

1 Ramsey Fringes Overview:

Following a double pulse, the population of the excited state is:

$$P_2 = 4 \sin^2 \theta \sin^2 \frac{\Omega' \tau}{2} \left\{ \cos \frac{\Omega' \tau}{2} \cos \frac{\Delta_0 T}{2} - \cos \theta \sin \frac{\Omega' \tau}{2} \sin \frac{\Delta_0 T}{2} \right\}$$

under the assumption that initially,

$$\begin{pmatrix} C_1(0) \\ C_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The final state vector is:

$$\begin{pmatrix} C_1(2\tau + T) \\ C_2(2\tau + T) \end{pmatrix} = \rho_2 D \rho_1 \begin{pmatrix} C_1(0) \\ C_2(0) \end{pmatrix}$$

where $|C_1|^2 + |C_2|^2 = 1$ for all value of time, and ρ_1 and ρ_2 are propagators associated with Pulse 1 and Pulse 2 (both with width τ), respectively. D is a propagator associated with the field-free evolution of duration T .

Specifically, in the interaction representation:

$$\rho_1 = e^{-i\bar{\Delta}\tau} \begin{pmatrix} e^{-i\frac{\Delta_0}{2}\tau} \left(\cos \frac{\Omega'\tau}{2} + i \cos \theta \sin \frac{\Omega'\tau}{2} \right) & i e^{-i(\frac{\Delta_0}{2}\tau + \phi_0)} \sin \theta \sin \frac{\Omega'\tau}{2} \\ i e^{i(\frac{\Delta_0}{2}\tau + \phi_0)} \sin \theta \sin \frac{\Omega'\tau}{2} & e^{i\frac{\Delta_0}{2}\tau} \left(\cos \frac{\Omega'\tau}{2} - i \cos \theta \sin \frac{\Omega'\tau}{2} \right) \end{pmatrix}$$

$$\rho_2 = e^{-i\bar{\Delta}\tau} \begin{pmatrix} e^{-i\frac{\Delta_0}{2}\tau} \left(\cos \frac{\Omega'\tau}{2} + i \cos \theta \sin \frac{\Omega'\tau}{2} \right) & i e^{-i(\frac{\Delta_0}{2}\tau + \phi_0)} \sin \theta \sin \frac{\Omega'\tau}{2} e^{-i\Delta_0(\tau+T)} \\ i e^{i(\frac{\Delta_0}{2}\tau + \phi_0)} \sin \theta \sin \frac{\Omega'\tau}{2} e^{i\Delta_0(\tau+T)} & e^{i\frac{\Delta_0}{2}\tau} \left(\cos \frac{\Omega'\tau}{2} - i \cos \theta \sin \frac{\Omega'\tau}{2} \right) \end{pmatrix}$$

and

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note that D , in the interaction representation, is the identity matrix. This is different from Ramsey's original approach in which the state vector does evolve and change during the delay time T . The angle θ is defined as:

$$\sin \theta = \frac{\Omega_0^*}{\Omega'}$$

and

$$\cos \theta = \frac{\Delta_0 + \Delta_d}{\Omega'}$$

where Ω_0^* is the complex conjugate of the Rabi rate, and Ω' can be defined as the "effective Rabi rate."

$$\Omega' = \sqrt{|\Omega_0^*| + (\Delta_0 + \Delta_d)^2}$$

2 Detailed Derivation for P_f

In the interaction representation,

$$i \begin{pmatrix} \dot{a}_i(t) \\ \dot{a}_f(t) \end{pmatrix} = \begin{pmatrix} \Delta_i & -\frac{\Omega_0^*}{2} e^{-i\Delta_0 t} \\ -\frac{\Omega_0}{2} e^{i\Delta_0 t} & \Delta_f \end{pmatrix} \begin{pmatrix} a_i(t) \\ a_f(t) \end{pmatrix} \quad (1)$$

where Δ_i is the ac Stark shift in the $|i\rangle$ state and Δ_f is the ac Stark shift in the $|f\rangle$ state. We first solve for a_i :

$$i\ddot{a}_i = \Delta_i \dot{a}_i - \frac{\Omega_0^*}{2} e^{-i\Delta_0 t} \dot{a}_f + \frac{i\Delta_0}{2} \Omega_0^* e^{-i\Delta_0 t} a_f, \quad (2)$$

where

$$a_f = \left(\frac{\Omega_0^*}{2} e^{-i\Delta_0 t} \right)^{-1} (\Delta_i a_i - i\dot{a}_i) = \frac{2}{\Omega_0^*} e^{i\Delta_0 t} (\Delta_i a_i - i\dot{a}_i). \quad (3)$$

