigma_1 - 1)}$$

we find that

$$\mathcal{P} = \prod_{\ell}^{N} \sigma_1(\ell) = e^{i\pi/2 \sum_n (\sigma_1(n) - 1)}$$

$$= e^{-i\pi/2 \sum_n (2\Psi^\dagger(n)\Psi(n))}$$

$$= e^{-i\pi N_\Psi} = (-1)^{N_\Psi}$$

where

$$N_\Psi = \sum_{n=1}^{N} \Psi^\dagger(n)\,\Psi(n) = \text{total fermion number associated w/ Dirac field } \Psi$$

$\Rightarrow$ $\mathcal{P}$ is the fermion parity of the state.

$\mathcal{P} = -1$ @ odd N_Ψ

$\mathcal{P} = 1$ @ even N_Ψ.

⟹ ~~But note that~~ periodic solution in the even sector & anti-periodic solutions in the odd sector are physically irrelevant & should be discarded

↳ This is key to proving why, in the thermodynamic limit, the model has $\boxed{2}$ degenerate gnd states at low temp (T, α) and only $\boxed{1}$ in high-T state (disordered).

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(d) Solution by Fourier Transform

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The rest of the details here can be found in notes on "Quantum Ising Chain for beginners"

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Reading notes

April 4, 2021

$\boxed{\text{Chapter 10: Gauge Theories}}$