Derive Laplacian. $x = r\sin\theta\cos\varphi \quad y = r\sin\theta\sin\varphi \quad z = r\cos\theta$

$r = (x^2+y^2+z^2)^{1/2} \quad \theta = \cos^{-1}\left(\frac{z}{r}\right) \quad \varphi = \tan^{-1}\left(\frac{y}{x}\right)$

$$\boxed{\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} + \frac{\partial\varphi}{\partial x}\frac{\partial}{\partial\varphi}}$$

$\sin\theta\cos\varphi$

$$\frac{\partial r}{\partial x} = \frac{x}{(x^2+y^2+z^2)^{1/2}} = \frac{x}{r}; \quad \frac{\partial\theta}{\partial x} = \frac{-1}{\sqrt{1-\frac{z^2}{r^2}}}\cdot\frac{-xz}{r^3} = \frac{(-r)}{\sqrt{x^2+y^2}}\cdot\frac{(-xz)}{r^3} = \frac{\cos\theta\cos\varphi}{r}$$

$$\frac{\partial\varphi}{\partial x} = \frac{1}{1+y^2/x^2}\cdot\left(\frac{y}{x}\right)' = \frac{-y^2}{x^2+y^2} = \frac{-r\sin\theta\sin\varphi}{r^2\sin^2\theta} = \frac{-\sin\varphi}{r\sin\theta}$$

$$\boxed{\frac{\partial}{\partial x} = (\sin\theta\cos\varphi)\frac{\partial}{\partial r} + \left(\frac{1}{r}\cos\theta\cos\varphi\right)\frac{\partial}{\partial\theta} + \left(-\frac{1}{r}\frac{\sin\varphi}{\sin\theta}\right)\frac{\partial}{\partial\varphi}}$$

What is $\frac{\partial^2}{\partial x^2}$? → think of operator... + product rule...

$$\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = (\sin\theta\cos\varphi)\frac{\partial}{\partial r}\left[\frac{\partial}{\partial x}\right] + \left(\frac{1}{r}\cos\theta\cos\varphi\right)\frac{\partial}{\partial\theta}\left[\frac{\partial}{\partial x}\right] + \left(\frac{-1}{r}\frac{\sin\varphi}{\sin\theta}\right)\frac{\partial}{\partial\varphi}\left[\frac{\partial}{\partial x}\right]$$

$$= \sin\theta\cos\varphi\frac{\partial}{\partial r}\left[\sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta} + \left(\frac{-1}{r}\frac{\sin\varphi}{\sin\theta}\right)\frac{\partial}{\partial\varphi}\right]$$

$$+\frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta}\left[\sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta} + \left(\frac{-1}{r}\frac{\sin\varphi}{\sin\theta}\right)\frac{\partial}{\partial\varphi}\right]$$

$$-\frac{1}{r}\frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\left[\sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta} + \left(\frac{-1}{r}\frac{\sin\varphi}{\sin\theta}\right)\frac{\partial}{\partial\varphi}\right]$$

Product rule

$$\frac{\partial^2}{\partial x^2} = \sin\theta\cos\varphi\left[\sin\theta\cos\varphi\frac{\partial^2}{\partial r^2} - \frac{1}{r^2}\cos\theta\cos\varphi\frac{\partial}{\partial\theta} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial^2}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial\varphi} - \frac{1}{r}\frac{\sin\varphi}{\sin\theta}\frac{\partial^2}{\partial\theta\partial\varphi}\right] +$$

$$\frac{1}{r}\cos\theta\cos\varphi\left[\cos\theta\cos\varphi\frac{\partial}{\partial r} + \sin\theta\cos\varphi\frac{\partial^2}{\partial\theta\partial r} + \left(\frac{-1}{r^2}\right)\sin\theta\cos\varphi\frac{\partial}{\partial\theta} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial^2}{\partial\theta^2} + \frac{1}{r}\frac{\sin\varphi\cdot\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\varphi} +\right.$$

$$\left. + \left(\frac{-1}{r}\right)\frac{\sin\varphi}{\sin\theta}\frac{\partial^2}{\partial\theta\partial\varphi}\right] - \frac{1}{r}\frac{\sin\varphi}{\sin\theta}\left[-\sin\theta\sin\varphi\frac{\partial}{\partial r} + \sin\theta\cos\varphi\frac{\partial^2}{\partial\varphi\partial r} - \frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial^2}{\partial\varphi\partial\theta}\right.$$