From Eq. (1), we also have

$$\begin{aligned} \dot{a}_f &= i^{-1} \left(-\frac{\Omega_0}{2} e^{i\Delta_0 t} a_i + \Delta_f a_f \right) \\ &= i^{-1} \left[-\frac{\Omega_0}{2} e^{i\Delta_0 t} a_i + \Delta_f (\Delta_i a_i - i\dot{a}_i) \left(\frac{2}{\Omega_0^*} e^{i\Delta_0 t} \right) \right] \\ &= \frac{2i}{\Omega_0^*} e^{i\Delta_0 t} \left(\frac{|\Omega_0|^2}{4} a_i - \Delta_f \Delta_i a_i + i\Delta_f \dot{a}_i \right). \end{aligned} \quad (4)$$

Therefore,

$$\begin{aligned} \ddot{a}_i &= -i\Delta_i \dot{a}_i + \frac{i\Omega_0^*}{2} e^{-i\Delta_0 t} \left[\frac{2i}{\Omega_0^*} e^{i\Delta_0 t} \left(\frac{|\Omega_0|^2}{4} a_i - \Delta_f \Delta_i a_i + i\Delta_f \dot{a}_i \right) \right] \\ &\quad + \frac{\Delta_0}{2} \Omega_0^* e^{-i\Delta_0 t} \frac{2}{\Omega_0^*} e^{i\Delta_0 t} (\Delta_i a_i - i\dot{a}_i) \\ &= -i\Delta_i \dot{a}_i - \frac{|\Omega_0|^2}{4} a_i + \Delta_f \Delta_i a_i - i\Delta_f \dot{a}_i + \Delta_0 \Delta_i a_i - i\Delta_0 \dot{a}_i \\ &= -i(\Delta_0 + \Delta_i + \Delta_f) \dot{a}_i - \left[\frac{|\Omega_0|^2}{4} - \Delta_i(\Delta_f + \Delta_0) \right] a_i \\ &= -i(\Delta_0 + \Delta_i + \Delta_f) \dot{a}_i - \frac{1}{4} (|\Omega_0|^2 - 4\Delta_i \Delta_f - 4\Delta_i \Delta_0) a_i. \end{aligned} \quad (5)$$

We obtain the first second-order homogeneous differential equation:

$$\ddot{a}_i + i(\Delta_0 + \Delta_i + \Delta_f) \dot{a}_i + \left[\frac{|\Omega_0|^2}{4} - \Delta_i(\Delta_f + \Delta_0) \right] a_i = 0. \quad (6)$$

Let a guess solution be $a_i(t) = a_0 e^{i\omega t}$. The characteristic equation is:

$$\begin{aligned} -\omega^2 + i(\Delta_0 + \Delta_i + \Delta_f) \omega + \frac{1}{4} (|\Omega_0|^2 - 4\Delta_0 \Delta_i - 4\Delta_i \Delta_f) &= 0 \\ -\omega^2 - (\Delta_0 + \Delta_i + \Delta_f) \omega + \frac{1}{4} (|\Omega_0|^2 - 4\Delta_0 \Delta_i - 4\Delta_i \Delta_f) &= 0. \end{aligned} \quad (7)$$

Solving the quadratic equation (7) and obtain w :

$$\begin{aligned}\omega &= -\frac{\Delta_0 + \Delta_f + \Delta_i}{2} \pm \frac{1}{2}\sqrt{(\Delta_0 + \Delta_f + \Delta_i)^2 + |\Omega_0|^2 - 4\Delta_0\Delta_i - 4\Delta_i\Delta_f} \\ &= -\frac{\Delta_0 + \Delta_f + \Delta_i}{2} \\ &\quad \pm \frac{1}{2}\sqrt{(\Delta_0 + \Delta_f + \Delta_i)^2 + \Delta_0^2 + |\Omega_0|^2 + (\Delta_i - \Delta_f)^2 - 2\Delta_0(\Delta_i - \Delta_f)}\end{aligned}\quad (8)$$

