QUANTUM FIELD THEORY

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Preface. These notes come from Prof. Patton's QFT notes, Peskin-Schröder QFT, and "Quantum field theory & condensed matter" - Shankar

Enjoy!

Conventions

- $\hbar = c = 1$
- $[\text{length}] = [\text{time}] = [\text{energy}]^{-1} = [\text{mass}]^{-1}$
- $\eta_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(1\ -1\ -1\ -1)$
- $p^2 = p_\mu p^\mu = E^2 - |p|^2 = mc^2 = m$
- $x_\mu = g_{\mu\nu} x^{\nu} = (x^0, -\vec{x})$
- $x^\mu = g^{\mu\nu} x_\nu = (x^0, \vec{x})$
- $\partial_\mu = \dfrac{\partial}{\partial x^\mu} = \left(\dfrac{\partial}{\partial x^0}, \nabla\right)$
- $\varepsilon^{0123} = +1\ ;\ \varepsilon_{0123} = -1$

  $\varepsilon^{1230} = -1$
- $E = i\dfrac{\partial}{\partial x^0}\ ,\ \vec{p} = -i\vec{\nabla}$
- $p^\mu = i\partial^\mu$

- $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- $$\sigma^i \sigma^j = \delta^{ij} + i \varepsilon^{ijk} \sigma^k$$

- $\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; $\sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

- Heaviside step fn:

$$\theta(x) = \begin{cases} 0 & x \not< 0 \\ 1 & x > 0 \end{cases}$$

- Dirac delta fn: $\delta(x) = \frac{d}{dx} \theta(x)$

- n-dimensional Dirac δ-fn:

$$\int d^4x \, \delta^{(n)}(x) = 1$$

- FT:

$$f(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \tilde{f}(x)$$

$$\tilde{f}(x) = \int d^4k \, e^{ik\cdot x} f(x)$$

- $$\int d^4x \, e^{ik\cdot x} = (2\pi)^4 \delta^{(4)}(k)$$

- EM $\Phi = \frac{Q}{4\pi r}$ ← Coulomb potential

- Fine structure constant:

$$\alpha = \frac{e^2}{4\pi} = \frac{e^2}{4\pi\hbar c} \approx \frac{1}{137}$$

- Maxwell's eqn:

$$\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0 \;, \quad \partial_\mu F^{\mu\nu} = e j^\nu$$

where

$$A^\mu = (\Phi, \vec{A}) \;; \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Elements of classical Field Theory

▣ Lagrangian Field Theory:

Lag-density

$$S = \int \underline{L}\, dt = \int \mathcal{L}(\phi, \partial_\mu \phi)\, d^4x$$

$$(\underline{L} = \mathcal{L}\, d^3x)$$

(Lagrangian)

Principle of least action:

$$0 = \delta S$$

$$= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi)\right\}$$

(E-L eqn)

$$= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\delta\phi + \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\right]\right\}$$

$$\Rightarrow \quad \boxed{0 = \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right] - \frac{\partial \mathcal{L}}{\partial \phi}}$$

FTC → term vanishes @ boundary

Euler - Lagrange Equations

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right\}$$

Ex $\mathcal{L} = \phi^2$: $\frac{\partial \mathcal{L}}{\partial \phi} = 2\phi$; $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0 \Rightarrow 2\phi = 0$

$$\left.\begin{array}{l} \mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi) \\ \Rightarrow \frac{\partial \mathcal{L}}{\partial \phi} = 0 ; \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 2\partial^\mu \phi \end{array}\right\} \Rightarrow \partial^\mu \phi = 0.$$

Ex Klein-Gordon field: (real)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$$

- $\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$

- $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi.$

relativistic particle of mass m.

E-L $\Rightarrow -m^2 \phi = \partial_\mu \partial^\mu \phi = \Box \phi$

$$\rightarrow \boxed{(\Box + m^2)\phi = 0}$$

Klein-Gordon Eqn.

Ex $\phi = e^{-ip \cdot x} \Rightarrow (-p^2 + m^2) = 0 \Rightarrow p^2 = m^2$

Noether's Thm

Thm

$$\boxed{\text{For every continuous symmetry, there exists a conserved current } j^\mu \text{ which implies a local conservation law}}$$

$$\partial_0 j^0 = -\vec{\nabla}\cdot\vec{j} \quad \Leftrightarrow \quad \partial_\mu j^\mu = 0$$

Ex Conservation of charge ...

$$Q = \int j^0 \, d^3x$$

(gauss' law)

$$\Rightarrow \frac{\partial Q}{\partial t} = \int \frac{dj^0}{dt} d^3x = -\int \vec{\nabla}\cdot\vec{j} \, d^3x$$

$$= -\oint \vec{J} \, d^2s$$

//

Idea Consider continuous transf. → infinitesimally (local)

$$(\star) \qquad \phi(x) \to \phi'(x) = \phi(x) + \underbrace{\alpha}_{\text{small}} \Delta\phi(x)$$

$(\star)$ is a symmetry if EOM invariant under $(\star)$.

$\Leftrightarrow$ S is invariant.

$\Rightarrow$ $\mathcal{L}$ must be invariant, up to $\alpha \partial_\mu J^\mu(x)$.

for some J^μ.

Let us compare this expectation for $\Delta\mathcal{L}$ to the result obtained by varying the fields ...

$$\alpha\Delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}(\alpha\Delta\phi) + \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\partial_\mu(\alpha\Delta\phi)$$

$$= \alpha\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) + \alpha\underbrace{\left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right\}\right)}_{0}\Delta\phi$$

$$\Rightarrow \quad \Delta\mathcal{L} = \partial_\mu\left\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right\}$$

<u>So</u> $\dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi$ is the desired J^μ.

So that $\partial_\mu j^\mu(x) = 0$ where

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - J^\mu .$$

<u>Ex</u> <u>massless KG field</u>

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)$$

Consider transformation:

$$\phi(x) \longrightarrow \phi(x) + \alpha \quad , \Delta\phi = 1$$

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi = \partial^\mu\phi .$$

<u>Check</u> $\partial_\mu j^\mu = \partial_\mu\partial^\mu\phi = \Box\phi = 0$ since

$(m^2 + \Box)\phi = 0$ & $m = 0$

Ex Complex KG field

$$\mathcal{L} = (\partial_\mu \phi^+)(\partial^\mu \phi) - m^2 \phi^\dagger \phi .$$

again, EOM is

$$(m^2 + \Box)\phi = 0.$$

Symmetry: $\phi \rightarrow e^{i\alpha}\phi$.

For infinitesimal transf we have:

$$\alpha D\phi = i\alpha\phi \qquad \text{(Taylor expand)}$$
$$\alpha D\phi^+ = -i\alpha\phi^+$$

One can check that

$$j^\mu = i\left[(\partial^\mu\phi^+)\phi - \phi^+(\partial^\mu\phi)\right]$$

is the conserved current.

$\boxed{\text{Ex}}$

Space-time translation:

Consider infi. translation:

$$x^\mu \rightarrow x^\mu - a^\mu$$

in field transform. this becomes:

$$\phi(x) \rightarrow \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi(x)$$

Lagrangian is a scalar $\Rightarrow$ must transform the same way:

$$\mathcal{L}(x) \rightarrow \mathcal{L}(x) + a^\mu \partial_\mu \mathcal{L} = \mathcal{L} + a^\nu \partial_\mu (\delta^\mu_\nu \mathcal{L})$$

Compare this to the eqn:

$$\mathcal{L} \rightarrow \mathcal{L} + a \partial_\mu J^\mu ,$$

we have $J^\mu = \delta^\mu_\nu \mathcal{L}$

$\Rightarrow$ apply them, we find:

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\partial_\nu \phi) - \delta^\nu_\mu \mathcal{L}$$

make μ explicit...

$$\boxed{T^\nu_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \phi - \delta^\nu_\mu \mathcal{L}}$$

STRESS-ENERGY TENSOR. (or Energy-Momentum tensor)

<u>Conserved charge</u> is the Hamiltonian...

$$H = \int T^{00} d^3x = \int \mathcal{H} d^3x \quad (\text{time-translation})$$

momentum $\longrightarrow$ $p^i = \int T^{0i} d^3x = - \int \pi \partial_i \phi \, d^3x$ (spatial translation)

Ex Klein-Gordon field again...

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 .$$

$$T^{00} = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)}\partial^0\phi - g^{00}\mathcal{L} = \cdots$$

$$= \frac{1}{2}(\partial_0\phi)(\partial^0\phi) - \frac{1}{2}(\partial_i\phi)(\partial^i\phi) + \frac{1}{2}m^2\phi^2$$

so

$$H = \int T^{00}\,d^3x = \int \left\{ \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}(\vec\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right\} d^3x$$

note: all terms are positive ... (sum of squares)
→ can't fall into arbitrary negative energy.

$$\boxed{\text{THE KLEIN-GORDON FIELD as HARMONIC OSCILLATOR}}$$

promote: ϕ, π to operators & impose suitable commutation relations...

Recall ....

$$[q_i, p_j] = i\delta_{ij}$$

$$[q_i, q_j] = [p_i, p_j] = 0 .$$

Harmonic oscillator: $H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2$

Ladder operators:

- annihilation: $a = \frac{1}{\sqrt{2}}\left(q\sqrt{m\omega} + ip\frac{1}{\sqrt{m\omega}}\right)$
- creation: $a^\dagger = \frac{1}{\sqrt{2}}\left(q\sqrt{m\omega} - ip\frac{1}{\sqrt{m\omega}}\right)$
- $a^\dagger a = \frac{1}{\omega}H - \frac{1}{2} \Leftrightarrow H = \omega\left(a^\dagger a + \frac{1}{2}\right)$

  \# operator...
- $|0\rangle$, $a|0\rangle = 0$.

  $|n\rangle = \frac{1}{\sqrt{n!}}\underbrace{a^\dagger \ldots a^\dagger}_{n}|0\rangle$
- $[a^\dagger a, a] = -a$
- $[a^\dagger a, a^\dagger] = a^\dagger$
- $[H, a] = -\omega a$; $[H, a^\dagger] = \omega a^\dagger$

  a lowers by ω

  $a^\dagger$ raises by ω
- $Ha^\dagger|n\rangle = (E_n + \omega)a^\dagger|n\rangle$
- $H|0\rangle = \frac{1}{2}\omega|0\rangle \rightarrow E_0 = \frac{1}{2}\omega$
- $E_n = \left(n + \frac{1}{2}\right)\omega$

For continuum systems ... commutation relations become ..

$$\boxed{\begin{aligned} [\phi(x), \pi(y)] &= i\delta^{(3)}(x-y) \\ [\phi(x), \phi(y)] &= [\pi(x), \pi(y)] = 0. \end{aligned}}$$

Next, the Hamiltonian is now also an operator...
To find spec(H), Fourier transf $\phi(x)$

$$\rightarrow \quad \phi(x,t) = \int \frac{d^3p}{(2\pi)^3} e^{ip\cdot x} \phi(\vec{p},t)$$

Recall KG eqn: $(m^2 + \Box)\phi = 0$

$$\Rightarrow \left\{ \frac{\partial^2}{\partial t^2} + (|p|^2 + m^2) \right\} \phi(\vec{p},t) = 0$$

$\Rightarrow$ This is the same eqn as SHO with freq

$$\omega_p = \sqrt{|\vec{p}|^2 + m^2}.$$

Now, $H_{SHO} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2\phi^2 \qquad (m=1)$

$\rightarrow$ know spectrum! $\quad (n + \frac{1}{2})\omega.$

$$\phi = \frac{1}{\sqrt{2\omega}}(a^\dagger + a) \quad ; \quad p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger)$$

$$[a, a^\dagger] = 1.$$

Since it's more convenient to work in position space

$$\boxed{\begin{aligned} \phi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p e^{ip\cdot x} + a_p^+ e^{-ip\cdot x} \right) \\ \pi(x) &= \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_p}{2}} \left(a_p e^{ip\cdot x} - a_p^+ e^{ipx} \right) \end{aligned}}$$

note

$$\begin{cases} a_p \text{ goes with } e^{+ip\cdot x} \\ a_p^+ \text{ goes with } e^{-ip\cdot x}. \end{cases}$$

- Easy to show that

$$\begin{cases} [\phi(\vec{x}), \phi(\vec{y})] = 0 = [\pi(\vec{x}), \pi(\vec{y})] \\ [\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y}). \end{cases}$$

- Can re-arrange...

$$\begin{cases} \phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_p + a_{-p}^+) e^{ip\cdot x}. \\ \pi(x) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_p}{2}} (a_p - a_{-p}^+) e^{+ip\cdot x}. \end{cases}$$

→ get commutation relation between $a_{\vec{p}}$:

$$\boxed{[a_p ; a_{p'}^+] = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}')}$$

Now, can check that

$$[\phi(x), \pi(x')] = \int \frac{d^3p\, d^3p'}{(2\pi)^6} \left(\frac{-i}{2}\right) \sqrt{\frac{\omega_{p'}}{\omega_p}} \times e^{i(p\cdot x + p'\cdot x')} \times$$

$$\left([a^\dagger_{-p}, a_{p'}] - [a_p, a^\dagger_{-p'}] \right)$$

$$= i\delta^{(3)}(x - x') \quad \checkmark$$

- Now, can express Hamiltonian in terms of ladder ops

Recall that $\longrightarrow$ KG field, last time

$$H = \int d^3x \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial^0 \phi - g^{00} \mathcal{L} \right\}$$

$$= \int d^3x \left\{ \frac{1}{2} (\partial_0 \phi)^2 + \frac{1}{2} (\nabla\phi)^2 + \frac{1}{2} m^2 \phi^2 \right\}$$

To quantize, need to define π ... Turns out that

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial^0 \phi(x) \longrightarrow \left(\text{like } p = \frac{\partial \mathcal{L}}{\partial \dot{q}}\right)$$

So ...

$$H = \int d^3x \left\{ \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla\phi)\cdot(\nabla\phi) + \frac{1}{2} m^2 \phi^2 \right\}$$

with $\pi(x) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_p}{2}} (a_p - a^\dagger_{-p}) e^{ip\cdot x}$ we get

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_p + a^\dagger_{-p}) e^{ip\cdot x}$$

results in $\delta(p-p')$ $\Rightarrow p = p'$

$$H = \int d^3x \int \frac{d^3p\, d^3p'}{(2\pi)^6}\, e^{i(p+p')\cdot x} \Big\{ -\frac{\sqrt{w_p w_{p'}}}{4}(a_p - a^\dagger_{-p})(a_{p'} - a^\dagger_{-p'})$$

$$+ \frac{-p\cdot p' + m^2}{4\sqrt{w_p w_{p'}}}(a_p + a^\dagger_{-p})(a_{p'} + a^\dagger_{-p'}) \Big\}$$

Some $\delta^{(3)}$ will appear...

$$= \int \frac{d^3p}{(2\pi)^3}\, w_p \left(a^\dagger_p a_p + \frac{1}{2}[a_p, a^\dagger_p] \right)$$

So

$$\boxed{H = \int \frac{d^3p}{(2\pi)^3}\, w_p \left(a^\dagger_p a_p + \frac{1}{2}[a_p, a^\dagger_p] \right)}$$

$\rightarrow$ can evaluate commutators...

$$\boxed{[H, a^\dagger_p] = w_p a^\dagger_p \ ; \ [H, a_p] = -w a_p}$$

With H, can find momentum operator...

KG field $\longrightarrow$ from $P^i = \int d^3x\, T^{0i} = -\int \pi \partial_i \phi\, d^3x$, we get

$$\boxed{\begin{aligned} \vec{P} &= -\int d^3x\, \pi(x) \nabla \phi(x) \\ &= \int \frac{d^3p}{(2\pi)^3}\, \vec{p}\, a^\dagger_p a_p \end{aligned}}$$

$a^\dagger_p$ creates momentum $\vec{p}$ & energy $\underset{E_p}{w_p} = \sqrt{|p|^2 + m^2}$.

Excitation: $a^\dagger_p a^\dagger_q \ldots |0\rangle \equiv$ "particles".

$\hookrightarrow$ <u>each excitation $a^\dagger_p$ is a particle.</u>

$\Rightarrow$ get particle statistics...

Consider 2-particle state... $a_p^\dagger a_q^\dagger |0\rangle$.

Since $[a_p^\dagger, a_q^\dagger] = 0$, we have

$$\boxed{a_p^\dagger a_q^\dagger |0\rangle = a_q^\dagger a_p^\dagger |0\rangle}$$

$\rightarrow$ Klein Gordon particles follow Bose-Einstein stats.

- <u>Normalization</u> $\langle 0|0\rangle = 1$.

$$|p\rangle \propto a_p^\dagger |0\rangle .$$

This normalization $\longrightarrow$ $\langle q|p\rangle = (2\pi)^3 \delta^{(3)}(q-p)$ is NOT Lorentz inv

Pf Under a Lorentz boost $\rightarrow$ $p_3' = \gamma(p_3 + \beta E)$

$E' = \gamma(E + \beta p_3)$

using the identity $\rightarrow$ $\delta(f(x) - f(x_0)) = \dfrac{1}{|f'(x_0)|}\delta(x - x_0)$

<u>we compute</u>: $\delta^{(3)}(p-q) = \delta^{(3)}(p'-q') \cdot \left(\dfrac{dp_3'}{dp_3}\right)$

$\underbrace{\delta(p_1-q_1)\delta(p_2-q_2)}_{\text{same}}\underbrace{\delta(p_3-q_3)}_{\text{boosted}}$

$$= \delta^{(3)}(p'-q') \cdot \gamma\left(1 + \beta \frac{dE}{dp_3}\right)$$

$$= \delta^{(3)}(p'-q') \frac{\gamma}{E}(E + \beta p_3)$$

$$= \delta^{(3)}(p'-q')(E'/E)$$

So $\boxed{\delta^{(3)}(p-q) = \delta^{(3)}(p'-q')\left(\frac{E'}{E}\right)}$ $\rightarrow 1 \Leftrightarrow E = E_p = E'$

For normalization to work $\rightarrow$ use $\underset{\overset{\|\|}{w_p}}{E_p}$, not E.

$\rightarrow$ define: $\boxed{|p\rangle = \sqrt{2E_p}\, a_p^\dagger |0\rangle}$

so that $\boxed{\langle p|q\rangle = 2E_p (2\pi)^3 \delta^{(3)}(p-q)}$

Completeness relation...

1 particle $\rightarrow$ $\boxed{1 = \int \frac{d^3p}{(2\pi)^3} |p\rangle \frac{1}{2E_p} \langle p|}$

Interpret $\phi(x)|0\rangle$ .... we know that $a_p^\dagger$ creates momentum p & energy $E_p = w_p$.

What about operator $\phi(x)$?

$$\phi(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot x} |p\rangle \quad , \quad a_p^\dagger|0\rangle = \frac{1}{\sqrt{2E_p}}|p\rangle$$

Pf. By defn...

annihilates.

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x}\right)$$

$$\rightarrow \phi(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot x}|0\rangle \qquad \square$$

$\rightarrow$ $\phi(x)|0\rangle$ is a lin. superposition of single-particle states

~~Note HW1, HW2 easy~~

that have well-defn momentum.

When nonrelativistic $\rightarrow$ $E_p \simeq$ Constant!

$\Rightarrow$ [$\phi(x)$ acting on the vacuum, "creates a particle at position x^μ".]

$\hookrightarrow$ Confirm this by computing.

$$\langle 0| \phi(x) |p\rangle = \langle 0| \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{p'}}} \left(a_{p'} e^{ip'\cdot x} + a_{p'}^\dagger e^{-ip'\cdot x}\right) \sqrt{2E_p}\, a_p^\dagger |0\rangle$$

$$\boxed{\langle 0| \phi(x) |p\rangle = e^{ip\cdot x}}.$$

$\hookrightarrow$ Interpretation: position-space representation of the single-particle wfn of the state $|p\rangle$, just like ...

$\boxed{\langle x|p\rangle \propto e^{ip\cdot x}}$ in QM!

$\langle 0| \phi(x) \sim \langle x| \ldots$ (don't take this literally, etc).

Note HW1, HW2 are easy, so we'll skip for now.

ep 14, 2020

THE KLEIN - GORDON FIELD IN SPACETIME

Last time → we quantized KG field in the Schrödinger picture.

→ Now, switch to Heisenberg picture.

Recall... Schrödinger picture:

$\mathcal{U}(t) = e^{-iHt}$ is the time evolution.

$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$. → state evolve in time

→ In the Heisenberg picture, ... Operators evolve in time

$$\mathcal{O}(t) = \mathcal{U}^\dagger(t)\,\mathcal{O}(0)\,\mathcal{U}(t).$$

So that

$$\langle \psi_1 | \mathcal{O}(t) | \psi_1 \rangle = \langle \psi_1(t) | \mathcal{O} | \psi_1(t) \rangle$$

↓ Heisenberg ↓ Schrödinger.

→ make the operators ϕ & π time-dependent like this

$$\boxed{\begin{aligned} \phi(x) &\to \phi(x,t) = e^{iHt}\,\phi(x)\,e^{-iHt} \\ \pi(x) &\to \pi(x,t) = e^{iHt}\,\pi(x)\,e^{-iHt} \end{aligned}}$$

Recall Heisenberg eqn of motion $i\frac{\partial}{\partial t}\mathcal{O} = [\mathcal{O}, H]$

which gives, upon substituting in $\phi(x,t)$, $\pi(x,t)$

$$i\frac{\partial}{\partial t}\phi(x,t) = \left[\phi(x,t), \int d^3x' \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right\}\right]$$

$(\phi \leftrightarrow \phi)$

$\rightarrow$ only nontrivial term is 1^{st}.

$$= \int d^3x' \left(i\delta^{(3)}(x-x')\,\pi'(x,t) \right)$$

$$= i\pi(x,t) \Rightarrow \boxed{\frac{\partial}{\partial t}\phi(x,t) = \pi(x,t)}$$

and

$$i\frac{\partial}{\partial t}\pi(x,t) = \left[\pi(x,t), \int d^3x' \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right\}\right]$$

(integrate by parts here)

$$= \int d^3x' \left(-i\delta^{(3)}(x-x')(-\nabla^2 + m^2)\,\phi(x',t) \right)$$

$$= -i(-\nabla^2 + m^2)\,\phi(x,t)$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}\pi(x,t) = (m^2 - \nabla^2)\,\phi(x,t)}$$

Combining these 2 results we get...

$$\boxed{\frac{\partial^2}{\partial t^2}\phi(x,t) = (\nabla^2 - m^2)\,\phi(x,t)}$$

$\hookrightarrow$ rearranging this gives

$$\boxed{(\Box + m^2)\,\phi(x,t) = 0} \longrightarrow \text{just the KG eqn...}$$

- Now, can better understand the time dependence of $\phi(x)$, $\pi(x)$ by writing them in terms of creation & annihilation ops.

<u>Recall</u>: $H a_p = a_p (H - E_p)$ → from comm. rule.

$$\Longrightarrow \begin{pmatrix}\text{proof by}\\ \text{induction}\end{pmatrix} \quad \boxed{H^n a_p = a_p (H-E_p)^n}$$

Similarly, $\boxed{H^n a_p^\dagger = a_p^\dagger (H+E_p)^n}$

→ So, we have

$$\boxed{e^{iHt} a_p e^{-iHt} = a_p e^{-iE_p t}} \to \text{from above} \ldots$$

and

$$\boxed{e^{iHt} a_p^\dagger e^{-iHt} = a_p^\dagger e^{+iE_p t}}$$

→ Now ... we want to write $\phi(x,t)$ in terms of these operators. (since $\phi(x)$, $\pi(x)$ is a combo of a & $a^\dagger$)

- We know that $\phi(x,t) = e^{iHt} \phi(x) e^{-iHt}$.

and from before ...

$$\phi(x) = \phi(x,0) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x} \right)$$

Substitute this into $\phi(x,t) = e^{iHt}\phi(x) e^{-iHt}$ we find ...

$$\phi(x,t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_p e^{-iE_pt} e^{ip\cdot x} + a_p^\dagger e^{iE_pt} e^{-ip\cdot x} \right\}$$

now, note that $p^0 = E_p$

$$\Rightarrow p\cdot x = E_p \cdot x^0 - \vec{p}\cdot\vec{x} \Big|_{p^0=E_p} = E_pt - \vec{p}\cdot\vec{x}\Big|_{p^0=E_p}$$

So...

$$\boxed{\phi(x,t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_p e^{-i\vec{p}\cdot\vec{x}} + a_p^\dagger e^{+i\vec{p}\cdot\vec{x}} \right\}\Bigg|_{p^0=E_p}}$$

Similarly, we can find

$$\boxed{\pi(x,t) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_p}{2}} \left(a_p e^{-i\vec{p}\cdot\vec{x}} - a_p^\dagger e^{+i\vec{p}\cdot\vec{x}} \right)\Bigg|_{p^0=E_p}}$$

from defn or from $\pi(x,t) = \frac{\partial}{\partial t}\phi(x,t)$.

Note we can also do everything, but starting from P and not H. But we won't worry about that.

Causality Note that causality is broken ~~when~~ without the presence of fields.

$\rightarrow$ QFT resolves causality problem.

In our present formalism, still in Heisenberg picture...

...the amplitude for a particle to propagate from y to x is given by

$$\langle 0|\phi(x)\phi(y)|0\rangle$$

"Two-point correlation function"

Let $\boxed{\mathcal{D}(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle}$

$$\mathcal{D}(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle$$

$$= \langle 0| \text{ in terms of } a_p \text{ \& } a_p^\dagger \ldots |0\rangle$$

$$= \langle 0| a_p a_q^\dagger |0\rangle =$$

$$= (2\pi)^3 \delta^{(3)}(p-q)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x-y)}$$

More explicitly...

@ $p^0 = E_p$
$p^{0\prime} = E_{p'}$

$$\mathcal{D}(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{\sqrt{2E_p}} \frac{1}{\sqrt{2E_{p'}}} \langle 0| e^{-ipx} e^{+ip'y} a_p a_{p'}^\dagger |0\rangle$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^6} \left(\frac{1}{\sqrt{2E_p}} \frac{1}{\sqrt{2E_{p'}}}\right) e^{-ip\cdot(x-y)} (2\pi)^3 \delta^{(3)}(p-p')$$

$$\mathcal{D}(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \cdot e^{-ip\cdot(x-y)} \quad \square.$$

Note that this integral is Lorentz-invariant.

→ Now, let us evaluate this integral for some particular value of $x-y$...

(1) Suppose that $x-y = (t, 0, 0, 0)$, then

$$\mathcal{D}(x-y) = \frac{4\pi}{(2\pi)^3}\int_0^\infty dp\, \frac{p^2}{2\sqrt{p^2+m^2}}\, e^{-i\sqrt{p^2+m^2}\,t}$$

(timelike)

$$= \frac{1}{4\pi^2}\int_m^\infty dE\,\sqrt{E^2-m^2}\; e^{-iEt}$$

$$\underset{t\to\infty}{=} e^{-imt}.$$ → dominated by region above $p \approx 0$.

(2) Suppose that $x-y = (0, \vec{x}-\vec{y}) = (0, \vec{r})$ then

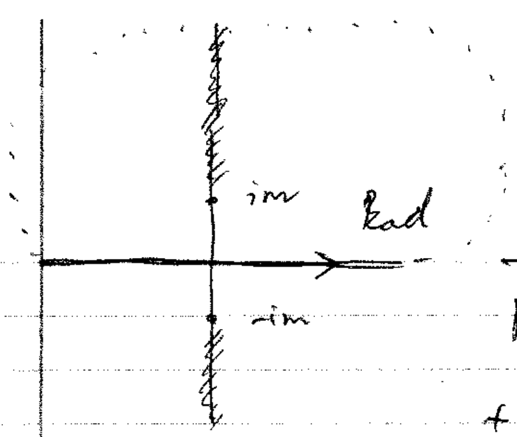
$$\mathcal{D}(x-y) = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_p}\, e^{i\vec{p}\cdot\vec{r}}$$

$$= \frac{2\pi}{(2\pi)^3}\int_0^\infty dp\,\frac{p^2}{2E_p}\,\frac{e^{ipr}-e^{-ipr}}{ipr}$$

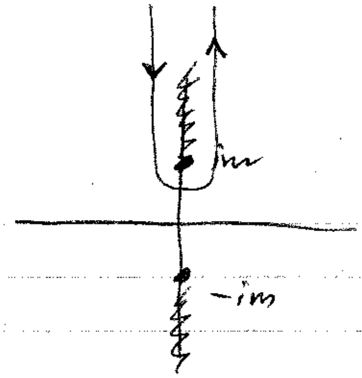
$$= \frac{-i}{2(2\pi)^2 r}\int_{-\infty}^\infty dp\,\frac{p\,e^{ipr}}{\sqrt{p^2+m^2}}$$

→ branch cut @ $\pm im$ (singularity)...

→ must change contour.. z wick rotate



push
contour
+ Wick
rotate ...



$\rho = -ip$
(Wick rotate)

To get $$\mathcal{D}(x-y) = \frac{1}{4\pi^2 r}\int_m^\infty d\rho \, \frac{\rho e^{-\rho r}}{\sqrt{\rho^2 - m^2}}$$

$$\Rightarrow \quad \boxed{\mathcal{D}(x-y) \underset{r\to\infty}{=} \frac{e^{-mr}}{r}} \longrightarrow \underline{\text{non zero!}}$$

(more details about integrals like this can be found in Zee's QFT in a nutshell ...)

What does it mean for $\mathcal{D}(x-y)$ to be nonzero when $x-y$ is spacelike?

We saw that when $(x-y)^\mu (x-y)_\mu = -(\vec{x}-\vec{y})^2 < 0$ is spacelike, cannot have causality between x & y.

$$\mathcal{D}(x-y) \neq 0 \Rightarrow \text{??? paradox?}$$

→ __NO!__ To discuss causality, we should ask not whether particles can propagate over spacelike intervals ...

... but whether a "__measurement__" at one point can affect a measurement at another point whose separation from the first is spacelike -

→ No violation of causality b/c no signal can be transmitted → no info is exchanged.

→ Suppose we have a local measurement @ x, call this $\phi(x)$. & a local measurement @ y, called $\phi'(y)$

So long as $[\phi(x), \phi'(y)] = 0$, the 2 measurements don't affect one another.

~ measure the field ϕ @ x & @ y,

If $[\phi(x), \phi(y)] = 0$ when $(x-y)^2 < 0$ then we're good

$$\cancel{\rightarrow [\phi(x), \phi(y)] = \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_{p'}}} \left\{ e^{-ip\cdot x}[a_p, a_{p'}^\dagger] e^{ip'\cdot y} + e^{ip\cdot x}[a_p^\dagger, a_{p'}] e^{-ip'\cdot y} \right\}}$$

$$\rightarrow [\phi(x), \phi(y)] = \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_{p'}}} \times \left\{ \left[(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x}), (a_{p'} e^{-ip'\cdot y} + a_{p'}^\dagger e^{ip'\cdot y}) \right] \right.$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_{p'}}} \left\{ e^{-ip\cdot x} \underbrace{[a_p, a_{p'}^\dagger]}_{(2\pi)^3 \delta^3(p-p')} e^{ip'\cdot y} + e^{ip\cdot x} \underbrace{[a_p^\dagger, a_{p'}]}_{-(2\pi)^3 \delta^3(p-p')} e^{-ip'\cdot y} \right\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \left(\frac{1}{2E_p}\right) \left\{ e^{-ip(x-y)} - e^{-ip(y-x)} \right\} = D(x-y) - D(y-x)$$

So $[\phi(x), \phi(y)] = D(x-y) - D(y-x)$.

Since $D(y-x)$ is Lorentz invariant, can take

$y-x \to x-y$. (possible b/c $(x-y)^2 < 0$)

Note that when $(x-y)^2 > 0 \to$ there's no continuous transf that takes $y-x \to x-y$

$\to$ so this is only possible because $(x-y)^2 < 0$ (spacelike).

So $D(y-x) = D(x-y)$

So, $[\phi(x), \phi(y)] = 0$ ✓

The Klein-Gordon Propagator

~~Let's look at $[\phi(x), \phi(y)]$ in more details ...~~

~~$[\phi(x), \phi(y)]$ is just a number~~

~~$\Rightarrow$ can write $[\phi(x), \phi(y)] \equiv \langle 0|[\phi(x), \phi(y)]|0\rangle$~~

~~$\to \langle 0|[\phi(x), \phi(y)]|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left\{ e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right\}$~~

~~(assuming $x^0 > y^0$)~~

Δ Poles @ $p^2 = m^2$ ⇓ $p^0 = \pm E_p$

~~$= \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{1}{2E_p} e^{-ip\cdot(x-y)} \Big|_{p^0 = E_p} - \frac{1}{2E_p} e^{-ip(x-y)} \Big|_{p^0 = -E_p} \right.$~~

The Klein-Gordon Propagator

→ Before looking at this... need to look at Green's Function & Contour Integrals...

→ Consider function

$$\frac{1}{(p^0_0 - E_p)(p^0_0 + E_p)} \qquad E_p = \sqrt{|p|^2 + m^2}.$$

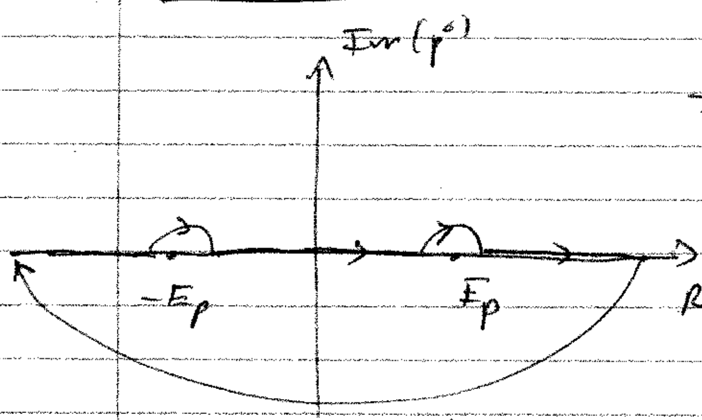
→ Poles at $p^0_0 = \pm E_p$.

Look at FT:

$$\boxed{f(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} \frac{dp^0}{(p^0 - E_p)(p^0 + E_p)} e^{-ip^0 t}}$$

→ How to integrate this?

If $t > 0$ → ~~xxxxxxxxxxxxxxxx~~



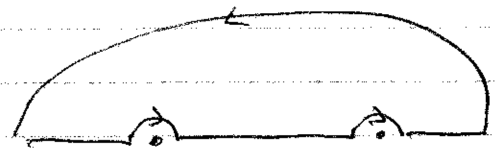
$$\to \quad f(t) = (-2\pi i) \sum \mathrm{Res}(f)$$

$$= (-2\pi i) \left\{ \frac{e^{iE_p t}}{2\pi(-2E_p)} + \frac{e^{-iE_p t}}{2\pi(2E_p)} \right\}$$

↑ close contour...
but clockwise → reverse sign

→ $$\boxed{f(t) = \frac{-i}{2E_p}\left(e^{+iE_p t} - e^{-iE_p t} \right)} \qquad (t > 0)$$

If $t < 0$ close contour above poles



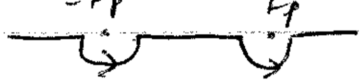
$\rightarrow \quad f(t) = 0.$

$\rightarrow$ So, altogether, we have...

$$\boxed{\begin{aligned} f(t) &= \int \frac{1}{2\pi} \frac{dp^0}{(p^0 - E_p)(p^0 + E_p)} e^{-ip^0 t} \\ &= \Theta(t) \left(\frac{-i}{2E_p} \right) \left(e^{-iE_p t} - e^{+iE_p t} \right) \\ &\text{where } \Theta(t) \text{ is the Heaviside Step fn...} \\ \Theta(t) &= \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \end{aligned}}$$

$\rightarrow$ Retarded / Forward Propagating Green's fn

Suppose the contour is taken as



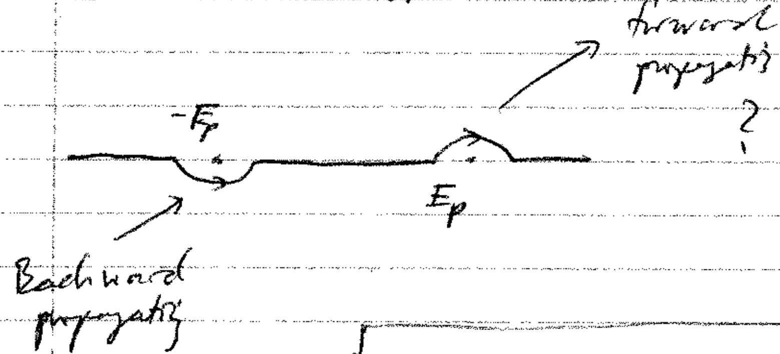
, then we'll get

$$t > 0 \rightarrow f(t) = 0$$

$$t < 0 \rightarrow f(t) = \Theta(-t) \frac{i}{2E_p} \left(e^{-iE_p t} - e^{iE_p t} \right)$$

$\rightarrow$ Advanced / Backward Propagating Green's fn.

☒ What about



$$\rightarrow \boxed{f(t) = \Theta(t)\left(\frac{-i}{2E_p}\right)e^{-iE_p t} + \Theta(-t)\left(\frac{-i}{2E_p}\right)e^{+iE_p t}}$$

↑ Time-ordered Green's fn.

With this, we can study the commutator $[\phi(x), \phi(y)]$

Consider this quantity...

$$\langle 0|\,[\phi(x),\phi(y)]\,|0\rangle = \int \frac{d^3p}{(2\pi)^3}\underbrace{\frac{1}{2E_p}\left(e^{-ip\cdot(x-y)} - e^{ip(x-y)}\right)}$$

$$= \int \frac{d^3p}{(2\pi)^3}\left\{\frac{1}{2E_p}\underset{\substack{\text{pole} \\ @\, p_0 = E_p}}{e^{-ip(x-y)}} + \frac{1}{-2E_p}\underset{\substack{\text{pole } @ \\ p_0 = -E_p}}{e^{-ip(x-y)}}\right\}$$

4D integral →

$$\underset{x^0 > y^0}{=} \int \frac{d^3p}{(2\pi)^3}\underbrace{\int \frac{dp^0}{2\pi i}\,\frac{-1}{p^2-m^2}\,e^{-ip(x-y)}}_{f(t)\text{ before, where}}$$

$$(p^0 - E_p)(p^0 + E_p) = p^{0\,2} - |p|^2 - m^2 = p^2 - m^2 .$$

Now, call ...

$$\boxed{D_R(x-y) \equiv \theta(x^0-y^0)\langle 0|[\phi(x),\phi(y)]|0\rangle}$$

Then

$$(\Box+m^2)D_R(x-y) = \Box D_R(x-y) + m^2 D_R(x-y)$$

$$= (\Box\theta(x^0-y^0))\langle 0|[\phi(x),\phi(y)]|0\rangle$$

$$+ 2(\partial_\mu\theta(x^0-y^0))\partial^\mu\langle 0|[\phi(x),\phi(y)]|0\rangle$$

$$+ \theta(x^0-y^0)(\Box+m^2)\underbrace{\langle 0|[\phi(x),\phi(y)]|0\rangle}_{\text{constant}}$$

$$= -\delta(x^0-y^0)\langle 0|[\pi(x),\phi(y)]|0\rangle$$

$$+ 2\delta(x^0-y^0)\langle 0|[\pi(x),\phi(y)]|0\rangle + 0$$

$$= \delta(x^0-y^0)\langle 0|[\pi(x),\phi(y)]|0\rangle$$

needs verification (easy)

$$= -i\delta^{(4)}(x-y)$$

$$-i\delta^{(3)}(x-y)$$

So

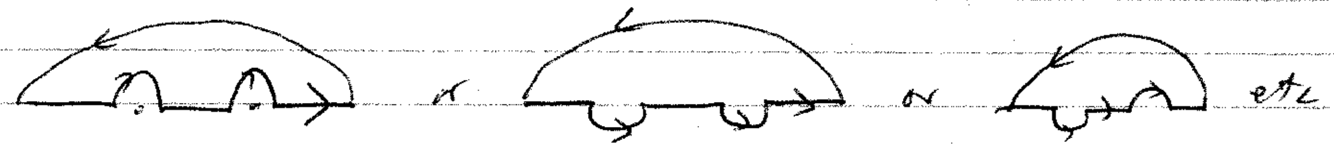
$$\boxed{(\Box+m^2)D_R(x-y) = -i\delta^{(4)}(x-y)}$$

$\longrightarrow$ $D_R(x-y)$ is a Green's fn of the Klein-Gordon operator.

