§2.1

local properties of toric varieties

1. $U_\sigma = \mathrm{spec}(\mathbb{C}[S_\sigma])$

$\varphi: \mathrm{specm}(\mathbb{C}[S_\sigma]) \to \mathrm{Hom}_{sg}(S_\sigma, \mathbb{C})$

$\because \varphi$ is isomorphism

(1) Given a point $x \in \mathrm{specm}(\mathbb{C}[S_\sigma])$, there is a corresponding $\mathbb{C}$-algebra homomorphism $f_x : \mathbb{C}[S_\sigma] \to \mathbb{C}$

Then we define $\varphi_x = \varphi(x) \in \mathrm{Hom}_{sg}(S, \mathbb{C})$ by $\varphi_x(u) \coloneqq f_x(\chi^u)$

(2) Given a semigroup homomorphism $f: S_\sigma \to \mathbb{C}$

Then there is a corresponding homomorphism $\tilde{f}: \mathbb{C}[S_\sigma] \to \mathbb{C}$ where $\tilde{f}(\chi^u) = f(u)$

Then $\varphi^{-1}(f)$ is defined by the corresponding $x \in \mathrm{specm}(\mathbb{C}[S_\sigma])$ of $\tilde{f}$

2. a distinguish point in U_σ is defined by $x_\sigma : S_\sigma \to \mathbb{C}$

$$u \mapsto \begin{cases} 1 & \text{if } u \in \sigma^\perp \\ 0 & \text{otherwise} \end{cases}$$

- x_σ is well-defined
- σ spans $N_\mathbb{R} \iff$ ~~$\mathbb{R}$~~ $\sigma^\perp = \{0\}$

$\iff x_\sigma$ is a fixed point of the action of T_N on U_σ

(1) 3. the singularity of X_σ

σ spans $N_{\mathbb{R}}$ and $\sigma^\perp = \{0\}$ the corresponding $f \in \mathrm{Hom}(\mathbb{C}[S_\sigma], \mathbb{C})$ to the point x_σ is

$$f : \mathbb{C}[S_\sigma] \to \mathbb{C}$$
$$\chi^u \mapsto \begin{cases} 1 & \text{if } u = 0 \\ 0 & \text{otherwise} \end{cases}$$

the corresponding ideal m to the point x_σ is $\ker(f)$, that is

$m = \bigoplus_{u \neq 0} \mathbb{C}\chi^u$ $\qquad$ $m = I(\chi^u)$ u is nonzero in S_σ

$m^2 = (\chi^u)$, u are sums of two elements of $S_\sigma \setminus \{0\}$

the cotangent space $\triangleq m/m^2$

$\dim(m/m^2) = |\mathcal{H}|$ Hilbert basis

has a basis of $\overline{\chi}^u$ for those u in $S_\sigma \setminus \{0\}$ that are not sum of two such vectors

Thm: X_σ is nonsingular $\Leftrightarrow$ $\dim(m/m^2) = n$

$\Updownarrow$

dim

(necessary?)

Lemma 3. Let σ be a strongly convex rational cone ~~and~~ and $\dim(\sigma) = N_{\mathbb{R}}$

then $\dim(T_{x_\sigma}(U_\sigma)) = |\mathcal{H}|$

$\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational cone

Thm: U_σ is nonsingular if and only if σ is generated by part of a basis for the lattice N, in which case

$$U_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k} \quad k = \dim(\sigma)$$

pf: "Let" $n = \dim N_{\mathbb{R}}$ lemma 1: σ is strongly convex $\Leftrightarrow \dim U_\sigma = n$ (ref: D.A. Cox Thm 1.2.18)

According to lemma 1,

$\dim U_\sigma = n = \dim N_{\mathbb{R}}$

"$\Leftarrow$" Let $\sigma = \text{cone}(e_1, \cdots, e_k) \subseteq N_{\mathbb{R}}$ where $\{e_i\}_{i=1}^{r}$ is part of the basis of N and $r \leq n$