Let $\bar{\Delta} = (\Delta_i + \Delta_f)/2$ and $\Delta_d = \Delta_f - \Delta_i$, this gives

$$\omega = -\frac{\Delta_0}{2} - \bar{\Delta} \pm \frac{1}{2}\sqrt{|\Omega_0|^2 + (\Delta_0 + \Delta_d)^2}\quad (9)$$

Next, let the “effective Rabi rate” be Ω' , defined as

$$\Omega' = \sqrt{|\Omega_0|^2 + (\Delta_0 + \Delta_d)^2}.\quad (10)$$

The general solution to eq. (7) is:

$$\begin{aligned}a_i &= a_+ e^{i\omega_+ t} + a_- e^{i\omega_- t} \\ &= e^{-i\bar{\Delta}t} e^{-i\frac{\Delta_0}{2}t} \left(a_+ e^{i\frac{\Omega'}{2}t} + a_- e^{-i\frac{\Omega'}{2}t} \right)\end{aligned}\quad (11)$$

So,

$$a_i = e^{-i\bar{\Delta}t} e^{-i\frac{\Delta_0}{2}t} \left(A \cos \frac{\Omega't}{2} + B \sin \frac{\Omega't}{2} \right)\quad (12)$$

Next, we solve for a_f . From eq. (3):

$$\begin{aligned}a_f &= \frac{2}{\Omega_0^*} e^{i\Delta_0 t} (\Delta_i a_i - i\dot{a}_i) \\ &= \frac{2}{\Omega_0^*} e^{i\Delta_0 t} \left[\Delta_i e^{-i\bar{\Delta}t} e^{-i\frac{\Delta_0}{2}t} \left(A \cos \frac{\Omega't}{2} + B \sin \frac{\Omega't}{2} \right) - i\dot{a}_i \right] \\ &= \frac{2}{\Omega_0^*} e^{i\frac{\Delta_0}{2}t} \left[\Delta_i e^{-i\bar{\Delta}t} \left(A \cos \frac{\Omega't}{2} + B \sin \frac{\Omega't}{2} \right) - i\dot{a}_i \right]\end{aligned}\quad (13)$$

where

$$\begin{aligned}-i\dot{a}_i &= (-i)^2 \left(\bar{\Delta} + \frac{\Delta_0}{2} \right) e^{-i\bar{\Delta}t} e^{-i\frac{\Delta_0}{2}t} \left[\left(A \cos \frac{\Omega't}{2} + B \sin \frac{\Omega't}{2} \right) \right. \\ &\quad \left. + -i\frac{\Omega'}{2} \left(-A \sin \frac{\Omega't}{2} + B \cos \frac{\Omega't}{2} \right) \right] \\ &= e^{-i\bar{\Delta}t} e^{-i\frac{\Delta_0}{2}t} \left[-\left(\bar{\Delta} + \frac{\Delta_0}{2} \right) \left(A \cos \frac{\Omega't}{2} + B \sin \frac{\Omega't}{2} \right) \right. \\ &\quad \left. + i\frac{\Omega'}{2} \left(A \sin \frac{\Omega't}{2} - B \cos \frac{\Omega't}{2} \right) \right].\end{aligned}\quad (14)$$

Assume that at $t = 0$, $A = a_i(0)$ and

$$B = i \frac{\Omega_0^*}{\Omega'} a_f(0) + i \frac{\Delta_d + \Delta_0}{\Omega'} a_i(0). \quad (15)$$

So, from Eq. (12):

$$a_i(t) = e^{-i(\bar{\Delta} + \frac{\Delta_0}{2})t} \left\{ a_i(0) \left[\cos \frac{\Omega' t}{2} + i \frac{\Delta_0 + \Delta_d}{\Omega'} \sin \frac{\Omega' t}{2} \right] + a_f(0) \frac{i\Omega_0^*}{\Omega'} \sin \frac{\Omega' t}{2} \right\} \quad (16)$$