$$\left. - \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\varphi} - \frac{1}{r}\frac{\sin\varphi}{\sin\theta}\frac{\partial^2}{\partial\varphi^2}\right] \cdot \boxed{S_0}\ \frac{\partial^2}{\partial x^2} = \left[\sin^2\theta\cos^2\varphi\frac{\partial^2}{\partial r^2} - \frac{1}{r^2}\cos\theta\sin\theta\cos^2\varphi\frac{\partial}{\partial\theta} +\right.$$

$$\left.\frac{1}{r}\sin\theta\cos\theta\cos^2\varphi\frac{\partial^2}{\partial r\partial\theta} + \frac{1}{r^2}\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi} - \frac{1}{r}\cos\varphi\sin\varphi\frac{\partial^2}{\partial r\partial\varphi}\right] + \left[\frac{1}{r}\cos^2\theta\cos^2\varphi\frac{\partial}{\partial r} + \frac{1}{r}\sin\theta\cos\theta\cos^2\varphi\frac{\partial^2}{\partial\theta\partial r}\right.$$

$$\left. - \frac{1}{r^2}\sin\theta\cos\theta\cos^2\varphi\frac{\partial}{\partial\theta} + \frac{1}{r^2}\cos^2\theta\cos^2\varphi\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2}\frac{\sin\varphi\cos\varphi\cos^2\theta}{\sin^2\theta}\frac{\partial}{\partial\varphi} - \frac{1}{r^2}\frac{\cos\theta\cos\varphi\sin\varphi}{\sin\theta}\frac{\partial^2}{\partial\theta\partial\varphi}\right] + \left[\frac{1}{r}\sin^2\varphi\frac{\partial}{\partial r}\right.$$

$$\left. - \frac{1}{r}\sin\varphi\cos\varphi\frac{\partial^2}{\partial\varphi\partial r} + \frac{1}{r^2}\frac{\cos\theta\sin^2\varphi}{\sin\theta}\frac{\partial}{\partial\theta} - \frac{1}{r^2}\frac{\cos\theta\cos\varphi\sin\varphi}{\sin\theta}\frac{\partial^2}{\partial\varphi\partial\theta} + \frac{1}{r^2}\frac{\cos\varphi\sin\varphi}{\sin^2\theta}\frac{\partial}{\partial\varphi} + \frac{1}{r^2}\frac{\sin^2\varphi}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]$$

$$\frac{\partial \varphi}{\partial y} = \frac{1 \cdot \cos\varphi}{r \sin\theta}$$

$$\boxed{\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial y}\frac{\partial}{\partial \varphi}} \quad \text{where } \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta\sin\varphi \; ; \; \frac{\partial \theta}{\partial y} = \frac{1}{r}\cos\theta\sin\varphi$$

$$\hookrightarrow \frac{\partial}{\partial y} = \sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial \theta} + \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial \varphi}$$

$$\hookrightarrow \frac{\partial^2}{\partial y^2} = \sin\theta\sin\varphi\frac{\partial}{\partial r}\left[\frac{\partial}{\partial y}\right] + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial \theta}\left[\frac{\partial}{\partial y}\right] + \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial \varphi}\left[\frac{\partial}{\partial y}\right]$$

$$= \sin\theta\sin\varphi\frac{\partial}{\partial r}\left[\sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial \theta} + \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial \varphi}\right] + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial \theta}\left[\sin\theta\sin\varphi\frac{\partial}{\partial r} + \right.$$

$$\left.\frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial \theta} + \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial \varphi}\right] + \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial \varphi}\left[\sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial \theta} + \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial \varphi}\right]$$