Then $\sigma^\vee = \text{Cone}(e_1^*, \cdots, e_k^*, \pm e_{k+1}^*, \cdots \pm e_n^*)$

and $\mathcal{H} = \{e_1^*, \cdots, e_k^*, \pm e_{k+1}^*, \pm e_n^*\}$

Therefore $\mathbb{C}[S_\sigma]$

$$= \mathbb{C}[\chi^{e_1^*}, \cdots, \chi^{e_k^*}, \chi^{\pm e_{k+1}^*}, \cdots, \chi^{\pm e_n^*}]$$

$$= \mathbb{C}[x_1, \cdots, x_k, x_{k+1}^{\pm 1}, \cdots, x_n^{\pm 1}]$$

and $U_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$

lemma 2: If $U_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$, U_σ is smooth ?

"$\Rightarrow$" If U_σ is nonsingular

(1) Suppose $\dim(\sigma) = n$.

nonsingularity definition? Since x_σ is nonsingular in U_σ, $\dim(T_{x_\sigma}(U_\sigma)) = \dim U_\sigma = n$.

On the other hand,

$\dim(T_{X_\sigma}(U_\sigma)) = |\mathcal{H}|$ lemma 3

Thus $n = |\mathcal{H}| \geq |\{\text{edges } \rho \subseteq \sigma^\vee\}| \geq n$

$\geq$ holds since each minimal generators of edge ρ is contained in $\mathcal{H}$

$\geq$ holds since σ is strongly convex $\Rightarrow \dim \sigma^\vee = n$

Then $\sigma^\vee$ has n edges and $\mathcal{H}$ consists of the minimal generators of these edges

Since $M = \mathbb{Z}S_\sigma$ and S_σ is generated by $\mathcal{H}$

the n edges form a basis of M

Then σ is generated by a basis of N

and $U_\sigma \cong \mathbb{C}^n$

(2) suppose $\dim(\sigma) = k < n$

Let $N_\sigma = \sigma \cap N + (-\sigma) \cap N$

N is saturated

$\Rightarrow N_\sigma$ is saturated

$\Rightarrow N/N_\sigma$ is torsion-free

$\Rightarrow N = N_\sigma \oplus N'$

with $\dim(N_\sigma) = k$

and $\dim(N') = n-k$

and $\sigma = \sigma' \oplus \{0\}$

$\Rightarrow M = M' \oplus M''$

and $S_\sigma = S_{\sigma'} \oplus M''$

$((\sigma')^\vee \cap M')$

$z\bar{m} = 0$

$\Leftrightarrow zm \in N_\sigma$

$\Rightarrow m \in N_\sigma$

$\Rightarrow \bar{m} = 0$

$\mathbb{C}[S_\sigma] \cong \mathbb{C}[S_{\sigma'}] \otimes_{\mathbb{C}} \mathbb{C}[M'']$

$\Rightarrow U_\sigma = \mathrm{spec}(\mathbb{C}[S_\sigma])$

$\simeq \mathrm{Spec}(\mathbb{C}[S_{\sigma'}] \otimes_{\mathbb{C}} \mathbb{C}[M''])$

$\stackrel{?}{=} U_{\sigma'} \times T_{N'} = U_{\sigma'} \times (\mathbb{C}^*)^{n-k}$

Since U_σ is nonsingular,

then $U_{\sigma'}$ is nonsingular

$\Rightarrow \sigma'$ is generated by a basis of N_σ

and $U_{\sigma'} \cong \mathbb{C}^k$

$\Rightarrow U_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$, σ is generated by part of the basis of N

Example:

$N = \mathbb{Z}^2$ and σ is generated by e_2 and $me_1 - e_2$ $(m \geq 2)$

$A_\sigma = \mathbb{C}[S_\sigma] = \mathbb{C}[X, XY, XY^2, \cdots, XY^m]$

Setting $X = U^m$ and $Y = V/U$

We have

$A_\sigma = \mathbb{C}[U^m, U^{m-1}V, \cdots, UV^{m-1}, V^m]$
$\subset \mathbb{C}[U, V]$