Since $D_R(x-y) = 0$ @ $x^0 < y^0$

$\Rightarrow$ $D_R(x-y) \equiv$ "Retarded" Green's fn

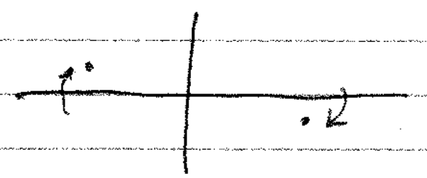
Now ... As we have seen, there are many ways to take the contour ...



$\longrightarrow$ Use the Feynman prescription instead

$$\boxed{D_F(x-y) \equiv \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\varepsilon} e^{-ip\cdot(x-y)}}$$

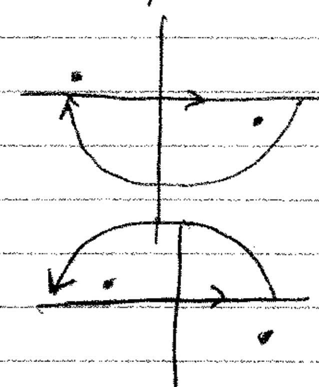
$\rightarrow$ Convenient! B/c now poles are $p^0 = \pm(E_p - i\varepsilon)$



$\longrightarrow$ when $x^0 > y^0$ $\longrightarrow$ take contour below

when $x^0 < y^0$ $\longrightarrow$ take contour above.

$\rightarrow$ get same expression but with $x \leftrightarrow y$.



So,

$$\boxed{\begin{aligned} D_F(x-y) &= \begin{cases} D(x-y) & x^0 > y^0 \\ D(y-x) & x^0 < y^0 \end{cases} \\ &= \theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0-x^0)\langle 0|\phi(y)\phi(x)|0\rangle \\ &\equiv \langle 0| T\{\phi(x)\phi(y)\}|0\rangle \end{aligned}}$$

"time-ordering" symbol.

Time-ordering symbol $\Rightarrow$ instructs us to place the operators that follows in order with the latest to the left.

$\rightarrow$ apply $(\Box + m^2)$ to last line, get D_F is Green's fn of Klein-Gordon Operator.

$\hookrightarrow$ $D_F(x-y)$ is called the "Feynman Propagator" for a Klein-Gordon operator...

$\hookrightarrow$ propagation amplitude

$\rightarrow$ But we can't much calculation at this point just yet.

$\rightarrow$ B/c we're only looked at the free KG theory

$\rightarrow$ field eqn in this case is linear : there are no interactions...

$\rightarrow$ this theory is too simple to make any predictions...

$\rightarrow$ need perturbation ....

One kind of interaction the ~~str~~ can als be solved ...

$\hookrightarrow$

Particle Creation by a classical Source

Consider the source $j(x)$

Recall... free field: $(\Box + m^2)\phi = 0$

→ now... $(\Box + m^2)\phi = j(x)$ fixed fn in space time.

$j(x)$ is nonzero only for a finite time interval

The associated Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) - \frac{1}{2}m^2\phi^2 + j(x)\phi(x)$$

If $j(x)$ is turned on for only a finite time, it is easiest to solve...

Before $j(x)$ is turned on, $\phi(x)$ has the form

$$\phi_0(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x}\right)$$

With a source...

$$\phi(x) = \phi_0(x) + i\int d^4y \, D_R(x-y) j(y)$$

⋮

We won't worry about this for now...

Some problems & Insights

(1) Classical EM (no sources) follows from the action

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

(a) Identify $\begin{cases} E^i = -F^{0i} \\ \varepsilon^{ijk} B^k = -F^{ij} \end{cases}$

→ Derive the E-L eqn (Maxwell's eqn)

→ easy to show that

$$\boxed{\partial_\mu F^{\mu\nu} = 0} \; ; \quad \vec{\nabla} \cdot \vec{E} = 0 \quad (\nu = 0)$$

$$-\partial_t \mathcal{E} + \vec{\nabla} \times \vec{E} = 0 \quad (\nu = i)$$

(2) Complex scalar field

$$S = \int d^4x \left(\partial_\mu \phi^+ \partial^\mu \phi - m^2 \phi^+ \phi \right)$$

Derive E-L eqn:

$$\boxed{\begin{aligned} &i\partial_t \phi^+ - \frac{1}{2m} \nabla^2 \phi^+ = 0 \\ &-i\partial_t \phi - \frac{1}{2m} \nabla^2 \phi = 0 \end{aligned}}$$

now... write $\phi \Rightarrow e^{-i\theta}\phi$, $\phi^\dagger \to e^{i\theta}\phi^\dagger$

$\sim \phi - i\theta\phi$ $\qquad\qquad \sim \phi^\dagger + i\theta\phi^\dagger$

$\to \Delta\phi \sim -i\theta$ $\qquad\qquad \Delta\phi^\dagger \sim i\phi^\dagger$

So that

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)}\Delta\phi^\dagger$$

conserved current.

↳ can find conjugate momenta:

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \to \text{... compute ...}$$

→ can get Hamiltonian... $\qquad$ there's a Ham. formula in book, but we won't worry abt this.

$$\mathcal{H} = \int d^3x \left(\pi^\dagger\pi + (\nabla\phi)^\dagger(\nabla\phi) + m^2\phi^\dagger\phi\right)$$

(3) If we take $(x-y)^2 = -r^2$ → can explicitly write

$$\langle 0|\phi(x)\phi(y)|0\rangle = D(x-y) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip(x-y)}$$

↳ when $(x-y)^2 < -r^2 \to D(x-y)$ can be written ~~as~~ in terms of Bessel Functions...

THE DIRAC FIELD

(1) Lorentz Invariance in Wave Eqn

→ Lorentz transformation... $\boxed{x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu}$

→ what happens to $\phi(x)$ under Λ?

we require that $\phi'(x') = \phi(x)$

$$\rightarrow \boxed{\phi(x) \longrightarrow \phi'(x) = \phi(\Lambda^{-1}x)} \text{ so that}$$

$$\phi'(\Lambda x) = \phi(\Lambda^{-1}\Lambda x) = \phi(x) \checkmark$$

→ what about $\partial_\mu \phi(x)$?

Under transform --- $\partial_\mu \phi(x) \longrightarrow \partial_\mu(\phi(\Lambda^{-1}x))$

$$\rightarrow \boxed{\partial_\mu(\phi(\Lambda^{-1}x)) = (\Lambda^{-1})^\nu{}_\mu \, (\partial_\nu \phi)(\Lambda^{-1}x)}$$

Recall that ...

$$(\Lambda^{-1})^\rho{}_\mu (\Lambda^{-1})^\sigma{}_\nu \, g^{\mu\nu} = g^{\rho\sigma}$$

$$\rightarrow (\partial_\mu \phi(x))^2 \longrightarrow g^{\mu\nu}(\partial_\mu \phi'(x))(\partial_\nu \phi'(x))$$

$$= g^{\mu\nu}[\partial_\mu(\phi(\Lambda^{-1}x))][\partial_\nu(\phi(\Lambda^{-1}x))]$$

$$= g^{\mu\nu}\{(\Lambda^{-1})^\rho{}_\mu \partial_\rho \phi\}\{(\Lambda^{-1})^\sigma{}_\nu \partial_\sigma \phi\}(\Lambda^{-1}x)$$

$$= (\partial\phi)^2(\Lambda^{-1}x) \longleftarrow = g^{\rho\sigma}\partial_\rho\phi\, \partial_\sigma\phi \, (\Lambda^{-1}x) \rightarrow \text{evaluated}$$

<u>Recall</u>

$$\boxed{\begin{aligned} \phi(x) &\longrightarrow \phi(\Lambda^{-1}x) \\ \partial_\mu \phi(x) &\longrightarrow (\Lambda^{-1})^\nu{}_\mu (\partial_\nu \phi)(\Lambda^{-1}x) \\ (\partial_\mu \phi(x))^2 &\longrightarrow (\partial_\mu \phi)^2(\Lambda^{-1}x) \end{aligned}}$$

<u>So</u> it is clear that

$$\boxed{\mathcal{L} \longrightarrow \mathcal{L}(\Lambda^{-1}x)}$$

Lagrangian is Lorentz - invariant.

→ The action $S = \int d^4x\, \mathcal{L}$ is <u>also</u> Lorentz inv.

→ also clear that EOM <u>is also Lorentz inv.</u>

$$(\Box + m^2)\, \phi'(x) = \left((\Lambda^{-1})^\mu{}_\nu \partial_\mu (\Lambda^{-1})^{\sigma\mu} \partial_\sigma + m^2\right) \phi(\Lambda^{-1}x)$$

$$= (\partial^\sigma \partial_\sigma + m^2)\, \phi(\Lambda^{-1}x)$$

$$= 0 \quad \checkmark$$

Sep 20, 2020

→ How do we find Lorentz - invariant theories, in general?

→ for simplicity, restrict attention to lin. transf

→ $\phi_a = v \in \mathbb{C}^n$, matrix giving Lorentz transform Λ.

→ $\boxed{\Phi_a(x) \Rightarrow M_{ab}(\Lambda)\, \Phi_b(\Lambda^{-1}x)}$

$n \times n$

→ the most general nonlinear transf can be built out of linear ones ⇒ suffices to consider M only.

↳ for short, write $\Phi \longmapsto M(\Lambda)\Phi$.

→ What are the possible allowed $M(\Lambda)$?

Obs $\{\Lambda\}$ form a group $M(\Lambda'')M(\Lambda) = M(\Lambda')$

$\Lambda''\Lambda = \Lambda'$

→ the correspondence between Λ & M must be preserved under multiplication.

$\{\Lambda\}$ Lorentz group $\longleftrightarrow$ $\{M(\Lambda)\}$ → n-dim representation of the Lorentz group

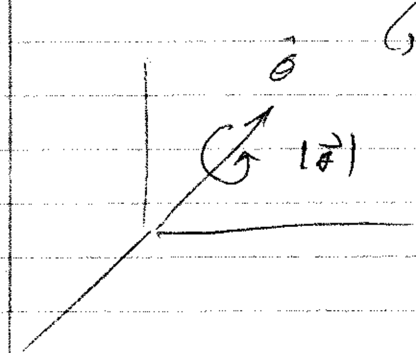
↳ $\boxed{?}$ What are the finite-dim matrix reps of the Lorentz group?

Ex in QM... spin $\frac{1}{2}$ → $\{M\}$ are the 2×2 unitary matrices with determinant 1.

→ $\boxed{U = e^{-i\theta^i\sigma^i/2}}$ → $\sum_i \theta^i \sigma^i/2$

3 arbitrary parameters

& Pauli matrices.



$\{U(\vec\theta) = e^{-i\vec\theta\cdot\vec\sigma/2}\}$

→ In the case for arbitrary spin representations...

$$\mathcal{U}(\vec{\theta}) = e^{-i\vec{\theta}\cdot\vec{J}} \quad \text{where } \vec{J} = (J^1, J^2, J^3)$$

$$\text{and } [J^i, J^j] = i\sum_l \epsilon^{jkl} J^l$$

→ Check that this works for spin 1/2:

$$\left[\frac{\sigma^j}{2}, \frac{\sigma^k}{2}\right] = i\sum_l \epsilon^{jkl}\frac{\sigma^l}{2} \quad ✓$$

→ for spinless particles... $\psi(\vec{x})$ can be decomposed into orbital angular momentum states $J = 0, 1, 2, \ldots$
(no intrinsic spin ⇒ $J = L$)

- $\vec{J} = \vec{L} = \vec{x} \times \vec{p} = \vec{x} \times (-i\vec{\nabla})$
- $J^j = i\sum_l \epsilon^{jkl} x^k \nabla^l$
- $\nabla^l = -\nabla_l = -\dfrac{\partial}{\partial x^l}$

But the cross product is special to 3D case.

→ write operator as antisymmetric tensor...

$$\boxed{J^{ij} = -i(x^i\nabla^j - x^j\nabla^i)}$$ → reproduces the cross product.

So that $J^3 = J^{12}$, etc.

→ generalize to 4D: → 6 operators that generate { 3 boosts, 3 rotations, of the Lorentz group.

$$\boxed{J^{\mu\nu} = +i(x^\mu\partial^\nu - x^\nu\partial^\mu)}$$

→ Spatial Rotations: $J^{jk} = i(x^j \partial^k - x^k \partial^j)$

→ Lorentz boosts along x^j axis: $J^{0j} = i(x^0 \partial^j - x^j \partial^0)$

→ Now, want to get commutation rules...
→ compute the commutators of differential ops to get

$$\boxed{[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\sigma} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho}\right)}$$

Ex 3 rotations:

$J^{12} = -J^{21}$
$J^{23} = -J^{32}$
$J^{13} = -J^{31}$

3 boosts

$J^{01} = -J^{10}$
$J^{02} = -J^{20}$
$J^{03} = -J^{30}$

→ 6 total generating matrices...

Ex Consider the 4×4 matrices $(J^{\mu\nu})_{\alpha\beta}$ where μ, ν label which of the 6 matrices, while α, β label the component / matrix element...

$$\boxed{(J^{\mu\nu})_{\alpha\beta} = i\left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha\right)}$$

Can verify that $(J^{\mu\nu})_{\alpha\beta}$ satisfies the comm. relation...

→ These are matrices that act on ordinary Lorentz 4-vectors...
to see this...

→ Look at elements of the Lorentz group...

$$\mathcal{U}(\omega_{\mu\nu}) = \operatorname{Exp}\left[\frac{-i}{2} \omega_{\mu\nu} J^{\mu\nu} \right]$$

infinitesimally →
$$\simeq \mathbb{I} + \frac{-i}{2} \omega_{\mu\nu} J^{\mu\nu}$$
$$\simeq \delta^\alpha_\beta + \frac{-i}{2} \omega_{\mu\nu} (J^{\mu\nu})^\alpha{}_\beta$$

So, infinitesimally...

$$V^\alpha \longrightarrow \delta^\alpha_\beta + \frac{-i}{2} \omega_{\mu\nu} (J^{\mu\nu})^\alpha{}_\beta V^\beta$$

→ $\omega_{\mu\nu}$ is an anti-symmetric tensor that gives the infinitesimal angles...

$V_\alpha, V_\beta \longrightarrow$ 4-vectors...

Ex 1 When $\omega_{12} = -\omega_{21} = \theta$, $\omega_{\mu\nu} = 0$ else, we get

$$[V^\mu] \longrightarrow \begin{array}{c|c} & \begin{matrix} t & x & y & z \end{matrix} \\ \begin{matrix} t \\ x \\ y \\ z \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\theta & 0 \\ 0 & \theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} [V^\mu]$$

→ Infinitesimal $\boxed{\text{ROTATION on } xy \text{ plane.}}$

Ex 2 When $\omega_{01} = -\omega_{10} = \beta \Rightarrow$ get
$\omega_{\mu\nu} = 0$ else

$$[V^\mu] \xrightarrow{\sim} \begin{pmatrix} 1 & \beta & & \\ \beta & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} [V^\mu] \rightarrow \boxed{\text{BOOST along } x}$$

THE DIRAC EQUATION

→ Now that we have seen one f.d. representation of the Lorentz group

→ need to develop formalism for finding all other ~~formalisms~~ representations...

(Problem 3.1)

→ focus on spin $\frac{1}{2}$ systems...

→ In this case, use Dirac's trick ~~due to~~ -

Suppose we had a set of 4 $n \times n$ matrices γ^μ satisfying:

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \times I}$$

Then we could write down an n-dim representation of the Lorentz algebra...

$$\boxed{S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]}$$

These matrices satisfy the commutation relation...

→ $[S^{\mu\nu}, S^{\rho\sigma}] = i(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho})$

→ Verify that this trick works in 3D Euclidean space:

in which case, $\boxed{\gamma^j = i\sigma^j}$ → $\boxed{\{\gamma^i, \gamma^j\} = -2\delta^{ij}}$

→ The matrices representing the Lorentz algebra are then

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\sum_k \varepsilon^{ijk}\sigma^k = J^l$$

Which is what we saw before $\Rightarrow$ angular momentum.

$$\left\{ \begin{array}{l} J^3 = S^{12} = \frac{1}{2}\sigma^3 \\ J^2 = S^{31} = \frac{1}{2}\sigma^2 \\ J^1 = S^{23} = \frac{1}{2}\sigma^1 \end{array} \right\}$$

$\rightarrow$ now, want $S^{\mu\nu}$ for 4D Minkowski space...

$\rightarrow$ matrices γ^μ must be at least 4×4.

$\rightarrow$ suffices to write one explicit realization of the Dirac algebra since all reps are unitarily equiv

Ex

$$\boxed{\gamma^0 = \begin{pmatrix} 0 & \mathbb{I}_{2\times 2} \\ \mathbb{I}_{2\times 2} & 0 \end{pmatrix} ; \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}}$$

"Weyl" / "Chiral" representation.

$\rightarrow$ In this case, the boost & rotation generators are...

Boosts $\rightarrow$ $S^{0i} = \frac{i}{4}[\gamma^0, \gamma^i] = \frac{-i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$

(sum) sum

Rotations $\rightarrow$ $S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \sum_k \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \equiv \sum_k \frac{1}{2}\varepsilon^{ijk}\Sigma^k$

$\boxed{\text{Digression: Group theory \& Representation Theory}}$

Why are we interested in this?

→ Recall that we want to look at ~~all~~ transformation under which the ~~Lagrangian~~ Actions is "invariant"

→ In particular, we want S to be Lorentz invariant

→ can consider this simple Lorentz transformation

$$\begin{cases} \phi(x) \longrightarrow \phi(\Lambda^{-1} x) \\ \text{i.e. } \phi(x^\mu) \rightarrow \phi(\Lambda^{-1\,\mu}{}_\nu x^\nu). \end{cases}$$

→ (check that S is invariant)

→ But this is very simple ... ⇒ There are many more transforms that leave S Lorentz invariant.

→ How do we find all of them?

⇒ For simplicity, we'll just restrict ourselves to linear ~~combinations~~ transformations

→ look at transformations of the form

$$\phi_a(x) \longrightarrow \sum_b M_{ab}(\Lambda)\, \Phi_b(\Lambda^{-1} x)$$

→ more succinctly ...

$$\boxed{\phi \rightarrow M(\Lambda)\,\phi}$$

These matrices M must be "nice" in the sense that M must obey...

$$\boxed{\phi \rightarrow M(\Lambda'')M(\Lambda)\phi = M(\Lambda''\Lambda)\phi}$$

This says that $\{M\}$ (the collection of M's) must be a <u>representation</u> of the Lorentz group.

<u>What??</u> So... recall that $\{\Lambda\}$ is a collection of Lorentz transforms, and they form a group

$$\rightarrow \quad \boxed{\{\Lambda\} \equiv \text{Lorentz group}}$$

<u>A representation</u> π is a function satisfying the property of a group G

$$\pi(g_1)\,\pi(g_2) = \pi(g_1 g_2)$$

$$\underset{\in G}{\uparrow} \qquad \underset{\in G}{\uparrow} \qquad \underset{\in G}{\uparrow}$$

With this, it is clear that

$$\boxed{\{\Lambda\}\ \text{Lorentz group} \Longrightarrow \{M\}\ \text{is a representation of}\ \{\Lambda\}}$$

So... what are these M_i's?

→ <u>Ex</u> Rotation group for spin $\frac{1}{2}$ particles

For spin - $\frac{1}{2}$, the most important nontrivial representation is the 2D representation.

↳ these are unitary matrices with det = 1
(2x2)

⇒ In general: $\mathcal{U} = e^{-i\,\vec{\sigma}\cdot\vec{\theta}/2}$

$\vec{\sigma}$ → Pauli matrices
$\vec{\theta}$ → angle.

For infinitesimal rotations, we can write

$$\mathcal{U} = 1 - \frac{i}{\hbar}\vec{J}\cdot\vec{\theta} \equiv 1 - \vec{J}\cdot\vec{\theta}$$

$\{\mathcal{U}\}$ form a Lie-algebra of the L-group.

$\vec{J}$ here are the "generator" of the Lie algebra

when $\{\mathcal{U}\}$ is a representation of the rotational group, we identify

$$\vec{J} \leftrightarrow \frac{\vec{\sigma}}{2}$$

→ $\vec{J}$ as the quantum angular momentum operator

→ satisfies the commutation relation

$$[J^i, J^j] = i\varepsilon^{ijk} J^k$$

like the generators of SO(3), namely the Pauli matrices...

→ finite rotations are formed by matrix exp.

$$R = \exp\left[-i\theta^i\sigma^i\right]$$

Sep 27, 2020

Back to present problem...

to get generators of the Lie algebra of the Lorentz group, first look at how the angular momentum operators are written in 4D:

(3D) $\vec{J} = \vec{x}\times\vec{p} = \vec{x}\times(-i\vec{\nabla})$

(4D) $\boxed{J^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)}$

with commutation relation:

$$\boxed{[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\right)}$$

→ any matrices that are ~~to~~ represent this algebra must obey the same comm. relation.

→ look at matrices of the form

$$(J^{\mu\nu})_{\alpha\beta} = i\left(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha\right)$$

→ by symmetry, μ,ν ~~label~~ label which of the six matrices we want;

→ α,β label components.

The Dirac Eqn

What are the representations of the Lorentz group?

, especially for spin-$\frac{1}{2}$?

<u>Dirac's trick</u> if we have a set of 4 $n \times n$ matrices γ^μ which satisfies:

Dirac algebra $\rightarrow$ $$\boxed{\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \times I_{n\times n}}$$

Then the n-dim representation of the Lorentz algebra:

$$\boxed{S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]}$$

$\rightarrow$ In other words, $S^{\mu\nu}$ satisfies:

$$[S^{\mu\nu}, S^{\rho\sigma}] = i(g^{\nu\rho}S^{\mu\sigma} + g^{\mu\sigma}S^{\nu\rho} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho})$$

(*) Note that this trick works ~~rein~~ in any dim.

e.g. take $\gamma^j = i\sigma^j$ so that $\{\gamma^i, \gamma^j\} = -2\delta^{ij}$

$\Rightarrow$ $S^{ij} = \frac{1}{2}\varepsilon^{ijk}S^k$ $\rightarrow$ just as before.

2D representation of the rotation group.

Spin $\frac{1}{2}$: $J^3 = J^{12} = \frac{1}{2}\sigma^3$; $J^2 = \frac{1}{2}\sigma^2 = S^{31}$, $J^1 = S^{23} = \frac{1}{2}\sigma^1$

One such representation for the Dirac algebra is

$$\boxed{\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}_{4\times 4} \; ; \; \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}_{4\times 4}}$$

Weyl / chiral representation.

get

Boosts $\quad S^{0i} = \frac{i}{4}[\gamma^0, \gamma^i] = \frac{-i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$

Rotations

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} = \frac{1}{2}\varepsilon^{ijk}\Sigma^k$$

Hermitian Defn not ~~unitary~~, but Ψ is also classical field, not a wfn

All 4-component field Ψ that transforms under boosts & rotations according to → is called a Dirac spinor

S^{ij} are Hermitian
S^{0i} are anti-Hermitian

→ fine b/c Ψ is a classical field, not a ~~q~~ wfn.

Now, what is the field eqn for ψ?

$\rightarrow$ try $(\Box + m^2)\psi = 0 \quad \leftarrow$ KG field eqn.

But this obviously works because the representation are block-diagonal...

$\rightarrow$ need a stronger equation that implies the KG eqn but also contains additional info.

To do this, look at transformation of γ matrices

In an expression we can think of...

$$[\cdots]\Lambda_{\frac{1}{2}}^{-1}\left[4\times4\right]\Lambda_{\frac{1}{2}}\left[\vdots\right] \qquad \tfrac{1}{2}: \text{for spin } \tfrac{1}{2}$$

where $\Lambda_{\frac{1}{2}} = \exp\left\{\frac{-i}{2}\omega_{\mu\nu}S^{\mu\nu}\right\}$

$$\simeq 1 - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}$$

$$\Rightarrow [\gamma^\mu] \rightarrow [\Lambda_{\frac{1}{2}}^{-1}][\gamma^\mu][\Lambda_{\frac{1}{2}}]$$

$$= \left(1 + \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\gamma^\mu\left(1 - \frac{i}{2}\omega_{\alpha\beta}S^{\alpha\beta}\right)$$

$= \ldots$ (some terms of higher order cancelled ...)

$$= \gamma^\mu - \frac{i}{2}\omega_{\alpha\beta}\underbrace{[\gamma^\mu, S^{\alpha\beta}]}_{?}$$

where a quick computation shows that

$$\boxed{[\gamma^\mu, S^{\alpha\beta}] = (J^{\alpha\beta})^\mu{}_\nu \gamma^\nu}$$

where

$$J^{\alpha\beta} = +i\left(g^{\alpha\mu}\delta^\beta_\nu - g^{\beta\mu}\delta^\alpha_\nu\right)$$

So...

$$\boxed{\gamma^\mu \longrightarrow \gamma^\mu - \frac{i}{2}\omega_{\alpha\beta}(J^{\alpha\beta})^\mu{}_\nu \gamma^\nu} = \Lambda^{-1}_{\frac{1}{2}}\gamma^\mu\Lambda_{\frac{1}{2}}$$

→ γ^μ transforms like 4-vectors...!

⇒ γ^μ are invariant under simultaneous rotations of their vectors & spinor indices...

→ can treat "μ" on γ^μ as a vector index!

→ can dot γ^μ into ∂_μ to form a Lorentz-inv. differential operator...

Dirac eqn

$$\boxed{(i\gamma^\mu\partial_\mu - m)\Psi = 0}$$

Check that this is Lorentz-inv:

Let $\Psi(x) \longrightarrow \Lambda_{\frac{1}{2}}\Psi(\Lambda^{-1}x)$ then

$$i\gamma^\mu\partial_\mu\Psi \to \left(i\gamma^\mu\Lambda^{+1}_{\frac{1}{2}}\right)\left(\partial_\mu\left(\Psi(\Lambda^{-1}x)\right)\right)$$

$$= i\Lambda_{\frac{1}{2}}\left(\Lambda^{-1}_{\frac{1}{2}}\gamma^\mu\Lambda_{\frac{1}{2}}\right)\cdot(\Lambda^{-1})^\alpha{}_\mu(\partial_\alpha\Psi)(\Lambda^{-1}x)$$

same Lorentz transform

$$= i\Lambda_{\frac{1}{2}} (\Lambda)^{\mu}{}_{\nu} \gamma^{\nu} \cdot (\Lambda^{-1})^{\alpha}{}_{\mu} \ (\partial_\alpha \Psi)(\Lambda^{-1}x)$$

$$= i\Lambda_{\frac{1}{2}} \gamma^{\mu} \underbrace{(\Lambda)^{\mu}{}_{\nu} (\Lambda)^{\alpha}{}_{\mu}}_{\delta^{\alpha}_{\nu}} (\partial_\alpha \Psi)(\Lambda^{-1}x)$$

$$= i\Lambda_{\frac{1}{2}} \gamma^{\mu} \partial_{\mu} \Psi(\Lambda^{-1}x)$$

$$\Rightarrow \quad i\gamma^{\mu}\partial_{\mu}\Psi(x) \to \Lambda_{\frac{1}{2}} i\gamma^{\mu}\Psi(\Lambda^{-1}x)$$

$\to$ transforms the same way as $\Psi(\Lambda^{-1}x)$.

Cleaver way:

Let $[i\gamma^{\mu}\partial_{\mu} - m]\Psi(x) \to [i\gamma^{\mu}(\Lambda^{-1})^{\nu}{}_{\mu}\partial_{\nu} - m]\Lambda_{\frac{1}{2}}\Psi(\Lambda^{-1}x)$

$$= \Lambda_{\frac{1}{2}} \underbrace{\Lambda^{-1}_{\frac{1}{2}} [i\gamma^{\mu}\Lambda^{-1}{}^{\nu}{}_{\mu}\partial_{\nu} - m]\Lambda_{\frac{1}{2}}}\Psi(\Lambda^{-1}x)$$

$$= \Lambda_{\frac{1}{2}} \left\{ i\Lambda^{\mu}{}_{\sigma}\gamma^{\sigma}(\Lambda^{-1})^{\nu}{}_{\mu}\partial_{\nu} - m \right\} \Psi(\Lambda^{-1}x)$$

$$= \Lambda_{\frac{1}{2}} \left\{ i\gamma^{\nu}\partial_{\nu} - m \right\} \Psi(\Lambda^{-1}x)$$

$$= 0 \quad \checkmark$$

Now, can show that Dirac eqn implies KG eqn:

$$0 = (i\gamma^{\mu}\partial_{\mu} - m)\Psi$$

$$\to \quad 0 = (-i\gamma^{\mu}\partial_{\mu} - m)(+i\gamma^{\nu}\partial_{\nu} - m)\Psi$$

$$= (\gamma^{\mu}\gamma^{\nu}\partial_{\nu}\partial_{\mu} + m^2)\Psi \quad = \ldots$$

$$= \left(\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2 \right) \psi$$

$$= \left[\frac{1}{2} \left(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu \right) \partial_\mu \partial_\nu + m^2 \right] \psi$$

$$= \left[\frac{1}{2} \{ \gamma^\mu, \gamma^\nu \} \partial_\mu \partial_\nu + m^2 \right] \psi$$

KG eqn.

$$= \left(g^{\mu\nu} \partial_\mu \partial_\nu + m^2 \right) \psi = \left(\Box + m^2 \right) \psi = 0$$

What is the Lagrangian for the Dirac theory?

→ need a way to multiply two Dirac spinors to get a Lorentz scalar.

$\psi^\dagger \psi$ doesn't work b/c under a boost,

$$\psi^\dagger \Delta_{\frac{1}{2}}^\dagger \Delta_{\frac{1}{2}} \psi \neq \psi^\dagger \psi \quad \text{since } \Delta_{\frac{1}{2}} = \exp\{-i \omega_{\mu\nu} S^{\mu\nu}\}$$

not unitary .. since not all $S^{\mu\nu}$ are Herm.

→ to fix this, define

$$\boxed{\bar{\psi} = \psi^\dagger \gamma^0}$$

Then under infinitesimal transform, get

$$\bar{\psi} \longrightarrow \psi^\dagger \Delta_{\frac{1}{2}}^\dagger \gamma^0 \cong \psi^\dagger \left(1 + \frac{i}{2} \omega_{\mu\nu} (S^{\mu\nu})^\dagger \right) \gamma^0$$

when ~~...~~ $\mu \neq 0$ & $\nu \neq 0$, $(S^{\mu\nu})^\dagger = (S^{\mu\nu})$

$$\& \left(S^{\mu\nu} \leftrightarrow \gamma^0 \right)$$

When $\mu = 0$ or $\nu = 0$, $(S^{\mu\nu})^\dagger = -S^{\mu\nu}$

& $S^{\mu\nu}$ anti-commutes w/ γ^0.

$$\rightarrow \quad \bar{\Psi} \rightarrow \Psi^\dagger \left(1 + \frac{i}{2} w_{\mu\nu} (S^{\mu\nu})^\dagger\right) \gamma^0$$

$$= \underbrace{\Psi^\dagger \gamma^0}\left(1 + \frac{i}{2} w_{\mu\nu} S^{\mu\nu}\right)$$

$$= \bar{\Psi}\left(1 + \frac{i}{2} w_{\mu\nu} S^{\mu\nu}\right) = \bar{\Psi} \Lambda_{\frac{1}{2}}^{-1} \text{ as desired}$$

$$\rightarrow \quad \boxed{\bar{\Psi} \rightarrow \bar{\Psi} \Lambda_{\frac{1}{2}}^{-1}}$$

and so $\boxed{\bar{\Psi}\Psi = \Psi^\dagger \gamma^0 \Psi}$ is a Lorentz scalar.

Similarly, can show that

$\boxed{\bar{\Psi}\gamma^\mu \Psi}$ is a Lorentz vector.

$\rightarrow$ the correct Lorentz-inv. Dirac Lagrangian is

$$\boxed{\mathcal{L}_{Dirac} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi}$$

$$\begin{cases} \text{E-L eqn for } \bar{\Psi} \text{ gives } (i\gamma^\mu \partial_\mu - m)\Psi = 0 \\ \text{E-L eqn for } \Psi \text{ gives } -i\partial_\mu \bar{\Psi}\gamma^\mu - m\bar{\Psi} = 0 \end{cases}$$

WEYL SPINOR

Recall that

$$\boxed{\begin{aligned} S^{0i} &= \frac{-i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \\ S^{ij} &= \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \end{aligned}}$$

Since block-diagonal $\Rightarrow$ Dirac representation of the Lorentz group is reducible.

$\rightarrow$ Can form 2-D representations by considering each block separately.

$\rightarrow$ write $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ $\begin{matrix} \rightarrow \text{left-handed} \\ \rightarrow \text{RH} \end{matrix}$ Weyl spinors

Under infinitesimal boost $\vec{\beta}$ & rotation $\vec{\theta}$, these transform as

$$\boxed{\begin{aligned} \psi_L &\rightarrow \left(1 - i\vec{\theta}\cdot\vec{\sigma}/2 - \vec{\beta}\cdot\vec{\sigma}/2\right)\psi_L \\ \psi_R &\rightarrow \left(1 - i\vec{\theta}\cdot\vec{\sigma}/2 + \vec{\beta}\cdot\vec{\sigma}/2\right)\psi_R \end{aligned}}$$

Recall that $\left(\tanh(\vec{\beta}) = \frac{|v|}{c}\right)$.

$\rightarrow$ Transf of ψ_R is equiv to transf of ψ_L^*

By noting that

$$\psi_L^* \longrightarrow \left(1 + i\tilde{\theta}\cdot \frac{\vec{\sigma}^*}{2} - \tilde{\beta}\cdot\frac{\vec{\sigma}^*}{2}\right)\psi_L^*$$

noting that $\underset{\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}}{\sigma^2}\vec{\sigma}^* = -\vec{\sigma}\,\underset{\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}}{\sigma^2}$ $\quad(\sigma^2 = \sigma^y)$

we find:

$$\sigma^2\psi_L^* \longrightarrow \sigma^2\left[1 + i\tilde{\theta}\cdot\frac{\vec{\sigma}^*}{2} - \tilde{\beta}\cdot\frac{\vec{\sigma}^*}{2}\right]\psi_L^*$$

$$= \underbrace{\left[1 - i\tilde{\theta}\cdot\frac{\vec{\sigma}}{2} + \tilde{\beta}\cdot\frac{\vec{\sigma}}{2}\right]}_{\text{like } \psi_R \text{ transform.}}\sigma^2\psi_L^*$$

So $\sigma^2\psi_L^*$ transform like ψ_R...

With $\psi = \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}$, the Dirac eqn has form.

$$\boxed{(i\gamma^\mu\partial_\mu - m)\psi = 0 \iff \begin{pmatrix}-m & i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\\ i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla}) & -m\end{pmatrix}\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} = 0}$$

When $m = 0$, the eqns for ψ_L & ψ_R decouple to give us

$$\left.\begin{cases} i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\psi_L = 0 \\ i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\psi_R = 0\end{cases}\right\} \longrightarrow \text{Weyl eqns.}$$

→ important for neutrinos & weak force studies...

For convenience let us define...

$$\boxed{\sigma^\mu \equiv (1, \vec{\sigma}) \quad ; \quad \bar{\sigma}^\mu = (1, -\vec{\sigma})}$$

so that

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \qquad \sigma^\mu = (1, \sigma^1, \sigma^2, \sigma^3)$$

With this, can simplify notation. Dirac eqn becomes...

$$\boxed{\begin{pmatrix} -m & i\sigma\cdot\partial \\ i\bar{\sigma}\cdot\partial & -m \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0}$$

$i\sigma\cdot\partial = i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})$

$i\bar{\sigma}\cdot\partial = i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})$

& the Weyl eqns become:

$$\boxed{\begin{aligned} (i\bar{\sigma}\cdot\partial)\psi_L &= 0 \\ (i\sigma\cdot\partial)\psi_R &= 0 \end{aligned}}$$

Free-particle solution of Dirac Eqn

Since Dirac field ψ satisfies KG eqn, ψ can be written as a lin. comb. of plane waves:

$$\psi(x) = u(p)\, e^{-ip\cdot x} \quad , \quad p^2 = m^2$$

so that

$$p^0 = \sqrt{\vec{p}^2 + m^2} = E_{\vec{p}}.$$

Look only solutions with positive frequency ... that is
$E_p = p^0 > 0$ ....

ψ solves Dirac eqn $\Rightarrow$ $(i\gamma^\mu \partial_\mu - m)\psi = 0$

$(\Rightarrow)$ $(i\gamma^\mu \partial_\mu - m)\, u(p)\, e^{-ip\cdot x} = 0$

$$\Rightarrow \quad \boxed{(i\gamma^\mu p_\mu - m)\, u(p) = 0}$$

go to rest frame $\Rightarrow$ $p = p_0 = (m, \vec{0})$. The soln for generic p can be obtained by boosting with $\Lambda_{\frac{1}{2}}$.

In rest frame, we have

$$(i\gamma^\mu p_\mu - m)\, u(p) \longrightarrow (m\gamma^0 - m)\, u(p_0) = m(\gamma^0 - 1)\, u(p_0) = 0$$

$$\Rightarrow \quad m \begin{pmatrix} -I & I \\ I & -I \end{pmatrix} u(p_0) = 0$$

$$\Rightarrow \quad \boxed{u(p_0) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}} \quad \text{two-component spinor}$$

just a factor with norm. constraint.

$$\xi^\dagger \xi = 1.$$

<u>What are these ξ?</u>

Look at rotation generators ..

$$\boxed{S^{ij} = \frac{1}{2} \varepsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}}$$

In particular, $S^z = S^{12} = \frac{1}{2}\begin{pmatrix} \sigma^z & 0 \\ 0 & \sigma^z \end{pmatrix}$

So if $\left\{ \begin{array}{l} \xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \text{gives spin-}z = \frac{+1}{2} \\ \xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \text{gives spin-}z = \frac{-1}{2} \end{array} \right\}$

Now, we're in rest frame, so $p^\mu = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Now, boost to frame where particle has velocity...

$\vec{v} = v \cdot \vec{z}$. Let $\underbrace{\tanh(\eta)}_{\text{"rapidity"}} = \frac{v}{c}$.

Then

$$\begin{pmatrix} E \\ p^3 \end{pmatrix} = p^\mu = \underbrace{\begin{pmatrix} \cosh\eta & 0 & 0 & \sinh(\eta) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh(\eta) \end{pmatrix}}_{\Lambda_{\frac{1}{2}} \rightarrow \text{ just the Lorentz transform.}} \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m\cosh\eta \\ 0 \\ 0 \\ m\sinh\eta \end{pmatrix}$$

(infinitesimal $\Lambda_{\frac{1}{2}}$)

$\rightarrow$ In this frame, $\left\{ \begin{array}{l} E = m\cosh\eta \\ p^3 = m\sinh\eta \end{array} \right.$

Now, apply the same boost to $u(p)$ ...

$$u(p) = \Lambda_{\frac{1}{2}} \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix} \qquad \left(\Lambda_{\frac{1}{2}}\right) = \exp\left(\frac{-i}{2} \omega_{\mu\nu} S^{\mu\nu}\right)$$

$$= \exp\left(\frac{-1}{2} \eta \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\right) \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

σ^3 ... $i/(\sigma^3 \quad \sigma^3)$

So, infitesimally...

$$\exp\left\{ \frac{-1}{2}\eta \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix} \right\} \simeq \begin{pmatrix} \cosh \eta/_2 - \sinh \eta/_2\, \sigma^3 & 0 \\ 0 & \cosh\eta/_2 + \sinh\frac{\eta}{2}\,\sigma^3 \end{pmatrix}$$

So that

$$u(p) \simeq \sqrt{m} \begin{pmatrix} \left(\cosh \eta/_2 - \sinh \eta/_2\, \sigma^3\right) \xi \\ \left(\cosh \eta/_2 + \sinh \eta/_2\, \sigma^3\right) \xi \end{pmatrix} \ldots$$

Simplify... note that

$$\left(\cosh \eta/_2 - \sinh \eta/_2\, \sigma^3\right)^2 = \ldots$$

$$= \cosh \eta - \sinh \eta\, \sigma^3$$

$$= E/_m - P^3/_m\, \sigma^3 = \frac{p\cdot\sigma}{m}$$

$$= \frac{p_\mu \sigma^\mu}{m} \quad \text{where } \sigma^\mu = (1, \vec{\sigma})$$

$p_\mu \sigma^\mu/_m$

So...

$$\left(\cosh \eta/_2 - \sinh \eta/_2\, \sigma^3\right) = \sqrt{\frac{p\cdot\sigma}{m}}$$

&

$$\left(\cosh \eta/_2 + \sinh \eta/_2\, \sigma^3\right) = \sqrt{\frac{p\cdot\bar{\sigma}}{m}}$$

So...

$$\boxed{u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\; \xi \\ \sqrt{p\cdot\bar{\sigma}}\; \xi \end{pmatrix}}$$

$\longrightarrow$ compact & valid for any arbitrary direction of $\vec{p}$.

