SCATTERING THEORY

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Chapter 7 in Sakurai

1) Lippmann - Schwinger Equation

Assume time - independence.

$$H = H_0 + V = \frac{p^2}{2m} + V .$$

Without scatterer, $V = 0 \Rightarrow$ eigenstate is just free particle $|p\rangle$

$V \neq 0 \Rightarrow$ eigenstate changes.

But if collision is elastic, eigenstate changes, but eigenvalue doesn't.

→ Let $|\phi\rangle$ be eigenstate of H_0: $\boxed{H_0|\phi\rangle = E|\phi\rangle}$

→ Want to solve $\boxed{(H_0 + V)|\psi\rangle = E|\psi\rangle}$

- Both H_0 & V have continuous energy spectra
- Want $|\psi\rangle$ s.t. $|\psi\rangle \to |\phi\rangle$ as $V \to 0$.

→ "Solution" looks like

$$\boxed{|\psi\rangle = \frac{1}{E - H_0} V|\psi\rangle + |\phi\rangle}$$

$\frac{1}{E - H_0}$ is singular, so we make it slightly complex ...

$$\boxed{|\psi^{(\pm)}\rangle = |\phi\rangle + \frac{1}{E - H_0 \pm i\varepsilon} V|\psi^{(\pm)}\rangle}$$

→ Lippmann - Schwinger Eqn

For position basis; we have

$$\langle x | \psi^{(\pm)} \rangle = \langle x | \phi \rangle + \int d^3x' \left\langle x \left| \frac{1}{E - H_0 \pm i\varepsilon} \right| x' \right\rangle \langle x' | V | \psi^{(\pm)} \rangle$$

We know that $\langle x | \phi \rangle = \dfrac{e^{i\vec{p}\cdot\vec{x}/\hbar}}{(2\pi\hbar)^{3/2}}$

Now want to evaluate the kernel of the integral eqn

$$\boxed{G_{\pm}(x, x') = \frac{\hbar^2}{2m} \left\langle x \left| \frac{1}{E - H_0 \pm i\varepsilon} \right| x' \right\rangle}$$

Claim: $G_{\pm}(x, x') = \dfrac{-1}{4\pi} \dfrac{e^{\pm ik|x - x'|}}{|x - x'|}$ where $E = \dfrac{\hbar^2 k^2}{2m}$

Pf//

$$\frac{\hbar^2}{2m} \left\langle x \left| \frac{1}{E - H_0 \pm i\varepsilon} \right| x' \right\rangle = \frac{\hbar^2}{2m} \int d^3p' \int d^3p'' \langle x | p' \rangle$$

$$\cdot \left\langle p' \left| \underbrace{\frac{1}{E - H_0 \pm i\varepsilon}}_{\downarrow\ (P^2/2m)} \right| p'' \right\rangle \langle p'' | x' \rangle$$

Use

$$\left\langle p' \left| \frac{1}{E - p^2/2m \pm i\varepsilon} \right| p'' \right\rangle = \frac{\delta^{(3)}(p' - p'')}{E - (p')^2/2m \pm i\varepsilon}$$

So

$$\frac{\hbar^2}{2m} \left\langle x \left| \frac{1}{E - H_0 \pm i\varepsilon} \right| x' \right\rangle = \frac{\hbar^2}{2m} \int \frac{d^3p'}{(2\pi\hbar)^3} \frac{e^{ip'\cdot(x - x')/\hbar}}{[E - p'^2/2m \pm i\varepsilon]}$$

Now, $E = \hbar^2 k^2/2m$. set $p' = \hbar q$, get

$$\frac{1}{(2\pi)^3}\int_0^\infty q^2 dq \int_0^{2\pi} d\phi \int_{-1}^{+1} \frac{d(\cos\theta)\, e^{i|q|\cdot|x-x'|\cos\theta}}{k^2 - q^2 \pm i\varepsilon}$$

$$= \frac{-1}{8\pi}\frac{1}{i|x-x'|}\int_{-\infty}^{\infty} \frac{dq\, q\,\left(e^{iq|x-x'|} - e^{-iq|x-x'|}\right)}{q^2 - k^2 \mp i\varepsilon}$$

(Residue Thm)

$$= \frac{-1}{4\pi}\frac{e^{\pm ik|x-x'|}}{|x-x'|}$$

by noting that the integrand has poles $q = \pm k\sqrt{1 \pm \left(\frac{i\varepsilon}{k^2}\right)}$

$\simeq k \pm i\varepsilon'$

So $G_\pm(x,x') = \frac{-1}{4\pi}\frac{e^{ik|x-x'|}}{|x-x'|}$

But this is just the Green's function for the Helmholtz eqn.

$$\boxed{(\nabla^2 + k^2)\, G_\pm(x,x') = \delta^{(3)}(x-x')}$$

With $G_\pm$, we have

$$\langle x|\psi^{(\pm)}\rangle = \langle x|\phi\rangle - \frac{2m}{\hbar^2}\int d^3x' \frac{e^{\pm ik|x-x'|}}{4\pi|x-x'|}\langle x'|V|\psi^{(\pm)}\rangle$$

= sum of wfn of incident wave $\langle x|\phi\rangle$ and a term that represents the effect of scattering.