Product rule

$$\frac{\partial^2}{\partial y^2} = \sin\theta\sin\varphi\left[\sin\theta\sin\varphi\frac{\partial^2}{\partial r^2} - \frac{1}{r^2}\cos\theta\sin\varphi\frac{\partial}{\partial \theta} + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial^2}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial \varphi} + \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial^2}{\partial r\partial\varphi}\right]$$

$$+\frac{1}{r}\cos\theta\sin\varphi\left[\cos\theta\sin\varphi\frac{\partial}{\partial r} + \sin\theta\sin\varphi\frac{\partial^2}{\partial\theta\partial r} + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial^2}{\partial\theta^2} - \frac{1}{r}\sin\theta\sin\varphi\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial^2}{\partial\theta\partial\varphi}\right.$$

$$\left.- \frac{1}{r}\frac{\cos\varphi\cdot\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\varphi}\right] + \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\left[\sin\theta\cos\varphi\frac{\partial}{\partial r} + \sin\theta\sin\varphi\frac{\partial^2}{\partial\varphi\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta} + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial^2}{\partial\varphi\partial\theta}\right.$$

$$\left.- \frac{1}{r}\frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial\varphi} + \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial^2}{\partial\varphi^2}\right] \quad \text{So} \quad \frac{\partial^2}{\partial y^2} = \left[\sin^2\theta\sin^2\varphi\frac{\partial^2}{\partial r^2} - \frac{1}{r^2}\cos\theta\sin\theta\sin^2\varphi\frac{\partial}{\partial\theta} + \right.$$

$$\left.\frac{1}{r}\cos\theta\sin\theta\sin^2\varphi\frac{\partial^2}{\partial r\partial\theta} - \frac{1}{r^2}\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi} + \frac{1}{r}\sin\varphi\cos\varphi\frac{\partial^2}{\partial r\partial\varphi}\right] + \left[\frac{1}{r}\cos^2\theta\sin^2\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\theta\sin^2\varphi\frac{\partial^2}{\partial r\partial\theta}\right.$$

$$\left.+\frac{1}{r^2}\cos^2\theta\sin^2\varphi\frac{\partial^2}{\partial\theta^2} - \frac{1}{r^2}\sin\theta\cos\theta\sin^2\varphi\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\cos\theta\sin\varphi\cos\varphi}{\sin\theta}\frac{\partial^2}{\partial\theta\partial\varphi} - \frac{1}{r^2}\frac{\cos\varphi\sin\varphi\cos^2\theta}{\sin^2\theta}\frac{\partial}{\partial\varphi}\right] +$$

$$\left[\frac{1}{r}\cos^2\varphi\frac{\partial}{\partial r} + \frac{1}{r}\sin\varphi\cos\varphi\frac{\partial^2}{\partial\varphi\partial r} + \frac{1}{r^2}\frac{\cos\theta\cos^2\varphi}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\cos\varphi\cos\theta\sin\varphi}{\sin\theta}\frac{\partial^2}{\partial\varphi\partial\theta} + \frac{-1}{r^2}\frac{\sin\varphi\cos\varphi}{\sin^2\theta}\frac{\partial}{\partial\varphi} + \frac{1}{r^2}\frac{\cos^2\varphi}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]$$

$$\boxed{\frac{\partial}{\partial z} = \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial z}\frac{\partial}{\partial \varphi}} \quad \text{where } \frac{\partial r}{\partial z} = \frac{z}{r} = \cos\theta, \; \frac{\partial \theta}{\partial z} = \frac{-1}{r}\sin\theta, \; \frac{\partial \varphi}{\partial z} = 0$$

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta} + 0 \quad \text{So} \quad \frac{\partial^2}{\partial z^2} = \cos\theta\frac{\partial}{\partial r}\left[\frac{\partial}{\partial z}\right] - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\left[\frac{\partial}{\partial z}\right] = \cos\theta\frac{\partial}{\partial r}\left[\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right]$$

$$- \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\left[\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right] = \cos\theta\left[\cos\theta\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{r}\sin\theta\frac{\partial^2}{\partial r\partial\theta}\right] - \frac{1}{r}\sin\theta\left[-\sin\theta\frac{\partial}{\partial r} + \right.$$

$$\left.\cos\theta\frac{\partial^2}{\partial\theta\partial r} - \frac{1}{r}\cos\theta\frac{\partial}{\partial\theta} - \frac{1}{r}\sin\theta\frac{\partial^2}{\partial\theta^2}\right] = \left[\cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\cos\theta\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{r}\cos\theta\sin\theta\frac{\partial^2}{\partial r\partial\theta}\right]$$

$$+\left[\frac{1}{r}\sin^2\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\cos\theta\frac{\partial^2}{\partial\theta\partial r} + \frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2}\sin^2\theta\frac{\partial^2}{\partial\theta^2}\right]$$