Fact $(p\cdot\sigma)(p\cdot\bar{\sigma}) = (p^0)^2 - \vec{p}^{\,2} = p^2 = m^2.$

Now, back to example...

$$p = (E, 0, 0, p^3)$$

$$\Rightarrow \quad p\cdot\sigma = \cdots = \begin{pmatrix} E-p^3 & 0 \\ 0 & E+p^3 \end{pmatrix}$$

so $\quad \sqrt{p\cdot\sigma} = \begin{pmatrix} \sqrt{E-p^3} & 0 \\ 0 & \sqrt{E+p^3} \end{pmatrix}$

and $\quad \sqrt{p\cdot\bar{\sigma}} = \begin{pmatrix} \sqrt{E+p^3} & 0 \\ 0 & \sqrt{E-p^3} \end{pmatrix}$

Pick $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then (spin $\frac{1}{2}$)

$$u(p) = \begin{pmatrix} \sqrt{E-p^3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sqrt{E+p^3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

Pick $\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ then (spin $-\frac{1}{2}$)

$$u(p) = \begin{pmatrix} \sqrt{E+p^3}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \sqrt{E-p^3}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

In the massless limit, $E \to p^3 \quad \left(E^2 = \sqrt{m^2c^{\cancel{4}} + (p^3)^{\cancel{2}}}\right)$

$$\Rightarrow \boxed{\begin{aligned} u(p) &= \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \sqrt{2E}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \quad \text{spin } \frac{1}{2} \\ u(p) &= \begin{pmatrix} \sqrt{2E}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix} \quad \text{spin } \frac{-1}{2} \end{aligned}}$$

These states: $u(p) = \sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$, $u(p) = \sqrt{2E}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ are eigenstates of the helicity operator

$$\boxed{h \equiv \hat{p}\cdot\vec{S} = \sum_i \frac{1}{2}\hat{p}_i \begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix}} = \frac{1}{2}\hat{p}\cdot\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix}$$

When $\begin{cases} h = \frac{1}{2} \Rightarrow \text{call Right-handed} \\ h = \frac{-1}{2} \Rightarrow \text{call Left-handed}\end{cases}$

Note: Dirac helicity is frame-dependent (for massive particle) — since can boost so that momentum is in the opposite direction.

(This can't happen for massless particles).

Back to Weyl's eqn:

$$\begin{cases} i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\psi_L = i(\bar{\sigma}\cdot\partial)\psi_L = 0 \\ i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\psi_R = i(\sigma\cdot\partial)\psi_R = 0 \end{cases}$$

Plug $\psi = u(p)e^{-ip\cdot x}$ in, $\partial_0 \longrightarrow -iE$

$\vec{\nabla} \longrightarrow i\vec{p}$.

So, with $m=0$, $\vec{p} = E\hat{p}$.

$$\Rightarrow \text{get} \begin{cases} (E + E\hat{p}\cdot\sigma)\psi_L = 0 \Rightarrow (E)(1+2h)\psi_L = 0 \quad \Rightarrow h = \frac{-1}{2} \\ (E - E\hat{p}\cdot\sigma)\psi_R = 0 \Rightarrow (E)(1-2h)\psi_R = 0 \quad \longrightarrow h = \frac{1}{2} \end{cases}$$

$$\Rightarrow \begin{cases} \psi_L \text{ is left-handed} \\ \psi_R \text{ is right-handed} \end{cases}, \text{ as expected}$$

$\#$

Recap... $\psi(x) = u(p)\, e^{-ip\cdot x}$ $(p^0 > 0)$ $\rightarrow$ positive frequency

$$\rightarrow \quad u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix} \longrightarrow \text{spinor.}$$

when $\psi(x) = u(p)\, e^{-ip\cdot x}$ $(p^0 < 0)$ $\longrightarrow$ negative frequency

$$\Rightarrow \quad (u(p)) = \cdots = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ -\sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix}$$

$\#$

Now, note that ($p^0 > 0$ again)

$$u^\dagger u = \left(\xi^\dagger\sqrt{p\cdot\sigma},\ \xi^\dagger\sqrt{p\cdot\bar\sigma}\right)\cdot\begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix}$$

$$= \xi^\dagger\left[(p\cdot\sigma) + (p\cdot\bar\sigma)\right]\xi$$

$\longrightarrow$ depends on p!

$$\Rightarrow \quad u^\dagger u = 2E_p\, \xi^\dagger\xi$$

$\rightarrow$ $u^\dagger u$ is not a Lorentz-inv scalar just like $\psi^\dagger\psi$.

$\Rightarrow$ to make one such Lorentz-inv scalar, define

$$\boxed{\bar u(p) = u^\dagger(p)\,\gamma^0} \quad \Rightarrow \quad \boxed{\bar u u = 2m\,\xi^\dagger\xi} \quad \text{Lorentz-inv (indep of } \vec p)$$

So, wish ~~de~~ $\bar{u}u = u^\dagger \gamma^0 u = 2m\underbrace{\xi^\dagger\xi}_{1} = 2m$

$\rightarrow$ convenient to choose ONB spinors. ξ^1 & ξ^2.

This gives 2 linearly indep solution for $u(p)$:

$$\boxed{u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} \qquad s = 1,2}$$

Normalize:

$$\bar{u}^r(p)\,u^s(p) = 2m\,\delta^{rs} \Leftrightarrow u^{r\dagger}(p)\,u^s(p) = 2E_p\,\delta^{rs}$$

For the negative-freq solns, we get

$$\bar{v}^r(p)\,v^s(p) = -2m\,\delta^{rs} \Leftrightarrow v^{r\dagger}(p)\,v^s(p) = +2E_p\,\delta^{rs}$$

and

v, u are orthogonal to each other...

$$\bar{u}^r(p)\,v^s(p) = \bar{v}^r(p)\,u^s(p) = 0$$

Finally, talk about $\boxed{\text{Spin sums}}$

$\rightarrow$ useful when evaluating Feynman diagrams.

$\rightarrow$ when we need to sum all spin-$\frac{1}{2}$ polarizations

Since $\{\xi^s\}$ form an ONB,

$$\sum_{s=1,2} \xi^s \xi^{s\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using this, we find that

$$\sum_{s=1,2} u^s(p)\,\bar{u}^s(p) = \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix} \cdot \left(\xi^{s\dagger}\sqrt{p\cdot\bar\sigma},\ \xi^{s\dagger}\sqrt{p\cdot\sigma} \right)$$

$$= \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix} \cdot \left(\xi^{s\dagger}\sqrt{p\cdot\sigma},\ \xi^{s\dagger}\sqrt{p\cdot\bar\sigma} \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

"completeness" $\longrightarrow$

$$= \begin{pmatrix} p\cdot\sigma & m \\ m & p\cdot\bar\sigma \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar\sigma & m \end{pmatrix}$$

$$\left(\begin{aligned} \sqrt{p\cdot\bar\sigma}\,\sqrt{p\cdot\sigma} &= \cancel{\tfrac{1}{2}\left((p\cdot\bar\sigma)+(p\cdot\sigma)\right)^2 - (p\cdot\sigma - p\cdot\bar\sigma)} = m \\ &= \sqrt{(p\cdot\bar\sigma)(p\cdot\sigma)} = \sqrt{\left((p^0,\vec p)\cdot(1,-\vec\sigma)\right)\left((p^0,\vec p)\cdot(1,\vec\sigma)\right)} \\ &= \sqrt{(p^0)^2 - \vec p^{\,2}} = m. \end{aligned} \right)$$

So,

$$\boxed{\begin{aligned} \sum_{s=1,2} u^s(p)\,\bar{u}^s(p) &= \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar\sigma & m \end{pmatrix} = \underbrace{p\cdot\gamma}_{\not p} + m I \\ \sum_{s=1,2} v^s(p)\,\bar{v}^s(p) &= \begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar\sigma & -m \end{pmatrix} = \underbrace{p\cdot\gamma}_{\not p} - m I \end{aligned}}$$

Feynman's slash notation

→ The combo $\gamma \cdot p$ occur so often that Feynman introduced the notation:

$$\boxed{\not{p} \equiv \gamma^{\mu} p_{\mu} = p_{\mu} \gamma^{\mu}}$$

#

Exercise

Recall that $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$

Let ψ_L^* be the complex conjugate of ψ_L. The Majorana eqn is given by

$$i\bar{\sigma} \cdot \partial \psi_L(x) - im\sigma^2 \psi_L^*(x) = 0$$

where

$$\sigma^2 = \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

and

$$\bar{\sigma} = (1, -\vec{\sigma})$$

$$m = \text{Majorana mass.}$$

(a) Show that $i\bar{\sigma} \cdot \partial \psi_L(x) - im\sigma^2 \psi_L^*(x) = 0$ is inv under infinitesimal rotation.

(b) Show that $i\bar{\sigma} \cdot \partial \psi_L(x) - im\sigma^2 \psi_L^*(x) = 0$ is inv under infinitesimal boosts.

(a) In general, infinitesimal Lorentz transforms on ψ_L has the form

$$\psi_L(x) \longrightarrow \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\psi_L(x)$$

→ Rotation has the form:

$$\psi_L(x) \longrightarrow \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\psi_L(x)$$

$$\Rightarrow \sigma^2\psi_L^*(x) \longrightarrow \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\sigma^2\psi_L(x) \quad \text{(notes)}$$

Lorentz transform:

$$\psi_L(x) \rightarrow \Lambda_{\frac{1}{2}}\psi_L(\Lambda^{-1}x)$$

$$\partial_\mu\psi_L(x) \longrightarrow (\Lambda^{-1})^\alpha_{\ \mu}\,\partial_\alpha\psi_L(\Lambda^{-1}x)$$

→ put these together ...

$$\psi_L(x) \longrightarrow \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\psi_L(\Lambda^{-1}x)$$

$$\sigma^2\psi_L^*(x) \longrightarrow \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\sigma^2\psi_L^*(\Lambda^{-1}x)$$

$$\rightarrow \quad -im\sigma^2\psi_L^*(x) \rightarrow -im\left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\sigma^2\psi_L^*(\Lambda^{-1}x)$$

Next, $i\bar{\sigma}\cdot\partial\psi_L(x) = i\bar{\sigma}^\mu\partial_\mu\psi_L(x)$

$$\Rightarrow i\bar{\sigma}^\mu(\Lambda^{-1})^\alpha_{\ \mu}\,\partial_\alpha\psi_L(\Lambda^{-1}x)\left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)$$

$$= i\bar{\sigma}^\mu\left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)(\Lambda^{-1})^\alpha_{\ \mu}\,\partial_\alpha\psi_L(\Lambda^{-1}x)$$

use trick: multiply:

$$1 \equiv \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\left(1 + i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right) \quad \text{(rot x inv-rot)}$$

$$\Rightarrow \to i\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\overbrace{\left(1+i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^{\mu}\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)}$$

$$\times (\Lambda^{-1})^{\alpha}_{\ \mu}\,\partial_\alpha \Psi_L(\Lambda^{-1}x)$$

$$= i\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\boxed{?}\,(\Lambda^{-1})^{\alpha}_{\ \mu}\,\partial_\alpha \Psi_L(\Lambda^{-1}x)$$

what is $\boxed{?}$

$$\to \boxed{?} = \left(1+i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^{\mu}\left(1-i\vec{\theta}\cdot\vec{\sigma}/2\right)$$

$$\simeq \bar{\sigma}^{\mu} + i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\bar{\sigma}^{\mu} - i\vec{\theta}\cdot\bar{\sigma}^{\mu}\frac{\vec{\sigma}}{2}$$

$$= \bar{\sigma}^{\mu} - \frac{i}{2}\vec{\theta}\left[\bar{\sigma}^{\mu},\vec{\sigma}\right]$$

can show that

$$= \bar{\sigma}^{\mu} - i\theta\,[J_4^{ij}]^{\mu}_{\ \nu}\,\bar{\sigma}^{\nu}$$

$$\downarrow$$

$$i(g^{i\mu}\delta^{j}_{\nu} - g^{j\mu}\delta^{i}_{\nu})$$

$$\Rightarrow \boxed{?} = (\Lambda_4)^{\mu}_{\ \nu}\,\bar{\sigma}^{\nu} \to \bar{\sigma}^{\mu} \text{ transforms like 4-vector}$$

$$\Rightarrow i\bar{\sigma}\cdot\partial\Psi_L(x) \Rightarrow i\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\Lambda^{\mu}_{\ \nu}\,\bar{\sigma}^{\nu}(\Lambda^{-1})^{\alpha}_{\ \mu}\,\partial_\alpha\Psi_L(\Lambda^{-1}x)$$

$$= i\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\delta^{\alpha}_{\nu}\,\bar{\sigma}^{\nu}\,\partial_\alpha\Psi_L(\Lambda^{-1}x)$$

$$= i\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^{\nu}\,\partial_\nu\Psi_L(\Lambda^{-1}x)$$

$$= i\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}\cdot\partial\Psi_L(\Lambda^{-1}x) \checkmark$$

$\rightarrow \quad i\bar{\sigma}\cdot\partial\psi_L(x) - im\,\sigma^2\psi_L^*(x) = 0$

$\rightarrow$ leads to infinitesimal rotation...

$$\left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\Big\{ \underbrace{i\bar{\sigma}\cdot\partial\psi_L(\Lambda^{-1}x) - im\,\sigma^2\psi_L^*(\Lambda^{-1}x)}_{=0} \Big\} = 0$$

$\Rightarrow$ done! So Majorana eqn is invariant under infinitesimal rotations...

(b) Boosts (proceed in a similar way...)

Key

$$\left(1 - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^\mu\left(1 + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)$$

$$= \bar{\sigma}^\mu - \frac{1}{2}\vec{\beta}\{\bar{\sigma}^\mu, \vec{\sigma}\}$$

$$\rightarrow \quad \bar{\sigma}^\mu \rightarrow \bar{\sigma}^\mu - i\vec{\beta}\,[J_4^{0i}]^\mu{}_\nu\,\bar{\sigma}^\nu$$

$$\rightarrow \quad \left(1 - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^\mu\left(1 + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right) = (\Lambda)^\mu{}_\nu\,\bar{\sigma}^\nu \quad \text{as before}$$

so plug this back into the original eqn.

~~Sep 28, 2020~~ $\boxed{\text{Dirac Matrices \& Dirac Field Bilinears}}$

Oct 2, 2020 Recall that $\bar{\psi}\psi$ is Lorentz scalar...

Recall that $\bar{\psi}\gamma^\mu\psi$ is also a 4-vector.

$\Rightarrow$ $\boxed{?}$ Consider $\bar{\psi}\Gamma\psi$, where Γ is any 4×4

→ can we decompose into terms that have definite transformation properties under the Lorentz group?

↳ Γ can be written as combo of 16-element basis defined by

$$
\begin{array}{lll}
1: & \mathbb{1} & \to 1 \\
4: & \gamma^\mu & \to 4C2 \\
6: & \gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu,\gamma^\nu] \equiv \gamma^{[\mu}\gamma^{\nu]} & \to 4C3 \\
4: & \gamma^{\mu\nu\sigma} = \gamma^{[\mu}\gamma^\nu\gamma^{\sigma]} & \to 4C2 \\
1: & \gamma^{\mu\nu\sigma\rho} = \gamma^{[\mu}\gamma^\nu\gamma^\sigma\gamma^{\rho]} & \to 4C2
\end{array}
$$

16 total.

→ all are anti-symmetric products.

→ Each set of matrices transform as an antisym tensor of successively higher rank.

→ Introduce

$$\boxed{\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3}$$

$$= \frac{-i}{4!}\varepsilon^{\mu\nu\sigma\rho}\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\rho$$

$0123 \to 1$
$1023 \to -1$

↳ totally anti-symmetric

Note that → $\boxed{(\gamma^5)^2 = \mathbb{1}}$

$$\boxed{\begin{aligned} (\gamma^5)^\dagger &= -i(\gamma^3)^\dagger \cdots (\gamma^0)^\dagger \\ &= +i\gamma^3\gamma^2\gamma^1\gamma^0 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5 \end{aligned}}$$

also

$$\boxed{\begin{aligned} \{\gamma^5, \gamma^\mu\} &= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu + i\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 \quad (-1) \\ &= 0 \end{aligned}}$$

and thus

$$\boxed{[\gamma^5, S^{\mu\nu}] = [\gamma^5, \tfrac{i}{4}[\gamma^\mu, \gamma^\nu]] = 0}$$

⇒ Eigenstates of γ^5 with different eigv don't mix under Lorentz transform.

→ In basis, can write...

$$\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \begin{matrix} \to \text{ for } \Psi_L \text{ (left-hd)} \\ \to \text{ for } \Psi_R \text{ (right-hd)} \end{matrix}$$

→ a Dirac spinor with only L/R component is an eigenstate of γ^5 with eigv $(-1)/(1)$.

With γ^5, can rewrite the table of 4×4 matrices as

$\mathbb{1}$	scalar	1
γ^μ	vector	4
$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$	tensor	6
$\gamma^\mu\gamma^5$	pseudo vector	4
γ^5	pseudo scalar	1
		16

pseudo-vector/scalar is due to the fact that they transform like vector/scalar, BUT with an additional sign change under parity-transf under Lorentz transf

Ex Parity transf: $\vec{x} \to -\vec{x}$

$$\hookrightarrow (x^0, x^i) \to (x^0, -x^i)$$

If instead $(x^0; \vec{x}) \to -(x^0; -\vec{x}) = (-x^0; \vec{x})$ under parity, we call this a <u>pseudo-vector</u>

$\to$ pseudo vect/scalar flips sign under parity transf.

$\to$ From vector & pseudo-vector we can form 2 currents out of Dirac field bilinears...

$$\boxed{\begin{array}{ll} j^\mu(x) = \bar{\Psi}(x)\gamma^\mu\Psi(x) & \to \text{vector current} \\ j^{\mu 5}(x) = \bar{\Psi}(x)\gamma^\mu\gamma^5\Psi(x) & \to \text{pseudo vector current} \end{array}}$$

Assume that Ψ satisfies Dirac eqn... $\bar{\Psi} = \Psi^\dagger\gamma^0$

$\to i\not{\partial}\Psi = m\Psi$ & $-i\not{\partial}\bar{\Psi} = m\bar{\Psi}$ (since $\bar{\gamma}^\mu = \gamma^\mu$)

$\to$ compute div of these currents...

$$\partial_\mu j^\mu = (\partial_\mu\bar{\Psi})\gamma^\mu\Psi + \bar{\Psi}\gamma^\mu(\partial_\mu\Psi)$$

$$= im\bar{\Psi}\Psi + \bar{\Psi}(-im\Psi) = 0$$

$$\to \boxed{\partial_\mu j^\mu = 0}$$

→ j^μ is always <u>conserved</u> if $\psi(x)$ satisfies Dirac eqn

→ It is the Noether current for the symmetry

$$\psi(x) \to e^{i\alpha}\psi(x)$$

<u>Similarly</u>

$$\partial_\mu j^{\mu 5} = (\partial_\mu \bar\psi)\gamma^\mu\gamma^5\psi + \bar\psi\gamma^\mu\gamma^5\partial_\mu\psi$$

$$= (\partial_\mu\bar\psi)\gamma^\mu\gamma^5\psi + (-1)\bar\psi\gamma^5\gamma^\mu\partial_\mu\psi$$

$$= im\bar\psi\gamma^5\psi + (-1)(-i)m\bar\psi\gamma^5\psi$$

→ $\boxed{\partial_\mu j^{\mu 5} = 2im\bar\psi\gamma^5\gamma}$ → "axial vector current

→ if $m=0$ then $\partial_\mu j^{\mu 5}$ is conserved.

→ When $m=0$, $j^{\mu 5}$ is the Noether current for the symmetry

$$\psi(x) \to e^{i\alpha\gamma^5}\psi(x).$$

(we worry about ~~that~~ the rest of this section in ~~Weyl~~ Peskin's ...)

QUANTIZATION OF THE DIRAC FIELD

→ now, ready to construct quantum theory of the Dirac field.

Recall Lagrangian:

$$\mathcal{L} = \bar{\Psi}(i\not{\partial} - m)\Psi = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi .$$

→ Canonical momentum conjugate to Ψ is

$$\frac{\partial \mathcal{L}}{\partial(\partial_0\Psi)} = i\bar{\Psi}\gamma^0 = i\Psi^\dagger .$$

→ Hamiltonian is given by

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial(\partial_0\Psi)}\partial_0\Psi - \mathcal{L} = \cancel{i\bar{\Psi}\gamma^0\partial_0\Psi} - \cancel{i\bar{\Psi}\gamma^0\partial_0\Psi}$$

$$- i\bar{\Psi}\vec{\gamma}\cdot\vec{\nabla}\Psi + m\bar{\Psi}\Psi$$

$$= -i\bar{\Psi}\vec{\gamma}\cdot\vec{\nabla}\Psi + m\bar{\Psi}\Psi$$

Thus, $$\boxed{H = \int \mathcal{H}\, d^3x = \int d^3x\, \bar{\Psi}(-i\vec{\gamma}\cdot\vec{\nabla} + m)\Psi}$$

→ now let's figure out the commutators to make this a quantum field theory...

→ HOW **NOT** TO QUANTIZE THE DIRAC FIELD

This won't work!

<u>Guess</u> $[\psi_a(\vec{x}), i\psi_b^\dagger(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y})\,\delta_{ab}$

spinor components $(a, b = 1, 2, 3, 4)$

i.e. $$[\psi_a(\vec{x}), \psi_b^\dagger(\vec{y})] = \delta^{(3)}(\vec{x}-\vec{y})\,\delta_{ab}$$

In matrix notation...

$$[\psi_a(\vec{x}), \psi_a^\dagger(\vec{y})] = \mathbb{1}_{4\times 4}\,\delta^{(3)}(\vec{x}-\vec{y})$$

$$\psi_a \rightarrow \begin{bmatrix} \cdot \\ \vdots \\ \cdot \end{bmatrix} \qquad \psi_a^\dagger \rightarrow [\,-\,-\,-\,]$$

Also guess $[\psi_a(\vec{x}), \psi_b(\vec{y})] = 0$

$= [\psi_a^\dagger(\vec{x}), \psi_b^\dagger(\vec{y})] = 0$

Note that

$$[\psi(\vec{x}), \bar{\psi}(\vec{y})] = [\psi(x), \psi^\dagger(y)\gamma^0]$$

$$= [\psi(x), \psi^\dagger(y)]\gamma^0 = \gamma^0\delta^{(3)}(\vec{x}-\vec{y})$$

With these... we recall that for bosons we wrote...

(real field) $$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left\{\vec{a}_p + \vec{a}^\dagger_{-p}\right\} e^{i\vec{p}\cdot\vec{x}} \quad (\text{FT})$$

↳ For <u>complex</u> field, we get

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left\{\vec{a}_p + \vec{b}^\dagger_{-p}\right\} e^{+i\vec{p}\cdot\vec{x}}$$

In the case of Dirac field, need spin degree of freedom...

Try ...

$$\Psi(\vec{x}) = \sum_{r=1,2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_{\vec{p}}^{r} u^r(\vec{p}) + b_{-\vec{p}}^{r} v^r(-\vec{p}) \right\} e^{+i\vec{p}\cdot\vec{x}}$$

Spin degrees of freedom

Fourier components : $\psi(x) = u(p) e^{ip}$

2 $$\psi^\dagger(\vec{x}) = \sum_{r=1,2} \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_{\vec{p}}^{r\dagger} u^{r\dagger}(\vec{p}) + b_{-\vec{p}}^{r\dagger} v^{r\dagger}(-\vec{p}) \right\} e^{-i\vec{p}\cdot\vec{x}}$$

Recall that u, v also solves Dirac eqn in the sense that (in momentum space...)

$$\boxed{p^\mu \gamma_\mu u^r(\vec{p}) = m u^r(p) \quad \& \quad p^\mu \gamma_\mu v^r(\vec{p}) = -m v^r(p)}$$

we can try the commutators ...

$$[a_{\vec{p}}^{r}, a_{\vec{p}'}^{s\dagger}] = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p}-\vec{p}')$$

$$[b_{\vec{p}}^{r}, b_{\vec{p}'}^{s}] = -(2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p}-\vec{p}')$$

$$[a_{\vec{p}}^{r}, a_{\vec{p}'}^{s}] = 0$$

$\vdots$

the rest are all zero ...

we find that

as desired...

$$[\Psi_a(\vec{x}), \Psi_b(\vec{y})] = 0 = [\Psi_a^\dagger(\vec{x}), \Psi_b^\dagger(\vec{y})]$$

We also find that

$$\{\Psi_a(\vec{x}), \Psi_b^\dagger(\vec{y})\} = \delta_{ab}\, \delta^{(3)}(\vec{x}-\vec{y}).$$

as desired.

With these ... we can try computing the Hamiltonian...

$$\mathcal{H} = \int d^3x \left[-i\bar{\Psi}\vec{\gamma}\cdot\vec{\nabla}\Psi + m\bar{\Psi}\Psi \right]$$

$$= \int d^3x \left\{ \Psi^\dagger \gamma^0 \left[-i\vec{\gamma}\cdot\vec{\nabla} + m \right] \Psi \right\}$$

just coeffs

Now, with $p^\mu \gamma_\mu u^r(p) = m u^r(p)$

$$\longrightarrow \quad (\vec{p}\cdot\vec{\gamma} + m)\, u^r(p) = p^0\gamma^0 u^r(p) = E_p \gamma^0 u^r(p)$$

Similarly, b/c $p^\mu \gamma_\mu v^{(r)}(p) = -m v^{(r)}(p)$

$$\longrightarrow \quad (\vec{p}\cdot\vec{\gamma} + m)\, v^r(p) = -E_p \gamma^0 v^{(r)}(p).$$

So....

$$\rightarrow \quad [-i\vec{\gamma}\cdot\vec{\nabla} + m]\Psi = [-i\vec{\gamma}\cdot\vec{\nabla} + m] \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[a_p^r u_p^r + b_{-p}^{r\dagger} v^r \right] e^{i\vec{p}\cdot\vec{x}}$$

$$= \gamma^0 \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ E_p\, a_p^r u^r(\vec{p}) - E_{p}\, b_{-p}^{r\dagger} v^r(-\vec{p}) \right\} e^{i\vec{p}\cdot\vec{x}}$$

So...

$$\mathcal{H} = \int d^3x \left\{ \Psi^\dagger \gamma^0 \gamma^0 \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ \ldots \right\} e^{i\vec{p}\cdot\vec{x}} \right\}$$

plug in ...

$$\rightarrow \mathcal{H} = \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} E_p \left\{ a_p^{r\dagger} a_p^r - \underbrace{b_{+p}^r b_{+p}^{r\dagger}}_{\substack{\parallel \\ b_{+p}^{r\dagger} b_{+\vec{p}}^r + \cancel{\text{const}}}} \right\}$$

$\boxed{!}$

→ By creating more and more particles with $b^\dagger$, we can lower the energy indefinitely

→ This is <u>bad</u>...

→ $\boxed{\text{So we should use Fermi-Dirac statistics instead → anti-commutators instead of commutators...}}$

→ Require that

$$\boxed{\left\{ a_{\vec{p}}^r , a_{\vec{q}}^{s\dagger} \right\} = \left\{ b_{\vec{p}}^r , b_{\vec{q}}^{s\dagger} \right\} = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p}-\vec{q})}$$

no longer bosonic!

& all other anti-commutators are zero...

When this is true, we find that

$$\boxed{\begin{aligned} \{ \psi_a(\vec{x}) , \psi_b^\dagger(\vec{y}) \} &= \delta^{(3)}(\vec{x}-\vec{y})\, \delta_{ab} \\ \{ \psi_a(\vec{x}) , \psi_b(\vec{y}) \} &= \{ \psi_a^\dagger(x) , \psi_b^\dagger(y) \} = 0 \end{aligned}}$$

where we're using

$$\psi(x) = \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[a_{\vec{p}}^r u^r(\vec{p}) + b_{-\vec{p}}^{r\dagger} v^r(-\vec{p}) \right] e^{i\vec{p}\cdot\vec{x}}$$

Compute the Hamiltonian again, we find that

$$\mathcal{H} = \int \frac{d^3p}{(2\pi)^3} \sum_{r=1,2} E_{\vec{p}} \left(a_{\vec{p}}^{r\dagger} a_{\vec{p}}^{r} - \underbrace{b_{-\vec{p}}^{r} b_{-\vec{p}}^{r\dagger}}_{-b_{-\vec{p}}^{r\dagger} b_{-\vec{p}}^{r} + \cancel{\text{const}}} \right)$$

$$\Rightarrow \boxed{\mathcal{H} = \int \frac{d^3p}{(2\pi)^3} \sum_{r=1,2} E_p \left\{ a_{\vec{p}}^{r\dagger} a_{\vec{p}}^{r} + b_{-\vec{p}}^{r\dagger} b_{-\vec{p}}^{r} \right\}}$$

now good, b/c E is bdd below...

→ also can compute...

$$\boxed{\vec{P} = \int \frac{d^3p}{(2\pi)^3} \sum_{r=1,2} \vec{p} \left(a_{\vec{p}}^{r\dagger} a_{\vec{p}}^{r} + b_{+\vec{p}}^{r\dagger} b_{+\vec{p}}^{r} \right)}$$

To avoid sign confusion, we will usually write...

$$\boxed{\Psi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{r=1,2} \left(a_{\vec{p}}^{r} u^r(\vec{p}) e^{+i\vec{p}\cdot\vec{x}} + b_{\vec{p}}^{r\dagger} v^r(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} \right)}$$

As a Heisenberg field,

$$\boxed{\Psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{r=1,2} \left(a_p^r u^r(\vec{p}) e^{-ip\cdot x} + b_p^{r\dagger} v^r(\vec{p}) e^{ip\cdot x} \right)}$$

where:

- a_p^r : annihilates particles
- $a_p^{r\dagger}$: creates particles
- b_p^r : annihilates anti-particles
- $b_p^{r\dagger}$: creates anti-particles.

Vacuum state as $|0\rangle$ where

$$\begin{cases} a_p^s|0\rangle = 0 \\ b_p^s|0\rangle = 0 \end{cases}$$

Define one-particle excitation state w/ covariant norm:

$$\boxed{|\vec{p},s\rangle = \sqrt{2E_p}\; a_p^{s\dagger}|0\rangle}$$

so that

$$\boxed{\langle \vec{p},s|\vec{q},r\rangle = \sqrt{2E_p}\sqrt{2E_q}\;\delta^{(3)}(\vec{p}-\vec{q})\,\delta^{rs}}$$

Now, look at Lorentz transform ...

$$\psi(x) \xrightarrow{\text{Lortz}} \psi'(x) = \Lambda_{\frac{1}{2}}\,\psi(\Lambda_4^{-1}x)$$

recall that with $\begin{cases} \omega_{12} = -\omega_{21} = \theta \\ S^{\mu\nu} = \frac{1}{2}[\gamma^\mu,\gamma^\nu] \end{cases}$ $\rightsquigarrow \exp\{-i\omega_{\mu\nu}S^{\mu\nu}/2\} \simeq \mathbb{1} - \frac{i}{2}\vec{\theta}\cdot\vec{\Sigma}$

$$= \mathbb{1} - \frac{i}{2}\vec{\theta}\cdot\begin{pmatrix}\vec{\sigma} & \\ & \end{pmatrix}$$

$\to$ and $\psi(\Lambda_4^{-1}x) \simeq [1 - \vec{\theta}\cdot\vec{J}]\,\psi(x)$

$$\vec{J} = \vec{x}\times(-i\vec{\nabla})$$

So ~~With~~ $\psi \to \psi + \delta\psi$ where

$$\delta\psi = \psi' - \psi = \left(-\frac{i}{2}\vec{\theta}\cdot\vec{\Sigma}\right)\psi(x) - \vec{\theta}\cdot(\vec{x}\times\vec{\nabla})\,\psi(x)$$

By Noether's thm,

$$\vec{J}_{total} \text{ (total spin)} = \int \frac{d^3x}{2}\left[\psi^\dagger(-i\vec{x}\times\vec{\nabla})\psi + \frac{1}{2}\psi^\dagger\vec{\Sigma}\psi\right].$$

We won't worry abt the rest of this section about propagators

↳ we'll come back to them later when looking at Feynman diagrams.

$\boxed{\text{DISCRETE SYMMETRIES OF THE DIRAC THEORY}}$

Basically, we have

Parity —— Time Reversal

Charge conjugation

Recall that ~~we~~ before, we looked at implementation of <u>continuous</u> Lorentz transform.

→ found that $\forall \Lambda \in$ Lorentz group

$\exists\, \mathcal{U}(\Lambda)$ unitary for which

$$\mathcal{U}(\Lambda)\psi(x)\mathcal{U}^{-1}(\Lambda) = \Lambda_{\frac{1}{2}}^{-1}\psi(\Lambda x).$$

→ Now, we'll talk about <u>Discrete</u> symmetries on the Dirac field.

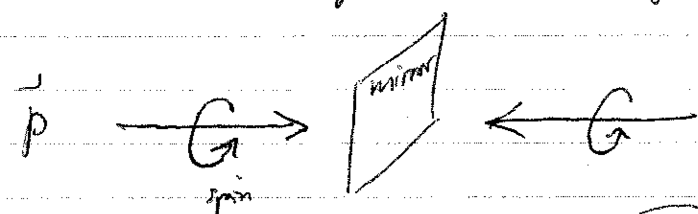
Apart from continuous Lorentz transforms, there are other spacetime-transformations for which the Lagrangian might remain invariant –

→ e.g. $\left\{ \begin{array}{c} \text{time-reversal} \\ \text{parity} \end{array} \right\}$.

Parity (P) : flips direction of <u>spatial</u> vectors

$$\boxed{P : (t, \vec{x}) \longrightarrow (t, -\vec{x})}.$$

⇝ mirror sym → change the handedness.



⇝ <u>Note</u> momentum flips sign, but spin is unchanged.

Time-reversal

$$\boxed{T : (t, \vec{x}) \to (-t, \vec{x})}$$

P, T don't belong to the "proper" Lorentz group $L_+^{\uparrow}$

→ the full Lorentz group breaks into 4 disconnected subsets ...

↳

$$\begin{tikzcd} L_+^\uparrow \arrow[r, leftrightarrow, "P"] \arrow[d, leftrightarrow, "T"] & L_-^\uparrow = PL_+^\uparrow \arrow[d, leftrightarrow, "T"] \\ L_+^\downarrow = TP_+^\uparrow \arrow[r, leftrightarrow, "P"] & L_-^\downarrow = PTL_+^\uparrow \end{tikzcd}$$

"orthochronous"

"non-orthochronous"

"proper" "improper"

$\boxed{\text{charge conjugation}}$ → interchange particles & anti-particles.

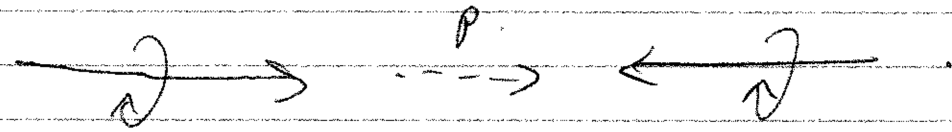
↳ non-space-time.

Let's look at $\boxed{\text{Parity}}$.

Note that because $P: (t, \vec{x}) \to (t, -\vec{x})$

→ momentum flips sign

<u>but</u> not spin! ⇒ what is P? As an operator?



As an operator on creation / annihilation ops, we want

$$\boxed{P^\dagger a_{\vec{p}}^s P = a_{-\vec{p}}^s \quad \& \quad P^\dagger b_{\vec{p}}^s P = b_{-\vec{p}}^s}$$

where, as discussed, P must be unitary.

$$PP^\dagger = P^\dagger P = \mathbb{1}.$$

Taking adjoint, get

$$\boxed{P^\dagger a^{s\dagger}_{\vec p} P = a^{s\dagger}_{-\vec p} \quad \& \quad P^\dagger b^{s\dagger}_{+\vec p} P = b^{s\dagger}_{-\vec p}}$$

But these might be too restrictive ... we can get better constraints by requiring that

$$P^\dagger a^s_{\vec p} P = \eta_a\, a^s_{-\vec p} \quad \& \quad P^\dagger b^s_{\vec p} P = \eta_b\, b^s_{-\vec p}$$

as long as $|\eta_a|^2 = |\eta_b|^2 = \mathbb{1}$ are "phases"!

Why? b/c ultimately, all observables will have fermion operators in pairs and the phases η_a, η_b will cancel:

$$\left\{\begin{aligned} P^\dagger a^{s'\dagger}_{\vec p} a^s_{\vec p} P &= a^{s'\dagger}_{-\vec p}\, a^s_{-\vec p} \\ P^\dagger b^{s'\dagger}_{\vec p}\, b^{s'}_{\vec p} P &= b^{s\dagger}_{-\vec p}\, b^s_{-\vec p}. \end{aligned}\right\}$$

With this, let's ~~see~~ implement parity condition on $\psi(x)$

$\longrightarrow P^\dagger \psi P = \,?$ (to find out what these η_a, η_b must be ...)

$$P^\dagger \psi(x) P = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1,2} \Big(\eta_a a^s_{-\vec{p}}\, u^s(p) e^{-i\vec{p}\cdot\vec{x}} + \eta_b^* b^{s\dagger}_{-\vec{p}}\, v^s(\vec{p}) e^{i\vec{p}\cdot\vec{x}} \Big)$$

<u>Define</u>

$$\begin{cases} \tilde{p} = (E_p, -\vec{p}) \\ \tilde{x} = (t, -\vec{x}). \end{cases}$$

<u>Note that</u>

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\, \xi^s \\ \sqrt{p\cdot\bar{\sigma}}\, \xi^s \end{pmatrix} \quad \text{where } \sigma = (\mathbb{1}, \vec{\sigma}), \quad \bar{\sigma} = (\mathbb{1}, -\vec{\sigma})$$

$$= \begin{pmatrix} \sqrt{\tilde{p}\cdot\bar{\sigma}}\, \xi^s \\ \sqrt{\tilde{p}\cdot\sigma}\, \xi^s \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \overbrace{\begin{pmatrix} \sqrt{\tilde{p}\cdot\sigma}\, \xi^s \\ \sqrt{\tilde{p}\cdot\bar{\sigma}}\, \xi^s \end{pmatrix}}^{u^s(-\vec{p})} = \overbrace{\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}}^{\gamma^0} u^s(-\vec{p})$$

$$\Rightarrow \boxed{u^s(p) = \gamma^0 u^s(-\vec{p})}$$

and

$$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\, \xi^s \\ -\sqrt{p\cdot\bar{\sigma}}\, \xi^s \end{pmatrix} = \dots = \begin{pmatrix} 0 & -\mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} v^s(-\vec{p})$$

$$\Rightarrow \boxed{v^s(p) = -\gamma^0 v^s(p)}$$

With these, we find that

$\tilde{p}\cdot\vec{x} = p\cdot\tilde{x}$

$$P^\dagger \psi(x) P = \gamma^0 \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1,2} \left(\eta_a a^s_{-p} u^s(-p) e^{-i\tilde{p}\cdot\tilde{x}} + \eta_b^* b^{s\dagger}_{-p} v^s(-p) e^{i\tilde{p}\cdot\tilde{x}} \right)$$

now, notice that if $\eta_a = \eta_b^*$ then it's "nice":

$$\boxed{(\eta_a = \eta_b^*) \Rightarrow P\psi^\dagger(x)P = \eta_a \gamma^0 \psi(\tilde{x})} \to P\text{-transf in final form}$$

$\to$ suffices to choose $\eta_a = 1 = -\eta_b$.

relative sign between fermions & antifermions ...

Now, useful to know how various Dirac field bilinears transform under parity ...