$U_\sigma = \mathrm{Spec}(A_\sigma)$ is the cone over the rational normal curve of degree m

$A_\sigma \hookrightarrow \mathbb{C}[U, V]$

$\Phi : \mathbb{C}^2 \hookrightarrow U_\sigma$

$\Phi : \mathbb{C}^2 \to \mathbb{C}^{m+1}$ $\quad \Phi(\mathbb{C}^2) = U_6 = V(I)$

$(s,t) \mapsto (s^m, s^{m-1}t, \cdots, st^{m-1}, t^m)$

$I \subseteq \mathbb{C}[x_0, \cdots, x_d)$

$\parallel$

$(x_i x_{j+1} - x_{i+1} x_j \mid 0 \le i < j \le d-1)$

I is homogeneous

$\Phi : \mathbb{P}^1 \to \mathbb{P}^{m}$

$(s:t) \mapsto (s^m : s^{m-1}t : \cdots : t^m)$

$\Phi(\mathbb{P}^1) = \hat{V}$ projective variety

$\hat{V} = (U_6 \setminus \{0\}) / \mathbb{C}^*$

§2.2 surfaces, quotient singularities

Δ Continue the example above

$6 = \text{cone}(e_2, me_1 - e_2)$

$6^\vee = \text{cone}(e_1^*, e_1^* + me_2^*)$

$\mathcal{H}(S_6) = \{e_1^*, e_1^* + e_2^*, \cdots e_1^* + me_2^*\}$

$A_6 = \mathbb{C}[S_6] = \mathbb{C}[X, XY, \cdots, XY^m]$

and $U_6 = \text{spec}(\mathbb{C}[S_6])$

, Let $G = \mu_m = \{m^{th}$ roots of units$\} \subseteq \mathbb{C}$

Claim: $U_6 = \mathbb{C}^2/G$ is a quotient variety where G acts on $\mathbb{C}^2$ is given by $G \times \mathbb{C}^2 \to \mathbb{C}^2$

$(\zeta, (u,v)) \mapsto \zeta \cdot (u,v) = (\zeta u, \zeta v)$

Let $N' \subseteq N$ be a sublattice, and ~~N's~~ the basis of N' is $\{e_1, me_1 - e_2\}$

Then $M' = \mathrm{Hom}(N', \mathbb{Z})$ and the basis of M' is $\{\frac{1}{m} e_1^*, e_2^* + \frac{1}{m} e_1^*\}$

Let $\sigma' = \sigma$, but regarded in N'

Since σ' is generated by the basis of N', then $U_{\sigma', N'} \cong \mathbb{C}^2$.

$\underline{\text{To prove } U_{\sigma, N} = U_{\sigma', N'}/G}$

$$\mathcal{H}(S_\sigma) = \{e_1^*, e_1^* + e_2^*, \cdots e_1^* + me_2^*\}$$

$$\mathcal{H}(S_{\sigma'}) = \{\tfrac{1}{m} e_1^*, \tfrac{1}{m} e_1^* + e_2^*\}$$

$$A_\sigma = \mathbb{C}[X, XY, \cdots, XY^m] = \mathbb{C}[U^m, U^{m-1}V, \cdots, UV^{m-1}, V^m]$$

$$A_{\sigma'} = \mathbb{C}[U, UY] = \mathbb{C}[U, V]$$

$$(U = X^{\frac{1}{m} e_1^*}, U^m = X, V = UY)$$

G acts on $A_{\sigma'}$ is given by

$$\zeta : F \longmapsto F(\zeta U, \zeta V) \quad (\zeta \in G)$$

Then $A_\sigma = (A_{\sigma'})^G$ and $U_{\sigma, N} = U_{\sigma', N'}/G$

$$= \mathbb{C}^2/G$$

4. Arbitrary singular 2-d affine toric variety ~~(toric surface)~~

prop: Let $\sigma \subseteq N_{\mathbb{R}} \cong \mathbb{R}^2$ be a 2-d strongly convex cone. Then there exist a basis e_1, e_2 for N s.t