From Eq. (13) and (14), we obtain an expression for $a_f(t)$:

$$\begin{aligned} a_f(t) &= \frac{2}{\Omega_0^*} e^{-i\bar{\Delta}t} e^{i\frac{\Delta_0}{2}t} \left\{ \Delta_i \left(a_i(0) \cos \frac{\Omega' t}{2} + \left(\frac{i\Omega_0^*}{\Omega'} a_f(0) + i \frac{\Delta_d + \Delta_0}{\Omega'} a_i(0) \right) \sin \frac{\Omega' t}{2} \right) \right. \\ &\quad \left. - \left(\bar{\Delta} + \frac{\Delta_0}{2} \left(a_i \cos \frac{\Omega' t}{2} + \left(\frac{i\Omega_0^*}{\Omega'} a_f(0) + \frac{i}{2} \frac{\Delta_d + \Delta_0}{\Omega'} a_i(0) \right) \sin \frac{\Omega' t}{2} \right) \right) \right. \\ &\quad \left. i \frac{\Omega'}{2} \left(a_i \sin \frac{\Omega' t}{2} - \left(\frac{i\Omega_0^*}{\Omega'} a_f(0) + \frac{i}{2} \frac{\Delta_d + \Delta_0}{\Omega'} a_i(0) \right) \cos \frac{\Omega' t}{2} \right) \right\} \\ &= \frac{2}{\Omega_0^*} e^{-i\bar{\Delta}t} e^{i\frac{\Delta_0}{2}t} \left\{ \left(\Delta_i - \bar{\Delta} - \frac{\Delta_0}{2} \right) \left(A \cos \frac{\Omega' t}{2} + B \sin \frac{\Omega' t}{2} \right) \right. \\ &\quad \left. \frac{i\Omega'}{2} \left(A \sin \frac{\Omega' t}{2} - B \cos \frac{\Omega' t}{2} \right) \right\}. \end{aligned} \quad (17)$$

Now, notice that

$$\Delta_i - \bar{\Delta} = \Delta_i - \frac{\Delta_i + \Delta_f}{2} = -\frac{\Delta_d}{2}. \quad (18)$$

So,

$$\begin{aligned} a_f(t) &= e^{-i\bar{\Delta}t} e^{i\frac{\Delta_0}{2}t} \left\{ A \left(-\frac{\Delta_d + \Delta_0}{\Omega_0^*} \cos \frac{\Omega' t}{2} + \frac{i\Omega'}{\Omega_0^*} \sin \frac{\Omega' t}{2} \right) \right. \\ &\quad \left. - B \left(\frac{\Delta_d + \Delta_0}{\Omega_0^*} \sin \frac{\Omega' t}{2} + \frac{i\Omega'}{\Omega_0^*} \cos \frac{\Omega' t}{2} \right) \right\} \\ &= e^{-i\bar{\Delta}t} e^{i\frac{\Delta_0}{2}t} \left\{ a_i(0) \left(-\frac{\Delta_d + \Delta_0}{\Omega_0^*} \cos \frac{\Omega' t}{2} + \frac{i\Omega'}{\Omega_0^*} \sin \frac{\Omega' t}{2} \right) \right. \\ &\quad \left. - \left(i \frac{\Omega_0^*}{\Omega'} a_f(0) + i \frac{\Delta_d + \Delta_0}{\Omega'} a_i(0) \right) \left(\frac{\Delta_d + \Delta_0}{\Omega_0^*} \sin \frac{\Omega' t}{2} + \frac{i\Omega'}{\Omega_0^*} \cos \frac{\Omega' t}{2} \right) \right\} \\ &= e^{-i\bar{\Delta}t} e^{i\frac{\Delta_0}{2}t} \left\{ a_i \sin \frac{\Omega' t}{2} \left(\frac{i\Omega'}{\Omega_0^* \Omega'} - \frac{i(\Delta_d + \Delta_0)^2}{\Omega_0^* \Omega'} \right) \right. \\ &\quad \left. a_f(0) \left(\cos \frac{\Omega' t}{2} - i \frac{\Delta_d + \Delta_0}{\Omega'} \sin \frac{\Omega' t}{2} \right) \right\}. \end{aligned} \quad (19)$$

Next, note that

$$\Omega'^2 = (\Delta_0 + \Delta_d)^2 + |\Omega_0|^2 = (\Delta_0 + \Delta_d)^2 + \Omega_0 \Omega_0^*. \quad (20)$$

So

$$a_f(t) = e^{-i\bar{\Delta}t} e^{i\frac{\Delta_0}{2}t} \left\{ a_i(0) \frac{i\Omega_0}{\Omega'} \sin \frac{\Omega't}{2} + a_f(0) \left(\cos \frac{\Omega't}{2} - i \frac{\Delta_d + \Delta_0}{\Omega'} \sin \frac{\Omega't}{2} \right) \right\}. \quad (21)$$