Recall ... 5 of them:

$$\bar{\psi}\psi, \quad \bar{\psi}\gamma^\mu\psi, \quad i\bar{\psi}[\gamma^\mu, \gamma^\nu]\psi$$

$$\bar{\psi}\gamma^\mu\gamma^5\psi, \quad i\bar{\psi}\gamma^5\psi.$$

to find these, first compute: $P\bar{\psi}(x)P$ ...

$$P^\dagger\bar{\psi}(x)P = P^\dagger\psi^\dagger(x)\gamma^0 P = (P^\dagger\psi P)^\dagger\gamma^0 \qquad (\gamma^{0\dagger} = \gamma^0)$$

$$= \eta_a^* (\gamma^0\psi(\tilde{x}))^\dagger \gamma^0 = \eta_a^* \psi^\dagger(\tilde{x})\gamma^0\gamma^0$$

$$\to \boxed{P^\dagger\bar{\psi}P = \eta_a^* \bar{\psi}(\tilde{x})\gamma^0}$$

With this---

$$P^\dagger \underset{(x)}{\bar\Psi} \underset{(x)}{\Psi} P = \underbrace{P^\dagger \bar\Psi(x) P}\overset{\mathbb{1}}{\widetilde{P P^\dagger}} \Psi(x) P$$

$$= \eta_a^* \bar\Psi(\tilde{x}) \gamma^0 \eta_a \gamma^0 \Psi(\tilde{x})$$

$$= |\eta_a|^2 \bar\Psi(\tilde{x}) \Psi(\tilde{x})$$

Scalar $\boxed{P^\dagger \bar\Psi \Psi P(x) = \bar\Psi\Psi(\tilde{x})}$ (scalar) bilinear.

can also show ---

$$\boxed{\begin{aligned} P^\dagger \underbrace{\bar\Psi(x) \gamma^\mu \Psi} P &= \bar\Psi \gamma^0 \gamma^\mu \gamma^0 \Psi(\tilde{x}) \\ \text{(vector)} \quad &= \begin{cases} + \bar\Psi \gamma^\mu \Psi(\tilde{x}) & \mu = 0 \\ - \bar\Psi \gamma^\mu \Psi(\tilde{x}) & \mu = 1,2,3 \end{cases} \end{aligned}}$$

$$P^\dagger (i\bar\Psi\gamma^5\Psi) P = i\bar\Psi(\tilde{x})\gamma^0\gamma^5\gamma^0\Psi(\tilde{x}) = -i\bar\Psi\gamma^5\Psi(\tilde{x})$$

pseudo scalar ↳ (−)

~~$\left\{ \begin{array}{ll} - \bar\Psi \gamma^\mu \gamma^5 \Psi(\tilde{x}) & \mu = 0 \\ \bar\Psi \gamma^\mu \gamma^5 \Psi(\tilde{x}) & \mu = 1,2,3 \end{array} \right.$~~

$$P^\dagger \bar\Psi \gamma^\mu \gamma^5 \Psi(x) P = \bar\Psi(\tilde{x}) \gamma^0 \gamma^\mu \gamma^5 \gamma^0 \Psi(\tilde{x})$$

pseudo vector.

$$= \begin{cases} - \bar\Psi \gamma^\mu \gamma^5 \Psi(\tilde{x}) & \mu = 0 \\ + \bar\Psi \gamma^\mu \gamma^5 \Psi(\tilde{x}) & \mu = 1,2,3 \end{cases}$$

(−)

<u>Note</u> the relative sign: $\boxed{-\eta_a = +\eta_b^*}$ is important.

for the relationship between fermions & anti-fermions

<u>Consider</u> ~~and~~ fermion-antifermion state...

$$a_p^{s\dagger} b_q^{s\dagger} |0\rangle \xrightarrow{P} P\left(a_p^{s\dagger} b_q^{s\dagger} |0\rangle\right)$$

$$= P^\dagger \left(a_p^{s\dagger} b_q^{s\dagger}\right) P |0\rangle$$

$$= \underbrace{P^\dagger a_p^{s\dagger} P}P^\dagger b_q^{s\dagger} P |0\rangle$$

$$= \eta_a a_{-p}^{s\dagger} \eta_b b_{-q}^{s\dagger} |0\rangle$$

$$= -(\eta_b \eta_b^*) a_{-p}^{s\dagger} b_{-q}^{s\dagger} |0\rangle$$

$$= - a_{-p}^{s\dagger} b_{-q}^{s\dagger} |0\rangle$$

$\rightarrow$ a state containing a fermion-antifermion pair gets an extra (-1) under parity transform.

TIME REVERSAL

if T is unitary & $[T, H] = 0$

$$\rightarrow T^\dagger e^{iHt} T = e^{iHt} T^\dagger T = e^{iHt}$$

$\rightarrow$ no good...

what if $T^\dagger H T = -H$? or $\{T, H\} = 0$?

But this is no good either since implies that H is unbounded ...

→ Assume this ...

$\boxed{\text{"Time-reversal is conjugate-linear/anti-linear"}}$

Assume:

T is unitary

$$T^\dagger c T = c^* \qquad (c \in \mathbb{C}).$$

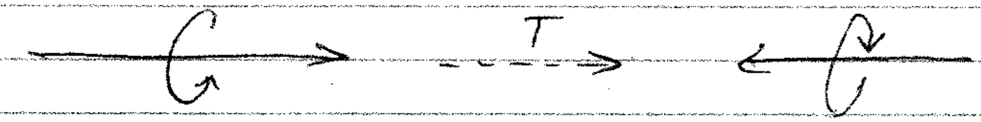
$$[T, H] = 0$$

With these

$$T^\dagger e^{-iHt} T = e^{iHt} \quad ✓$$

→ time reversal: momentum & spin are reversed

↳ like watching a movie played backwards.



Flipping momentum is easy.

what abt flipping spinor ξ? We know that

In some basis ...

$$\xi(\uparrow) = \begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{pmatrix} \xrightarrow{\text{basis}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\xi(\downarrow) = \begin{pmatrix} -e^{-i\phi}\sin\theta/2 \\ \cos\theta/2 \end{pmatrix} \xrightarrow{\text{basis}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Let $\xi^s = (\xi(\uparrow), \xi(\downarrow))$ for $s = 1, 2$ & define

reversed spin

$$\boxed{\xi^{-s} = -i\sigma^2 (\xi^s)^*}$$

σ^y

this is the flipped spinor ...

It is clear that

$$\xi^{-s} = -i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} (\xi(\uparrow), \xi(\downarrow))^*$$

$$= (\xi(\downarrow), -\xi(\uparrow))^*$$

where

$$\left\{ \begin{array}{l} \xi^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \xi^{1*} \\ \\ \xi^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \xi^{2*} \end{array} \right.$$

→ This is convenient since our time reversal op. involves complex conjugation ...

→ Can show:
$$\boxed{u^{-s}(-\vec{p}) = \begin{pmatrix} \sqrt{\tilde{p}\cdot\sigma}\,(-i\sigma^2)\xi^{s*} \\ \sqrt{\tilde{p}\cdot\bar{\sigma}}\,(-i\sigma^2)\xi^{s*} \end{pmatrix}}$$

So if we use the identity ...

$$\sqrt{\tilde{p}\cdot\sigma}\,\sigma^2 = \sigma^2\sqrt{p\cdot\sigma^*}$$

(prove using $\sigma^2\vec{\sigma} = -\vec{\sigma}^*\sigma^2$

then we set

$$u^{-s}(\tilde{p}) = \begin{pmatrix} \sqrt{\tilde{p}\cdot\sigma}\,(-i\sigma^2)\,\xi^{s*} \\ \sqrt{\tilde{p}\cdot\bar{\sigma}}\,(-i\sigma^2)\,\xi^{s*} \end{pmatrix} = \begin{pmatrix} (-i\sigma^2)\sqrt{\vec{p}\cdot\sigma^*}\,\xi^{s*} \\ (-i\sigma^2)\sqrt{\vec{p}\cdot\bar{\sigma}^*}\,\xi^{s*} \end{pmatrix}$$

$$= (-i)\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}[u^s(p)]^* = -\gamma^1\gamma^3[u^s(p)]^*$$

$\sigma^2 = \sigma^y$

$$\Rightarrow \boxed{u^{-s}(\tilde{p}) = -\gamma^1\gamma^3[u^s(p)]^*}$$ element-wise complex conjugation

Similarly, ...

$$\boxed{v^{-s}(\tilde{p}) = -\gamma^1\gamma^3[v^s(p)]^*}$$

in this relation, v^{-s} contains $\boxed{\xi^{-(-s)} = -\xi^s}$

a 360° flip introduces a $(-)$ sign.

~~[illegible]~~

Now we can define time reversal operation on the creation - annihilation operators...

spins don't matter here... $\rightarrow$

$$\boxed{T^\dagger a^s_{\vec p} T = a^{-s}_{-\vec p} \quad \& \quad T^\dagger b^s_{\vec p} T = b^{-s}_{-\vec p}}$$

→ flip spin

→ flip momentum

where we now define

$$\begin{cases} a^{-s}_{-\vec p} = (a^{\downarrow}_{-\vec p}, -a^{\uparrow}_{-\vec p}) \\ b^{-s}_{-\vec p} = (b^{\downarrow}_{-\vec p}, -b^{\uparrow}_{-\vec p}) \end{cases}$$

just like what we did with $\xi^{-s} = (\xi(\downarrow), -\xi(\uparrow))$

if
$$\begin{cases} a^s_{\vec p} = (a^{\uparrow}_{\vec p}, a^{\downarrow}_{\vec p}) \\ b^s_{\vec p} = (b^{\uparrow}_{\vec p}, b^{\downarrow}_{\vec p}) \end{cases}$$

analogous to what we did before...

With this, let's evaluate $T^\dagger \Psi(x) T$:

$$\rightarrow T^\dagger \Psi(x) T = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1,2} T^\dagger \left(a^s_{\vec p} u^s(p) e^{-ip\cdot x} + b^{s\dagger}_p v^s(p) e^{+ip\cdot x} \right) T$$

$$= \int \frac{d^3 p}{(2\pi)^3 \sqrt{2E_p}} \sum_{s=1}^{2} \Big\{ a^{-s}_{-\vec p} [u^s(p)]^* e^{ip\cdot x} + b^{-s\dagger}_{-p} [v^s(p)]^* e^{-ip\cdot x} \Big]$$

where under T, $\quad = \gamma^1\gamma^3 \Psi(x_T)$, $x_T = (-t, \vec x)$

- $a^s_p \xrightarrow{T} a^{-p}_{-p}$
- ~~$u^{-s}(\tilde p) = -\gamma^1\gamma^3 [u^s(p)]^*$~~
- ~~$\rightarrow \gamma^1\gamma^3 u^{-s}(\tilde p) = [u^s(p)]^*$~~
- $T^\dagger e^{-ip\cdot x} T = e^{+ip\cdot x}$; $T^\dagger u^s_p T = [u^s_p]^*$

Marginal notes: "spins don't matter here..." with arrow to the boxed equation; "flip spin" and "flip momentum" at right.

note sign change here

Because $\{u^s(p)\}^* = \gamma_1\gamma_3 u^{-s}(\tilde{p})$, we have

$$T^\dagger \psi(x) T = \gamma^1\gamma^3 \int \frac{d^3\tilde{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\tilde{p}}}} \sum_{s=1}^{2} \left\{ a^{-s}_{\tilde{p}} u^{-s}(\tilde{p}) e^{i\tilde{p}(t,-\vec{x})} + b^{-s}_{\tilde{p}} v^{-s}(\tilde{p}) e^{-i\tilde{p}(t,-\vec{x})} \right\}$$

$$= \gamma^1\gamma^3 \psi(-t, x) \qquad = -\tilde{p}(-t, \vec{x})$$

$$\Rightarrow \boxed{T^\dagger \psi(x,t) T = \gamma^1\gamma^3 \psi(x,-t)}$$

Next, can check the action of T on bilinears...

$$T^\dagger \bar{\psi} T = T^\dagger \psi^\dagger \gamma^0 T = T^\dagger \psi^\dagger T \gamma^0 \quad \text{real}$$

$$= (\gamma^1\gamma^3\psi(x_T))^\dagger \gamma^0 = \psi^\dagger(x_T)\gamma^{3\dagger}\gamma^{1\dagger}\gamma^0$$

$$\qquad\qquad -\gamma^3 \quad -\gamma^1$$

$$= \psi^\dagger(x_T)\gamma^3\gamma^1\gamma^0$$

$$= +\psi^\dagger(x_T)\gamma^0\underbrace{\gamma^3\gamma^1}$$

$$\Rightarrow \boxed{T^\dagger \bar{\psi} T = -\bar{\psi}(x_T)\gamma^1\gamma^3}$$

with this, can compute the rest....

<u>Scalar</u> $\boxed{T\bar{\psi}\psi(x) T = \bar{\psi}\underbrace{(-\gamma^1\gamma^3)(\gamma^1\gamma^3)}_{\mathbb{1}}\psi(x_T) = \bar{\psi}(x_T)\psi(x_T)}$

<u>Pseudoscalar</u> $\rightarrow$ get $(-)$ "pseudo"

$$\boxed{T^\dagger i\bar{\psi}\gamma^5\psi T = -i\bar{\psi}(-\gamma^1\gamma^3)\gamma^5(\gamma^1\gamma^3)\psi(x_T) = -i\bar{\psi}(x_T)\gamma^5\psi(x_T)}$$

Vector

$$\boxed{\begin{aligned} T^\dagger \underset{(x)}{\bar\Psi \gamma^\mu \Psi} T &= \bar\Psi(-\gamma^1\gamma^3)(\gamma^\mu)^* (\gamma^1\gamma^3)\Psi \\ &= \begin{cases} +\bar\Psi\gamma^\mu\Psi(x_T) & \mu = 0 \\ -\bar\Psi\gamma^\mu\Psi(x_T) & \mu = 1,2,3. \end{cases} \end{aligned}}$$

This makes sense... Recall that $\bar\Psi\gamma^0\Psi$ is the charge density

↳ $\bar\Psi\gamma^0\Psi$ should be the same under T -

as we saw: $T^\dagger\bar\Psi\gamma^0\Psi T = \bar\Psi\gamma^0\Psi$.

but current density (time-dep) must reverse si[gn]

$\rightarrow \quad T^\dagger\bar\Psi\gamma^i\Psi T = -\bar\Psi\gamma^i\Psi$ ✓.

$\boxed{\text{Charge Conjugation - Matter-anti-matter flip}}$

↳ $\begin{cases}\text{anti particles \& particles are swapped.} \\ \text{spin \& momentum are the same.}\end{cases}$

Let $\left\{\begin{aligned} C^\dagger a_p^s C &= b_p^s \\ C^\dagger b_p^s C &= a_p^s. \end{aligned}\right\} \rightarrow$ ignore phases...

How should C act on $\Psi(x)$?

First, look at relation...

$$(v^s(p))^* = \begin{pmatrix} \sqrt{p\cdot\sigma}\,(-i\sigma^2)\xi^{s*} \\ \sqrt{p\cdot\bar\sigma}\,(-i\sigma^2)\xi^{s*} \end{pmatrix}^* = \begin{pmatrix} -i\sigma^2\sqrt{p\cdot\bar\sigma^*}\,\xi^{s*} \\ i\sigma^2\sqrt{p\cdot\sigma^*}\,\xi^* \end{pmatrix}^*$$

$$= \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix} =$$

$\longrightarrow$ Get

$$\boxed{\begin{aligned} u^s(p) &= -i\gamma^2(v^s(p))^* \\ v^s(p) &= -i\gamma^2(u^s(p))^* \end{aligned}}$$

$\longrightarrow$

$$C^\dagger\psi(x)C = \int\frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\sum_{s=1}^{2}\Big\{-i\gamma^2 b_p^s(v^s(p))^* e^{-ip\cdot x} - i\gamma^2 a_p^{s\dagger}(u^s(p))^* e^{ip\cdot x}\Big)$$

$$= -i\gamma^2\psi^*(x) = -i\gamma^2(\psi^\dagger)^T = -i(\bar\psi\gamma^0\gamma^2)^T$$

$\Rightarrow$ $\boxed{C^\dagger\psi(x)C = -i(\bar\psi\gamma^0\gamma^2)^T}$ $\rightarrow$ C is a unitary op.

On bilinears... first, find $\qquad \bar\psi^\dagger = (\psi^\dagger)^\dagger\gamma^0 = \psi\gamma^0$

$$\boxed{\begin{aligned} C^\dagger\bar\psi(x)C &= C^\dagger\psi^\dagger\gamma^0C = C^\dagger\psi^\dagger C\gamma^0 = -i\psi^T\gamma^2\gamma^0 \\ &= (-i\gamma^2\psi)^T\gamma^0 = (-i\gamma^0\gamma^2\psi)^T \end{aligned}}$$

Next ....

$$C^\dagger\bar\psi\psi C = \left(-i(\gamma^0\gamma^2\psi)^T\right)\left(-i(\bar\psi\gamma^0\gamma^2)^T\right) = \cdots =$$

$$= -\left[(-i\bar\psi\gamma^0\gamma^2)(-i\gamma^0\gamma^2\psi)\right]^T = +\bar\psi\gamma^0\gamma^2\gamma^0\gamma^2\psi$$

$$= -+\bar\psi\gamma^0\gamma^2\gamma^2\gamma^0\psi = +\bar\psi\psi$$

So... $\boxed{C^\dagger \bar{\Psi}\Psi C = \bar{\Psi}\Psi}$ → scalar

vector $\boxed{C^\dagger i\bar{\Psi}\gamma^5\Psi C = i(-i\gamma^0\gamma^2\Psi)^T\gamma^5(-i\bar{\Psi}\gamma^0\gamma^2)^T = i\bar{\Psi}\gamma^5\Psi}$

pseudo-scalar $C^\dagger\bar{\Psi}\gamma^\mu\Psi C = -\bar{\Psi}\gamma^\mu\Psi$

pseudo vector. $C^\dagger\bar{\Psi}\gamma^\mu\gamma^5\Psi C = +\bar{\Psi}\gamma^\mu\gamma^5\Psi$

(I'll skip the derivations... to save time)

Summary

	$\bar{\Psi}\Psi$	$i\bar{\Psi}\gamma^5\Psi$	$\bar{\Psi}\gamma^\mu\Psi$	$\bar{\Psi}\gamma^\mu\gamma^5\Psi$	$\bar{\Psi}\sigma^{\mu\nu}\Psi$	∂_μ
P	$+1$	-1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu(-1)^\nu$	$(-1)^\mu$
T	$+1$	-1	$(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu(-1)^\nu$	$-(-1)^\mu$
C	$+1$	$+1$	-1	$+1$	-1	$+1$
CPT	$+1$	$+1$	-1	-1	$+1$	-1

Notice that

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi$$ is invariant under C, P, T separately

→ in general, can't build a Lorentz inv QFT with a Hermitian Hamiltonian that violates CPT!

Invariance under CPT is required for any Lorentz invariant local Hermitian op.

$\boxed{\text{Problem 5}}$

↳ (to be continued...)

$\boxed{\text{Correlation functions for Dirac fields}}$

$\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle \rightsquigarrow$ Dirac propagation amplitudes...

only "a" term contributes

only "a^+" term contributes

Recall...

$$\to \psi_A(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{s=1,2}\left\{ a^s_{\vec{p}}\, u^s_A(p)\, e^{-ip\cdot x} + \cancel{b^{s\dagger}_{\vec{p}}\, v^s_A(p)\, e^{ip\cdot x}} \right\}$$

$\langle 0|$

$$\to \bar{\psi}_B(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{s=1,2}\left\{ \cancel{b^s_{\vec{p}}\, \bar{v}^s_B(p)\, e^{-ip\cdot x}} + a^{s\dagger}_{\vec{p}}\, \bar{u}^s_B(p)\, e^{ip\cdot x} \right\}$$

$|0\rangle$

where $\{a^r_{\vec{p}}, a^{s\dagger}_{\vec{q}}\} = \{b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}\} = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\delta^{rs}$

$$\to \langle 0|\psi_A(x)\bar{\psi}_B(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\underbrace{\sum_s u^s_A(p)\bar{u}^s_B(p)}_{(\not{p}+m)_{AB}} e^{-ip(x-y)}$$

$$= (i\not{\partial}_x + m)_{AB}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip(x-y)}$$

$$\boxed{\langle 0|\psi_A(x)\bar{\psi}_B(x)|0\rangle = (i\not{\partial}_x + m)_{AB}\, \mathcal{D}(x-y)}$$

$$\langle 0|\bar{\psi}_B(y)\psi_A(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\underbrace{\sum_s v_A^s(p)\bar{v}_B^s(p)}_{(\not{p}-m)_{AB}} e^{-ip(x-y)}$$

b terms contribute

$b^\dagger$ terms contribute

$$= \int \frac{d^3p}{(2\pi)^3}\left(\frac{1}{2E_p}\right)(\not{p}-m)_{AB}\, e^{-ip(x-y)}$$

$$\boxed{\langle 0|\bar{\psi}_B(y)\psi_A(x)|0\rangle = -(i\not{\partial}_x+m)_{AB}\,D(y-x)}$$

Feynman Propagator

$$S_F^{AB}(x-y) = \begin{cases} \langle 0|\psi_A(x)\bar{\psi}_B(y)|0\rangle & \text{for } x^0>y^0 \\ -\langle 0|\bar{\psi}_B(y)\psi_A(x)|0\rangle & \text{for } x^0<y^0 \end{cases}$$

$$= \langle 0|T\{\psi_A(x),\bar{\psi}_B(y)\}|0\rangle$$

--- time-ordering ---

where $T\{\psi_A(x),\bar{\psi}_B(y)\}$

$$= \theta(x^0-y^0)\,\psi_A(x)\bar{\psi}_B(y)$$

$$-\,\theta(y^0-x^0)\,\bar{\psi}_B(y)\psi_A(x)$$

$\boxed{\text{minus sign for Fermions}}$

Let's check the calculations...

$$\langle 0|\psi_A(x)\bar{\psi}_B(y)|0\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\frac{1}{\sqrt{2E_{p'}}}\langle 0|\left\{\sum_s a^s_{A\vec{p}}\, u^s_A(p)e^{-ip\cdot x} + \cancel{b^{s\dagger}_{A\vec{p}}\, v^s_A(p)e^{-ip\cdot x}}\right\}$$

$$\times\left\{\cancel{\sum_s b^s_{B\vec{p}'}\,\bar{v}^s_B(p')e^{-ip'\cdot y}} + a^{s\dagger}_{B\vec{p}'}\,\bar{u}^s_B(p')e^{ip'\cdot y}\right\}|0\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{2}\frac{1}{\sqrt{E_pE_{p'}}}\langle 0|\sum_s a^s_{A\vec{p}}\, u^s_A(p)\sum_s a^{s\dagger}_{B\vec{p}'}\,\bar{u}^s_B(p')e^{-i(p\cdot x - p'\cdot y)}|0\rangle$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^6}\frac{1}{2}\frac{1}{\sqrt{E_pE_{p'}}}\sum_s \langle 0|u^s_A(p)\,\bar{u}^s_B(p')|0\rangle\,(2\pi)^3\delta^{(3)}(p-p') \times e^{i(p\cdot x - p'\cdot y)}$$

$$= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\underbrace{\sum_s u^s_A(p)\,\bar{u}^s_B(p)}\, e^{-ip(x-y)}$$

$$(\not{p}+m)_{AB} = (\gamma^\mu p_\mu + m)_{AB} \quad \left(\underline{\text{spin sum}}\ \underline{\text{relations}}\right)$$

$$= (i\not{\partial}_x + m)_{AB}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip(x-y)}. \checkmark$$

Similarly, we can get the other relations too...

Oct 5, 2020

(1) Recall Dirac bispinor field...

$$\psi(x) = \sum_{r=1}^{2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[a^r_{\vec{p}}\, u^r(\vec{p}) + b^{r\dagger}_{-\vec{p}}\, v^r(-\vec{p}) \right] e^{i\vec{p}\cdot\vec{x}}$$

Use $\{ a^r_{\vec{p}}, a^{s\dagger}_{\vec{p}'} \} = \{ b^r_{\vec{p}}, b^{s\dagger}_{\vec{p}'} \} = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p}-\vec{p}')$

and all other anti-comm $= 0$, derive the following:

$$\{ \psi_a(\vec{x}) , \psi_b^\dagger(\vec{y}) \} = \delta_{ab}\, \delta^{(3)}(\vec{x}-\vec{y})$$

(2) The momentum operator is the Noether charge associated with spatial translation...

$$\vec{P} = -i \int d^3\vec{x}\; \psi^\dagger(x)\, \vec{\nabla} \psi(\vec{x})$$

Show that

$$\vec{P} = \sum_{r=1}^{2} \int \frac{d^3 p}{(2\pi)^3}\, \vec{p} \left(a^{r\dagger}_{\vec{p}} a^r_{\vec{p}} + b^{r\dagger}_{\vec{p}} b^r_{\vec{p}} \right)$$

Oct 10, 2020

(1) Well...

$$\{ \psi_a(x), \psi_b^\dagger(y) \}$$

$$= \psi_a(x)\, \psi_b^\dagger(y) + \psi_b^\dagger(y)\, \psi_a(x)$$

$$\cancel{\sum_{r=1}^{2} \int \frac{d^3 p_a\, d^3 p_b}{(2\pi)^6\, 2\sqrt{E_{p_a} E_{p_b}^*}}\, e^{i(\vec{p}_a\cdot\vec{x} - \vec{p}_b\cdot\vec{y})}}$$

To keep things clean...

$\psi_a(x)\,\psi_b^\dagger(y) + \psi_b^\dagger(x)\psi_a(x)$ which involves the factor...

$$\sum_{r=1}^{2}\sum_{s=1}^{2}\left[\hat{a}^r_{p_a} u^r(p_a) + \hat{b}^{r\dagger}_{-p_a} v^r(-p_a)\right]\left[(\hat{a}^s_{p_b} u^s(p_b))^\dagger + (\hat{b}^{s\dagger}_{-p_b} v^s(-p_b))^\dagger\right]$$

$$+\sum_{r=1}^{2}\sum_{s=1}^{2}\left[(\hat{a}^s_{p_b} u^s(p_b))^\dagger + (\hat{b}^{s\dagger}_{-p_b} v^s(-p_b))^\dagger\right]\left[\hat{a}^r_{p_a} u^r(p_a) + \hat{b}^{r\dagger}_{-p_a} v^r(-p_a)\right]$$

$$= \sum_{r,s=1}^{2} \left\{ \hat{a}^r_{p_a}, \hat{a}^{s\dagger}_{p_b} \right\} u^r(p_a) u^{s\dagger}(p_b) + \left\{ \hat{b}^r_{p_a}, \hat{b}^{\dagger s}_{p_b} \right\} v^r(p_a) v^s(p_b) \}$$

$$= \sum_{r,s=1}^{2} (2\pi)^3 \delta^{r,s} \delta^{(3)}(p_a - p_b) \left\{ u^r(p_a) u^{s\dagger}(p_b) + v^r(-p_a) v^{s\dagger}(-p_b) \right\}$$

$$\Rightarrow \left\{ \psi_a(x), \psi_b^\dagger(y) \right\}$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} \sum_{r=1}^{2} \left\{ u^r(p) u^{r\dagger}(p) + v^r(-p) v^{r\dagger}(-p) \right\}$$

Now, we want to convert $u^{s\dagger} \to \bar{u}$

$\to$ need γ^0. In particular, recall that $\gamma^0\gamma^0 = \mathbb{1}$

from page 63 $\Rightarrow$ and $\boxed{u^{s\dagger}(p)\gamma^0 = \bar{u}^s(p)}$, we have

$$\left\{ \psi_a(x), \psi_b^\dagger(y) \right\} = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} \sum_{r=1}^{2} \left\{ u^r_p u^{r\dagger}_p \gamma^0 \gamma^0 + v^r_{-p} v^{r\dagger}_{-p} \gamma^0 \gamma^0 \right\}$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} \sum_{r=1}^{2} \left\{ u^r_p \bar{u}^r_p + v^r_{-\vec{p}} \bar{v}^r_{-\vec{p}} \right\} \gamma^0 \quad \left(\text{spin sum}\right)$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} \left(\underbrace{p_+\cdot\gamma}_{E_p\gamma^0 - \cancel{\vec{p}\cdot\vec{\gamma}}} + \cancel{m\mathbb{I}} + \underbrace{p_-\cdot\gamma}_{E_p\gamma^0 + \cancel{\vec{p}\cdot\vec{\gamma}}} - \cancel{m\mathbb{I}} \right) \gamma^0 \Big|_{\delta_{a,b}}$$

(102)

recall that only $\vec{p} \to -\vec{p}$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} \left[\cancel{2 p\cdot\gamma}\right]\gamma^0$$

Rather $$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} \left\{ E_p\cancel{\gamma^0} - \cancel{\vec{p}\cdot\vec{\gamma}} + E_p\cancel{\gamma^0} + \cancel{\vec{p}\cdot\vec{\gamma}} \right\}_{ab} \cancel{\gamma^0}$$

$$= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}(\vec{x}-\vec{y})} = \delta_{ab}\,\delta^{(3)}(\vec{x}-\vec{y}).$$

So $$\boxed{\{\psi_a(x),\ \psi_b^\dagger(y)\} = \delta_{ab}\,\delta^{(3)}(x-y)}$$

---

(2) Let $$\vec{P} = -i\int d^3\vec{x}\ \psi^\dagger(\vec{x})\,\vec{\nabla}\psi(\vec{x}).$$

Show that

$p = (p^0, \vec{p})$

$$\vec{P} = \sum_{i=1}^{2} \int \frac{d^3p}{(2\pi)^3}\, \vec{p}\left(a^{r\dagger}_{\vec{p}} a^r_{\vec{p}} + b^{r\dagger}_{\vec{p}} b^r_{\vec{p}}\right).$$

Well.

$ip\cdot x = \ldots + i\vec{p}\cdot\vec{x}$

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} e^{ip\cdot x} \sum_{r=1}^{2} \left\{ a^r_{\vec{p}} u^r_{\vec{p}} + b^{r\dagger}_{-\vec{p}} v^r_{-\vec{p}} \right\}.$$

So

$$\nabla\psi(x) = \sum_{r=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (i\vec{p})$$

(2) $$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(a^s_{\vec{p}} u^s_p e^{-ip\cdot x} + b^{s\dagger}_{\vec{p}} v^s_p e^{ip\cdot x} \right)$$

$$\rightarrow \nabla \psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s (i\vec{p}) \left\{ a^s_{\vec{p}} u^s_p e^{-ip\cdot x} - b^{s\dagger}_{\vec{p}} v^s_p e^{ip\cdot x} \right\}$$

$$\psi^\dagger(x) = \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \sum_{r=1}^{2} \left\{ b^r_q v^{\dagger r}(q) e^{-iqx} + a^{r\dagger}_q u^{\dagger r}_q e^{iqx} \right\}$$

$$\rightarrow \int d^3x (-i) \psi^\dagger \nabla \psi$$

$$= \int d^3x \int \frac{d^3q}{(2\pi)^3} \frac{d^3p}{(2\pi)^3} \frac{1}{2} \frac{1}{\sqrt{E_p E_q}} \underbrace{e^{ix(p-q)}}_{\delta^{(3)}(p-q)} \vec{p} \times \sum_s \times \sum_r$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \frac{1}{E_p} \vec{p} \left\{ \sum_{s,r=1}^{2} \left(a^{r\dagger}_{\vec{p}} a^s_{\vec{p}} \underbrace{u^{\dagger r}_p u^s_p}_{2E_p \delta^{rs}} - b^r_{\vec{p}} b^{s\dagger}_{\vec{p}} v^{\dagger r}_p v^s_p \right) \right\}$$

other uv terms = 0

$$= \int \frac{d^3p}{(2\pi)^3} \frac{2E_p}{2E_p} \vec{p} \left(a^{r\dagger}_p a^r_p - b^r_p b^{r\dagger}_p \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \vec{p} \left(a^{r\dagger}_p a^r_p - b^r_p b^{r\dagger}_p \right)$$

Finally ... $\{ b^r_p, b^{r\dagger}_p \} = (2\pi)^3 \delta^{rr}(\vec{p}-\vec{p})$ ignore ...

$$\Rightarrow -b^r_p b^{r\dagger}_p = b^{r\dagger}_p b^r_p - (2\pi)^3 \delta^{rr} \delta(\vec{p}-\vec{p})$$

$$\Rightarrow \vec{P} = \int d^3x \, \psi^\dagger(x)(-i\nabla)\psi(x) \quad \text{momentum op}$$

$$\boxed{\vec{P} = \int \frac{d^3p}{(2\pi)^3} \vec{p} \left(a^{r\dagger}_p a^r_p + b^{r\dagger}_p b^r_p \right)}$$

More problems

(1) Let $\mathcal{U}$ be the following unitary op:

$$\mathcal{U} = \exp\left\{ \frac{-i\pi}{2} \sum_{r=1}^{2} \int \frac{d^3p}{(2\pi)^3} \left(a_{\vec p}^{r\dagger} - b_{\vec p}^{r\dagger}\right)\left(a_{\vec p}^{r} - b_{\vec p}^{r}\right) \right\}$$

investigate the effect of $\mathcal{U}$ on $a_{\vec p}^r$ & $b_{\vec p}^r$. I.e. compute

$$\mathcal{U}^\dagger a_{\vec p}^r \mathcal{U} \quad \& \quad \mathcal{U}^\dagger b_{\vec p}^r \mathcal{U} .$$

What type of transform does $\mathcal{U}$ produce?

Well...

$$\mathcal{U}^\dagger a_{\vec p}^r \mathcal{U} = \underbrace{\exp\left\{ +\frac{i\pi}{2}\left(\sum_{r=1}^{2} \int \frac{d^3p}{(2\pi)^3} \left(a_{\vec p}^{r\dagger} - b_{\vec p}^{r\dagger}\right)\left(a_{\vec p}^{r} - b_{\vec p}^{r}\right)\right)^\dagger \right\}}_{X^\dagger}$$

$$\cdot\, a_{\vec p}^r \,\cdot\, \underbrace{\exp\left\{ \frac{i\pi}{2} \sum_{r=1}^{2} \int \frac{d^3p}{(2\pi)^3} \left(a_{\vec p}^{r\dagger} - b_{\vec p}^{r\dagger}\right)\left(a_{\vec p}^{r} - b_{\vec p}^{r}\right) \right\}}_{X}$$

$\Rightarrow$ no good way to do this except for power series.

Recall that $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$, $e^{-x} = \sum_{n=0}^{\infty} \frac{1}{n!} (-x)^n$

So $\mathcal{U}^\dagger a_{\vec p}^r \mathcal{U} = \left(\sum_{n=0}^{\infty} \frac{(X^\dagger)^n}{n!} \right) a_{\vec p}^r \left(\sum_{m=0}^{\infty} \frac{(X)^m}{m!} \right) = \sum_{n,m=0}^{\infty} \frac{1}{n!\,m!} (X^\dagger)^n a_{\vec p}^r X^m$

but notice that $\mathcal{U}$ unitary iff X hermitian.

$$\rightarrow X^\dagger = X \rightarrow \mathcal{U}^\dagger a_{\vec p}^r \mathcal{U} = \sum_{n,m=0}^{\infty} \frac{1}{n!\,m!} X^n a_{\vec p}^r X^m .$$

Useful theorem:

$$e^X Y e^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \frac{1}{3!}[X,[X,[X,Y]]] + \cdots$$

& Identity relating comm & anti-comm:

$$[AB, C] = ABC - CAB$$

$$= ABC + ACB - ACB - CAB$$

$$= A\{B,C\} - \{A,C\}B.$$

We ~~get~~ will need to compute

$$\mathcal{U}^\dagger a_{\vec{q}}^r \mathcal{U} = a_{\vec{q}}^r + [X, a_{\vec{q}}^r] + \frac{1}{2!}[X,[X,a_{\vec{q}}^r]] + \ldots$$

→ need to compute

$$[X, a_{\vec{q}}^r] = X a_{\vec{q}}^r - a_{\vec{q}}^r X = \ldots = ?$$

$$= \Big(+\frac{i\pi}{2} \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \Big\{ (a_{\vec{p}}^{s\dagger} - b_{\vec{p}}^{s\dagger})(a_{\vec{p}}^s - b_{\vec{p}}^s)\, a_{\vec{q}}^r$$

$$- \Big\{ a_{\vec{q}}^r \left(a_{\vec{p}}^{s\dagger} - b_{\vec{p}}^{s\dagger}\right)\left(a_{\vec{p}}^s - b_{\vec{p}}^s\right) \Big\} \Big)$$

$$= \frac{i\pi}{2} \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \Big[\left(a_{\vec{p}}^{s\dagger} - b_{\vec{p}}^{s\dagger}\right) \underbrace{\{ a_{\vec{p}}^s - b_{\vec{p}}^s, a_{\vec{q}}^r \}}_{0}$$

$$- \{ a_{\vec{p}}^{s\dagger} - b_{\vec{p}}^{s\dagger}, a_{\vec{q}}^r \} \left(a_{\vec{p}}^s - b_{\vec{p}}^s\right) \Big]$$

$$= \frac{i\pi}{2} \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \Big[-(2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p} - \vec{q}) \left(a_{\vec{p}}^s - b_{\vec{p}}^s\right) \Big]$$

$$= -\frac{i\pi}{2}\left(a_{\vec{q}}^r - b_{\vec{q}}^r\right)$$

Next term,

$$[X,[X,a_q^r]] = [X, -\tfrac{i\pi}{2}(a_q^r - b_q^r)]$$

$$= -\tfrac{i\pi}{2}[X, a_q^r] + \tfrac{i\pi}{2}\{X, b_q^r\} = \ldots$$

$$= 2\left(\tfrac{i\pi}{2}\right)^2 (a_q^r - b_q^r)$$

So... $e^X a_p^r e^{-X} = a_q^r - \frac{i\pi}{2}(a_q^r - b_q^r) + 2\left(\frac{i\pi}{2}\right)^2 (a_q^r - b_q^r)$

what's next? $+ \,?$

each step $\rightarrow$ $+2\left(\frac{i\pi}{2}\right)$ & alt $(-)$ sign.

$$\rightarrow \quad \mathcal{U}^\dagger a_p^r \mathcal{U} = a_p^r - \frac{i\pi}{2}(a_p^r - b_p^r) + \frac{1}{2!}2\left(\frac{i\pi}{2}\right)^2 (a_p^r - b_p^r),$$

$$- \frac{1}{3!}4\left(\frac{i\pi}{2}\right)^3 (a_p^r - b_p^r) + \frac{1}{4!}\left(\frac{i\pi}{2}\right)^4 6\,(\ldots) + \ldots$$

$$= a_p^r \left\{1 - \frac{i\pi}{2} + \frac{1}{2!}2\left(\frac{i\pi}{2}\right)^2 - \frac{1}{3!}4\left(\frac{i\pi}{2}\right)^3 + \frac{6}{4}\left(\frac{i\pi}{2}\right)^4 + \cdots\right\}$$

check this!

$$+ \; b_p^r \left\{\frac{i\pi}{2} - \frac{1}{2!}2\left(\frac{i\pi}{2}\right)^2 + \frac{1}{3!}4\left(\frac{i\pi}{2}\right)^3 - \frac{6}{4}\left(\frac{i\pi}{2}\right)^4 + \cdots\right\}$$

$$= a_p^r \left\{1 - \frac{1}{2}\cdot 2\right\} + b_p^r \cdot \left\{\frac{1}{2}\cdot 2\right\}$$

$$= b_p^r \quad \rightarrow \quad \boxed{\mathcal{U}^\dagger a_p^r \,\mathcal{U} = b_p^r}$$

$\rightarrow$ <u>$\mathcal{U}$ corresponds to charge conjugation!</u>

(2) using Dirac annihilation & creation ops, construct P for which

$$P^\dagger a^r_{\vec p} P = a^r_{-\vec p} \quad \& \quad P^\dagger b^r_{\vec p} P = b^r_{-\vec p}.$$

Last time, we find that

$$[X, a^r_{\vec p}] = -\left(\frac{i\pi}{2}\right)\left(a^r_{\vec p} - b^r_{\vec p}\right) \rightarrow \text{target}$$

→ we want X for which

$$\begin{cases} [X, a^r_{\vec p}] = -\left(\frac{i\pi}{2}\right)\left(a^r_{\vec p} - a^r_{-\vec p}\right). \\ [X, b^r_{\vec p}] = -\left(\frac{i\pi}{2}\right)\left(b^r_{\vec p} + b^r_{-\vec p}\right). \end{cases}$$

→ Let

$$P = \exp\left\{-\frac{i\pi}{2}\sum_{r=1}^{2}\int\frac{d^3p}{(2\pi)^3}\left\{a^{r\dagger}_{\vec p}\left(a^r_{\vec p} - a^r_{-\vec p}\right) + b^{r\dagger}_{\vec p}\left(b^r_{\vec p} + b^r_{-\vec p}\right)\right\}\right\}$$

check this, like last time

→ should work! :)

Obv a^r_p only int. w/ 1st term $a^\dagger(\ )a = 0$

$aa^\dagger(\ ) = \delta(\ldots)$ ✓

Same with $b^r_{\vec p}$ ✓.

Interacting Fields & Feynman Diagrams

To get better description of the real world, need to include interactions in the theory.

To preserve causality, new terms may involve only products of fields at the same spacetime point.

↳ $\phi^4(x)$ ✓, but not $\phi(x)\phi(y)$.

→ $H_{int} = \int d^3x \, \mathcal{H}_{int}[\phi(x)] = -\int d^3x \, \mathcal{L}_{int}[\phi(x)]$

→ insist that H_{int} is a fn only of the fields, not of their derivatives.