$$\sigma = \mathrm{Cone}(e_2, me_1 - ke_2)$$

where $d > 0$, $0 \le k < d$, and

$$\gcd(d, k) = 1$$

~~Moreover, d, k are unique~~

Let $G = \mu_m \cong \mathbb{Z}/m\mathbb{Z} \cong N/N'$

Claim: $U_\sigma = \mathbb{C}^2/G$

G acts on $\mathbb{C}^2$ is given by

$$\zeta \cdot (x, y) = (\zeta x, \zeta^k y)$$

Let $N' \subseteq N$ and the basis of N' is $\{e_2, me_1 - ke_2\}$

Then $U_{\sigma', N'} = \mathbb{C}^2$

$M' = \mathrm{Hom}(N', \mathbb{Z})$, the basis of M' is $\{\frac{1}{m}e_1^*, e_2^* + \frac{k}{m}e_1^*\}$

~~$\mathcal{H}(S_\sigma) = \{ ie_1^* + je_2^* \mid j \le \frac{m}{k} i \}$~~ $i, j \in \mathbb{Z}_{\ge 0}$

$A_\sigma = \bigoplus \mathbb{C} \cdot X^i Y^j$, where $j \le \frac{m}{k} i$ $i, j \in \mathbb{Z}_{\ge 0}$

$X = \chi^{e_1^*}$, $Y = \chi^{e_2^*}$

$A_{\sigma'} \cong \mathbb{C}[U, U^k Y]$ where $U^m = X$, $V = U^k Y$

$= \mathbb{C}[U, V]$

G acts on $A_{\sigma'}$ is given by

$$\zeta : F \longmapsto F(\zeta U, \zeta^k V)$$

$$A_\sigma = \bigoplus \mathbb{C} \cdot U^{mi} \cdot (V/U^k)^j = \bigoplus \mathbb{C}\, U^{mi-kj} \cdot V^j$$

$$\zeta \cdot F = \Sigma\, c \cdot (\zeta U)^{mi-kj} \cdot (\zeta^k \cdot V)^j$$
$$= \Sigma\, c \cdot \mathcal{U}^{mi-kj} \cdot V^j$$

$$\Rightarrow A_\sigma = (A_{\sigma'})^G \quad \Rightarrow \quad U_\sigma = \mathbb{C}^2/G$$

$*$ "$\supseteq$" ?

△ General rank (Ref Cox ~~The~~ Prop 1.3.18)

prop: Let $N' \subset N$ be a sublattice of finite index with $G = N/N'$, and let $\sigma \subseteq N'_{\mathbb{R}} = N_{\mathbb{R}}$ be a strongly convex cone. Then

(a) There are natural isomorphisms

$$G \cong \mathrm{Hom}_{\mathbb{Z}}(M'/M, \mathbb{C}^*) = \ker(T_{N'} \to T_N)$$

(b) G acts on $\mathbb{C}[\sigma^\vee \cap M']$ with ring of invariants

$$\mathbb{C}[\sigma^\vee \cap M']^G = \mathbb{C}[\sigma^\vee \cap M]$$

(c) G acts on $U_{\sigma,N'}$ and $\phi : U_{\sigma,N'} \to U_{\sigma,N}$ is constant on G-orbits and induces

$$U_{\sigma,N'}/G \simeq U_{\sigma,N}$$

pf: (a) 1° Since N' has finite index in N, we have inclusion

$$N' \subseteq N \subseteq N_{\mathbb{Q}} \text{ and } M \subseteq M' \subseteq M_{\mathbb{Q}}$$

$0 \to N' \to N \to N/N' \to 0$ (Note 4)

$0 \to \mathrm{Hom}(N/N', \mathbb{Z}) \to M \xrightarrow{\varphi} M' \to \mathrm{Ext}^1(N/N', \mathbb{Z}) \to 0$

$M \times N \to \mathbb{Z}$ induces $M_{\mathbb{Q}} \times N_{\mathbb{Q}} \to \mathbb{Q}$

$\varphi: M \hookrightarrow M'$ injective

(since N/N' is a finite group)