Finally, let us put everything together in matrix form:

$$\begin{pmatrix} a_i(t) \\ a_f(t) \end{pmatrix} = \mathcal{M} \begin{pmatrix} a_i(0) \\ a_f(0) \end{pmatrix}, \quad (22)$$

where $\mathcal{M}(t)$ is the matrix

$$e^{-i\bar{\Delta}t} \begin{pmatrix} e^{-i\frac{\Delta_0}{2}t} \left(\cos \frac{\Omega't}{2} + i \frac{\Delta_d + \Delta_0}{\Omega'} \sin \frac{\Omega't}{2} \right) & e^{-i\frac{\Delta_0}{2}t} \frac{i\Omega_0^*}{\Omega'} \sin \frac{\Omega't}{2} \\ e^{i\frac{\Delta_0}{2}t} \frac{i\Omega_0}{\Omega'} \sin \frac{\Omega't}{2} & e^{i\frac{\Delta_0}{2}t} \left(\cos \frac{\Omega't}{2} - i \frac{\Delta_d + \Delta_0}{\Omega'} \sin \frac{\Omega't}{2} \right) \end{pmatrix} \quad (23)$$

Let's define more terms:

$$\begin{aligned} \Omega_0 &= |\Omega_0| e^{i\phi_0} \\ \Omega_0^* &= |\Omega_0| e^{-i\phi_0}. \end{aligned} \quad (24)$$

Therefore,

$$\begin{aligned} \cos \theta &= \frac{\Delta_0 + \Delta_d}{\Omega'} \\ \sin \theta &= \frac{|\Omega_0|}{\Omega'} = \frac{\Omega_0}{\Omega'} e^{i\phi_0}. \end{aligned} \quad (25)$$

At time τ , the matrix $\mathcal{M}(\tau)$ is:

$$e^{-i\bar{\Delta}\tau} \begin{pmatrix} e^{-i\frac{\Delta_0}{2}\tau} \left(\cos \frac{\Omega'\tau}{2} + i \cos \theta \sin \frac{\Omega'\tau}{2} \right) & ie^{-i\frac{\Delta_0}{2}\tau} e^{-i\phi_0} \sin \theta \sin \frac{\Omega'\tau}{2} \\ ie^{i\frac{\Delta_0}{2}\tau} e^{i\phi_0} \sin \theta \sin \frac{\Omega'\tau}{2} & e^{i\frac{\Delta_0}{2}\tau} \left(\cos \frac{\Omega'\tau}{2} - i \cos \theta \sin \frac{\Omega'\tau}{2} \right) \end{pmatrix} \quad (26)$$

Further simplification gives the matrix $\mathcal{M}(\tau)$:

$$e^{-i\bar{\Delta}\tau} \begin{pmatrix} e^{-i\frac{\Delta_0}{2}\tau} \left(\cos \frac{\Omega'\tau}{2} + i \cos \theta \sin \frac{\Omega'\tau}{2} \right) & ie^{-i(\frac{\Delta_0}{2}\tau + \phi_0)} \sin \theta \sin \frac{\Omega'\tau}{2} \\ ie^{i(\frac{\Delta_0}{2}\tau + \phi_0)} \sin \theta \sin \frac{\Omega'\tau}{2} & e^{i\frac{\Delta_0}{2}\tau} \left(\cos \frac{\Omega'\tau}{2} - i \cos \theta \sin \frac{\Omega'\tau}{2} \right) \end{pmatrix} \quad (27)$$

What is an interpretation of $\mathcal{M}(\tau)$? The matrix $\mathcal{M}(\tau)$ represents what a laser pulse of width τ and intensity Ω_0 does to an initial state vector $(a_i(0) a_f(0))^\top$.

Now, our goal is to derive the final state vector following a (i) a pulse of width τ , (ii) another a pulse of width τ after some wait time T . Let us call the propagator associated with the second pulse \mathcal{N} . The next step is to derive \mathcal{N} .

$$\begin{pmatrix} a_i(2\tau + T) \\ a_f(2\tau + T) \end{pmatrix} = \mathcal{N} \begin{pmatrix} a_i(\tau + T) \\ a_f(\tau + T) \end{pmatrix}. \quad (28)$$

We assume that, at $t = 0$, the probability amplitude of finding the atom in the ground state is 1 and in the excited state is 0:

$$\boxed{\begin{pmatrix} a_i(0) \\ a_f(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}} \quad (29)$$