→ Common ex in particle physics & CMT:

$$\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!}\phi^4}$$

<u>note</u> $\Pi(x)$ is still $\partial_0 \phi_0(x)$ since there are not any interaction terms involving $\partial_0 \phi$

λ: dimensionless "coupling constant".

→ In general, adding interactions produces divergences

However, no matter what the true physics looks like at high momenta or short distances, the low momentum / long distance physics is well-approximated by an "effective" FT.

with "renormalizable" interactions.

→ those interactions where coupling constant are has dimensions $\boxed{d \geq 0}$

$$[\text{Mass}]^d \text{ where } d \geq 0.$$

Ex $\quad -\frac{1}{2} m^2 \phi^2$ & $\frac{\lambda}{4!} \phi^4$ same dim

$$\rightarrow \lambda \sim [\text{mass}]^0 \rightarrow \text{Renormalizable.}$$

But $\quad -\frac{\lambda_6}{6!} \phi^6 \rightarrow$ <u>not</u> renormalizable.

since $\lambda \sim [\text{mass}]^{-2}$.

$\boxed{\text{Perturbation Expansion}}$

$$\text{Let } \quad H = H_0 + H_{int} \rightsquigarrow = \int d^3x \, \frac{\lambda}{4!} \phi^4(x)$$

H_0 ← KG, free

→ we will generate power series in λ.

At any t_0, we can write

$$\phi(t_0, \vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_{\vec{p}} \, e^{ip\cdot x} + a^\dagger_{\vec{p}} \, e^{-ip\cdot x} \right\}$$

$$\equiv \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ \underset{\substack{\parallel \\ a_{\vec{p}}}}{\tilde{a}_{\vec{p}}} \, e^{i\vec{p}\cdot\vec{x}} + \tilde{a}^\dagger_{\vec{p}} \, e^{-i\vec{p}\cdot\vec{x}} \right\}$$

where we've let $a_{\vec{p}}$ absorb $e^{\pm i E t_0}$

The Heisenberg field is then given by:

$$\rightarrow \boxed{\phi(t,\vec{x}) = e^{iH(t-t_0)}\,\phi(t_0,x)\,e^{-iH(t-t_0)}}$$

If there's no interaction then we have

$$\rightarrow \phi_{free}(t,\vec{x}) = e^{iH_0(t-t_0)}\,\phi(t_0,x)\,e^{-iH_0(t-t_0)}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left\{a_p e^{-ip\cdot x} + a_p^\dagger e^{+ip\cdot x}\right\}\Bigg|_{\substack{x^0 = t-t_0 \\ p^0 = E_{\vec{p}}}}$$

Define this to be $\phi_I(t,\vec{x})$, the interaction picture field

the interaction picture field $\equiv$ Heisenberg field when $\lambda = 0$.

Now, look at Heisenberg field...

$$\phi(t,\vec{x}) = e^{iH(t-t_0)}\,\phi(t_0,\vec{x})\,e^{-iH(t-t_0)}$$

$$= \underbrace{e^{iH(t-t_0)}e^{-iH_0(t-t_0)}}\,\phi_I(t,\vec{x})\,e^{+iH_0(t-t_0)}e^{-iH(t-t_0)}$$

$$= \mathcal{U}^\dagger(t,t_0)\,\phi_I(t,\vec{x})\,\mathcal{U}(t,t_0)$$

time evolution operator

$\rightarrow$ Evolve the operator as $\phi_I(t,\vec{x})$

OR

Evolve the states by $\mathcal{U}(t,t_0) \rightsquigarrow \mathcal{U}|\phi\rangle$ ...

→ now, we want to express $\mathcal{U}(t,t_0)$ entirely in ϕ_I

To do this, note that $\mathcal{U}(t,t_0)$ solves SE:

$$i\frac{\partial}{\partial t}\mathcal{U}(t,t_0) = e^{iH_0(t-t_0)}(H-H_0)e^{-iH(t-t_0)}$$

$$= e^{iH_0(t-t_0)} H_{int}\, e^{-iH(t-t_0)}$$

$$= \underbrace{e^{iH_0(t-t_0)} H_{int}\, e^{-iH_0(t-t_0)}}\, e^{iH_0(t-t_0)} e^{-iH(t-t_0)}$$

$$= H_I(t)\,\mathcal{U}(t,t_0)$$

where

$$H_I(t) = e^{iH_0(t-t_0)} \mathcal{H}_{int}\, e^{-iH_0(t-t_0)}$$

$$= \int d^3x\, \frac{\lambda}{4!}\phi_I^4 \quad \boxed{?}$$

$$= \int d^3x\, e^{iH_0(t-t_0)}\frac{\lambda}{4!}\phi^4\, e^{-iH_0(t-t_0)}$$

$$= \int d^3x\, \frac{\lambda}{4!}\phi_I^4 \quad ✓$$

→ this is the Hamiltonian in the interaction picture.

So since $\mathcal{U}$ solves the SE:

$$i\partial_t\,\mathcal{U}(t,t_0) = H_I(t)\,\mathcal{U}(t,t_0),$$

$\mathcal{U}$ must look like

$$\boxed{\mathcal{U}(t,t_0) \sim \exp\{-iH_I t\}}$$

More carefully, we actually have that

$$\boxed{\mathcal{U}(t,t_0) = \underset{\substack{\uparrow \\ \text{time-ordering symbol}}}{T} \exp\left\{ -i\int_{t_0}^{t} dt'\, H_I(t') \right\}}$$ Dyson's formula

Why T? Why ordering? $\Rightarrow$ B/c $H(t_1) \not\leftrightarrow H(t_2)$ when $t_1 \neq t_2$.

"T" puts the latest operators on the left.

hence $i\partial_t \mathcal{U}(t,t_0) = \underline{\underline{H_I(t)}}\, \mathcal{U}(t,t_0)$.

As a pwr series in λ:

$$\mathcal{U}(t,t_0) = 1 + (-i)\int_{t_0}^{t} dt'\, H_I(t') + \frac{(-i)^2}{2!}\int_{t_0}^{t}\int_{t_0}^{t} dt_1 dt_2\, T\{H_1(t_1) H_2(t_2)\}$$
$$+ \frac{(-i)^3}{3!}\int_{t_0}^{t}\int_{t_0}^{t}\int_{t_0}^{t} dt_1\, dt_2\, dt_3\, T\{H_1(t_1) H_2(t_2) H_3(t_3)\} + \ldots$$

Note "the time-ordering of the exponential is just ~~the time-ordering~~ the Taylor series of the terms time-ordered..."

$\rightarrow$ Now, we want to generalize $\mathcal{U}(t,t_0)$ to $\mathcal{U}(t,t')$

reference time

This generalization is natural...

$$\boxed{\mathcal{U}(t,t') \equiv T\left\{\exp\left[-i\int_{t'}^{t} dt''\, \mathcal{H}_I(t'')\right]\right\} \quad (t \geq t')}$$

Then we see that b/c both t, t' are variables...
we find:

$$\boxed{\begin{aligned} i\partial_t\, \mathcal{U}(t,t') &= \mathcal{H}_I(t)\,\mathcal{U}(t,t') \\ i\partial_{t'}\, \mathcal{U}(t,t') &= -\,\mathcal{U}(t,t')\,\mathcal{H}_I(t'). \end{aligned}}$$

and thus...

$$\boxed{\mathcal{U}(t,t') = e^{iH_0(t-t_0)}\, e^{-iH(t-t')}\, e^{-iH_0(t-t_0)}}$$

So $\mathcal{U}$ is unitary.

Further, for $t_1 \geq t_2 \geq t_3$,

$$\begin{aligned} \mathcal{U}(t_1,t_2)\,\mathcal{U}(t_2,t_3) &= \mathcal{U}(t_1,t_3) \\ \mathcal{U}(t_1,t_3)\,[\mathcal{U}(t_2,t_3)]^{\dagger} &= \mathcal{U}(t_1,t_2) \end{aligned}$$

Now, let $|0\rangle$ be gnd state of $\mathcal{H}_0$

$|\Omega\rangle$ be gnd state of $\mathcal{H}$

$|n\rangle$ be ~~gnd~~ label all $|E_n\rangle$ of $\mathcal{H}$.

Then, $(E_0 = \langle\Omega|H|\Omega\rangle)$

$$e^{-iHT}|0\rangle = e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_nT}|n\rangle\langle n|0\rangle$$

Assume $H_0|0\rangle = 0$ & consider $T\to\infty$ limit.

$\updownarrow$

$T \to \infty(1-i\varepsilon)$

Then e^{-iE_nT} dies slowest for $n=0$, and so ...

$$\to e^{-iHT}|0\rangle \to e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle$$

assume $\langle\Omega|0\rangle \neq 0$

So

$$\boxed{|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0T}\langle\Omega|0\rangle\right)^{-1} e^{-iHT}|0\rangle}$$

Now, since T large, we can shift it by a small constant ...

$$|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left\{e^{-iE_0(T+t_0)}\langle\Omega|0\rangle\right\}^{-1} e^{-iH(T+t_0)}|0\rangle$$

$$= \lim_{T\to\infty(1-i\varepsilon)} \left\{e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle\right\}^{-1} \underbrace{e^{-iH(t_0-(-T))}}_{\mathcal{U}(t_0,-T)}|0\rangle$$

$$= \lim_{T\to\infty(1-i\varepsilon)} \left\{e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle\right\}^{-1} e^{-iH(t_0-(-T))} \underbrace{e^{-iH_0(-T-t_0)}|0\rangle}_{=|0\rangle \text{ since } H_0|0\rangle = 0}$$

$$\Rightarrow |\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{\mathcal{U}(t_0,-T)|0\rangle}{e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle}$$

Similarly,

$$\boxed{\langle\Omega| = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\,\mathcal{U}(T,t_0)}{e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle}}$$

So, putting these together gives a correlation function...

For $x^0 > y^0 > t_0$, we have

$$\to \langle\Omega|\phi(x)\phi(y)|\Omega\rangle$$

$$= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\,\mathcal{U}(T,t_0)}{e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle} \overbrace{\left(\mathcal{U}(x^0,t_0)\right)^{\dagger}\phi_I(x)\,\mathcal{U}(x^0,t_0)}^{\phi(x)}$$

$$\times\left(\mathcal{U}(y^0,t_0)\right)^{\dagger}\phi_I(y)\,\mathcal{U}(y^0,t_0)\times$$

$$\frac{\mathcal{U}(t_0,-T)}{e^{-iE_0(T+t_0)}\langle\Omega|0\rangle}$$

$$= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\,\mathcal{U}(T,x^0)\phi_I(x)\,\mathcal{U}(x^0,y^0)\phi_I(y)\,\mathcal{U}(y^0,-T)|0\rangle}{|\langle 0|\Omega\rangle|^2\, e^{-iE_0(2T)}}$$

awkward...

so divide the whole thing by $1 = \langle\Omega|\Omega\rangle$

$$1 = \langle\Omega|\Omega\rangle = \frac{\langle 0|\,\overbrace{\mathcal{U}(T,t_0)\,\mathcal{U}(t_0,-T)}^{\mathcal{U}(T,-T)}|0\rangle}{|\langle 0|\Omega\rangle|^2\, e^{-iE_0(2T)}}.$$

to get (for $x^0 > y^0$)

$$\boxed{\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\,\mathcal{U}(T,x^0)\phi_I(x)\,\mathcal{U}(x^0,y^0)\phi_I(y)\,\mathcal{U}(y^0,-T)|0\rangle}{\langle 0|\,\mathcal{U}(T,-T)|0\rangle}}$$

So, we have that, by replacing U's with Dyson's formula (w/ time-ordering)

$$\boxed{\langle \Omega | T\{\phi(x)\phi(y)\} |\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0| T\{\phi_I(x)\phi_I(y)\exp\{-i\int_{-T}^{T} dt\, H_I(t)\}\} |0\rangle}{\langle 0| T\exp\{-i\int_{-T}^{T} dt\, H_I(t)\}|0\rangle}}$$

So, looks like the term

$\exp\left\{-i\int_{-T}^{T} dt\, H_I(t)\right\}$ ~~is important~~ & can be found, so

$\boxed{\text{Wick's theorem}}$

→ So, we have reduced the problem of calc. correlation functions to evaluating

$$\boxed{\langle 0| T\{\phi_I(x_1)\phi_I(x_2)\cdots\phi_I(x_n)\}|0\rangle}$$

→ This is the vacuum exp-value of time-ordered products of finite number of field operators.

$n=2 \Rightarrow$ get Feynman operator.

$n>2 \rightarrow$ can use brute force, but there are also ways to simplify calculations.

Now, we study $\boxed{\langle 0| T\{\phi_I(x)\phi_I(y)\}|0\rangle}$

Recall that

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$$\phi_I(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_{\vec p}\, e^{-ip\cdot x} + a^\dagger_{\vec p}\, e^{+ip\cdot x} \right\}$$

Call

$$\phi^+_I(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}}\, a_{\vec p}\, e^{-ip\cdot x}$$

and

$$\phi^-_I(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}}\, a^\dagger_{\vec p}\, e^{+ip\cdot x}$$

which is ~~useful~~ b/c

$$\phi^+_I(x)|0\rangle = 0, \qquad \langle 0|\phi^-_I(x) = 0.$$

only has annihilation ops — only has creation ops.

→ For $x^0 > y^0$,

$$T\{\phi_I(x)\,\phi_I(y)\} \overset{x^0>y^0}{=} \phi^+_I(x)\,\phi^+_I(y) + \phi^-_I(x)\,\phi^-_I(y) + \phi^+_I(x)\,\phi^-_I(y) + \phi^-_I(x)\,\phi^+_I(y)$$

$$= \phi^+_I(x)\,\phi^+_I(y) + \phi^-_I(y)\,\phi^+_I(x) + \phi^-_I(x)\,\phi^+_I(y) + \phi^-_I(x)\,\phi^-_I(y) + [\phi^+_I(x), \phi^-_I(y)]$$

every of these terms has the form $a^\dagger_{\vec p}\, a^\dagger_{\vec q}\, a_{\vec k}\, a_{\vec k}$

i.e. creation ops always on the left.

→ "Normal order" → has vanishing vacuum exp value.

What can we say about the commutator?

[just a number, there is no creation/annihilation op- in it!

$$[\phi_I^+(x), \phi_I^-(y)] = \langle 0| [\phi_I^+(x), \phi_I^-(y)] |0\rangle$$

$$= \langle 0| \phi_I^+(x)\phi_I^-(y) |0\rangle = \langle 0| \phi_I(x)\phi_I(y) |0\rangle .$$

With these, we can write

$$\boxed{T\{\phi_I(x)\phi_I(y)\} = \phi_I^+(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(y)\phi_I^+(x) + \phi_I^-(x)\phi_I^-(y) + \langle 0|\phi_I(x)\phi_I(y)|0\rangle}$$

Now, define the "normal ordering symbol" N

s.t. N takes the string a, $a^\dagger$'s and rearranges them so that $a^\dagger$'s are on the left

Ex.

$$\begin{cases} N(a_{\vec p}^\dagger a_{\vec p}) = a_{\vec p}^\dagger a_{\vec p} \\ N(a_{\vec p} a_{\vec p}^\dagger) = a_{\vec p}^\dagger a_{\vec p} \\ N(a_{\vec p} a_{\vec q} a_{\vec p}^\dagger) = a_{\vec p}^\dagger a_{\vec p} a_{\vec q} \end{cases}$$

ordering for $a_{\vec p}$, $a_{\vec q}$ doesn't matter since they commute.

→ Note N is not a well-defined mathematical operation

e.g. $N([a_{\vec p}, a_{\vec q}^\dagger]) \neq N((2\pi)^3\delta^{(3)}(\vec p - \vec q))$

→ it is only a lexicographic convention.

Now, let us consider general x^0, y^0, then

$$T\{\phi_I(x)\phi_I(y)\} = N\{\phi_I(x)\,\phi_I(y)\} + \begin{cases} [\phi_I^+(x), \phi_I^-(y)] & \text{for } x^0 > y^0 \\ [\phi_I^+(y), \phi_I^-(x)] & \text{for } x^0 < y^0 \end{cases}$$

→ Let us define the contraction of $\phi_I(x)$ & $\phi_I(y)$ as

$$\overbrace{\phi_I(x)\,\phi_I}(y) = \begin{cases} [\phi_I^+(x), \phi_I^-(y)] & x^0 \geq y^0 \\ [\phi_I^+(y), \phi_I^-(x)] & x^0 < y^0 \end{cases}$$

then notice that, from our previous discussion,

$$\overbrace{\phi_I(x)\,\phi_I}(y) = \begin{cases} \langle 0|\,\phi_I(x)\phi_I(y)\,|0\rangle & x^0 \geq y^0 \\ \langle 0|\,\phi_I(y)\phi_I(x)\,|0\rangle & y^0 > x^0 . \end{cases}$$

So,

Feynman propagator

$$\begin{aligned} \overbrace{\phi_I(x)\,\phi_I}(y) &= \langle 0|\,T\{\phi_I(x)\phi_I(y)\}\,|0\rangle \\ &= D_F(x-y) \rightsquigarrow \text{Feynman propagator.} \\ &= \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}\,e^{-ip(x-y)} \end{aligned}$$

With these, we have that

→ leaving out the I's subscripts

$$T\{\phi(x)\phi(y)\} = \mathcal{N}\{\phi(x)\phi(y) + \overbrace{\phi(x)\phi(y)}\}$$

$$\overbrace{\phi(x)\phi(y)} = \overbrace{\phi(y)\phi(x)}$$

$\rightarrow$ $T\{\phi(x)\phi(y)\} = \mathcal{N}\{\phi(x)\phi(y)\}$ + "contraction".

In fact, the generalization of this is called

<u>Wick's Theorem</u>

$$\boxed{\begin{aligned} &T\{\phi(x_1)\phi(x_2)\ldots\phi(x_m)\} \\ &= N\{\phi(x_1)\phi(x_2)\ldots\phi(x_m) + \text{all possible contractions}\}\end{aligned}}$$

<u>Ex</u> $T\{\phi_1\phi_2\phi_3\phi_4\}$ $\quad (\phi_n = \phi(x_n))$

$$\begin{aligned} = N\{&\phi_1\phi_2\phi_3\phi_4 + \\ &+ \overbrace{\phi_1\phi_2}\phi_3\phi_4 + \phi_1\overbrace{\phi_2\phi_3}\phi_4 + \phi_1\phi_2\overbrace{\phi_3\phi_4} \\ &\overbrace{\phi_1\phi_2\phi_3\phi_4} + \overbrace{\phi_1\phi_2\phi_3}\phi_4 + \phi_1\overbrace{\phi_2\phi_3\phi_4} + \overbrace{\phi_1\phi_2}\,\overbrace{\phi_3\phi_4} \\ &+ \underbrace{\overbrace{\phi_1\phi_2\phi_3}\phi_4} + \underbrace{\phi_1\overbrace{\phi_2\phi_3}\phi_4}\ \}\end{aligned}$$

what does $\mathcal{N}\{\overbrace{\phi_1\phi_2}\phi_3\phi_4\}$ mean?

$$N\{\overbrace{\phi_1\phi_2}\phi_3\phi_4\} = \overbrace{\phi_1\phi_2}\,\mathcal{N}\{\phi_3\phi_4\}$$

$$= \mathcal{D}_F(x_1 - x_2)\,\mathcal{N}\{\phi_3\phi_4\}.$$

Proof → prove by induction. $n=2$ is good (Feynman)

→ assume this holds for $n-1$.

Let $W(\phi_1 \cdots \phi_n) = N\{\phi_1\phi_2 \cdots \phi_n + \text{all possible contr.}\}$

To prove $W(\phi_1 \cdots \phi_n) = T\{\phi_1\phi_2 \cdots \phi_n\}$.

W/l/o/g: let $x_1^0 \geq x_2^0 \geq \ldots \geq x_n^0$.

Then $T\{\phi_1 \cdots \phi_n\} = \phi_1 T\{\phi_2 \cdots \phi_n\}$ since $n-1$ done.

$$= \phi_1 W(\phi_2 \cdots \phi_n)$$

So $T\{\phi_1 \cdots \phi_n\} = \underbrace{\phi_1^- W(\phi_2 \cdots \phi_n) + W(\phi_2 \cdots \phi_n)\phi_1^+}_{X}$

$$+ \underbrace{[\phi_1^+, W]}_{Y}$$

Let $X \equiv \phi_1^- W + W\phi_1^+$; $Y \equiv [\phi_1^+, W]$.

$X+Y$ are normal ordered: X contains all contractions in $W(\phi_1 \ldots \phi_n)$ & which does not contract ϕ_1 w/ anything.

Y contains all contractions in $W(\phi_1 \ldots \phi_n)$ which contracts ϕ_1 with something.

So $T(\phi_1\phi_2 \cdots \phi_n) = W(\phi_1 \cdots \phi_n)$.

(we won't worry too much abt this proof.

→ the main idea is the theorem itself).

$\square$

In any case, we have another way to explicitly write out the result of Wick's Thm:

$$\boxed{T\{\phi_1\phi_2\cdots\phi_n\} = N\left\{\exp\left[\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \overline{\phi_i\phi_j}\,\frac{\partial}{\partial\phi_i}\frac{\partial}{\partial\phi_j}\right]\phi_1\cdots\phi_n\right\}}$$

↑ we'll see why later on.

Feynman Diagrams

Wick's Thm allows ~~us~~ to write

$$\langle 0|\,T\{\phi_1\cdots\phi_n\}\,|0\rangle$$

in terms of sums and products of Feynman propagators.

→ Now, we will develop ~~a~~ diagrammatic expressions.

Recall that

$$T\{\phi_1\phi_2\phi_3\phi_4\} = N\{\phi_1\phi_2\phi_3\phi_4 + \text{all possible contractions}\}$$

But the only contribution to

$\langle 0|\,T\{\phi_1\phi_2\phi_3\phi_4\}\,|0\rangle$ is when <u>all</u> the ϕ's are contracted.

↳ This b/c whenever things are in normal order, the exp value vanishes. → $N(\overline{\phi_1\phi_2}\phi_3\phi_4) = \overline{\phi_1\phi_2}\,N(\phi_3\phi_4$

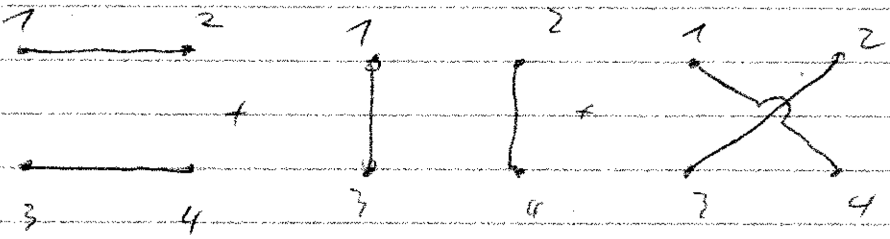
→ to "escape" from normal order, ϕ's have to be contracted

This means that

$$T\{\phi_1\phi_2\phi_3\phi_4\} = \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4$$

→ Can write this as Feynman diagrams...

$T\{\phi_1\phi_2\phi_3\phi_4\} =$



↳ $\boxed{\text{Interpretation}}$

Particles are created at 2 spacetime points, each propagates to one of the other points, then get annihilated.

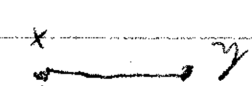
→ total amplitude of the process is the sum of the diagrams.

Well... what about something like...

$$\langle 0| T\left\{\phi(x)\phi(y)\exp\left\{-i\int_{-\infty}^{\infty} dt\, H_I(t)\right\}\right\} |0\rangle \;?$$

- Well... as a pwr series in λ, the lowest order term is

$$\langle 0| T\{\phi(x)\phi(y)\}|0\rangle = D_F(x-y).$$



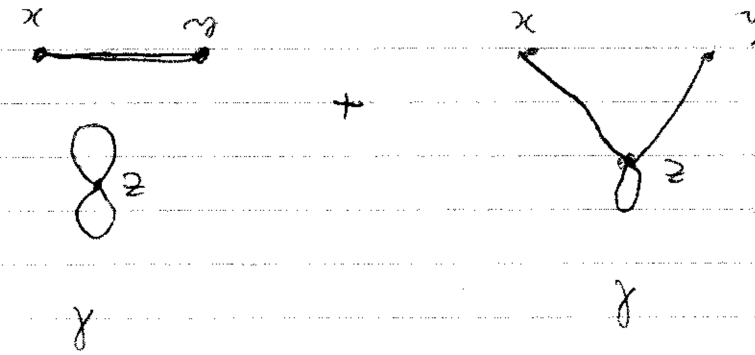
- 1st order $\langle 0| T\{\phi(x)\phi(y)(-i)\left(\int_{-\infty}^{\infty} dt\, H_I(t)\right)_{dt} |0\rangle$

$$= \langle 0| T\left\{\phi(x)\phi(y)(-i)\int d^4z\, \tfrac{\lambda}{4!}\phi^4(z)\right\}|0\rangle$$

$$= \dots \frac{-i\lambda}{4!}\int d^4z \,\langle 0|T\{\phi(x)\phi(y)\overbrace{\phi(z)\phi(z)\phi(z)\phi(z)}^{\phi^4(z)}\}|0\rangle$$

$$= \frac{-i\lambda}{4!}\int d^4z \Big\{ \phi(x)\phi(y)\cdot\{\phi(z)\phi(z)\phi(z)\phi(z) + \phi_z\phi_z\phi_z\phi_z + \phi_z\phi_z\phi_z\phi_z\}$$

$$+ \underbrace{\phi(x)\phi(y)\phi(z)\phi(z)\phi(z)\phi(z)}_{\text{12 terms that are identical}}\Big\}$$

= 

→ ' propagators

$$\int d^4z\; D_F(x-z)\,D_F(z-y)\,D_F(z-z)$$

↑ 12 of these.

$$\int d^4z\; D_F(x-y)\,D_F(z-z)\,D_F(z-z)$$

↳ each contraction D_F is a line.

each spacetime point is a dot.

→ but need to distinguish "external" and "internal" points.

"external" ↓ x, y "internal" ↓ z

Each internal point is associated w/ a factor of $-i\lambda\int d^4z \cdot$, with combinatorial factor...

How do we count these combinatorial factors?

Oct, 19, 2020

→ well... Each H_I has 4 ϕ_I's. $\phi(z)\phi(z)\phi(z)\phi(z)$,

→ interchanging the contraction "ends" will give the same amplitude...

→ so for each H_I we expect a factor of $4!$

→ cancels out the $\frac{1}{4!}$ in $\frac{\lambda}{4!}\phi^4$.

• In a diagram with more than one pair of H_I

We can exchange all the contraction ends of one H_I with contraction ends of the other H_I.

→ Since we integrate over all z_1, z_2 ⇒ these give the same amplitude.

→ For a diagram with n "internal vertices", (i.e. # of H_I's), we get a factor of $n!$ This cancels the $\frac{1}{n!}$ factor from Taylor series expansion of $\exp\{-i\int H_I(t)\,dt\}$.

→ ~~But there is a small subtlety~~ that is

For example... consider the λ^2 term:

$$\langle 0|\phi(x)\phi(y)\,\frac{1}{3!}\left(\frac{-i\lambda}{4!}\right)^3\int d^4z\,\phi\phi\phi\phi\int d^4w\,\phi\phi\phi\phi\int d^4u\,\phi\phi\phi\phi \qquad 16$$

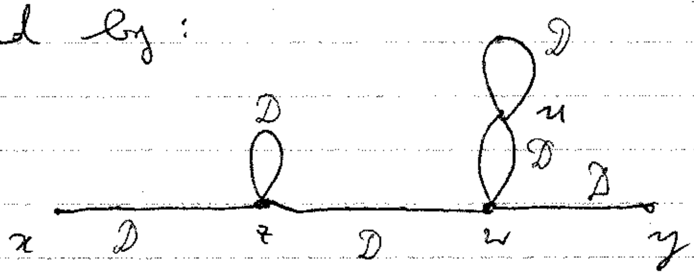
$$= \frac{1}{3!}\left(\frac{-i\lambda}{4!}\right)^3\int d^4z\,d^4w\,d^4u\,D_F(x-z)D_F(z-z)D_F(z-w)$$
$$D_F(w-y)D_F^2(w-u)D_F(u-u)$$

The number of contractions that give this same expression is

$$3! \times (4,3) \times (4,3,2,1) \times (4,3) \times (1/2)$$

- $3!$: interchange vertices
- $(4,3)$: placement of contractions into z vertex
- $(4,3,2,1)$: placement of contraction into w vertex
- $(4,3)$: " ... for u vertex
- $(1/2)$: interchange of w-u interaction

Represented by:



Now... there is a subtlety that goes with all of this, and that's <u>symmetry factors</u>.

Symmetry Factors

→ Best to consider the simplest diagram with the most general problem.

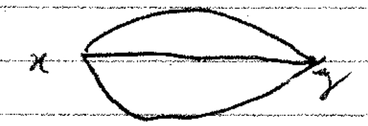
→ consider $\frac{\lambda}{3!}\phi^3$ theory ...

At 2^{nd} order in λ, $\langle 0|T\left\{\exp \int_{-\infty}^{\infty} \mathcal{H}_I(t)\,dt\right\}|0\rangle$

gives sth like

$$\int \langle 0|\phi(x)\phi(x)\phi(x)\,\phi(y)\phi(y)\phi(y)|0\rangle\, d^4x\, d^4y$$

↳ Feynman diagram is



$\mathcal{D}_F(x-y)\,\mathcal{D}_F(x-y)\,\mathcal{D}_F(x-y)$.

Naively we expect $2!$ from interchanging x-y.

and $3! \times 3! = 36$ from interchanging the ϕ's at x and ϕ's at y.

→ Expect 72. But are actually only 6:

$\phi_x^1 \;\; \phi_y^1 \;\; \rightarrow \;\; \left\{ \begin{array}{l} \phi_x\phi_x\phi_x\;\;\phi_y\phi_y\phi_y \\ \phi_x\phi_x\phi_x\;\;\phi_y\phi_y\phi_y \end{array} \right.$

Similarly for $\phi_x^1 - \phi_y^2$ $(\times 2)$

$\phi_x^1 - \phi_y^3$ $(\times 2)$

total $= 6$.

→ we have overshot by 12, because

$$\cancel{\phi_x\phi_x\phi_x\phi_y\phi_y\phi_y = \phi_x\phi_y\phi_x\phi_y\phi_y\phi_y}$$

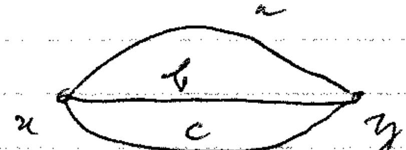
~~also interchanging x-y~~

(1) Simultaneously swap 1^{st} & 2^{nd} x
& 1^{st} & 2^{nd} y

$$\phi_x\phi_x\phi_x\phi_y\phi_y\phi_y = \phi_x\phi_x\phi_x\phi_y\phi_y\phi_y$$

(2) exchanging all x's & y's doesn't do anything either...

We can see what's going on by labelling the vertices & propagators ...



$a \leftrightarrow b$, $\updownarrow c \nearrow$ don't change the diagram

$x \leftrightarrow y$ also don't change the diagram

⇒ diagram has a permutation symmetry

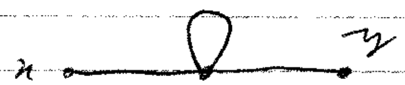
This has $2! \times 3! = 12$ elements; which is the factor we overshot with...

This is called the symmetry factor S.

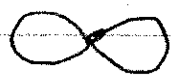
The number of diagrams is then

$$\frac{1}{s}(3!)(3!)(2!) = 6.$$

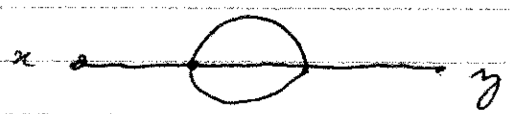
<u>Some examples</u>



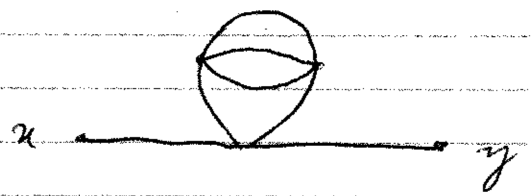
$s = 2 \qquad (x \leftrightarrow y).$



$s = 2 \cdot 2 \cdot 2 = 8$



$s = 3! = 6$



$s = 3! \cdot 2 = 12$

Now, we are ready to state the Feynman rules for position space...

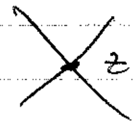
We'll start this rule lets us find

$$\langle 0 | T\left\{ \phi(x)\phi(y) \exp\left\{ -i\int_{-\infty}^{\infty} H_I(t)\,dt \right\} \right\} | 0 \rangle$$

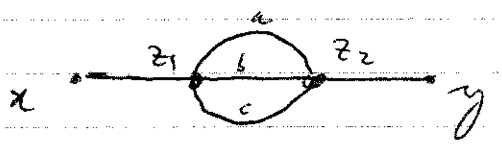
$$= \left(\text{sum all possible diagrams with two external points} \right)$$

where each diagram is built out of
- propagators
- vertices
- external pts

Feynman Rules for ϕ^4 theory

1. For each propagator $\underset{x}{\bullet}$———$\underset{y}{\bullet}$ $= D_F(x-y)$
2. For each vertex  $= (-i\lambda)\int d^4 z$
3. For each external point $\underset{x}{\bullet}$——— $= 1$
4. Divide by symmetry factor.

no $\frac{1}{4!}$ here

Ex 

$S = 3! = 6$
(permute the 3 propagators from $z_1 \to z_2$)

Amplitude:

$$\left(\frac{-i\lambda}{1}\right)^2 \cdot \frac{1}{6} \cdot \int d^4 z_1\, d^4 z_2\, D_F(x-z_1)\, D_F^3(z_1-z_2)\, D_F(z_2-y)$$

Interpretation

- Each of the vertex factor $(-i\lambda)$ is the amplitude for the emission and/or absorption of particles at a vertex.
- The integral $\int d^4 z$ instructs us to sum over all points where this process can occur.

→ this is the principle of superposition!

- $\int d^4 z$ is addition of amplitudes

↳ Feynman rules tell us to multiply the amplitudes for each independent part of the process.

Now, in most calculations, it is simpler to express the Feynman rules in terms of momentum

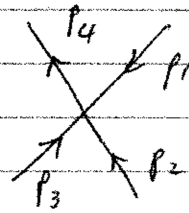
→ We want <u>momentum-space Feynman diagrams</u>

To do this, we write $D_F(x-y)$ in Fourier space

$$\boxed{D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon} e^{-ip\cdot(x-y)}}$$

Now, we represent this in a diagram by assigning a 4-momentum p to each propagator.

When 4 lines meet at a vertex, we get



$$\longrightarrow \int d^4z \, e^{-ip_1 z} e^{-ip_2 z} e^{-ip_3 z} e^{+ip_4 z}$$

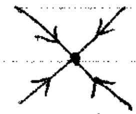
$$= (2\pi)^4 \delta^{(4)}(p_1+p_2+p_3-p_4)$$

i.e. <u>momentum is conserved</u> at each vertex.

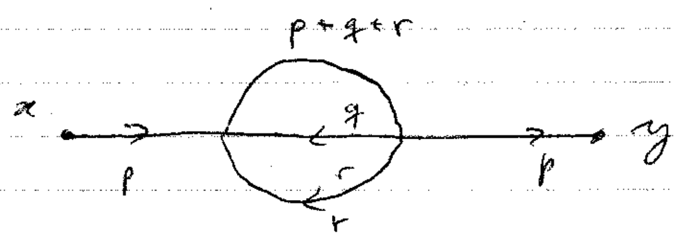
The delta functions can be used to perform integrals for the propagators...

→ From here we get momentum space Feynman rules.

Momentum-space Feynman rules

1. Each propagator $\xrightarrow[p]{\quad\quad}$ $= \dfrac{i}{p^2-m^2+i\varepsilon}$
2. Each vertex  $= -i\lambda$
3. Each external point $x \xleftarrow[p]{\quad\quad}$ $= e^{-ip\cdot x}$
4. Impose momentum conservation at each vertex
5. Integrate over each ~~e~~ undetermined momentum $$\int \frac{d^4p}{(2\pi)^4}$$
6. Divide by symmetry factor.

Ex



$$= \left(\frac{-i\lambda}{\cancel{4}}\right)^2 \frac{1}{6} \cdot \int \left\{ \left(\frac{i}{p^2-m^2+i\varepsilon}\right)^2 \frac{i}{(q+p+r)^2-m^2+i\varepsilon} \, \frac{i}{q^2-m^2+i\varepsilon} \right.$$

$$\left. \cdot \frac{i}{r^2-m^2+i\varepsilon} \right\} \frac{d^4q}{(2\pi)^4} \, \frac{d^4r}{(2\pi)^4}.$$

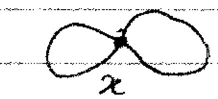
There is one subtlety, however...

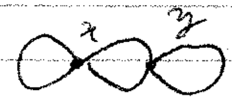
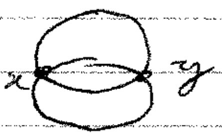
Consider diagrams without external vertices.

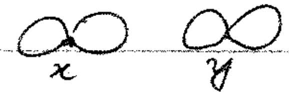
↳ these are diagrams from the term:

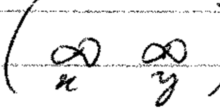
$$\langle 0|\, T\left\{\exp\left(-i\int_{-\infty}^{\infty} \mathcal{H}_I(t)\,dt\right)\right\}|0\rangle$$

→ these are called "vacuum diagrams".

At order λ we have ... 

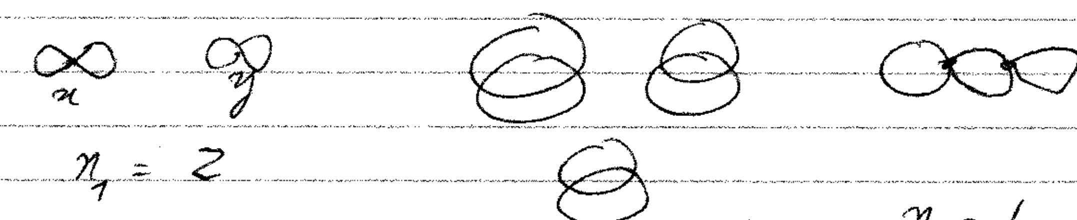
λ^2  and 

and () → disconnected diagram.

⇒ there is $S = 2$ for ()

In general, a vacuum diagram has connected subdiagrams V_i which appears n_i times

Ex $\quad V_1 \quad V_2 \quad V_3$



copies → $n_1 = 2$ $\quad n_3 = 1$ $\quad n_2 = 3$

The amplitude for the total diagram is the product:

$$\boxed{\prod_i \left(\frac{1}{n_i!} V_i^{n_i}\right)}$$

Now, ~~So~~, the sum over all ~~vacuum~~ diagrams can be written as

$$\boxed{\sum_{\substack{\text{all possible}\\\text{connected}\\\text{pieces}}} \sum_{\substack{\text{all}\\\{n_i\}}} \begin{pmatrix}\text{value of}\\\text{connected}\\\text{piece}\end{pmatrix} \times \left\{ \prod_i \frac{1}{n_i!} (V_i)^{n_i} \right\}}$$

The sum of the connected pieces factors out, giving

$$\boxed{= \left(\sum \text{connected} \right) \times \sum_{\substack{\text{all}\\\{n_i\}}} \left(\prod_i \frac{1}{n_i!} (V_i)^{n_i} \right)}$$

sum all the values of the connected pieces

Now...

$$\sum_{\{n_i\}} \left(\prod_i \frac{1}{n_i!} (V_i)^{n_i} \right)$$

$$= \prod_i \sum_{\{n_i\}} \frac{1}{n_i!} (V_i)^{n_i}$$

$$= \prod_i \exp(V_i)$$

$$= \exp\left(\sum_i V_i \right)$$

$$\boxed{\sum \text{all diagrams} = \sum \text{connected} \times \exp\left(\sum \text{disconnected} \right)}$$

Now, recall that the sum of all vacuum diagrams is just going to be

$$\exp\left(\sum_i V_i\right).$$

$$\Rightarrow \langle 0| T\left\{ \exp\left(-i\int_{-\infty}^{\infty} \mathcal{H}_I(t)\, dt\right)\right\} |0\rangle = \exp\left(\sum_i V_i\right)$$

and as we have argued...

$$\langle 0| T\left\{ \phi_I(x)\phi(y) \exp\left\{-i\int_{-\infty}^{\infty} \mathcal{H}_I(t)\, dt\right\}\right\} |0\rangle$$

$$= \left(\sum \text{connected}\right) \times \exp\left(\sum_i V_i\right).$$

$\Rightarrow$ And so we have that

$$\langle\Omega| T\{\phi(x)\phi(y)\}|\Omega\rangle = \lim \frac{\langle 0| T\left((\phi(x)\phi(y)) \{\exp \int iH(t)dt\}\right) |0\rangle}{\langle 0| T\left(\exp(-i\int H(t)dt)\right)}$$

$= \sum'$ all connected diagrams with 2 external pts.