Hence we have

$$\langle M'/M, N/N' \rangle \to \mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{C}^*$$

$$\langle [m], [n] \rangle \mapsto [\langle m, n \rangle] \mapsto \exp(2\pi i [\langle m, n \rangle])$$

(If $[m'] = [m]$, $[n, n']$

$m' = m + m_1 \quad n' = n + n_1 \quad m_1 \in M, \; n_1 \in N'$

$\langle m', n' \rangle = \langle m, n \rangle + \underbrace{\langle m, n_1 \rangle}_{\in \mathbb{Z}} + \underbrace{\langle m_1, n_1 \rangle}_{\in \mathbb{Z}} + \underbrace{\langle m_1, n \rangle}_{\in \mathbb{Z}}$

$\Rightarrow G = N/N' \cong \mathrm{Hom}_{\mathbb{Z}}(M'/M, \mathbb{C}^*)$

2° Since $T_N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$, applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ to

$$0 \to M \to M' \to M'/M \to 0$$

gives

$$1 \to \mathrm{Hom}_{\mathbb{Z}}(M'/M, \mathbb{C}^*) \to T_{N'} \to T_N \to 1$$

This is exact since $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ is left exact and $\mathbb{C}^*$ is divisble

Thus $\ker(T_{N'} \to T_N) = \mathrm{Hom}_{\mathbb{Z}}(M'/M, \mathbb{C}^*)$

(2) $G = N/N'$ acts on $\mathbb{C}[M']$ by,

$$v \cdot \chi^{u'} = \exp(2\pi i \langle u', v \rangle) \cdot \chi^{u'}$$

$(v \in N, \; u' \in M')$ confirm this is an action

① $v \in N' \quad v \cdot \chi^{u'} = \chi^{u'}$

② $(v+v') \cdot \chi^{u'} = (v \cdot \chi^{u'})(v' \chi^{u'})$

$(*) \quad \mathbb{C}[M']^G = \mathbb{C}[M]$

To prove $(*)$, let

N-basis $= \{e_1, \cdots, e_n\}$

N'-basis $= \{m_1 e_1, \cdots, m_n e_n\}$ $(m_i \in \mathbb{Z}_{>0})$ ~~(m_n)~~

then M-basis $= \{\pm e_1^*, \cdots, \pm e_n^*\}$

M'-basis $= \{\pm \frac{1}{m_1} e_1^*, \cdots, \pm \frac{1}{m_n} e_n^*\}$

and $\mathbb{C}[M] = \mathbb{C}[\chi^{\pm e_1^*}, \cdots, \chi^{\pm e_n^*}]$

$\mathbb{C}[M'] = \mathbb{C}[\chi^{\pm \frac{1}{m_1} e_1^*}, \cdots, \chi^{\pm \frac{1}{m_n} e_n^*}]$

Let $X_i = \chi^{e_i^*}$, $U_i = \chi^{\frac{1}{m_i} e_i^*}$ and $(U_i)^{m_i} = X_i$

then $\mathbb{C}[M] = \mathbb{C}[X_1, X_1^{-1}, \cdots, X_n, X_n^{-1}]$,

$\mathbb{C}[M'] = \mathbb{C}[U_1, U_1^{-1}, \cdots, U_n, U_n^{-1}]$

Let ~~$\zeta \in G$~~ $\underset{N/N'}{G} \ni \zeta = a_1 e_1 + \cdots + a_n e_n$

$(a_1, \cdots a_n) \in \bigoplus \mathbb{Z}/m_i\mathbb{Z}$

then $\forall\, U_1^{l_1} \cdots U_n^{l_n} \in \mathbb{C}[M']$

$\zeta \cdot (U_1^{l_1} \cdots U_n^{l_n})$

$= \exp\left(2\pi i \left(\sum_i \frac{a_i l_i}{m_i}\right)\right) \cdot U_1^{l_1} \cdots U_n^{l_n}$

Therefore $\forall\, X_1^{l_1} \cdots X_n^{l_n} = U_1^{m_1 l_1} \cdots U_n^{m_n l_n} \in \mathbb{C}[M]$