This gives the state amplitudes after time τ :

$$\begin{aligned} a_i(\tau) &= e^{-i\frac{\Delta_0}{2}\tau} \left\{ e^{-i\frac{\Delta_0}{2}\tau} \left(\cos \frac{\Omega'\tau}{2} + i \cos \theta \sin \frac{\Omega'\tau}{2} \right) \right\} \\ a_f(\tau) &= e^{-i\bar{\Delta}\tau} \left\{ i e^{i(\frac{\Delta_0}{2}\tau + \phi_0)} \sin \theta \sin \frac{\Omega'\tau}{2} \right\} \end{aligned} \quad (30)$$

The derivation of \mathcal{N} should be quite similar to that of \mathcal{M} . However, we should also take into account the wait time T . It turns out that we only need to add the extra terms $e^{i\Delta_0 t}$ and $e^{-i\Delta_0 t}$ to the off-diagonals. These terms represent how the state vector evolves over the “rest-time” T . The matrix $\mathcal{N}(T)$ has the following form:

$$\boxed{e^{-i\bar{\Delta}\tau} \begin{pmatrix} e^{-i\frac{\Delta_0}{2}\tau} \left(\cos \frac{\Omega'\tau}{2} + i \cos \theta \sin \frac{\Omega'\tau}{2} \right) & i e^{-i(\frac{\Delta_0}{2}\tau + \phi_0)} \sin \theta \sin \frac{\Omega'\tau}{2} e^{-i\Delta_0 T} \\ i e^{i(\frac{\Delta_0}{2}\tau + \phi_0)} \sin \theta \sin \frac{\Omega'\tau}{2} e^{i\Delta_0 T} & e^{i\frac{\Delta_0}{2}\tau} \left(\cos \frac{\Omega'\tau}{2} - i \cos \theta \sin \frac{\Omega'\tau}{2} \right) \end{pmatrix}} \quad (31)$$

So, the final state vector, as represented by the initial state vector and $\mathcal{M}(\tau)$ and $\mathcal{N}(\tau + T)$ is:

$$\begin{pmatrix} a_i(2\tau + T) \\ a_f(2\tau + T) \end{pmatrix} = \mathcal{N}(\tau + T) \mathcal{M}(\tau) \begin{pmatrix} a_i(0) \\ a_f(0) \end{pmatrix} = \mathcal{N}(\tau + T) \mathcal{M}(\tau) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (32)$$

Since we're only interested in the final state probability amplitude, we can ignore

the initial state amplitude:

$$\begin{aligned}
 a_f(2\tau + T) &= e^{-i\bar{\Delta}\tau} \left\{ i e^{i(\frac{\Delta_0}{2}\tau + \phi_0)} \sin \theta \sin \frac{\Omega'\tau}{2} e^{i\Delta_0(\tau+T)} a_i(\tau) \right. \\
 &\quad \left. e^{i\frac{\Delta_0}{2}\tau} \left(\cos \frac{\Omega'\tau}{2} - i \cos \theta \sin \frac{\Omega'\tau}{2} \right) a_f(\tau) \right\} \\
 &= e^{-i\bar{\Delta}\tau} \left\{ i e^{i(\frac{\Delta_0}{2}\tau + \phi_0)} \sin \theta \sin \frac{\Omega'\tau}{2} e^{i\Delta_0(\tau+T)} \right. \\
 &\quad \times e^{-i\frac{\Delta_0}{2}\tau} \left[e^{-i\frac{\Delta_0}{2}\tau} \left(\cos \frac{\Omega'\tau}{2} + i \cos \theta \sin \frac{\Omega'\tau}{2} \right) \right] \\
 &\quad \left. e^{i\frac{\Delta_0}{2}\tau} \left(\cos \frac{\Omega'\tau}{2} - i \cos \theta \sin \frac{\Omega'\tau}{2} \right) e^{-i\bar{\Delta}\tau} \left[i e^{i(\frac{\Delta_0}{2}\tau + \phi_0)} \sin \theta \sin \frac{\Omega'\tau}{2} \right] \right\}. \tag{33}
 \end{aligned}$$

Further simplification gives:

$$\begin{aligned}
 a_f(2\tau + T) &= i e^{-2i\bar{\Delta}\tau} e^{i(\Delta_0\tau + \phi_0)} \sin \theta \sin \frac{\Omega'\tau}{2} \left\{ e^{iT(\Delta_0 - \Delta_i)} \times \right. \\
 &\quad \left. \left(\cos \frac{\Omega'\tau}{2} + i \cos \theta \sin \frac{\Omega'\tau}{2} \right) + e^{-i\Delta_f\tau} \left(\cos \frac{\Omega'\tau}{2} - i \cos \theta \sin \frac{\Omega'\tau}{2} \right) \right\} \tag{34}
 \end{aligned}$$