More generally...

$$\boxed{\begin{gathered}\frac{\langle 0| T\left\{\phi(x_1)\dots\phi(x_n) \exp\left(-i\int_{-\infty}^{\infty} \mathcal{H}_I(t)\, dt\right)\right\} |0\rangle}{\langle 0| T\left\{\exp\left(-i\int_{-\infty}^{\infty} \mathcal{H}_I(t)\, dt\right)\right\} |0\rangle} \\ = \sum \text{connected diagrams with end points } x_1 \dots x_n\end{gathered}}$$

Cross section & the S-matrix

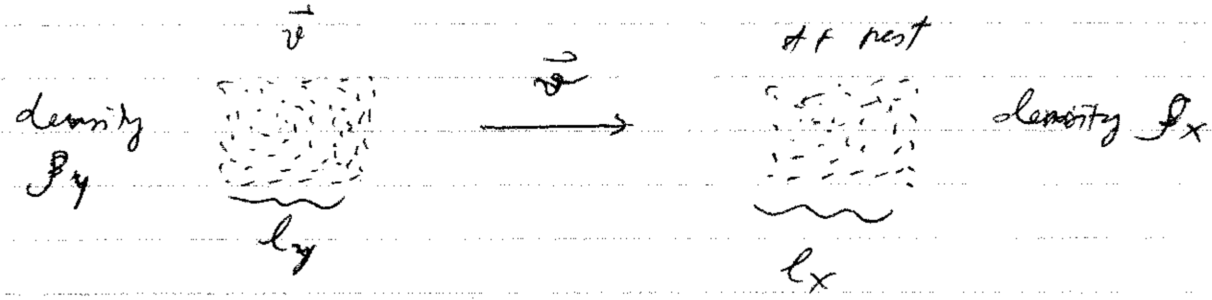
Now that we have a formula for computing the n-point correlation function...

→ Next task is to compute quantities that can be measured

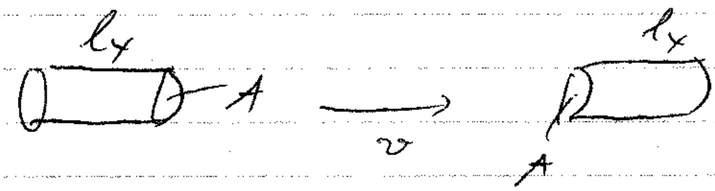
↳ cross section & decay rates.

The cross section:

Consider collision of 2 beams of particles with relatively well-defined momenta.



{ l_y, l_x are observed from rest.
{ ρ_y, ρ_x are densities at rest. Let A be the cross sectional area of overlap...



Total # of particles... $N_x = \rho_x l_x A$

$N_y = \rho_y l_y A$

→ Total # of scatterings is proportional to $N_x N_y$..

Let total number of scatterings be

$$N_X \cdot N_Y \cdot \left(\frac{\sigma}{A}\right)$$

probability one particular X particle & Y collide.

Call σ the effective area or "cross section" of the scattering process.

Let $N_X = 1$, then

$$\boxed{\sigma = \frac{\text{total \# scatterings}}{\rho_Y \cdot l_Y}}$$

For small time interval...

$$\sigma = \frac{\text{total \# scatterings} / \Delta T}{\rho_Y \cdot (l_Y / \Delta T)}$$

scattering rate

particle flux.

The differential cross section is the portion of σ in which the final particle momenta lie inside some window of momenta.

↳ write this as

$$\frac{d\sigma}{d^3\vec{p}_1 \cdots d^3\vec{p}_n}, \text{ so that}$$

final particle momenta.

$$\int \frac{d\sigma}{d^3\vec{p}_1 \cdots d^3\vec{p}_n} \, d^3\vec{p}_1 \cdots d^3\vec{p}_n = \int d\sigma = \sigma .$$

Now, if there are only 2 final particles then there are only two free parameters ...

<u>Why?</u> two spatial momenta → 6 params

4-momentum conservation → 4 ~~params~~ constraints

⇒ can take these two params to be orient. angles θ & ϕ

→ Then we can measure $\boxed{\frac{d\sigma}{d\Omega}(\theta, \phi)}$

where $d\Omega$ is the solid angle differential

$$\boxed{d\Omega = d\cos\theta \, d\phi}$$

"Differential cross section" refers to $\frac{d\sigma}{d\Omega}$

Let's look at an example ... Consider a periodic box with length L on all sides.

Spatial momentum modes are now discrete:

$$\vec{k} = \frac{2\pi}{L}(n_x, n_y, n_z) . \qquad n_i \in \mathbb{Z} .$$

there canon. relation:

$$\boxed{[a_{\vec k}, a^\dagger_{\vec k'}] = \delta_{\vec k, \vec k'} V} \longrightarrow V = L^3 \text{ (volume)}$$

as $V \to \infty$

$$\delta_{\vec k, \vec k'} \cdot V = \int_0^L \int_0^L \int_0^L dx_1\, dx_2\, dx_3\; e^{i(\vec k - \vec k') \cdot \vec x}$$

$$\to \int_0^\infty \int_0^\infty \int_0^\infty dx_1\, dx_2\, dx_3\; e^{i(\vec k - \vec k') \vec x}$$

$$= (2\pi)^3 \delta^{(3)}(\vec k - \vec k').$$

In this box...

$$\phi(x) = \sum_{\vec k} \frac{(2\pi/L)^3}{(2\pi)^3 \sqrt{2E_{\vec k}}} \left(a_{\vec k} e^{-ik\cdot x} + a^\dagger_{\vec k} e^{+ik\cdot x} \right)$$

$$= \sum_{k} \frac{1}{\sqrt{2E_k}} \left(a_{\vec k} e^{-ik\cdot x} + a^\dagger_{\vec k} e^{+ik\cdot x} \right)$$

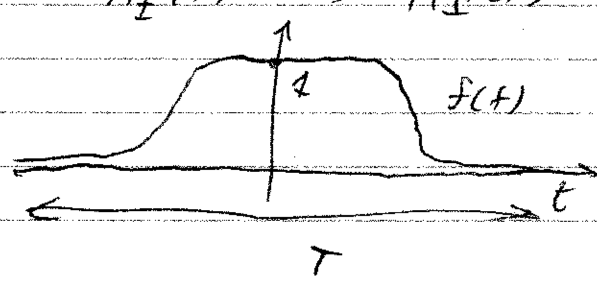
Oct 25, 2020

S-Matrix

Now imagine starting with free field theory...

at some early time then switching on the interaction slowly and then slowly switching off the interactions...

i.e. $H_I(t) \longrightarrow H_I(t) f(t)$ where $f(t)$ looks like



such that

Such that $\int_{-\infty}^{\infty} f(t)\, dt = T$, $\int_{-\infty}^{\infty} (f(t))^2 dt = T$

Let $S = T \exp\left\{ -i \int_{-\infty}^{\infty} dt \; H_I(t) f(t) \right\}$

> Define the S matrix as
>
> $$\langle \text{final} | S | \text{initial} \rangle$$
>
> where $|\text{initial}\rangle$ is a free particle state with momenta $\vec{k}_i^{\,I}$ & energies E_i^{I}
>
> and $|\text{final}\rangle$ is a free particle state with momenta $\vec{k}_f^{\,F}$ + energies $\vec{E}_i^{F}$

Now look at $S - I$:

$$\langle \text{final} | S - I | \text{initial} \rangle$$

- as $T \to \infty$, $V \to \infty$ we can make this amplitude as

$$\langle \text{final} | S - I | \text{initial} \rangle = i \cdot \mathcal{M} \cdot (2\pi)^4 \delta(E_{tot}^F - E_{tot}^I)\, \delta^{(3)}(\vec{k}_{tot}^{\,F} - \vec{k}_{tot}^{\,I})$$

fn of the momenta

- For finite T and finite V we instead have

$\langle \text{final} | S - I | \text{initial} \rangle$

$$= i \mathcal{M} \int_{-\infty}^{\infty} f(t)\, e^{i(E^F_{tot} - E^I_{tot})t} \cdot \delta_{\vec{k}^F_{tot}, \vec{k}^I_{tot}} \cdot V$$

So,

$$|\langle \text{final} | S - I | \text{initial} \rangle|^2 = |\mathcal{M}|^2 \cdot \delta_{\vec{k}^F_{tot}, \vec{k}^I_{tot}} \cdot V^2$$

$$\cdot \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t) f(t')\, e^{i(E^F_{tot} - E^I_{tot})(t'-t)}\, dt'\, dt}$$

as $T \to \infty$, this is some constant times $\delta(E^F_{tot} - E^I_{tot})$

What is this constant?
To get this → integrate w.r.t E^F_{tot} ..

$$\to \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{\left\{ \int_{-\infty}^{\infty} dE^F_{tot}\, e^{i(E^F_{tot} - E^I_{tot})(t'-t)} \right\}}_{2\pi\delta(t-t')} f(t) f(t')\, dt\, dt'$$

$$= 2\pi \int_{-\infty}^{\infty} f^2(t)\, dt = 2\pi \cdot T.$$

So the constant is $2\pi \cdot T$ and so

$$|\langle \text{final} | S - I | \text{initial} \rangle|^2 = |\mathcal{M}|^2 (2\pi)\, \delta(E^F_{tot} - E^I_{tot})\, \delta_{\vec{k}^F_{tot}, \vec{k}^I_{tot}}\, V^2 T$$

Now, we have been using relativistic normalizations

$$\begin{cases} \langle \text{initial} \,|\, \text{initial} \rangle = \prod_i \left(2E_i^I \cdot V\right) & \text{becomes } (2\pi)^3 \delta^{(3)}(0) \text{ as } V \to \infty \\ \langle \text{final} \,|\, \text{initial} \rangle = \prod_i \left(2E_i^F \cdot V\right) \end{cases}$$

To get transition probability per unit time...

$$\frac{\text{probability}}{\text{time}} = \frac{1}{T} \frac{\left|\langle \text{final} \,|\, S - I \,|\, \text{initial} \rangle\right|^2}{\langle \text{final} \,|\, \text{final} \rangle \langle \text{initial} \,|\, \text{initial} \rangle}$$

$$= \frac{|M|^2 (2\pi)\, \delta\left(E^F_{tot} - E^I_{tot}\right) \delta_{\vec{k}^F_{tot}, \vec{k}^F_{tot}} \cdot V^2}{\prod_i \left(2E_i^F \cdot V\right) \prod_i \left(2E_i^I \cdot V\right)}$$

As $V \to \infty$, $\delta_{\vec{k}^F_{tot}, \vec{k}^I_{tot}} \cdot V \Rightarrow (2\pi)^3 \delta^{(3)}\left(\vec{k}^F_{tot} - \vec{k}^I_{tot}\right)$

If we sum over final states in some window then we have

$$\underbrace{\sum_{\vec{k}_1^F \ldots \vec{k}^F_{n_F}} \frac{1}{\left(2E_1^F \cdot V\right)} \cdots \frac{1}{\left(2E^F_{n_F} \cdot V\right)}} \frac{|M|^2 (2\pi)^4 \delta\left(k^F_{tot} - k^F_{tot}\right) \cdot V}{\left(2E_1^I \cdot V\right) \cdots \left(2E^I_{n_I} \cdot V\right)}$$

As $V \to \infty$, $\dfrac{d^3 \vec{k}_1^F}{(2\pi)^3\, 2E_1^F} \cdots \dfrac{d^3 k^F_{n_F}}{(2\pi)^3\, 2E^F_{n_F}}$ $\quad \begin{cases} n_I = \#\text{ initial particles} \\ n_F = \#\text{ final particles} \end{cases}$

Consider single particle decay ... ($n_I = 1$)

The total decay rate is $\Gamma = \int d\Gamma$ where

$$d\Gamma = \frac{1}{2E^I}\left(\prod_{i=1}^{n_F} \frac{d^3\vec{k}_i^F}{(2\pi)^3 2E_i^F}\right)|\mathcal{M}|^2 (2\pi)^4 \delta^{(4)}(k_{tot}^F - k_{tot}^I)$$

For a 2-particle initial state, the cross section is given by

$$\sigma = \frac{\text{probability}}{\text{time} \cdot \text{flux density}}$$

flux density = relative velocity between beam and target × density of incoming beam in lab frame.

We have normalized probability for one incoming beam particle → density = $1/V$, and flux ...

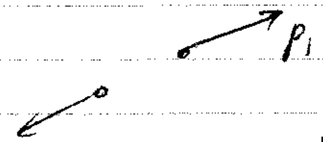
$$\text{flux} = \frac{|\vec{v}_A - \vec{v}_B|}{V}$$

$\vec{v}_A, \vec{v}_B$ = velocity of particles in lab frame.

So

$$d\sigma = \underbrace{\left(\prod_{i=1}^{n_F} \frac{d^3\vec{k}_i^F}{(2\pi)^3 2E_i^F}\right)(2\pi)^4\delta^{(4)}(k_{tot}^F - k_{tot}^I)}_{\text{call this } d\Pi_{n_F}} \frac{|\mathcal{M}|^2}{2E_A E_B|\vec{v}_A - \vec{v}_B|}$$

Now consider special case: 2 final particles ($n_p = 2$) in com frame



then $n_p = 2$ $\qquad p_1 = |\vec{p}_1|$

$$\to \int d\Pi_2 = \int \frac{dp_1\, p_1^2\, d\Omega}{(2\pi)^3\, 2E_1\, 2E_2} (2\pi)\, \delta(E_{com} - E_1 - E_2)$$

where final particle energies are $\begin{cases} E_1 = \sqrt{|\vec{p}_1|^2 + m_1^2} \\ E_2 = \sqrt{|\vec{p}_2|^2 + m_2^2} \end{cases}$

So

$$= \cancel{\int d\Pi_2 = \int d\Omega \frac{p_1^2}{16\pi^2 E_1 E_2}}$$

$$\to = \int d\Pi_2 = \int \frac{d\Omega}{16\pi^2} \int_0^\infty \frac{p_1^2\, \delta\left(-E_{com} + \sqrt{|\vec{p}_1|^2 + m_1^2} + \sqrt{|\vec{p}_2|^2 + m_2^2}\right)}{\sqrt{|\vec{p}_1|^2 + m_1^2} \cdot \sqrt{|\vec{p}_2|^2 + m_2^2}}\, dp_1$$

Recall that $\delta(f(x)) = \dfrac{\delta(x - x_0)}{|f'(x_0)|}$

$$\frac{dE_1}{dp_1} = \frac{d\sqrt{|\vec{p}_1|^2 + m_1^2}}{dp_1} = \frac{1}{2} \frac{2p_1}{\sqrt{|\vec{p}_1|^2 + m_1^2}} = \frac{p_1}{E_1}$$

Likewise $\dfrac{dE_2}{dp_2} = \ldots = \dfrac{p_2}{E_2}$.

So $$\int d\Pi_2 = \int \frac{d\Omega}{16\pi^2} \frac{p_1^2}{E_1 E_2 \left(\frac{p_1}{E_1} + \frac{p_2}{E_2}\right)} \Bigg|_{p_1 \text{ s.t. } E_{com} = E_1 + E_2}$$

$$= \int \frac{d\Omega}{16\pi^2} \frac{p_1}{E_1 + E_2} = \int \frac{d\Omega}{16\pi^2} \frac{p_1}{E_{com}}$$

So for 2 particles $\longrightarrow$ 2 particles ...

$$\left(\frac{d\sigma}{d\Omega}\right)_{com} = \frac{|\vec{p}^{\,final}| \cdot |\mathcal{M}|^2}{2E_A E_B |\vec{v}_A - \vec{v}_B| \cdot 16\pi^2 E_{com}} \qquad (E_{com} = E_A + E_B)$$

Now, it is conventional to define the T-matrix as

$$\boxed{S = 1 + iT}$$

Claim

$$\langle \vec{p}_1^{\,F}, \ldots, \vec{p}_{n_F}^{\,F} | iT | \vec{p}_A, \vec{p}_B \rangle$$

$$= \left({}_{final}\langle \vec{p}_1^{\,F}, \ldots \vec{p}_{n_F}^{\,F} | T \exp\left[i \int_{-\infty}^{\infty} dt\, H_I(t) \right] | \vec{p}_A, \vec{p}_B \rangle_{free} \right)_{\star}$$

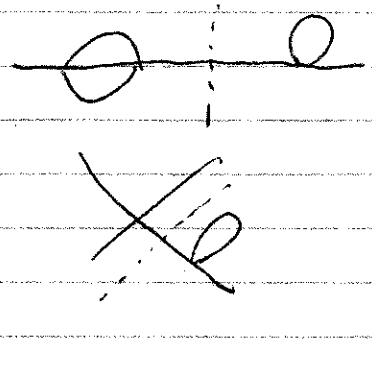
where $\star$ = connected diagrams only + "amputated" diagrams only

"amputated" $\equiv$ diagrams can't be broken into disconnected pieces by cutting one internal line

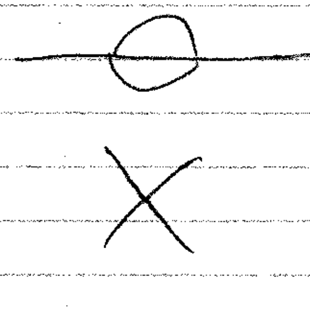
(i.e. 1 particle irreducible)

Ex

(not amputated)



(amputated)



The claim can't be proven now, but the idea is similar as before...

$$|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_0T}\langle\Omega|0\rangle\right)^{-1} e^{-iHT}|0\rangle$$

and we would like something similar...

$$|\vec{p}_1 \cdots \vec{p}_{n_F}\rangle \propto \lim_{T\to\infty(1-i\epsilon)} e^{-iHT}|\vec{p}_1,\ldots,\vec{p}_n\rangle_{free}$$

For now, we'll just take this claim as true...

→ Note that

$$\phi_I^+(x)|\vec{p}\rangle_{free} = \int \frac{d^3k}{(2\pi)^3\sqrt{2E_k}}\, a_{\vec{k}}\, e^{-ik\cdot x} \overbrace{\sqrt{2E_p}}^{\text{relativistic normalization}} a^\dagger_{\vec{p}}|0\rangle$$

$$= e^{-ip\cdot x}|0\rangle$$

We can think of taking the commutator of $\phi_I^+(x)$ with the $a_{\vec{p}}^\dagger$ from $|p\rangle_{free}$

→ suggests the notation

$$\phi_I(x)\,|\vec{p}\rangle_{free} = e^{-ip\cdot x}$$

Now drop the "free" subscript & finally...

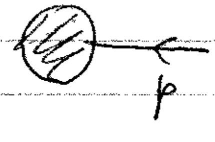
$$\langle\vec{p}|\Phi_I(x) = e^{+ip\cdot x}\langle 0| \quad \text{\& define}$$

$$\langle\vec{p}|\Phi(x) = e^{ip\cdot x}$$

→ Set Feynman Rules in position space with external lines

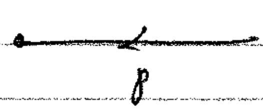
Propagator:  $D_F(x-y)$

Internal vertex:  $(-i\lambda)\int d^4z$

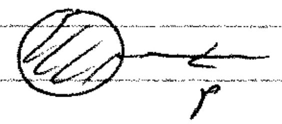
Each external line  $e^{-ip\cdot x}$

Divide by symmetry factor S.

Feynman Rules in momentum space with external line

Propagator:  $\dfrac{i}{p^2-m^2+i\epsilon}$

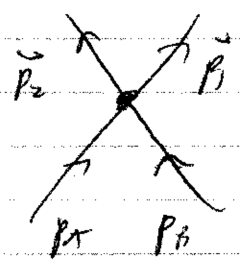
Internal vertex:  $-i\lambda$ & momentum conservation

External line  no extra factor (just 1)

Integrate over all unconstrained momenta & divide by S.

Ex $\langle \vec{p}_1, \vec{p}_2 | iT | \vec{p}_A, \vec{p}_B \rangle$ at lowest order...

well...



Feynman amplitude:

$i\mathcal{M} = -i\lambda$

So
$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{|\vec{p}^{\,final}|\,|\mathcal{M}|^2}{2E_A E_B\, |\vec{v}_A - \vec{v}_B|\, 16\pi^2 E_{cm}}$$

Let $p = |\vec{p}^{\,final}| = |\vec{p}_A| = |\vec{p}_B|$ all same since masses are all same

$$E_{cm} = 2E_A = 2\sqrt{p^2+m^2}$$
$$|\vec{v}_A - \vec{v}_B| = 2|\vec{v}_A| = \frac{2|\vec{p}_A|}{E_A} = \frac{2p}{E_A}.$$

So
$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{\lambda^2 p}{\frac{2p}{E_A}(2E_A)(2E_B)16\pi^2 E_{cm}} = \frac{\lambda^2}{64\pi^2 E_{cm}^2}$$

This is spherically symmetric, so

$$\sigma_{tot} = (4\pi)\frac{\lambda^2}{64\pi^2 E_{cm}^2}\cdot\frac{1}{2}$$ → particles in final state are identical so need a 1/2 factor...

→ $$\boxed{\sigma_{tot} = \frac{\lambda^2}{32\pi^2 E_{cm}^2}}$$ → our first QFT cross section!

Oct 26
2020

$\boxed{\text{Feynman Rules for Fermions}}$

So far we've discussed only the ϕ^4 theory...

→ need to generalize results to theories containing fermions.

→ need to generalize defn. of time-ordering & normal ordering symbols to include fermions...

Recall...

$$T\{\psi_a(x)\overline{\psi_b(y)}\} = \begin{cases} \psi_a(x)\overline{\psi_b(y)} & x^0 > y^0 \\ -\overline{\psi_b(y)}\psi_a(x) & x^0 < y^0 \end{cases}$$

The Feynman propagator, under this defn is

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon} e^{-ip\cdot(x-y)}$$

$$= \langle 0| T\{\psi_a(x)\overline{\psi_b(y)}\} |0\rangle$$

(Recall that $\not{p} = \gamma^\mu p_\mu$

Generalize of T to more than two fermion fields...

$$T\{\psi_1\psi_2\psi_3\psi_4\} = \begin{cases} \psi_1\psi_2\psi_3\psi_4 & \text{if } x_1^0 > x_2^0 > x_3^0 > x_4^0 \\ -\psi_2\psi_1\psi_3\psi_4 & \text{if } x_2^0 > x_1^0 > x_3^0 > x_4^0 \\ -\psi_3\psi_2\psi_1\psi_4 & \text{if } x_3^0 > x_2^0 > x_1^0 > x_4^0 \\ \vdots \end{cases}$$

Rule: $\times(-1)$ if odd # permutations.

$\times(+1)$ if even # permutations.

Similarly, for normal ordering symbol...

$$N\{a_{\vec{p}_1} a_{\vec{p}_2} a_{\vec{p}_3} a^{\dagger}_{\vec{p}_4}\} = (-1)^3 a^{\dagger}_{\vec{p}_4} a_{\vec{p}_1} a_{\vec{p}_2} a_{\vec{p}_3}$$

$\times(-1)$ if odd permutation of fields

$\times(+1)$ if even

With these, can generalize to get Wick's theorem...

1st case: 2 Dirac fields $T\{\psi_a(x)\overline{\psi}_b(y)\}$

$$T\{\psi_a(x)\overline{\psi}_b(y)\} = N\{\psi_a(x)\overline{\psi}_b(y)\} + \contraction{}{\psi}{{}_a(x)}{\overline{\psi}}\psi_a(x)\overline{\psi}_b(y)$$

where

$$\contraction{}{\psi}{{}_a(x)}{\overline{\psi}}\psi_a(x)\overline{\psi}_b(y) \equiv \begin{cases} \{\psi_a^+(x), \overline{\psi}_b^-(y)\} & x^0 > y^0 \\ -\{\overline{\psi}_b^-(y), \psi_a^+(x)\} & x^0 < y^0 \end{cases}$$

$$= \langle 0| T\{\psi_a(x)\overline{\psi}_b(y)\}|0\rangle$$

$$= S_F(x-y)$$

$$= -\contraction{}{\overline{\psi}}{{}_b(y)}{\psi}\overline{\psi}_b(y)\psi_a(x)$$

where recall that

$\psi^+, \bar{\psi}^+$ are the positive frequency part of $\psi, \bar{\psi}$

→ i.e. part with annihilation operators.

$\psi^-, \bar{\psi}^-$ ... " negative " creation operators.

Also note that:

$$\boxed{\overbrace{\psi_a(x)\psi_b(y)} = \overbrace{\bar{\psi}_a(x)\bar{\psi}_b(y)} = 0}$$

→ Just as we proved Wick's thm for bosons, we can show the same for fermions

$$T\{\psi_1\bar{\psi}_2\psi_3 \cdots\} = N[\psi_1\bar{\psi}_2\psi_3 \cdots + \text{all possible contractions}]$$

where we note that an expression such as

$$N[\psi_1\psi_2\bar{\psi}_3\bar{\psi}_4] = -\psi_1\bar{\psi}_3 N[\psi_2\bar{\psi}_4]$$

gets a minus sign since the $\bar{\psi}_3$ must leap over the ψ_2

Helpful hint: for any fully contracted quantity, count the number of times the contraction lines criss-cross tells you if the # of perm. is odd/even...

Ex $\psi_1\psi_2\psi_3\bar{\psi}_4\bar{\psi}_5\bar{\psi}_6$ → even

$\psi_1\psi_2\psi_3\bar{\psi}_4\bar{\psi}_5\bar{\psi}_6$ → odd.

$\boxed{\text{Yukawa theory}}$

↳ Next we consider the simplest theory with fermions.

$$\boxed{H_{yukawa} = H_{Dirac} + H_{Klein-Gordon} + \int d^3x\, g\bar{\Psi}\Psi\phi}$$

↑ simplified model of QED. We will carefully work out the rules of calculations for Yukawa theory before going to QED.

We will consider two-particle scattering reaction:

fermion (p) + fermion (k) ⟶ fermion (p') + fermion (k')

The leading contribution comes from the H_I^2 term of the S-matrix:

$$_0\langle p', k'|\, T\left\{ \frac{1}{2!}(-ig)\int d^4x\, \bar{\Psi}_I \Psi_I \phi_I\, (-ig)\int d^4y\, \bar{\Psi}_I \Psi_I \phi_I \right\} |p, k\rangle_0$$

Now use Wick's thm to reduce this to N-product of contractions → then act on uncontracted fields

Represent this as the contraction:

$$\Psi_I(x)|\vec{p}, s\rangle = \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}'}}} \sum_{s'} a^{s'}_{\vec{p}'} u^{s'}(p') e^{-ip'\cdot x} \underbrace{\sqrt{2E_{\vec{p}}}\, a^{s\dagger}_{\vec{p}}|0\rangle}_{|p,s\rangle}$$

$$= e^{-ip\cdot x} u^s(p)|0\rangle$$

↑ fermion state with momentum $\vec{p}$, spin s

Define:

So $\boxed{\psi_I(x)\,|\vec{p}, s\rangle = e^{-ip\cdot x}\, u^s(p)}$

Similarly,

$$\begin{cases} \bar{\psi}_I(x)\,|\vec{k}, s\rangle = e^{-ik\cdot x}\,\bar{v}^s(k) \\ \langle \vec{p}, s|\,\bar{\psi}_I(x) = e^{+ip\cdot x}\,\bar{u}^s(p) \\ \langle \vec{k}, s|\,\psi_I(x) = e^{+ik\cdot x}\,v^s(k) \end{cases}$$

So, typically, a contribution to the matrix element is the contraction ...

$$\langle \vec{p}', \vec{k}'|\,\frac{1}{2!}(-ig)\int d^4x\,\bar{\psi}\psi\phi\,(-ig)\int d^4y\,\bar{\psi}\psi\phi\,|\vec{p}, \vec{k}\rangle$$

Up to a (−) sign, the value of this quantity is

$$J = (-ig)^2 \int \frac{d^4q}{(2\pi)^4}\,\frac{i}{q^2 - m_\phi^2}\,(2\pi)^4\,\delta^{(4)}(p' - p + q)$$
$$\times\,(2\pi)^4\,\delta^{(4)}(k' - k - q)\,\bar{u}(p')\,u(p)\,\bar{u}(k')\,u(k)$$

$\phi - \phi$

$\vec{p}' \leftrightarrow \bar{\psi}$ $\quad \psi - p \quad$ $\vec{k}' - \bar{\psi}$ $\quad \psi - k$

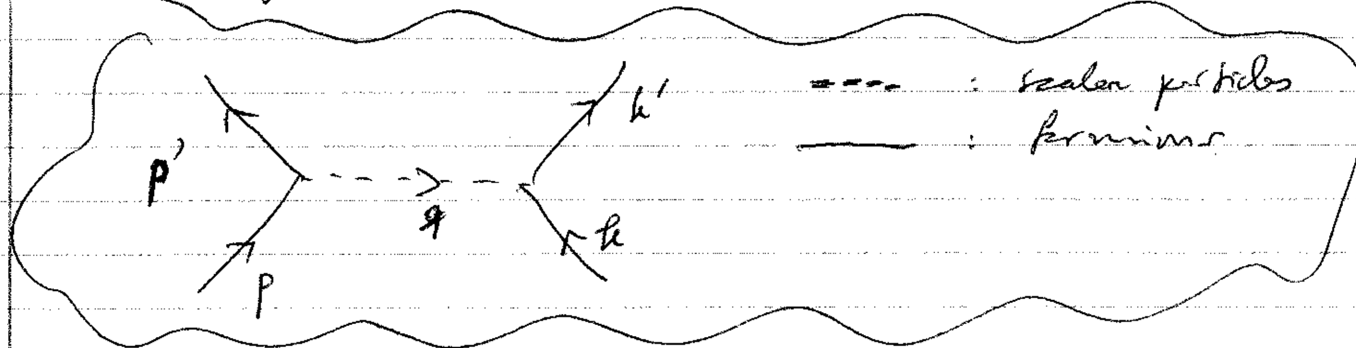
int. over all intermediate momenta.

where $q = p - p' = k' - k$.

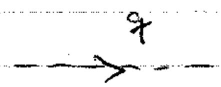
→ Upon using the δ functions, $J = iM(2\pi)^4\,\delta^{(4)}(\Sigma p)$

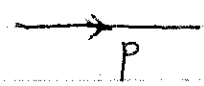
where $$iM = \frac{-ig^2}{q^2 - m_\phi^2}\,\bar{u}(p')\,u(p)\,\bar{u}(k')\,u(k)$$

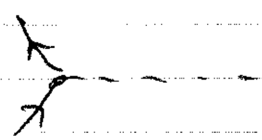
The Feynman diagram for this is ...

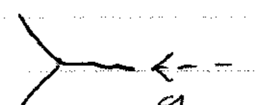


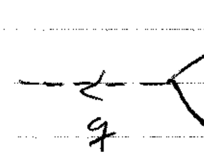
Feynman rules for fermions in momentum space

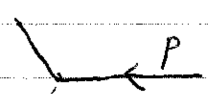
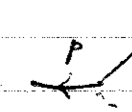
① Propagators: $\phi(x)\phi(y)$ =  $= \dfrac{i}{q^2 - m_\phi^2 + i\varepsilon}$

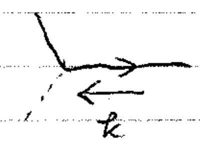
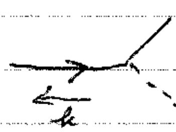
$\psi(x)\bar{\psi}(y)$ =  $= \dfrac{i(\not{p}+m)}{p^2 - m^2 + i\varepsilon}$

② Vertices  $= (-ig)$

③ External leg contractions: ⓐ $\phi|q\rangle$ =  $= 1$

ⓑ $\langle q|\phi$ =  $= 1$

ⓒ $\psi|\vec{p},s\rangle$ =  $= u^s(p)$ ⓓ $\langle\vec{p},s|\bar{\psi}$ =  $= \bar{u}^s(p)$
(fermions)

ⓔ $\bar{\psi}|k,s\rangle$ =  $= \bar{v}^s(k)$ ⓕ $\langle k,s|\psi$ = 
(anti fermion)

④ Mom. consv @ each vertex

⑤ Integrate over intermediate mom.

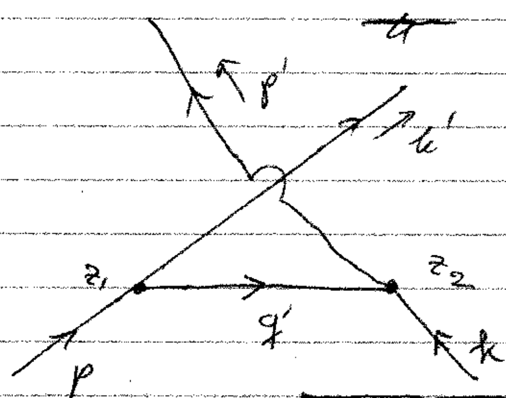
⑥ Figure out sign of diagram

Note $\left\{ \begin{array}{l} \text{initial states have momentum pointing } \underline{\text{in}} \\ \text{final} \underline{\qquad\qquad\qquad\qquad\qquad} \underline{\text{out}} \end{array} \right.$

external particle / antiparticle $\Longrightarrow$ 
same direction / opposite direction

Back to example: there are 2 criss-cross, so $(+1)$.

$\boxed{Ex}$



$$\underline{M} = \int d^4 z_1\, d^4 z_2 \, \langle k', p' |\, \bar{\Psi}_1 \Psi_1 \phi_1 \, \bar{\Psi}_2 \Psi_2 \phi_2 \,| \vec{p}, \vec{k} \rangle$$

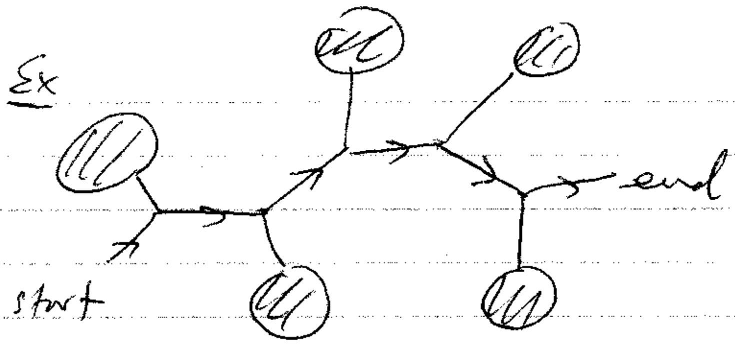
criss-cross $= 3 \Rightarrow (-1)$

$$\rightarrow \underline{M} = \underbrace{(-ig)^2}_{\text{2 vertices}} \underbrace{(-1)}_{3\text{ c-c}} \cdot \underbrace{\frac{i}{q'^2 - m_\phi^2 + i\varepsilon}}_{\phi-\phi} \big(\underbrace{\bar{u}(k')}_{\langle k'|\bar{\Psi}} \underbrace{u(p)}_{\Psi|p\rangle}\big)\big(\underbrace{\bar{u}(p')}_{\langle p'|\bar{\Psi}} \underbrace{u(k)}_{\Psi|k\rangle}\big)$$

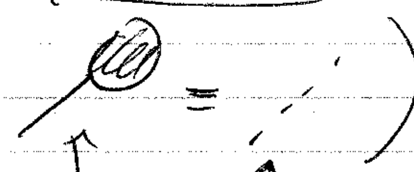
where $\underline{q' = p - k'}$.

{{{ Tips for each fermion line that doesn't close into loop, follow the particle number arrow to the end

Ex

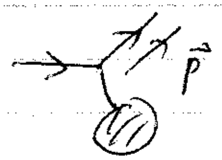


(note that



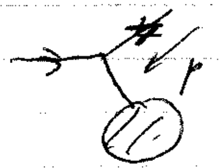
)

If the end is an outgoing fermion write down as



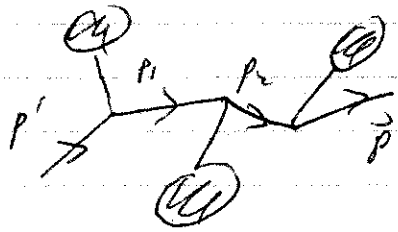
$\bar{u}(p)$

If the end is an incoming antifermion ---



$\bar{v}(p)$

Write down fermion propagators you encounter as you follow the particle number arrow backwards

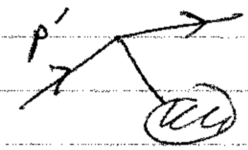


$$\bar{u}(p)\,\frac{i(\not{p}_2+m)}{p_2^2-m^2+i\epsilon}\cdot\frac{i(\not{p}_3+m)}{p_3^2-m^2+i\epsilon}\cdots$$

Note if $\underset{\longrightarrow\,\vec{p}}{\longrightarrow}$, $\dfrac{i(\not{p}+m)}{p^2-m^2+i\epsilon}$

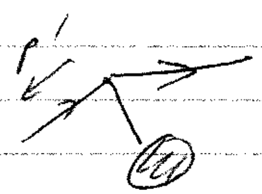
if $\underset{\longleftarrow\,\vec{p}}{\longrightarrow}$, $\dfrac{i(-\not{p}+m)}{p^2-m^2+i\epsilon}$

If the start is an incoming fermion...



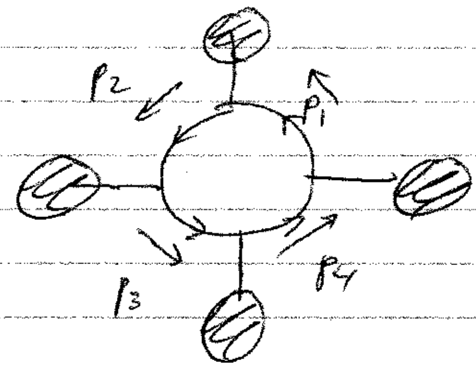
$u(p')$

If ... incoming antifermion



$v(p')$

If the fermion lines form a closed loop



$$= \bar\psi\psi\,\bar\psi\psi\,\bar\psi\psi\,\bar\psi\psi$$

$$= (-1)\,\mathrm{tr}\left\{\psi\bar\psi\,\psi\bar\psi\,\psi\bar\psi\,\psi\bar\psi\right\}$$

$$= (-1)\,\mathrm{tr}\left\{S_F\,S_F\,S_F\,S_F\right\}$$

→ a closed loop always gives a factor of (-1) & the <u>trace</u> of a product of Dirac matrices.

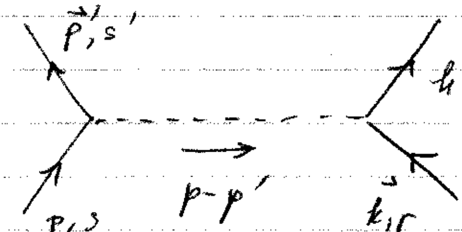
↳ "trace" because we sum over the spinor indices.

The Yukawa Potential

Consider non-relativistic scattering of 2 different fermions → interact via exchange of a scalar particle.

Ignore $\mathcal{O}(\vec{p}^{\,2}/m^2)$ corrections, moments are

$$\begin{cases} p = (m, \vec{p}) \quad , \quad k = (m, \vec{k}) \\ p' = (m, \vec{p}\,') \quad , \quad k' = (m, \vec{k}') \end{cases}$$



$= (m-m)^2 - (\vec{p}\,' - \vec{p})^2 + \mathcal{O}(\cdots)$

$$(p'-p)^2 = -|\vec{p}\,'-\vec{p}|^2 + \mathcal{O}(\vec{p}^{\,4})$$

$$u^s(p) = \sqrt{m}\begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix} \quad , \text{ etc.}$$

where $\xi^{s'\dagger}\xi^s = \delta_{ss'}$

<u>Spinor products</u>:
$$\begin{cases} \bar{u}^{s'}(p')u^s(p) = 2m\,\xi^{s'\dagger}\xi^s = 2m\,\delta^{ss'} \\ \bar{u}^{r'}(k')u^r(k) = 2m\,\xi^{r'\dagger}\xi^r = 2m\,\delta^{rr'} \end{cases}$$

→ the spin of each particle is conserved.

⇒ amplitude: $$i\mathcal{M} = \frac{ig^2}{(\vec{p}-\vec{p}\,')^2 + m_\phi^2}\left(\bar{u}^{s'}(p')u^s(p)\right)\left(\bar{u}^{r'}(k')u^r(k)\right)$$

$$\boxed{i\mathcal{M} = \frac{ig^2}{|\vec{p}\,'-\vec{p}|^2 + m_\phi^2}\, 2m\,\delta^{rr'}\, 2m\,\delta^{ss'}}$$

Okay so $\quad i\mathcal{M} = \dfrac{ig^2}{(\vec{p}-\vec{p}\,')^2 + m_\phi^2} \; 2m\,\delta^{ss'}\,2m\,\delta^{rr'}$

Compared with the Born approximation to the scattering amplitude in nonrelativistic ~~scattering~~ ~~[illegible]~~ ~~[illegible]~~ QM, in terms of the potential $V(x)$:

$$\langle p'|iT|p\rangle = -i\,\tilde{V}(\vec{q})\,(2\pi)\,\delta(E_{\vec{p}'} - E_{\vec{p}})$$

where $\vec{q} = \vec{p}\,' - \vec{p}$.

where does this come from?