$\zeta \cdot (U_1^{m_1 l_1} \cdots U_n^{m_n l_n})$

$= \exp\left(2\pi i \left(\sum_i a_i l_i\right)\right) \cdot U_1^{m_1 l_1} \cdots U_n^{m_n l_n}$

$= U_1^{m_1 l_1} \cdots U_n^{m_n l_n}$

$\Rightarrow \mathbb{C}[M] = \mathbb{C}[M']^G$

$\Rightarrow \mathbb{C}[G^\vee \cap M']^G = \mathbb{C}[G^\vee \cap M]$

(c) $\mathbb{C}[\sigma^\vee \cap M] = \mathbb{C}[\sigma^\vee \cap M']^G \subseteq \mathbb{C}[\sigma^\vee \cap M']$

gives a morphism

$$U_{\sigma,N'} \to U_{\sigma,N}$$

and G acts on $U_{\sigma,N'}$ is induced by action of $\mathbb{C}[\sigma^\vee \cap M']$

$$U_{\sigma,N'}/G \simeq U_{\sigma,N}$$

Moreover G acts on $T_{N'}$ and $T_{N'}/G \simeq T_N$

Apply the proposition to 2-d case

N - basis : e_1, e_2

N' - basis : $me_1 - ke_2, e_2$

M' - basis : $\frac{1}{m}e_1^*, e_2^* + \frac{k}{m}e_1^*$

M - basis : e_1^*, e_2^*

$$G = N/N' = \{v + N' \mid v = \zeta e_1 \quad 0 \le \zeta < m\}$$

$$M'/M = \{u + M \mid u = \tfrac{\zeta}{m} e_1^* \quad 0 \le \zeta < m\}$$

$$G = N/N' \cong \{e^{2\pi i \frac{\zeta}{m}} \mid 0 \le \zeta < m\}$$

$$G \times \mathbb{C}[M'] \to \mathbb{C}[M']$$

$v = \zeta e_1$ $\quad v \cdot \chi^{\frac{1}{m}e_1^*} = \exp(2\pi i \langle \tfrac{1}{m}e_1^*, \zeta e_1 \rangle) \chi^{\frac{1}{m}e_1^*}$

$$= \exp(2\pi i \tfrac{\zeta}{m}) \chi^{\frac{1}{m}e_1^*}$$

$$v \cdot \chi^{e_2^* + \frac{k}{m}e_1^*} = \exp(2\pi i \tfrac{\zeta}{m} \cdot k) \cancel{\chi^{\frac{1}{m}e_1^*}} \chi^{e_2^* + \frac{k}{m}e_1^*}$$

Apply the proposition to $\cancel{2-d}$ $n-d$

Given m and $a_1, \cdots, a_n$ to be the positive integers, the quotient ~~by the~~ $\mathbb{C}^n/\mu_m$ given by

$$\zeta \cdot (z_1, \cdots, z_n) = (\zeta^{a_1} z_1, \cdots \zeta^{a_n} z_n)$$

can be constructed as U_σ, by taking $N' = \sum_{i=1}^{n} \mathbb{Z} \cdot (1/a_i) \cdot e_i$

$$\subset N = N' + \mathbb{Z} \cdot (\tfrac{1}{m})(e_1 + \cdots + e_n)$$

and ~~the cone~~ $\sigma = \text{cone}(e_1, \cdots, e_n)$.

pf: Let $G = N/N' = \{N' + \frac{\zeta}{m}(e_1 + \cdots + e_n) \mid 0 \leq \zeta < m, \ \zeta \in \mathbb{Z}\}$

$$G \cong \exp(2\pi i \tfrac{\zeta}{m}) \cong \mu_m$$

N'-basis $\frac{1}{a_1} e_1, \cdots \frac{1}{a_n} e_n$

M'-basis $a_1 e_1^*, \cdots, a_n e_n^*$

$G \times \mathbb{C}[M'] \to \mathbb{C}[M']$ $\quad \mathbb{C}[M'] = \mathbb{C}[\chi^{a_1 e_1^*}, \cdots, \chi^{a_n e_n^*}]$