When V is <u>local</u> (i.e. V is diagonal in the x-representation)

i.e. $\langle x' | V | x'' \rangle = V(x') \, \delta^{(3)}(x' - x'')$

Then $\langle x' | V | \psi^{(\pm)} \rangle = \int d^3x'' \langle x' | V | x'' \rangle \langle x'' | \psi^{(\pm)} \rangle$

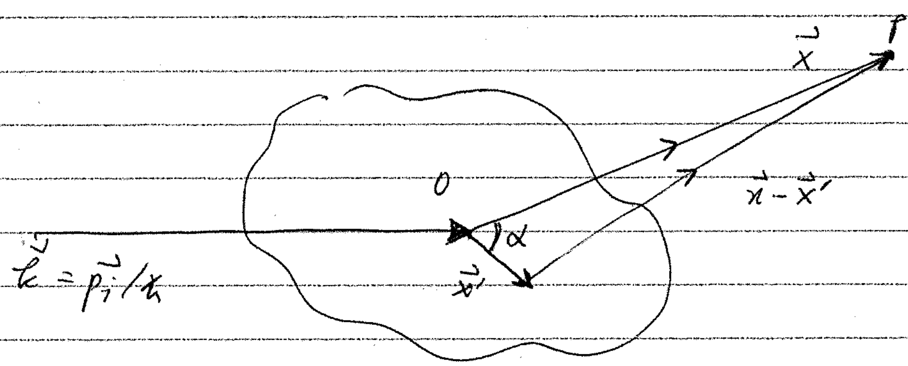
$= V(x') \langle x' | \psi^{(\pm)} \rangle$

and so, when V is <u>local</u>

$$\langle x | \psi^{(\pm)} \rangle = \langle x | \phi \rangle - \frac{2m}{\hbar^2} \int d^3x' \, \frac{e^{\pm ik|x-x'|}}{4\pi |x-x'|} V(x') \langle x' | \psi^{(\pm)} \rangle$$

What does this mean?

↳ we can set $|x| \gg |x'|$ where



Introducing $r = |x|$, $r' = |x'|$, $\alpha = \angle(x, x')$

when $r \gg r'$, $|x - x'| \simeq r - \hat{r} \cdot x'$

$\hat{r} = \dfrac{x}{|x|}$

Define $\vec{k}' \equiv k\hat{r}$. Then

$$e^{\pm ik|x-x'|} \simeq e^{\pm ikr} \cdot e^{\mp ik' \cdot x'} \quad \text{when } r \gg 1$$

and $\frac{1}{|x-x'|} \sim \frac{1}{r}$. Use $|k\rangle$ rather than $|p\rangle$ to remove extra $\hbar$'s ...

↳ $\vec{k} = \vec{p_i}/\hbar \Rightarrow$ get $\langle x|k\rangle = \dfrac{e^{ik\cdot x}}{(2\pi)^{3/2}}$.

• So, we have, in the ~~[illegible]~~ $|x| \gg |x'|$ limit...

$$\langle x|\psi^{(\pm)}\rangle \xrightarrow{\text{large } r} \langle x|k\rangle - \frac{1}{4\pi}\frac{2m}{\hbar^2}\frac{e^{ikr}}{r}\int d^3x' \cdot$$

$$e^{-ik'\cdot x'} V(x')\langle x'|\psi^{(+)}\rangle$$

$$\boxed{\langle x|\psi^{(\pm)}\rangle = \frac{1}{(2\pi)^{3/2}}\left[e^{ik\cdot x} + \frac{e^{ikr}}{r} f(k',k)\right]}$$

original plane wave in $\vec{k}$

plus outgoing spherical wave with amplitude $f(k';k)$

Here,

$$f(k',k) = \frac{-1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3\int d^3x' \underbrace{\frac{e^{-ik'\cdot x'}}{(2\pi)^{3/2}}}_{\langle k'|x'\rangle} V(x')\langle x'|\psi^{(+)}\rangle$$

$$= \frac{-1}{4\pi}(2\pi)^3\frac{2m}{\hbar^2}\langle k'|V|\psi^{(+)}\rangle$$

We won't worry about the backward propagating solution $\psi^{(-)}$, but it's easy to find out what it is by a similar approach.

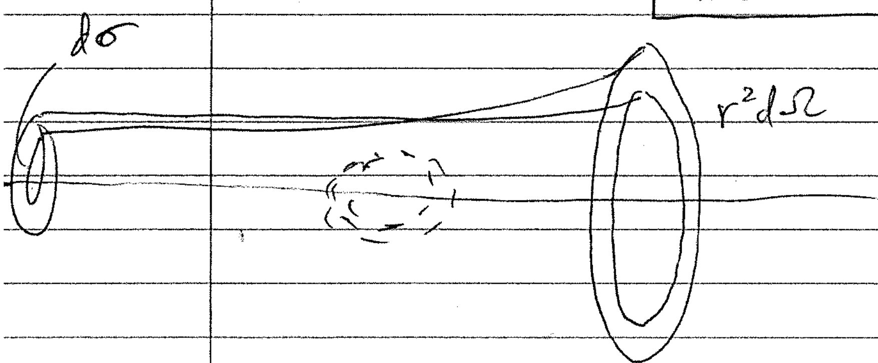
→ Need to obtain the differential cross section $\frac{d\sigma}{d\Omega}$

$$\frac{d\sigma}{d\Omega} d\Omega = \frac{\text{\# particles scattered into } d\Omega \text{ / time}}{\text{\# incident particles / area / time}}$$

$$= \frac{r^2 |j_{scatt}| \, d\Omega}{|j_{inc}|} = |f(k';k)|^2 d\Omega$$

and so

$$\boxed{\frac{d\sigma}{d\Omega} = |f(k';k)|^2}$$



② The Born Approximation.