(cf. Griffiths chapter 11 on the Born approximation)

check out Born's approximation to scattering amplitude in nonrelativistic quantum mechanics

Nov 1, 2020

So we found that

Amplitude $= i\mathcal{M} = \dfrac{ig^2}{|\vec{p}\,'-\vec{p}|^2 + m_\phi^2} \; 2m\,\delta^{rr'}\delta^{ss'}$

$\rightsquigarrow$ FT$\{V(r)\}$

$$= \langle p'|iT|p\rangle = -i\,\tilde{V}(\vec{q})\,(2\pi)\,\delta(E_{\vec{p}'} - E_{\vec{p}})$$

$(\vec{q} = \vec{p}\,' - \vec{p})$

$\to$ for Yukawa interaction.

$$\boxed{\tilde{V}(\vec{q}) = \frac{-g^2}{|\vec{q}|^2 + m_\phi^2}}$$

With $\tilde{V}(q)$, can take inv FT to get V...

$$V(\vec{x}) = \int \frac{d^3q}{(2\pi)^3} \frac{-g^2}{|q|^2 + m_\phi^2} e^{i\vec{q}\cdot\vec{x}}$$

$$= \frac{-g^2}{(4\pi^2)} \int_0^\infty \frac{dq}{1} q^2 \frac{e^{iqr} - e^{-iqr}}{iqr} \frac{1}{q^2 + m_\phi^2}$$

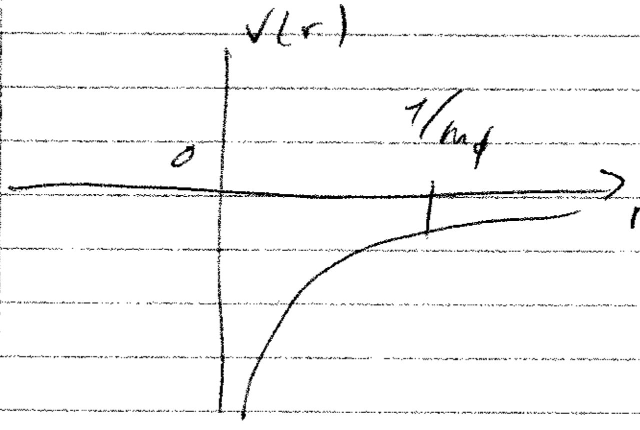
$$= \frac{-g^2}{4\pi^2 ir} \underbrace{\int_{-\infty}^{\infty} dq \frac{q e^{iqr}}{q^2 + m_\phi^2}}_{\text{evaluate w/ CIF}}$$

$$\Rightarrow \boxed{V(r) = \frac{-g^2}{4\pi} \frac{e^{-m_\phi r}}{r}} \rightsquigarrow \text{attractive Yukawa potential}$$

range $1/m_\phi = \frac{\hbar}{m_\phi c}$, the Compton wavelength of the exchanged boson. (Yukawa predicted the mass of the pion from this)

short-range

~~What CIF~~



Review of QUANTUM ELECTRODYNAMICS

To get to QED, we go from Yukawa theory by replace the scalar ~~potential~~ particle ϕ with a vector particle A_μ

& replace Yukawa Hamiltonian with

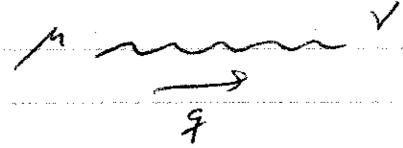
$$H_{int} = \int d^3x \; e\bar{\psi}\gamma^\mu\psi A_\mu .$$

→ Vector particle has spin 1

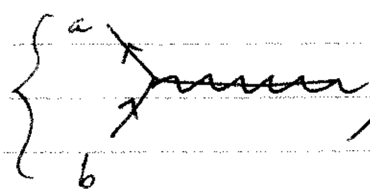
Recall that fermions come with spinor indices $u^s(p)$

→ **Photon** has a polarization vector $\varepsilon^\mu(p)$

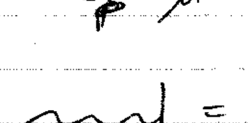
Feynman Rules (momentum space) → (derive later)

Photon propagator: 

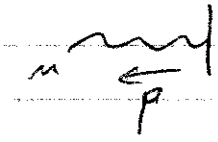
$\dfrac{-ig_{\mu\nu}}{q^2+i\varepsilon}$

New vertex 

$= -i\gamma^\mu_{ab}$

External photon lines $\quad A_\mu|\vec{p},\varepsilon\rangle$ = 

$= \varepsilon_\mu(p)$

$\langle\vec{p},\varepsilon|A_\mu$ = 

$= \varepsilon^*_\mu(p)$

To justify these rules... recall that in Lorentz gauge we have

$$\partial_\mu A^\mu = 0 \text{ , in which case}$$

$$\partial_\mu F^{\mu\nu} = 0 \Rightarrow \boxed{\partial_\mu \partial^\mu A^\nu = 0.}$$

So, each A^ν ($\nu = 0, 1, 2, 3$) satisfies the KG eqn with zero mass...

→ In $\vec{p}$ space... solutions are plane waves...

$$\left\{ \varepsilon_\mu(p)\, e^{-ip\cdot x} \right. \text{, where } p^2 = 0 \text{ (massless)}, \quad \varepsilon_\mu(p) = \text{4-vector}$$

In Fourier components...

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{r=0}^{3} \left(a^r_{\vec{p}}\, \varepsilon^r_\mu(p)\, e^{-ip\cdot x} + a^{r\dagger}_{\vec{p}}\, \varepsilon^{r*}_\mu(p)\, e^{ip\cdot x} \right)$$

$$\text{where } p^0 = E_p = \sqrt{|\vec{p}|^2}$$

where $r = 0, 1, 2, 3$ → basis for polarization vectors.

<u>But</u> notice that photons in the real world only have 2 polarizations...

→ More on this later...

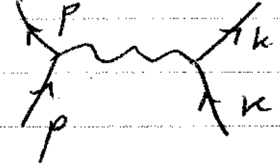
//

The Coulomb potential

Consider nonrelativistic scattering calculation of the previous section...

→ Look at scattering of 2 fermions of the same mass...

→ Leading-order contribution is... 2 new vertices

$$i\mathcal{M} = $$  $$= (-ie)^2\,\bar{u}(p')\gamma^\mu u(p)\,\frac{-ig_{\mu\nu}}{(p'-p)^2} \times \bar{u}(k')\gamma^\nu u(k)$$

photon line

In nonrelativistic limit...

$$\bar{u}^{s'}(p')\gamma^i u^s(p) = \left[\xi^{s'\dagger}\ \ \xi^{s'\dagger}\right]\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix}\begin{pmatrix}\xi^s\\ \xi^s\end{pmatrix}\times m$$

$$= 0 \quad \forall\, i = 1,2,3$$

and

$$\bar{u}^{s'}(p')\gamma^0 u^s(p) = \left(\xi^{s'\dagger}\ \ \xi^{s'\dagger}\right)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\xi^s\\ \xi^s\end{pmatrix}\times m$$

$$= 2m\,\delta^{ss'}$$

So... almost like the Yukawa case...

$$\boxed{i\mathcal{M} \simeq \frac{ie^2}{-(\vec{p}-\vec{p}\,')^2}\,(2m)^2\, \underset{\substack{\downarrow\\ 1}}{g_{00}}\,\delta^{ss'}\delta^{rr'}}$$

Same as Yukawa except the (−) sign & $m_\phi = 0$.

→ this potential is repulsive! → the potential is f.c. the Coulomb potential...

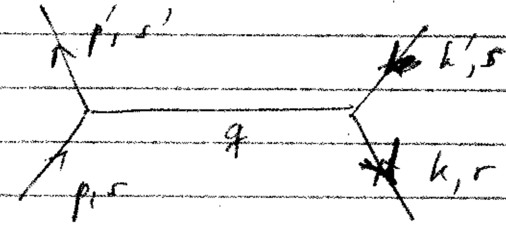
$$V(r) = \frac{e^2}{4\pi r} = \frac{\alpha}{r} \qquad \alpha = \frac{e^2}{4\pi} \sim \frac{1}{137}$$

(fine structure constant)

What about fermion-anti-fermion scattering?

for Yukawa & QED?

Yukawa:



$$|\vec{p}, s; \vec{k}, r\rangle = \sqrt{2E_{\vec{p}}}\sqrt{2E_{\vec{k}}}\; a^{s\dagger}_{\vec{p}}\, b^{r\dagger}_{\vec{k}}\, |0\rangle$$

$$\langle \vec{p}', s'; \vec{k}', r'| = \langle 0|\sqrt{2E_{\vec{p}'}}\sqrt{2E_{\vec{k}'}}\; b^{r'}_{\vec{k}'}\, a^{s'}_{\vec{p}'}$$

and so...

$$\langle \vec{k}', r'; \vec{p}', s'|\, \bar{\psi}\psi\phi\, \bar{\psi}\psi\phi\, |\vec{p}, s, \vec{k}, r\rangle \qquad \text{(3 criss-cross)}$$

For fermion...

$$\bar{u}^{s'}(p')\, u^{s}(p) = 2m\,\delta^{ss'}$$

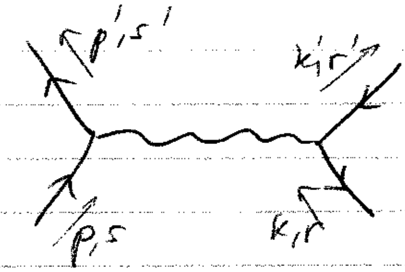
anti fermion

$$\bar{v}^{r}(k)\, v^{r'}(k') = \left(\xi^{r\dagger} \;\; -\xi^{r\dagger}\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi^{r'} \\ \xi^{r'} \end{pmatrix} \times m$$

$$= -2m\,\delta^{rr'}$$

So $V_{f\bar{f}}(r) = (-1)(-1)\, V_{pp}(r) = V_{pp}(r)$ (attractive)

$\boxed{\text{QED}}$



$\to$ has an overall (−1) sign

For fermion ... set $\bar{u}^{s'}(p')\gamma^\mu u^s(p) \to \bar{u}^{s'}(p')\gamma^0 u^s(p)$

$$\simeq 2m\,\delta^{s's}$$

For antifermion ...

$$\bar{v}^r(k)\gamma^\mu v^{r'}(k') \longrightarrow \bar{v}^r(k)\gamma^0 v^{r'}(k') = 2m\,\delta^{r'r}$$

& $\quad V_{f\bar{f}}(r) = (-1)V_{ff}(r) = -V_{ff}(r)$ (attractive)

Exchange particle	ff or $\bar{f}\bar{f}$	$f\bar{f}$
scalar (Yukawa)	attractive	attractive
vector (QED)	repulsive	attractive
tensor (gravity)	attractive	attractive

Some problems ...

$\boxed{\text{Problem}}$ (normal ordering & applications in QM & coherent state...)

→ look @ QSHO: $\mathcal{H} = \frac{1}{2}(p^2 + q^2)$

where $[p, q] = -i$, as usual...

$$a = \frac{1}{\sqrt{2}}(q + ip)$$

$$a^+ = \frac{1}{\sqrt{2}}(q - ip)$$

Let $|\Psi_0\rangle$ be the gnd state of $\mathcal{H}$ with normalization $\langle \Psi_0 | \Psi_0 \rangle = 1$. For any $z \in \mathbb{C}$, we define the coherent state

$$|z\rangle = c e^{za^+} |\Psi_0\rangle$$

where c is such that $\langle z | z \rangle = 1$.

(a) What is c?

well... $\langle z | z \rangle = 1 = \langle \Psi_0 | e^{z^* a} c^* c e^{za^+} | \Psi_0 \rangle$

Recall...

$(a^+)^n |\Psi_0\rangle = \sqrt{n!}\, |\Psi_n\rangle$

$$= |c|^2 \langle \Psi_0 | e^{z^* a} e^{za^+} | \Psi_0 \rangle$$

$$= |c|^2 \sum_{m,n} \langle \Psi_0 | \frac{(z^* a)^n}{n!} \frac{(za^+)^m}{m!} | \Psi_0 \rangle$$

$$= |c|^2 \sum_{m,n} (z^*)^n (z)^m \frac{1}{\sqrt{n!\, m!}} \langle \Psi_n | \Psi_m \rangle$$

$$= \sum_n \frac{|z|^2}{n!} |c|^2$$

$$= e^{|z|^2} |c|^2$$

$$\Rightarrow \boxed{c = \exp\left\{-|z|^2/2\right\}}$$

(b) What is $\langle z_1|z_2\rangle$?

Answer (easy to get)

$$\langle z_1|z_2\rangle = \exp\left\{\frac{-|z_1|^2-|z_2|^2}{2}\right\} \exp\left[z_1^* z_2\right]$$

(c) Show that $|z\rangle$ is eig. state of a & find eigv.

↳ plug in & use commutator rule for

$$[a, (a^\dagger)^n] = n(a^\dagger)^{n-1} \text{ to get}$$

$$a|z\rangle = z|z\rangle$$

(d) $\forall\ n, m \in \mathbb{Z}^+$, determine the normal-ordered expectation value

$$\langle z| : p^m q^n : |z\rangle$$

→ use $p = \frac{-i}{\sqrt{2}}(a - a^\dagger)$

$q = \frac{1}{\sqrt{2}}(a + a^\dagger)$

& then expand the normal ordering in binomial exp...

$$:p^m q^n: = \frac{(-i)^n}{2^{(n+m)/2}} \sum_{k=0}^{n}\sum_{\ell=0}^{m} \binom{n}{k}\binom{m}{\ell} (-a^\dagger)^k (a^\dagger)^\ell a^{n-k} a^{m-\ell}$$

$$= \frac{(-i)^n}{2^{(n+m)/2}} \sum_{k=0}^{n}\sum_{\ell=0}^{m} \binom{n}{k}\binom{m}{\ell} (a^\dagger)^{k+\ell} a^{n+m-k-\ell} (-1)^k$$

So ... with $a|z\rangle = z|z\rangle$ & $\langle z|a^\dagger = \langle z|z^*$

$$\langle z|:p^m q^n:|z\rangle = \frac{(-i)^n}{2^{(n+m)/2}} \sum_{k=0}^{n} \sum_{l=0}^{m} \binom{n}{k}\binom{m}{l} \langle z|(a^\dagger)^{k+l} a^{n+m-k-l}|z\rangle (-1)^k$$

$$= \underbrace{\cdots\cdots\cdots (z^*)^{k+l} z^{n+m-l-k} (-1)^k}$$

$$= \frac{(-i)^n}{2^{(n+m)/2}} (z-z^*)^n (z+z^*)^m$$

$$= \left[\sqrt{2}\, \mathrm{Im}(z)\right]^n \left[\sqrt{2}\, \mathrm{Re}(z)\right]^m$$

<u>More problems</u>

(1) Consider a universe with 1 dim in time & $d = 1, 2, 3$ for space.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_x)(\partial^\mu \phi_x) - \frac{1}{2} m_x^2 \phi_x^2 + \frac{1}{2}(\partial_\mu \phi_y)(\partial^\mu \phi_y) - \frac{1}{2} m_y^2 \phi_y^2$$
$$+ \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) - \frac{1}{2} M^2 \Phi^2 - \lambda \Phi \phi_x \phi_y$$

where $M > m_x + m_y$.

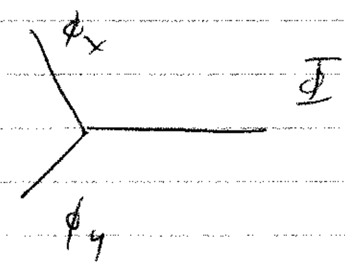
(a) $d = 3$: Find decay rate of the Φ particle in its COM frame to the lowest non vanishing order in λ.

Do the same for $d = 2, 1$.

Amplitude is given by

$$\langle \text{final} \,|\, S-1 \,|\, \text{initial} \rangle = i\mathcal{M}\,(2\pi)\,\delta(E^F_{tot} - E^I_{tot})\,(2\pi)^d\,\delta^{(d)}(\vec{k}^F_{tot} - \vec{k}^I_{tot})$$

To lowest nontrivial order...



$$(i\mathcal{M} = -i\lambda)$$

In COM, $\vec{k}^I_{tot} = 0$; $E^I_{tot} = M$

Decay rate: $\Gamma = \dfrac{\text{probability}}{\text{time}} = \dfrac{1}{2M}\displaystyle\int d\Pi_2\, |\mathcal{M}|^2$

where

$$\int d\Pi_2 = \int \frac{d^d\vec{k}_x}{(2\pi)^d\, 2E_x}\,\frac{d^d\vec{k}_y}{(2\pi)^d\, 2E_y}\,(2\pi)\,\delta(E^F_{tot} - M)\,(2\pi)^d\,\delta^{(d)}(\vec{k}^F_{tot})$$

$$= \int \frac{d^d\vec{k}_x\;\; k_x^{d-1}}{(2\pi)^d (2E_x)(2E_y)}\,(2\pi)\,\delta(E^F_{tot} - M)$$

where $E_i = \sqrt{\vec{k}_i^2 + m_i^2}$; $\dfrac{dE_i}{dk_i} = \dfrac{k_i}{E_i}$

$$\rightarrow \int d\Pi_2 = \int d\Omega\, \frac{k_x^{d-1}\,(2\pi)}{(2\pi)^d\cdot 4}\,\frac{1}{E_x E_y\left(\frac{k_x}{E_x} + \frac{k_y}{E_y}\right)} = \int d\Omega\, \frac{k_x^{d-2}}{(2\pi)^{d-1}\, 4M}$$

the value of k_x for which $E^F_{tot} = E_x + E_y = M$ is

$$E^F_{tot} = \frac{1}{2M}\sqrt{(M - m_x - m_y)(M - m_x + m_y)(M + m_x - m_y) \times (M + m_x + m_y)}$$

So $\Gamma = \left(\int d\Omega\right) \cdot \frac{\lambda^2}{8M} \frac{k_x^{d-2}}{(2\pi)^{d-1}}$

when $d=3$, $\Gamma = \frac{\lambda^2 k_x}{8\pi M^2}$

$d=2$, $\Gamma = \frac{\lambda^2}{8M^2}$.

$d=1$, $\Gamma = \frac{\lambda^2}{4k_x M^2}$.

More problems. Consider the same Lagrangian density ...

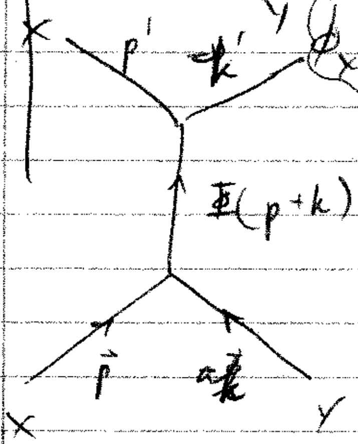
$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_x)(\partial^\mu\phi_x) - \frac{1}{2}m_x^2\phi_x^2 + \frac{1}{2}(\partial_\mu\phi_y)(\partial^\mu\phi_y) - \frac{1}{2}m_y^2\phi_y^2$$
$$+ \frac{1}{2}(\partial_\mu\Phi)(\partial^\mu\Phi) - \frac{1}{2}M^2\Phi^2 - \lambda\Phi\phi_x\phi_y.$$

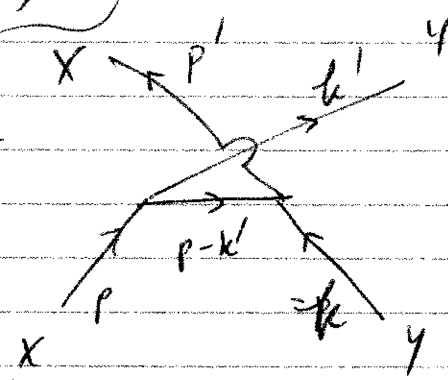
Find $\frac{d\sigma}{d\Omega}$ as a fn of $p=|\vec{p}|$, θ, λ, m_x, m_y, M

Consider $d=3$. In com frame consider elastic scattering of a ϕ_x + ϕ_y to lowest non vanishing order in λ.

$\phi_x + \phi_y \longrightarrow \phi_x + \phi_y$.

2 possibilities ...





$i\mathcal{M}_1 = (-i\lambda)^2 \frac{i}{(p+k)^2 - M^2 + i\varepsilon}$ $\qquad$ $i\mathcal{M}_2 = (-i\lambda)^2 \frac{i}{(p-k')^2 - M^2 + i\varepsilon}$

$$\rightarrow \quad iM = iM_2 + iM_1 = -i\lambda^2\left[\frac{1}{(p-k')^2 - M^2 + i\varepsilon} + \frac{1}{(p+k)^2 - M^2 + i\varepsilon}\right]$$

In com ... $p = (E_x, \vec{p})$ $\quad k = (E_y, -\vec{p})$

$p' = (E_x, \vec{p}')$ $\quad k' = (E_y, -\vec{p}')$

$|\vec{p}| = |\vec{p}|'$

where $E_i = \sqrt{\vec{p}_i^{\,2} + m_i^2}$, $E_{cm} = E_x + E_y$.

We have a formula that says ...

$$\left(\frac{d\sigma}{d\Omega}\right)_{cM} = \frac{|\vec{p}|\,|M|^2}{2E_x\, 2E_y\, |\vec{v}_x - \vec{v}_y|\,(16\pi^2)\, E_{cM}}$$

where $v_x = \vec{p}/E_x$, $v_y = -\vec{p}/E_y$

$$\rightarrow \quad |\vec{v}_x - \vec{v}_y| = |\vec{p}|\left(\frac{1}{E_x} + \frac{1}{E_y}\right)$$

$$\rightarrow \quad 2E_x 2E_y\, |\vec{v}_x - \vec{v}_y| = 4|\vec{p}|\,(E_x + E_y)$$

$$\rightarrow \quad \left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{|M|^2}{64\pi^2 (E_x + E_y)^2}$$

now ... $iM = -i\lambda^2\left(\frac{1}{(p-k')^2 - M^2 + i\varepsilon} + \frac{1}{(p+k)^2 - M^2 + i\varepsilon}\right)$

Using ...

$$(p-k')^2 = (E_x - E_y)^2 - (\vec{p}+\vec{p}')^2$$
$$= (E_x - E_y)^2 - (2\vec{p}^2 + 2\vec{p}^2\cos\theta)$$
$$= (E_x - E_y)^2 - 2\vec{p}^2(1+\cos\theta)$$

$$(p+k)^2 = (E_x + E_y)^2,$$

we get

$$|M|^2 = \lambda^4 \underbrace{\left\{ \frac{1}{(E_x-E_y)^2 - 2\vec{p}^2(1+\cos\theta) - M^2} + \frac{1}{(E_x+E_y)^2 - M^2} \right\}}$$

So
$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{\lambda^4}{64\pi^2 (E_x+E_y)^2}\Big[\qquad \Big]^2$$

So, explicitly...

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{\lambda^4}{64\pi^2\left[\sqrt{\vec{p}^2+m_x^2}+\sqrt{\vec{p}^2+m_y^2}\right]^2} \times$$
$$\left\{ \frac{1}{\left(\sqrt{\vec{p}^2+m_x^2}-\sqrt{\vec{p}^2+m_y^2}\right)^2 - 2\vec{p}^2(1+\cos\theta) - M^2} + \frac{1}{\left(\sqrt{\vec{p}^2+m_x^2}+\sqrt{\vec{p}^2+m_y^2}\right)^2 - M^2} \right)^2$$

Nov 8, 2020

Traces & Contraction of γ^μ's

Recall that in Weyl's representation...

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

$$\Rightarrow \quad \boxed{\mathrm{Tr}\,\gamma^\mu = 0}$$

- $$\mathrm{Tr}\,\gamma^\mu\gamma^\nu = \frac{1}{2}\mathrm{Tr}\,(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)$$

$$= \frac{1}{2}\mathrm{Tr}\,(2g^{\mu\nu}\cdot \mathbb{1}_{4\times 4}) = \boxed{4g^{\mu\nu} = \mathrm{Tr}\,\gamma^\mu\gamma^\nu}$$

- $$\mathrm{Tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = \mathrm{Tr}\left[2g^{\mu\nu}\gamma^\rho\gamma^\sigma - \gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma\right]$$

$$= \mathrm{Tr}\left(2g^{\mu\nu}\gamma^\rho\gamma^\sigma - \gamma^\nu(2g^{\mu\rho})\gamma^\sigma + \gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma\right)$$

$$= \mathrm{Tr}\left(2g^{\mu\nu}\gamma^\rho\gamma^\sigma - 2g^{\mu\rho}\gamma^\nu\gamma^\sigma + \gamma^\nu\gamma^\rho(2g^{\mu\sigma})\right.$$

$$\left. - \gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu\right)$$

$$\Rightarrow \quad \boxed{2\,\mathrm{Tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = \mathrm{Tr}\left(2g^{\mu\nu}\gamma^\rho\gamma^\sigma - 2g^{\mu\rho}\gamma^\nu\gamma^\sigma + 2g^{\mu\sigma}\gamma^\nu\gamma^\rho\right)}$$

$$\Rightarrow \quad \mathrm{Tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4g^{\mu\nu}g^{\rho\sigma} - 4g^{\mu\rho}g^{\nu\sigma} + 4g^{\mu\sigma}g^{\nu\rho}$$

Next... $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

$$\mathrm{Tr}\,\gamma^5 = \mathrm{Tr}\,\gamma^0\gamma^0\gamma^5 = \mathrm{Tr}\,\gamma^5\gamma^0\gamma^0 = -\mathrm{Tr}\,\gamma^5 \Rightarrow \boxed{\mathrm{Tr}\,\gamma^5 = 0}$$

Also, $\boxed{\mathrm{Tr}\left(\underbrace{\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_n}}_{n \text{ odd}}\gamma^5\right) = 0}$

since by moving γ^5 to the front, we get a factor of $(-1)^n$. if n odd then $(-1) \Rightarrow$ cancel to get 0

Similarly...

$$\boxed{\mathrm{Tr}\left(\gamma^{\mu_1}\cdots\gamma^{\mu_n}\right) = 0 \quad \text{if } n \text{ odd}}$$

<u>More results</u>

- $\mathrm{Tr}\left(\mathbb{1}_{4\times 4}\right) = 4$
- $\mathrm{Tr}\left(\gamma^{\mu_1}\cdots\gamma^{\mu_n}\right) = 0$, n odd
- $\mathrm{Tr}\left(\gamma^{\mu}\gamma^{\nu}\right) = 4g^{\mu\nu}$
- $\mathrm{Tr}\left(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\right) = 4g^{\mu\nu}g^{\rho\sigma} - 4g^{\mu\rho}g^{\nu\sigma} + 4g^{\mu\sigma}g^{\nu\rho}$
- $\mathrm{Tr}\left(\gamma^5\right) = 0$
- $\mathrm{Tr}\left(\underbrace{\gamma^{\mu_1}\cdots\gamma^{\mu_n}}_{\text{odd}}\gamma^5\right) = 0$
- $\mathrm{Tr}\left(\gamma^{\mu}\gamma^{\nu}\gamma^5\right) = 0$
- $\mathrm{Tr}\left(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\gamma^5\right) = -4i\epsilon^{\mu\nu\rho\sigma}$ completely antisymmetric

More useful contraction identities

- $\gamma_\mu\gamma^\mu = \gamma^\nu\gamma^\mu g_{\mu\nu} = \frac{1}{2}\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right) g_{\mu\nu}$
  $= \frac{1}{2}\, 2g^{\mu\nu} g_{\mu\nu} = 4.$

- $\gamma_\mu\gamma^\nu\gamma^\mu = \gamma_\mu\left(2g^{\nu\mu} - \gamma^\mu\gamma^\nu\right) = +2\gamma^\nu - 4\gamma^\nu = -2\gamma^\nu$

- $\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho}$
- $\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu$

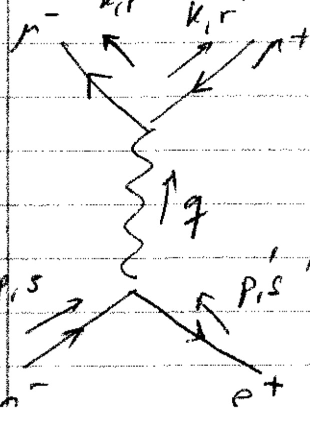
With this we're ready to look at Chapter 5

$\boxed{\text{Chapter 5: Elementary Processes of QED}}$

$e^+ e^- \longrightarrow \mu^+ \mu^-.$

one of the most simple processes in QED

$\longrightarrow$ we will calculate the unpolarized cross-section



$$iM = (-ie)^2\left(\bar v^{s'}(p')\gamma^\mu u^s(p)\right)\left(\bar u^r(k)\gamma^\nu v^{r'}(k')\right) \times \left(\frac{-ig_{\mu\nu}}{q^2+i\epsilon}\right)$$

$$= \frac{ie^2}{(p+p')^2}\left(\bar v^{s'}(p')\gamma^\mu u^s(p)\right)\left(\bar u^r(k)\gamma_\mu v^{r'}(k')\right)$$

→ need $|M|^2$. notice that

$$(\bar{v}\gamma^\mu u)^* = \overline{(\bar{v}\gamma^\mu u)} = \bar{u}\,\overline{\gamma^\mu}\,v = \bar{u}\gamma^\mu v.$$

Recall that $\bar{M} = \gamma^0 M^\dagger \gamma^0$

and so

$$\overline{\gamma^0} = \gamma^0\gamma^{0\dagger}\gamma^0 = \gamma^0$$

$$\overline{\gamma^i} = \gamma^0\gamma^{i\dagger}\gamma^0 = \gamma^i$$

$$\Rightarrow |M|^2 = \frac{e^4}{((p+p')^2)^2} \times \left(\bar{v}^{s'}(p')\gamma^\mu u^s(p)\right)\left(\bar{u}^r(k)\gamma_\mu v^{r'}(k')\right)$$
$$\times \left(\bar{u}^s(p)\gamma^\nu v^{s'}(p')\right)\left(\bar{v}^{r'}(k')\gamma_\nu u^r(k)\right)$$

$$= \frac{e^4}{((p+p')^2)^2}\left\{\bar{v}^{s'}(p')\gamma^\mu u^s(p)\bar{u}^s(p)\gamma^\nu v^{s'}(p')\right\}$$
$$\times\left\{\bar{u}^r(k)\gamma_\mu \bar{v}^{s'}(k')v^{r'}(k')\gamma_\nu u^r(k)\right\}$$

= ?

This simplifies when we throw away the spin info.

→ Want to compute $\frac{1}{2}\sum_s \frac{1}{2}\sum_{s'}\sum_r\sum_{r'} |M(s,s'\to r,r')|^2$

Recall completeness... $\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m$

$$\sum_s v^s(p)\,\bar{v}^s(p) = \not{p} - m$$

With this, we find that ...

$$\sum_{ss'rr'} \bar{v}^{s'}(p')\gamma^\mu u^s(p)\bar{u}^s(p)\gamma^\nu v^{s'}(p')$$

$$= \bar{v}(p')\gamma^\mu u(p)\bar{u}(p)\gamma^\nu v(p')$$

$$= \ell_\mu \quad ?$$

$$= \sum_{ss'} \bar{v}^{s'}_a(p')\gamma^\mu_{ab}\underbrace{u^s_b(p)\bar{u}^s_c(p)}\gamma^\nu_{cd}v^{s'}_d(p')$$

$$= (\not{p}'-m)_{da}\gamma^\mu_{ab}(\not{p}+m)_{bc}\gamma^\nu_{cd}$$

$$= \mathrm{Tr}\left\{(\not{p}'-m)\gamma^\mu(\not{p}+m)\gamma^\nu\right\}$$

So ...

$$\frac{1}{4}\sum_{\text{spins}}|M|^2 = \frac{e^4}{4(p+p')^4}\mathrm{Tr}\left\{(\not{p}'-m_e)\gamma^\mu(\not{p}+m_e)\gamma^\nu\right\} \times \mathrm{Tr}\left\{(\not{k}+m_\mu)\gamma_\mu(\not{k}'-m_\mu)\gamma_\nu\right\}$$

Now, look at each term...

$$\longrightarrow \mathrm{Tr}\left\{\gamma^\mu(\not{p}+m_e)\gamma^\nu(\not{p}'-m_e)\right\}$$

$$= \mathrm{Tr}\left\{\gamma^\mu \underset{\uparrow\, p_\alpha\gamma^\alpha}{\not{p}}\gamma^\nu \underset{\uparrow\, p'_\beta\gamma^\beta}{\not{p}'}\right\} - \mathrm{Tr}\left\{\gamma^\mu\gamma^\nu\right\}m_e^2$$

$$= 4p^\mu p'^\nu - 4g^{\mu\nu}p\cdot p' + 4p^\nu p'^\mu - 4g^{\mu\nu}m_e^2$$

$$= 4\left[p^\mu p'^\nu + p^\nu p'^\mu - g^{\mu\nu}(p\cdot p' + m_e^2)\right]$$

Similarly, ...

$$\mathrm{Tr}\left\{ \gamma_\mu (\not{k}' - m_\mu) \gamma_\nu (\not{k} + m_\mu) \right\}$$
$$= 4\left[k'_\mu k_\nu + k'_\nu k_\mu - g_{\mu\nu}(k'\cdot k + m_\mu^2) \right]$$

So...

$$\sum_{\text{spins}} \frac{1}{4} |M|^2 = \frac{4e^4}{q^4} \left\{ \begin{array}{l} 2(p\cdot k)(p'\cdot k') + 2(p\cdot k')(p'\cdot k) \\ -2(k\cdot k')(p\cdot p' + m_e^2) \\ -2(p\cdot p')(k'\cdot k + m_\mu^2) \\ +4(k'\cdot k + m_\mu^2)(p'\cdot p + m_e^2) \end{array} \right\}$$

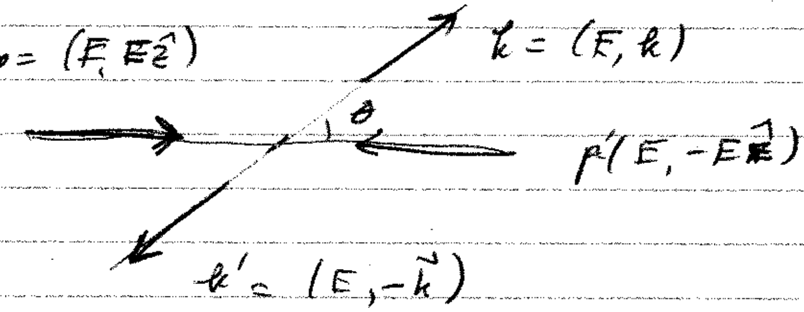
When energy is sufficiently high, can neglect ~~m_e~~ .. $m_e \longrightarrow$ set $m_e = 0 = \cancel{m}$

get ...

$$\boxed{\frac{1}{4}\sum_{\text{spins}} |M|^2 = \frac{4e^4}{q^4}\left\{ \begin{array}{l} 2(p'\cdot k')(p\cdot k) + 2(p\cdot k')(p'\cdot k) \\ + 2m_\mu^2 (p\cdot p') \end{array} \right\}}$$

To get a more explicit formula $\rightarrow$ go to com frame

(unpolarized)



$|\vec{k}| = \sqrt{E^2 - m_\mu^2}$

$k\cdot\hat{z} = |\vec{k}| \cos\theta .$

$$q^2 = (p+p')^2 = 2E^2 - 0 = 4E^2$$
$$p\cdot p' = E^2 + E^2 = 2E^2$$

$$p\cdot k = E^2 - E|\vec{k}|\cos\theta \qquad p\cdot k' = E^2 + E|\vec{k}|\cos\theta$$

$$p'\cdot k' = E^2 - E|\vec{k}|\cos\theta \qquad p'\cdot k = E^2 + E|\vec{k}|\cos\theta$$

So

$$\frac{1}{4}\sum_{\text{spins}} |M|^2 = \frac{8e^4}{16E^4}\Big\{E^2(E-|\vec{k}|\cos\theta)^2 + E^2(E+|\vec{k}|\cos\theta)^2 + 2m_\mu^2 E^2\Big\}$$

$$= \frac{e^4}{E^4}\Big\{E^2(E^2 + \underbrace{|\vec{k}|^2}_{(E^2-m_\mu^2)}\cos^2\theta) + m_\mu^2 E^2\Big\}$$

$$\Rightarrow \quad \frac{1}{4}\sum_{\text{spins}}|M|^2 = e^4\Big\{1 + \frac{m_\mu^2}{E^2} + \Big(1 - \frac{m_\mu^2}{E^2}\Big)\cos^2\theta\Big\}$$

now... for 2 body scattering into a 2-body system

$$\frac{d\sigma}{d\Omega} = \frac{1}{4E_A E_B|v_A - v_B|}\,\frac{|\vec{k}|}{16\pi^2 E_{CM}} \times \frac{1}{4}\sum_{\text{spins}}|M|^2$$

where $m_e \simeq 0$ and so e^- & e^+ are moving @ the speed of light

$$\rightarrow \quad |v_A - v_B| \sim 2c = 2$$

$E_{CM} = 2E_A = 2E_B = 2E$

So

$$\frac{d\sigma}{d\Omega} = \frac{1}{4E^2\cdot 2}\cdot\frac{\sqrt{E^2-m_\mu^2}}{16\pi^2 E_{CM}}\cdot e^4\Big\{1+\frac{m_\mu^2}{E^2} + \Big(1-\frac{m_\mu^2}{E^2}\Big)\cos^2\theta\Big\}$$

$$= \frac{e^4}{256\pi^2 E^2}\sqrt{1 - \frac{m_\mu^2}{E^2}}\,\Big\{1 + \frac{m_\mu^2}{E^2} + \Big(1 - \frac{m_\mu^2}{E^2}\Big)\cos^2\theta\Big\}$$

Integrate over $d\Omega$ gives...

$$\sigma_{tot} = \frac{e^4 \cdot 2\pi}{256\pi^2 E^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left\{2\left(1+\frac{m_\mu^2}{E^2}\right)+\frac{2}{3}\left(1-\frac{m_\mu^2}{E^2}\right)\right\}$$

$$= \underline{\frac{e^4}{64\pi E^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left\{\frac{4}{3}\left(1+\frac{1}{2}\frac{m_\mu^2}{E^2}\right)\right\}}$$

Conventionally, we write this as...

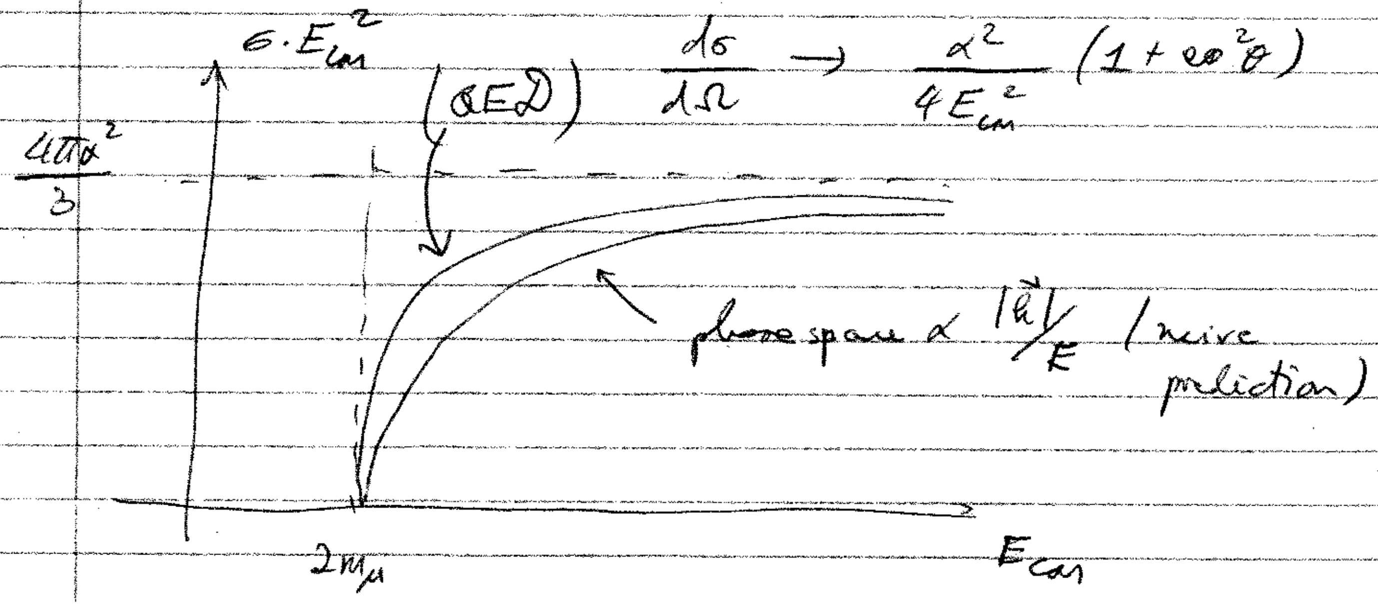
$$\sigma_{tot} = \frac{4\pi\alpha^2}{3E_{cm}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left(1+\frac{1}{2}\frac{m_\mu^2}{E^2}\right)$$

where $\alpha = e^2/4\pi$

when $E_{cm} < 2m_\mu \rightarrow$ cross section is zero b/c there isn't enough energy to produce a muon - antimuon pair.