$v \cdot \chi^{a_i e_i^*}$ $\quad v = [\frac{\zeta}{m}(e_1 + \cdots + e_n)] \in G$

$= \exp(2\pi i \langle \frac{\zeta}{m}(e_1 + \cdots + e_n), a_i e_i^* \rangle) \chi^{a_i e_i^*}$

$= \exp(2\pi i \frac{\zeta}{m} \cdot a_i) \chi^{a_i e_i^*}$

To non-affine toric variety
by making the group actions compatible
on affine open subvarieties.

- $\Delta = \{$ cones are generated by proper subsets of $\{v_0, v_1, \ldots, v_n\}\}$

where $v_1, \ldots, v_n$ linearly independent
and $v_0 = -v_1 - \cdots - v_n$

- Let $\cancel{N' = \sum_{i=1} \mathbb{Z} \cdot \frac{1}{d_i} e_i \quad N = N' + \mathbb{Z}\frac{1}{d_0} v_0}$ $\quad N = \sum_{i=1}^{n} \mathbb{Z} \frac{1}{d_i} v_i$

$\sigma_i = \mathrm{cone}(v_0, \cdots \hat{v}_i, \cdots v_n)$

$N_i = \sum_{\substack{k=1 \\ k \neq i}}^{n} \mathbb{Z} \frac{1}{d_i} v_i \subseteq N$

U_{σ_i} is the quotient of $\mathbb{C}^n$ by the cyclic group μ_{d_i} acting by
$\zeta($

Cor: If σ is simplex (simplicial),
then $U_\sigma \cong \mathbb{C}^n / G$
If Δ is a simplicial fan
then $X(\Delta)$ is an <u>orbifold</u>
(finite quotient singularities)

Date

To construct $\mathbb{P}(d_0, d_1, \cdots d_n)$

$\mathbb{P}(d_0, \cdots, d_n) \cong \mathbb{C}^{n+1}/\mathbb{C}^*$

$X_i = \mathbb{C}^{n+1} \setminus H_i \quad H_i : X_i = 0$

X_i is $\mathbb{C}^*$-invariant

$X_i /\!/ \mathbb{C}^* = \operatorname{spec}(A_i^G)$

$A_i^G = (\mathbb{C}[X_0, X_0^{-1}, X_1, \cdots, X_n])^{\mathbb{C}^*}$

$\mathbb{C}^* \curvearrowright \mathbb{C}[X_0, X_0^{-1}, X_1, \cdots X_n]$

$\mathbb{C}^* \curvearrowright \cancel{X_0^{d_0} \cdot (X_0^{-1})^{\beta_0} \cdot X_1^{d_1} \cdots X_n^{d_n}} \; X_0^{\alpha_0} \cdots X_n^{\alpha_n}$

$= \zeta^{d_0\alpha_0 + \alpha_1 d_1 + \cdots + \alpha_n d_n} \cdot X_0^{\alpha_0} \cdot X_1^{\alpha_1} \cdots X_n^{\alpha_n}$

$X_0^{\alpha_0} \cdots X_n^{\alpha_n}$ $\mathbb{C}^*$-invariant

$\Leftrightarrow \alpha_0 d_0 + \cdots + \alpha_n d_n = 0 \quad (\alpha_0 \in \mathbb{Z}, \alpha_1, \cdots, \alpha_n \in \mathbb{Z}_{\geq 0})$

- $\Delta = \{$cones are generated by proper subsets of $\{v_0, v_1, \cdots, v_n\}\}$

  where $v_0 + v_1 + \cdots + v_n = 0$

  $\sigma_0 = \{v_1, \cdots, v_n\}$

  Let $N_0 = \sum_{i=1}^{n} \frac{1}{d_i} v_i \subset N = N_0 + \mathbb{Z}\frac{1}{d_0} v_0$

  $= N_0 + \mathbb{Z}\frac{1}{d_0}(v_1 + \cdots + v_n)$

$N/N_0 = \mathbb{Z}/d_0\mathbb{Z}$

$M \subseteq M_0 = \sum_{i=1}^{n} \mathbb{Z} d_i v_i^*$

$N = \sum_{i=0}^{n} \mathbb{Z}\frac{1}{d_i} v_i \qquad \forall m \in M \quad m = \sum_{i=1}^{n} a_i d_i v_i^*$

satisfy $\sum_{i=1}^{n} a_i \frac{d_i}{d_0} \in \mathbb{Z}$

Toric morphism.