In order to calculate the transition probability $P_2 = |a_f|^2 = a_f^* a_f$, we have to find the complex conjugate of a_f . Consider this term:

$$E = e^{iT(\Delta_0 - \Delta_i)} \left(\cos \frac{\Omega'\tau}{2} + i \cos \theta \sin \frac{\Omega'\tau}{2} \right) + e^{-i\Delta_f\tau} \left(\cos \frac{\Omega'\tau}{2} - i \cos \theta \sin \frac{\Omega'\tau}{2} \right).$$

Let

$$a = \cos \frac{\Omega'\tau}{2} \tag{35}$$

$$b = \cos \theta \sin \frac{\Omega'\tau}{2} \tag{36}$$

It follows that

$$\begin{aligned}
 E &= e^{iT(\Delta_0 - \Delta_i)} (a + ib) + e^{-i\Delta_f\tau} (a - ib) \\
 &= [\cos((\Delta_0 - \Delta_i)\tau) + i \sin((\Delta_0 - \Delta_i)\tau)] (a + ib) \\
 &\quad + [\cos \Delta_f\tau - i \sin \Delta_f\tau] (a - ib) \\
 &= R + iI. \tag{37}
 \end{aligned}$$

The Real part R is:

$$\begin{aligned}
 R &= a \cos((\Delta_0 - \Delta_i)\tau) - b \sin((\Delta_0 - \Delta_i)\tau) + a \cos \Delta_f \tau - b \sin \Delta_f \tau \\
 &= 2a \cos\left(\frac{\tau(\Delta_0 - \Delta_i + \Delta_f)}{2}\right) \cos\left(\frac{\tau(\Delta_0 - \Delta_i - \Delta_f)}{2}\right) \\
 &\quad - 2b \sin\left(\frac{\tau(\Delta_0 - \Delta_i + \Delta_f)}{2}\right) \sin\left(\frac{\tau(\Delta_0 - \Delta_i - \Delta_f)}{2}\right) \\
 &= 2 \cos\left[T\left(\frac{\Delta_0}{2} - \bar{\Delta}\right)\right] \left[\cos \frac{\Omega' \tau}{2} \cos \frac{T(\Delta_0 + \Delta_d)}{2} - \cos \theta \sin \frac{\Omega' \tau}{2} \sin \frac{T(\Delta_0 + \Delta_d)}{2}\right].
 \end{aligned} \tag{38}$$

And deriving in a similar fashion, the imaginary part I is:

$$I = 2 \sin\left[T\left(\frac{\Delta_0}{2} - \bar{\Delta}\right)\right] \left[\cos \frac{\Omega' \tau}{2} \cos \frac{T(\Delta_0 + \Delta_d)}{2} - \cos \theta \sin \frac{\Omega' \tau}{2} \sin \frac{T(\Delta_0 + \Delta_d)}{2}\right]. \tag{39}$$

Therefore,

$$\begin{aligned}
 P_2 &= a_f^* a_f \\
 &= R^2 + I^2 \\
 &= 4 \sin^2 \theta \sin^2 \frac{\Omega' \tau}{2} \left[\cos \frac{\Omega' \tau}{2} \cos \frac{T(\Delta_0 + \Delta_d)}{2} - \cos \theta \sin \frac{\Omega' \tau}{2} \sin \frac{T(\Delta_0 + \Delta_d)}{2}\right]^2
 \end{aligned} \tag{40}$$

To complete our derivation and match our version with Ramsey's, we make one approximation:

$$\Delta_d = \Delta_f - \Delta_i \ll \Delta_0, \tag{41}$$

this basically says that the difference in the ac Stark shift between the states is much smaller than the detuning. This leaves us with:

$$\boxed{P_2 = 4 \sin^2 \theta \sin^2 \frac{\Omega' \tau}{2} \left(\cos \frac{\Omega' \tau}{2} \cos \frac{T \Delta_0}{2} - \cos \theta \sin \frac{\Omega' \tau}{2} \sin \frac{T \Delta_0}{2}\right)^2} \tag{42}$$