As $E_{cm} \rightarrow \infty$, $\sigma_{tot} \rightarrow \frac{4\pi\alpha^2}{3E_{cm}^2}$

$$\frac{d\sigma}{d\Omega} \rightarrow \frac{\alpha^2}{4E_{cm}^2}(1+\cos^2\theta)$$



Summary of method

(1) Draw Feynman diagram for the process

(2) Use Feynman rules to calculate $\mathcal{M}$

(3) Calculate $|\mathcal{M}|^2$, averaged or sum over all spins using the completeness relation.

(4) Evaluate traces using the trace theorems, collect terms & simplify as much as possible.

(5) Pick frame. Draw a picture of kinematics. Express all 4-momentum vectors in terms of variables such as E & θ.

(6) Plug $|\mathcal{M}|^2$ into cross section formula & integrate over all phase space variables to find σ_{tot}.

$\cancel{\quad}$

Production of Quarks – Antiquark pairs

An important cross section is

$e^+e^- \to$ hadrons

↑ any number of strongly interacting particles.

QCD → all hadrons are composed of Dirac fermions called "quarks" (which come in many "flavors")

↳ each quark has
- mass
- electric charge
- additional quantum number (color)

↳ Simplest process in QCD is

$$e^+ e^- \to q\bar{q} \quad \text{(quark – antiquark pair)}$$

To modify the $e^+e^- \to \mu^+\mu^-$ to get here, we need 3 things:

- Replace the muon charge e with $Q|e|$ → quark charge
- Count each quark 3 times, one for each color
- Include strong interaction within the produced quark – anti-quark ...

First 2 charges are easy: masses & charge are known.

u, c, t quarks: $Q = 2/3$

d, s, b quarks: $Q = -1/3$

→ finally since $M \propto$ charge^2 → insert Q^2.

→ counting: multiply everything by 3.

Finally in high energy limit, strong interaction can be neglected (Nobel 2004)

→ σ_{tot} for $e^+e^- \to q\bar{q}$ looks like...

$$\sigma_{tot} \longrightarrow R \equiv \frac{4\pi\alpha^2}{3E_{cm}^2} \cdot (3Q^2) \equiv (3Q^2)R$$

$$\frac{86.8 \text{ nbarns}}{(E_{cm} \text{ in GeV})^2}$$

→ Asymptotically,

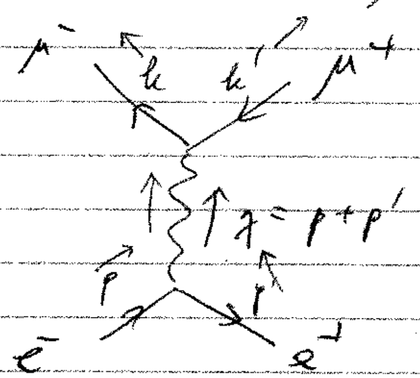
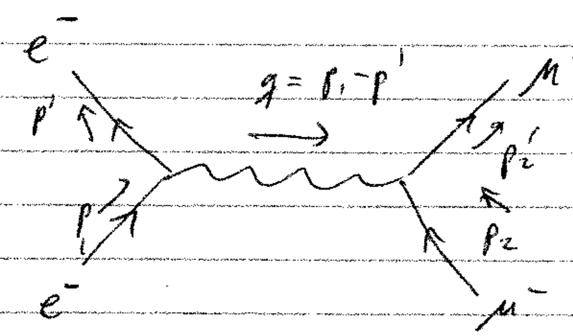
$$\sigma(e^+e^- \to \text{hadrons}) \xrightarrow[E\to\infty]{} 3\cdot\left(\textstyle\sum_i Q_i^2\right)R$$

(there's more to this ... but we won't be worrying about this too much

(~~&~~ a lot of experimental details)

Crossing Symmetry $e^+e^- \to \mu^+\mu^-$

Consider a related process: $e^-\mu^- \to e^-\mu^-$

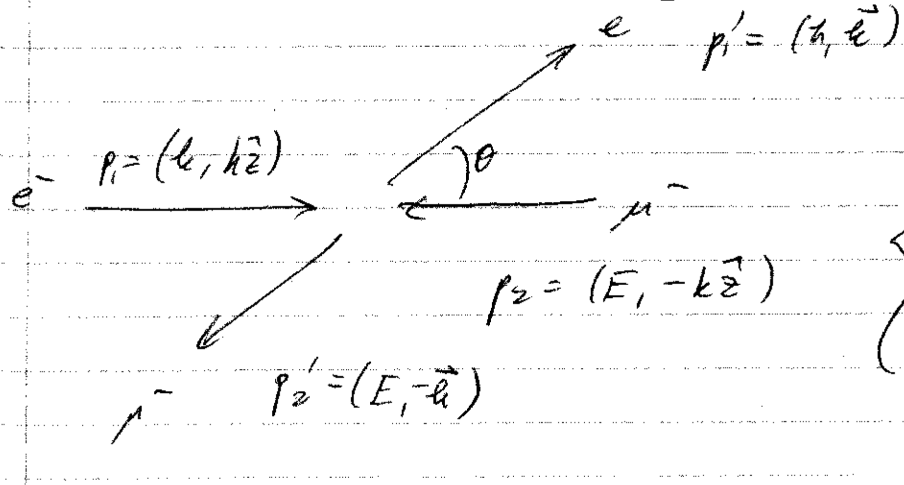
$e^+e^- \to \mu^+\mu^-$	$e^-\mu^- \to e^-\mu^-$
	
$\sum_{\text{spins}} \frac{1}{4}\|M\|^2 = \frac{8e^4}{q^4}\{(p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) + m_\mu^2(p\cdot p')\}$	$iM = \frac{(-ie)^2}{q^2}(\bar u(p_1')\gamma^\mu u(p_1)) \times (\bar u(p_2')\gamma^\nu u(p_2))$
	$\frac{1}{4}\sum_{\text{spins}}\|M\|^2 = \frac{8e^4}{q^2}\Big[(p_1\cdot p_2')(p_1'\cdot p_2) + (p_1\cdot p_2)(p_1'\cdot p_2') - m_\mu^2(p_1\cdot p_1')\Big]$

Compare...

$$p \longleftrightarrow p_1$$
$$k \longleftrightarrow p_2'$$
$$p' \longleftrightarrow -p_1'$$
$$k' \longleftrightarrow -p_2$$

$\longrightarrow$ incoming $\leftrightarrow$ outgoing; particle $\leftrightarrow$ antiparticle; momentum $\leftrightarrow$ $-$momentum

The kinematics here will be completely different.



$$\left\{ \begin{aligned} E^2 &= k^2 + m_\mu^2 \\ \vec{k} \cdot \hat{z} &= k \cos\theta \\ E + k &= E_{com}. \end{aligned} \right.$$

So evaluating the $p_1 \cdot p_2$, $p_1' \cdot p_2'$ --- combos gives

$$\frac{1}{4} \sum_{spins} |\mathcal{M}|^2 = \frac{2e^4}{k^2 (1-\cos\theta)^2} \left\{ (E+k)^2 + (E + k\cos\theta)^2 - m_\mu^2 (1 - \cos\theta) \right\}$$

$$\hookrightarrow \left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{|\mathcal{M}|^2}{64\pi^2 (E+k)^2}.$$

(which is obtained from the 1-massless particle limit of the 2-body - 2 body $\frac{d\sigma}{d\Omega}$) ...

So ...

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{\alpha^2}{2k^2 (E+k)^2 (1-\cos\theta)^2} \left\{ (E+k)^2 + (E + k\cos\theta)^2 - m_\mu^2 (1 - \cos\theta) \right\}$$

where $k = \sqrt{E^2 - m_\mu^2}$.

When $E \to \infty$, can set $m_\mu = 0$, and

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} \longrightarrow \frac{\alpha^2}{2E_{cm}^2 (1 - \cos\theta)^2} \left(4 + (1 + \cos\theta)^2\right)$$

note that $\left(\frac{d\sigma}{d\Omega} \propto \frac{1}{\theta^4} \right)$ (singular as $\theta \to 0$)

This is because of the photon propagator being nearly on shell ($q^2 \simeq 0$).

The same result can be seen in non-relativistic Rutherford scattering

↳ Divergent cross section is due to the fact that Coulomb force has ∞ range.

General Crossing Symmetry

→ The previous section is one example of crossing symmetry...

→ For a scalar particle...

$$\mathcal{M}(\phi(p) + X \to Y) = \mathcal{M}(X \to Y + \bar{\phi}(-p))$$

↳ flip sign of p and you get the new amplitude.

→ For fermion spinors, there is an additional minus sign for the unpolarized spin sums since

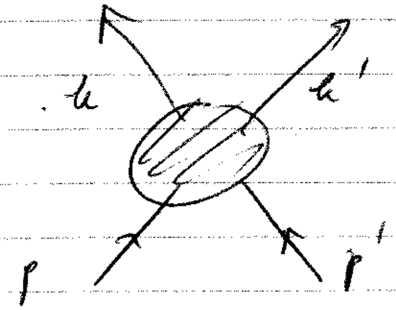
$$\sum_{\text{spins}} \bar{u}(p)\, u(p) = \not{p} + m \quad \text{while}$$

$$\sum_{\text{spins}} v(-p)\, \bar{v}(-p) = -\not{p} - m = -(\not{p} + m)$$

→ $\times(-1)$ for each flipped fermion as well...

Mandelstam variables. (Convenient for crossing symmetries)

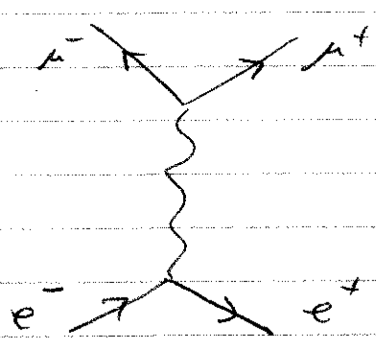
2 body → 2 body scattering...



$$\left\{ \begin{aligned} s &= (p+p')^2 = (k+k')^2 \\ t &= (k-p)^2 = (k'-p')^2 \\ u &= (k'-p)^2 = (k-p')^2 \end{aligned} \right\}$$

(Mandelstam variables)

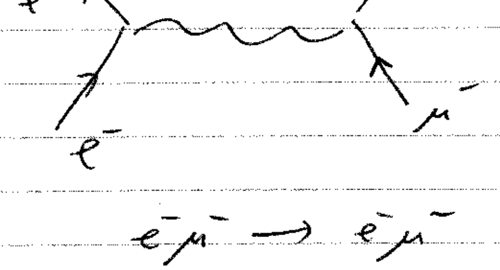
With this, we can write...



$$\Rightarrow \quad \frac{1}{4} \sum_{\text{spins}} |M|^2 = \frac{8e^4}{s^2} \left\{ \left(\frac{t}{2}\right)^2 + \left(\frac{u}{2}\right)^2 \right\}$$

$e^+e^- \to \mu^+\mu^-$

and



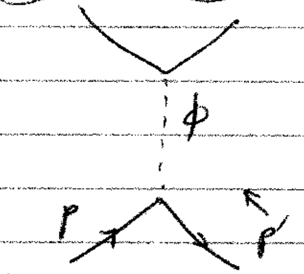
$$\Rightarrow \quad \frac{1}{4} \sum_{\text{spins}} |M|^2 = \frac{8e^4}{t^2} \left\{ \left(\frac{s}{2}\right)^2 + \left(\frac{u}{2}\right)^2 \right\}$$

$e^-\mu^- \to e^-\mu^-$

When a 2 body - 2 body diagram contains only <u>one</u> virtual particle...

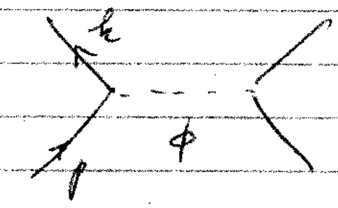
↳ conventionally described that particle as being in certain "channels".

s-channel:



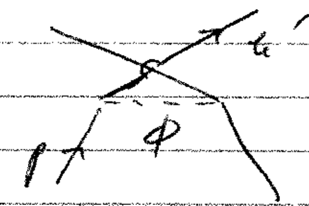
$$M \propto \frac{1}{s - m_\phi^2}$$

t-channel



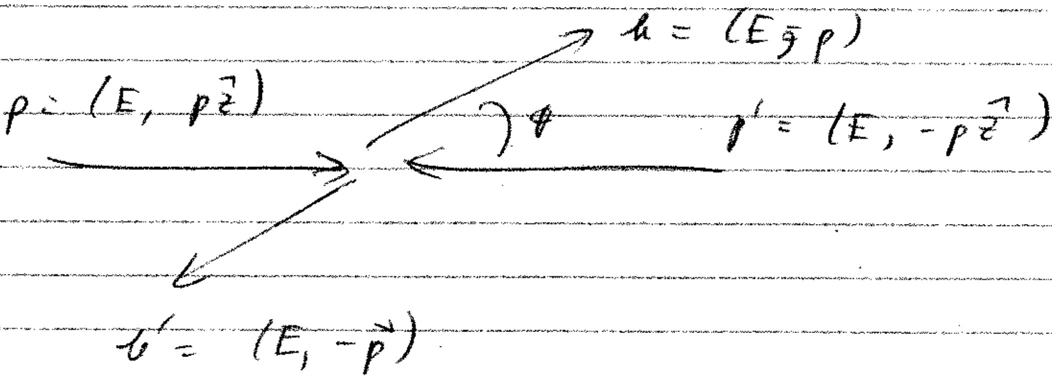
$$M \propto \frac{1}{t - m_\phi^2}$$

u-channel



$$M \propto \frac{1}{u - m_\phi^2}$$

To get better feel ... let's look at what happens in COM frame...



$$\begin{cases} \cdot\ s = (p+p')^2 = (k'+k)^2 = (2E)^2 = E_{cm}^2 \\ \cdot\ t = (k-p)^2 = -p^2\sin^2\theta - p^2(\cos\theta-1)^2 = -2p^2(1-\cos\theta) \\ \cdot\ u = (k'-p)^2 = -p^2\sin^2\theta - p^2(\cos\theta+1)^2 = -2p^2(1+\cos\theta) \end{cases}$$

~~[illegible]~~

We see that $t \to 0$ as $\theta \to 0$
$u \to 0$ as $\theta \to \pi$

Also note that ...

$$\boxed{s + t + u = \sum_{i=1}^{4} m_i^2}$$

square of mass of incoming / outgoing particles

this is true in general — just add everything up & use momentum conservation.

$$(p+p'-k'-k)^2 = 0.$$

<u>Compton scattering</u> $e^-\gamma \to e^-\gamma$

$\to$ X-ray & Γ-ray on electrons ...

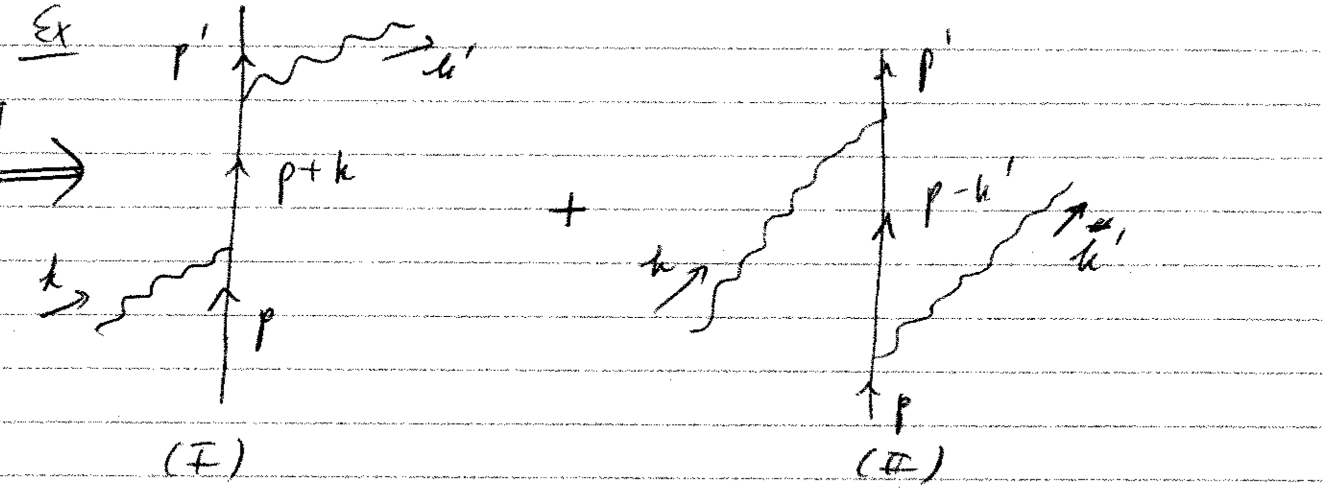
We will calculate σ_{tot} to lowest order in α.

(we will be using all the techniques we learned so far

+ how to deal with external photons.

Ex

2 diagrams @ lowest order $\Rightarrow$



Let $\varepsilon_\nu(k)$ + $\varepsilon_\nu(k')$ be the photon polarization vectors

$$iM = (I) + (II)$$

$$= \bar{u}(p')(-ie\gamma^\mu)\,\varepsilon^*_\mu(k')\,\frac{i(\not p + \not k + m)}{(p+k)^2 - m^2}\,(-ie\gamma^\nu)\,\varepsilon_\nu(k)\,u(p)$$

$$+\,\bar{u}(p')(-ie\gamma^\nu)\,\varepsilon_\nu(k)\,\frac{i(\not p - \not k + m)}{(p-k')^2 - m^2}\,(-ie\gamma^\mu)\,\varepsilon^*_\mu(k')\,u(p)$$

$$= -ie^2\,\varepsilon^*_\mu(k')\,\varepsilon_\nu(k)\,\bar{u}(p')\left\{\frac{\gamma^\mu(\not p + \not k + m)\gamma^\nu}{(p+k)^2 - m^2} + \frac{\gamma^\nu(\not p - \not k' + m)\gamma^\mu}{(p-k')^2 - m^2}\right]u(p)$$

now note that $p^2 = m^2 = 0$, $k^2 = 0$

$$\rightarrow\ (p+k)^2 - m^2 = 2p\cdot k$$

$$(p-k')^2 - m^2 = -2p\cdot k'$$

and finally ...

$$(\not p + m)\gamma^\nu u(p) = (2p^\nu - \gamma^\nu \not p + \gamma^\nu m)\, u(p)$$
$$= 2p^\nu u(p) - \underbrace{\gamma^\nu(\not p - m)\, u(p)}$$
$$= 2p^\nu u(p) \qquad 0 \text{ by defn of } u(p)$$

So...

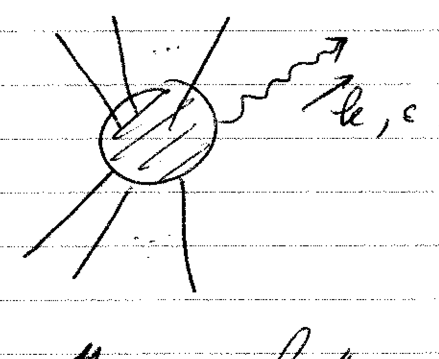
$$\boxed{iM = -ie^2 \varepsilon^*_\mu(k')\,\varepsilon_\nu(k)\,\bar u(p')\left\{\frac{\gamma^\mu \not k \gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{-\gamma^\nu \not k \gamma^\mu + 2\gamma^\nu p^\mu}{-2p\cdot k'}\right\} u(p)}$$

Next,... <u>Photon Polarization + Ward Identity</u>

Interaction for QED has the form

$e\int d^4x\; j^\mu A_\mu$ where j^μ is the conserved electric charge current.

Consider a process with outgoing photon with momentum k + polarization ε.



$$iM \sim iM^\mu(k)\,\varepsilon^*_\mu(k)$$

outgoing state

incoming state

where $M^\mu(k) \propto \int d^4x\; e^{ikx}\langle f|\, j^\mu(x)\,|i\rangle$.

Now, since $\partial_\mu j^\mu(x) = 0$ ...

$$k_\mu M^\mu(k) \propto \int d^4x \, e^{ik\cdot x} k_\mu \langle f | j^\mu(x) | i \rangle$$

$$= -i \int d^4x \, \partial_\mu (e^{ik\cdot x}) \langle f | j^\mu(x) | i \rangle$$

$$= i \int d^4x \, e^{ik\cdot x} \langle f | \partial_\mu j^\mu(x) | i \rangle$$

$$= 0.$$

→ This is an example of the Ward identity in QED.

→ when you replace the polarization vector for any external photon (incoming or outgoing) by the momentum k_μ, the amplitude vanishes.

$$\boxed{k_\mu M^\mu(k) = 0}.$$

more on this later...

→ This is essentially a statement of current conservation.

↳ which is a consequence of gauge symmetry of QED.

what can we say about the polarization sum

$$\sum_{\substack{\text{polarizations}\\ i}} \varepsilon_i^{*\mu}(k)\,\varepsilon_i^{\nu}(k) \; ?$$

↳ Consider an example -- Let $k^\mu = (k, 0, 0, k)$

→ 2 physical polarizations are usually taken as

$\varepsilon_1^\mu = (0, 1, 0, 0)$

$\varepsilon_2^\mu = (0, 0, 1, 0)$

$$\rightarrow \sum_i \varepsilon_i^{*\mu}\varepsilon_i^{\nu} = \begin{array}{c|c} & \nu \\ \hline \mu & \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix} \end{array}$$

But this is ugly... → can find a more familiar form...

Nov 9, 2020

Note that polarization enters calculation in the form

$$\sum_i \left|\varepsilon_{i\mu}^*(k)\, M^\mu(k)\right|^2$$

which is

$$\sum_i \varepsilon_i^{*\mu}(k)\,\varepsilon_i^{\nu}(k)\, M_\mu(k) M_\nu^*(k)$$

$$= |M_1(k)|^2 + |M_2(k)|^2 \qquad \overset{0}{\sim\!\sim}$$

$$= |M_1(k)|^2 + |M_2(k)|^2 + |M_3(k)|^2 - |M_0(k)|^2$$

since $k^\mu = (k, 0, 0, k)$

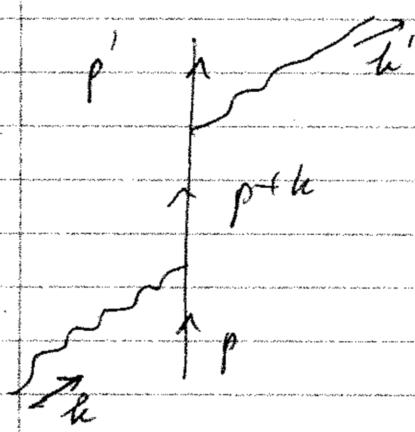
$$= -g_{\mu\nu} M^\mu(k) M^{\nu *}(k) \qquad k^\mu M_\mu = 0 \quad \text{(Ward identity)}$$

So we can take $\boxed{\sum_r \varepsilon_r^{*\mu}(k)\,\varepsilon_r^{\nu}(k) = -g^{\mu\nu}}$

even though this is not really an equality...

Now, back to Compton scattering...

↳ The Klein-Nishina Formula



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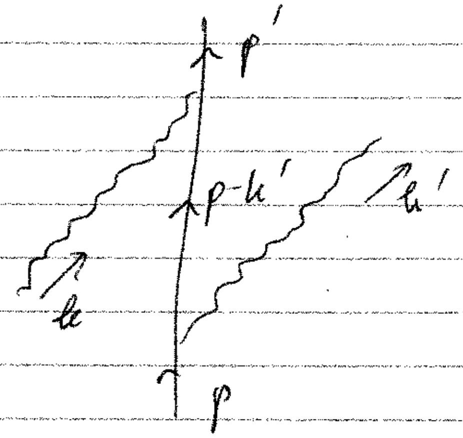


Recall that we found

$$i\mathcal{M} = -ie^2\,\varepsilon^*_\mu(k')\,\varepsilon_\nu(k)\,\bar u(p')\left\{\frac{\gamma^\mu \not k \gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{\gamma^\nu \not k' \gamma^\mu - 2\gamma^\nu p^\mu}{2p\cdot k'}\right\}u(p)$$

↳ now, we square, sum & divide by # of initial spins (2 electrons × 2 photons = 4)

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e^4}{4}\,\underbrace{g_{\mu\rho}\,g_{\nu\sigma}}\times \mathrm{tr}\Bigg\{(\not p'+m)\left\{\frac{\gamma^\mu \not k \gamma^\nu + 2\gamma^\mu p^\nu}{2\cdot p\cdot k} + \frac{\gamma^\nu \not k' \gamma^\mu - 2\gamma^\nu p^\mu}{2p\cdot k'}\right\}$$

$$\times\,(\not p+m)\times\left\{\frac{\gamma^\sigma \not k \gamma^\rho + 2\gamma^\rho p^\sigma}{2p\cdot k} + \frac{\gamma^\rho \not k' \gamma^\sigma - 2\gamma^\sigma p^\rho}{2p\cdot k'}\right\}\Bigg\}$$

(from polarization sums)

There's a lot of work & we don't do all details here...

Use: $\mathrm{tr}(\not p' \gamma^\mu \not k \gamma^\nu \not p \gamma_\nu \not k \gamma_\mu) = -2\,\mathrm{tr}(\not p' \gamma^\mu \not k \not p \not k \gamma_\mu)$

since $\gamma^\mu \not p \gamma_\mu = -2\not p$

next: $\gamma^\mu \not k \not p \not k \gamma_\mu = -2\not k \not p \not k$

so...

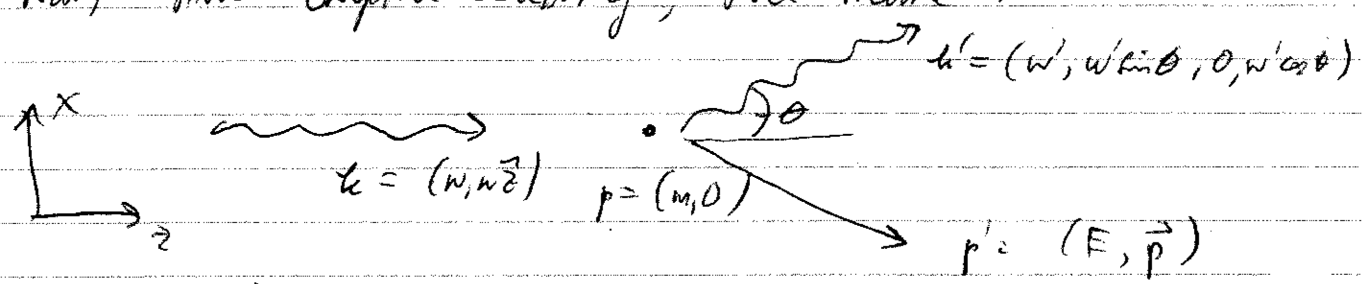
$$= +4\,\mathrm{tr}[\not p' \not k \not p \not k] = 4\cdot 4\big((p'\cdot k)(p\cdot k) - (p'\cdot p)\overset{0}{(k\cdot k)} + (p'\cdot k)(p\cdot k)\big)$$

$$= 32(p'\cdot k)(p\cdot k)$$

In the end, we get

$$\boxed{\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = 2e^4\left\{\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} + 2m^2\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right) + m^4\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right)^2\right\}}$$

Now, for Compton scattering, the picture is ...



$p'^2 = m^2$

$$\to m^2 = (p')^2 = (p+k-k')^2 = p^2 + \cancel{k^2} + \cancel{k'^2} + 2p\cdot(k-k') - 2k\cdot k'$$

$$= m^2 + 2p\cdot(k-k') - 2k\cdot k'$$

$$= m^2 + 2m(\omega - \omega') - 2\omega\omega'(1-\cos\theta)$$

So $\quad 2m(\omega-\omega') - 2\omega\omega'(1-\cos\theta) = 0$

$$\Rightarrow \boxed{\omega' = \frac{1}{\frac{1}{m}(1-\cos\theta)+\frac{1}{\omega}} = \frac{\omega}{1+\frac{\omega}{m}(1-\cos\theta)}}$$

Compton's formula for the shift in the photon wavelength.

→ Next, have to work out the phase space integrals

$$\int d\Pi_2 = \int \frac{d^3k'}{(2\pi)^3 2\omega'} \frac{d^3p'}{(2\pi)^3 2E'} (2\pi)^4 \delta^{(4)}(k'+p'-k-p)$$

$$= \int \frac{d^3k'}{(2\pi)^3 2\omega'} \frac{1}{2E'} (2\pi)\, \delta(\omega'+E'-\omega-m) \qquad (\vec{p}' = \vec{k}+\vec{p}-\vec{k}')$$

$$= \int \frac{d^3k'}{(2\pi)^3 2\omega' 2E'} (2\pi)\, \delta\left(\omega' + \sqrt{\omega^2+\omega'^2-2\omega\omega'\cos\theta+m^2} - \omega - m\right)$$

now use $\quad \delta(f(x)) = \dfrac{\delta(x-x_0)}{|f'(x_0)|}$

$$= \int \frac{\omega'^2\, d\omega'\, d\Omega}{(2\pi)^3 4\omega' E'} (2\pi) \frac{\delta\left(\omega' - \omega/(1+\omega/m(1-\cos\theta))\right)}{\left|1+\frac{\omega'-\omega\cos\theta}{E'}\right|} \quad \sqrt{\cdots}$$

$$= \frac{1}{8\pi}\int_{-1}^{1} \frac{d\cos\theta \cdot \omega'}{\left|1+\frac{\omega'-\omega\cos\theta}{E'}\right| E'} = \frac{1}{8\pi}\int_{-1}^{1}\frac{d\cos\theta\cdot\omega'}{|E'+\omega'-\omega\cos\theta|}$$

But $E' + \omega' =$ total energy $= \omega + m$ (initial energy)

So, we have...

$$\frac{1}{8\pi}\int_{-1}^{1}\frac{d\cos\theta\, \omega'}{|\omega+m-\omega\cos\theta|} = \frac{\omega'}{8\pi}\int\frac{d\cos\theta}{m+\omega(1-\cos\theta)}$$

now, $v_{\text{electron}} = 0$, so $|v_A - v_B| = 1$ ($v_A = 0$, $v_B = 1$)

So
$$\frac{d\sigma}{d\cos\theta} = \frac{1}{2\omega}\frac{1}{2m}\frac{\omega'}{8\pi(m+\omega(1-\cos\theta))}\frac{1}{4}\sum_{\text{spins}}|M|^2$$

Since $\dfrac{1}{m+\omega(1-\cos\theta)} = \dfrac{\omega'}{m\omega}$, we have

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{32\pi}\frac{\omega'^2}{m^2\omega^2}\left\{2e^4\left(\frac{p\cdot k'}{p\cdot k}+\frac{p\cdot k}{p\cdot k'}+2m^2\left(\frac{1}{p\cdot k}-\frac{1}{p\cdot k'}\right)+m^4\left(\frac{1}{p\cdot k}-\frac{1}{p\cdot k'}\right)^2\right\}\right.$$

Using $p\cdot k = m\omega$ & $p\cdot k' = m\cdot\omega'$

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{32\pi}\frac{\omega'^2}{m^2\omega^2}\left\{2e^4\left(\frac{\omega'}{\omega}+\frac{\omega}{\omega'}+2m\left(\frac{1}{\omega}-\frac{1}{\omega'}\right)+m^2\left(\frac{1}{\omega}-\frac{1}{\omega'}\right)^2\right)\right\}$$

$$= \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left\{\frac{\omega'}{\omega}+\frac{\omega}{\omega'}-\sin^2\theta\right\}$$

Klein-Nishina formula (1929)

when $\omega\to 0$, $\omega'/\omega \to 1$ &

$$\frac{d\sigma}{d\cos\theta} \to \frac{\pi\alpha^2}{m^2}(1+\cos^2\theta), \quad \sigma_{\text{tot}} = \frac{8\pi\alpha^2}{3m^2}$$

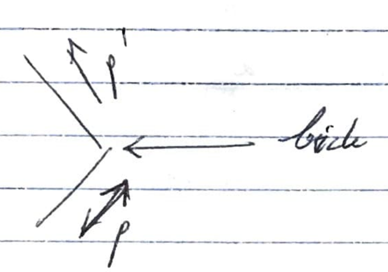
Thomson cross section for scattering of classical EM by free e^-

Radiative Corrections

Soft Brehmsstrahlung → low freq radiation when e^- undergoes sudden acceleration

Classical picture

@ $t = 0$, $x = 0$, e^- is given a momentum kick.



→ look at radiation of Maxwell's eqn... once we know the $j^\mu(x,t)$...

Recall for particle @ rest..

$$j^\mu = e\cdot(\text{particle density}, \vec{0}) = (1,0,0,0)\cdot e\cdot \delta^3(\vec{x})$$

$$\Rightarrow j^\mu(x) = \int dt' \, (1,0,0,0)^\mu \, e \, \delta^{(4)}(x - y(t'))$$

where $y(t') = (t', 0, 0, 0)$. world line of particle.

In general... $y^\mu(\tau)$

$$j^\mu(x) = e\int d\tau \, \frac{dy^\mu}{d\tau} \, \delta^{(4)}(x - y(\tau))$$

picks τ such that $y^0(\tau) = \tau$

At τ we have $\delta^{(3)}(\vec{x} - \vec{y}(\tau))$ and 4-velocity $\frac{dy^\mu(\tau)}{d\tau}$

check that j^μ is conserved...

Let $f(x)$ be fn such that $f(x) \to 0$ as $x \to \infty$

then $\int d^4x\, f(x)\, \partial_\mu j^\mu(x) = \int d^4x\, f(x)\, e \int d\tau \frac{dy^\mu}{d\tau} \partial_\mu \delta^{(4)}(x - y(\tau))$

$$= -e \int d\tau \frac{dy^\mu}{d\tau}(\tau) \left.\frac{\partial f(x)}{\partial x}\right|_{x = y(\tau)}$$

$$= -e\, f(y^\mu(\tau))\Big|_{\tau=-\infty}^{\tau=\infty} = 0 \checkmark$$

$\Rightarrow$ world line looks like...

$$y^\mu(\tau) = \begin{cases} \frac{p^\mu}{m}\tau & \tau < 0 \\ \frac{p'^\mu}{m} & \tau > 0 \end{cases}$$

$$\rightarrow j^\mu(x) = e \int_0^\infty d\tau \frac{p'^\mu}{m} \delta^{(4)}(x - p'/m\, \tau)$$
$$+ e \int_{-\infty}^0 d\tau \frac{p^\mu}{m} \delta^{(4)}(x - p/m\, \tau)$$

FT...

$$\tilde{j}^\mu(k) = ie \left\{ \frac{p'^\mu}{k \cdot p' + i\varepsilon} - \frac{p^\mu}{k \cdot p - i\varepsilon} \right\}$$

maxwell...

$$\partial_\mu \partial^\mu A^\nu = j^\nu \Rightarrow -k^2 \tilde{A}^\mu(k) = \tilde{j}^\mu(k)$$

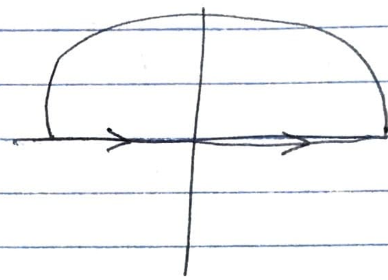
$$\Rightarrow \tilde{A}^\mu(k) = \frac{-ie}{k^2}\left(\frac{p'^\mu}{k \cdot p' + i\varepsilon} - \frac{p^\mu}{k \cdot p - i\varepsilon} \right)$$

So

$$A^\mu(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot x}\, \frac{(-ie)}{k^2}\left(\frac{p'^\mu}{k\cdot p' + i\epsilon} - \frac{p^\mu}{k\cdot p - i\epsilon}\right)$$

When $x^0 < 0$, momentum $= p^\mu$ ⇒ the term p'^μ cannot contribute.

→ $x^0 < 0$



Can have poles at $-|\vec{k}| \pm i\delta$ & $|\vec{k}| \pm i\delta$, but if pole at

$\mp|\vec{k}| + i\delta$ ⇒ contribution from p'^μ

⇒ both of these must be in the lower half plane

→ $x^0 < 0$ ⇒ residue @ $k\cdot p = i\epsilon$.

→ $$A^\mu(x) = \int \frac{d^3k}{(2\pi)^4}\, e^{+i\vec{k}\cdot\vec{x}}\, e^{-i\vec{k}\cdot\vec{p}/p^0\, t}\, (2\pi i)\, \frac{ie}{k^2}\, \frac{p^\mu}{p^0}$$

In rest frame ... $p^0 = m$, $\vec{p} = 0$

⇒ $$A^\mu(x) = e\int \frac{d^3k}{(2\pi)^4}\, e^{i\vec{k}\cdot\vec{x}}\, \frac{(1,0,0,0)}{|\vec{k}|^2}$$

↑

Coulomb potential ~~for~~ in the $\mu=0$ component & zero for other components.

→ Same thing with $x^0 > 0$ ($k\cdot p' = -i\epsilon$ ← residue)

~~Ettk~~ the interesting Bremsstrahlung radiation comes from the other 2 poles ... at $k^0 = |\vec k| - i\delta$
$k^0 = -|\vec k| - i\delta$

Residues give...

$$A^\mu_{rad}(x) = \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{-e}{2|\vec k|} e^{+i\vec k\cdot\vec x} \left(\frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p} \right)\Bigg|_{k^0=|\vec k|} e^{-i|\vec k|\cdot t} \right.$$

$$\left. + \frac{e}{2|\vec k|} e^{+i\vec k\cdot\vec x} \left(\frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p} \right)\Bigg|_{k^0=-|\vec k|} \cdot e^{+i|\vec k|\cdot t} \right\}$$

$$2^{nd} \text{ term} = \int \frac{d^3k}{(2\pi)^3} \frac{-e}{2|\vec k|} e^{-i\vec k\cdot\vec x} \left(\frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p} \right)\Bigg|_{k^0=|\vec k|} e^{+i|\vec k|\cdot t}$$

$$\|$$

complex conjugate of 1^{st} term...

$$\Rightarrow A^\mu_{rad}(x) = \mathrm{Re}\left\{ \int \frac{d^3k}{(2\pi)^3} \mathcal{Q}^\mu(\vec k)\, e^{+i\vec k\cdot\vec x} e^{-i|\vec k|\cdot t} \right\}$$

$$\downarrow$$

$$\frac{-e}{|\vec k|} \left(\frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p} \right)\Bigg|_{k^0=|\vec k|}$$

Now, recall that

$$E^i(x) = -F^{0i} = -\partial_0 A^i - \partial_i A^0 = -\partial_0 \vec A - \vec\nabla A^0$$

$$B^i(x) = \vec\nabla \times \vec A$$

Choose frame s.t. $p^0 = p'^0 = E$.

Let $k^\mu = (|\vec{k}|, \vec{k})$

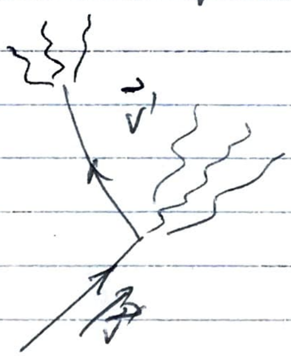
$p^\mu = (E, E\vec{v})$ $\quad p'^\mu = (E, E\vec{v}')$

then

$$\frac{1}{k \cdot p'} = \frac{1}{E|\vec{k}|(1 - \hat{k}\cdot\vec{v}')}$$

$$\frac{1}{k \cdot p} = \frac{1}{E|\vec{k}|(1 - \hat{k}\cdot\vec{v})}$$

→ Radiation peaked when $\vec{k}$ points in the same direction as $\vec{v}$ or $\vec{v}'$.



also the note that $k_\mu a^\mu = 0$.