So the Laplacian $\nabla^2 f = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} =$

$$\left[\sin^2\theta\cos^2\varphi\frac{\partial^2}{\partial r^2} - \frac{1}{r^2}\cos\theta\sin\theta\cos^2\varphi\frac{\partial}{\partial\theta} + \frac{1}{r^2}\sin\theta\cos\theta\cos^2\varphi\frac{\partial^2}{\partial r\partial\theta} + \frac{1}{r^2}\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi} - \frac{1}{r}\cos\varphi\sin\varphi\frac{\partial^2}{\partial r\partial\varphi}\right] + \left[\frac{1}{r}\cos^2\theta\cos^2\varphi\frac{\partial}{\partial r} + \frac{1}{r}\sin\theta\cos\theta\cos^2\varphi\frac{\partial^2}{\partial\theta\partial r} - \frac{1}{r^2}\sin\theta\cos\theta\cos^2\varphi\frac{\partial}{\partial\theta} + \frac{1}{r^2}\cos^2\theta\cos^2\varphi\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2}\frac{\sin\varphi\cos\varphi\cos^2\theta}{\sin^2\theta}\frac{\partial}{\partial\varphi} - \left[\frac{1}{r^2}\frac{\cos\theta\sin\varphi\sin\varphi}{\sin\theta}\frac{\partial^2}{\partial\theta\partial\varphi}\right] + \left[\frac{1}{r}\sin^2\varphi\frac{\partial}{\partial r} - \frac{1}{r}\sin\varphi\cos\varphi\frac{\partial^2}{\partial\varphi\partial r} + \frac{1}{r^2}\frac{\cos\theta\sin^2\varphi}{\sin\theta}\frac{\partial}{\partial\theta} - \frac{1}{r^2}\frac{\cos\theta\cos\varphi\sin\varphi}{\sin\theta}\frac{\partial^2}{\partial\varphi\partial\theta} + \frac{1}{r^2}\frac{\cos\varphi\sin\varphi}{\sin^2\theta}\frac{\partial}{\partial\varphi} + \frac{1}{r^2}\frac{\sin^2\varphi}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] + \left[\sin^2\theta\sin^2\varphi\frac{\partial^2}{\partial r^2} - \frac{1}{r^2}\cos\theta\sin\theta\sin^2\varphi\frac{\partial}{\partial\theta} + \frac{1}{r}\cos\theta\sin\theta\sin^2\varphi\frac{\partial^2}{\partial r\partial\theta} - \frac{1}{r^2}\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi} + \frac{1}{r}\sin\varphi\cos\varphi\frac{\partial^2}{\partial r\partial\varphi}\right] + \left[\frac{1}{r}\cos^2\theta\sin^2\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\theta\sin^2\varphi\frac{\partial^2}{\partial r\partial\theta} - \frac{1}{r^2}\sin\theta\cos\theta\sin^2\varphi\frac{\partial}{\partial\theta} + \frac{1}{r^2}\cos^2\theta\sin^2\varphi\frac{\partial^2}{\partial\theta^2} - \frac{1}{r^2}\frac{\cos\varphi\sin\varphi\cos^2\theta}{\sin^2\theta}\frac{\partial}{\partial\varphi} + \frac{1}{r^2}\frac{\cos\theta\sin\varphi\cos\varphi}{\sin\theta}\frac{\partial^2}{\partial\theta\partial\varphi}\right] + \left[\frac{1}{r}\cos^2\varphi\frac{\partial}{\partial r} + \frac{1}{r}\sin\varphi\cos\varphi\frac{\partial^2}{\partial\varphi\partial r} + \frac{1}{r^2}\frac{\cos\theta\cos^2\varphi}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\cos\varphi\cos\theta\sin\varphi}{\sin\theta}\frac{\partial^2}{\partial\varphi\partial\theta} - \frac{1}{r^2}\frac{\sin\varphi\cos\varphi}{\sin^2\theta}\frac{\partial}{\partial\varphi} + \frac{1}{r^2}\cos^2\varphi\frac{\partial^2}{\partial\varphi^2}\right] + \left[\cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\cos\theta\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r}\sin\theta\cos\theta\frac{\partial^2}{\partial r\partial\theta} + \frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2}\sin^2\theta\frac{\partial^2}{\partial\theta^2}\right]$$

Simplify →

$$\boxed{\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}}$$

↳

$$\boxed{\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\right] + \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right] + \frac{1}{r^2\sin^2\theta}\left[\frac{\partial^2}{\partial\varphi^2}\right]}$$