Suppose $\varphi : N' \to N$ is a homomorphism of lattices.

Δ' is a fan in $(N')_{\mathbb{R}}$

Δ is a fan in $(N)_{\mathbb{R}}$

Δ', Δ satisfy $\forall \sigma' \in \Delta', \exists \sigma \in \Delta$ s.t $\varphi_{\mathbb{R}}(\sigma') \subset \sigma$.

Then ① there is a morphism $\varphi_{\mathbb{R}} = \varphi \otimes 1$

$\varphi_* : X(\Delta') \to X(\Delta)$ s.t $N' \otimes \mathbb{R} \to N \otimes \mathbb{R}$

$\varphi_*(T_{N'}) \subseteq T_N$ and $n \otimes r \mapsto \varphi(n) \otimes r$

$\varphi_*|_{T_{N'}}$ is the map

$\varphi \otimes 1 : N' \otimes_{\mathbb{Z}} \mathbb{C}^* \to N \otimes_{\mathbb{Z}} \mathbb{C}^*$

② we have a commutative diagram

$$\begin{tikzcd} T_{N'} \times X(\Delta') \arrow[r] \arrow[d, "\varphi_*|_{T_{N'}} \times \varphi_*"'] & X(\Delta') \arrow[d, "\varphi_*"] \\ T_N \times X(\Delta) \arrow[r] & X(\Delta) \end{tikzcd}$$

① $\forall \sigma' \in \Delta, \exists \sigma \in \Delta$ s.t $\varphi_{\mathbb{R}}(\sigma') \subset \sigma$ φ

$\varphi : \sigma' \hookrightarrow \sigma$ induces a morphism

(1) ~~$\varphi : \mathbb{C}[M] \to \mathbb{C}[M']$~~ $\varphi_{\sigma'} : U_{\sigma'} \to U_\sigma \subset X(\Delta)$

Then we have $\varphi_* : X(\Delta') \to X(\Delta)$

by gluing $\varphi_{\sigma'}$ for all σ'

(2) if $\varphi_*|_{U_{\sigma'}} = \varphi_{\sigma'}$

Confirm: $\varphi_{\sigma_i}|_{U_{\sigma_i} \cap U_{\sigma_j}} = \varphi_{\sigma_j}|_{U_{\sigma_i \cap \sigma_j}}$

Obviously: $\varphi_{\{0\}} : T_{N'} \to T_N$

② $\varphi_*|_{T_{N'}}$ is a group morphism

$$\Rightarrow \begin{tikzcd} T_{N'} \times T_{N'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & T_{N'} \arrow[d] \\ T_N \times T_N \arrow[r] & T_N \end{tikzcd}$$

Since T_N is dense in U_σ

$$\Rightarrow \begin{tikzcd} T_{N'} \times U_{\sigma'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & U_{\sigma'} \arrow[d] \\ T_N \times U_\sigma \arrow[r] & U_\sigma \end{tikzcd}$$

The action is compatible with gluing

$$\Rightarrow \begin{tikzcd} T_{N'} \times X(\Delta') \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & X(\Delta') \arrow[d] \\ T_N \times X(\Delta) \arrow[r] & X(\Delta) \end{tikzcd}$$

prop: Any (seperated, normal) variety X with torus T_N. Then there exist a fan Σ in $N_{\mathbb{R}}$ s.t $X \simeq X_\Sigma$

prop: $N' \subseteq N$, $\Delta \subseteq N_{\mathbb{R}} = N'_{\mathbb{R}}$, Then
$\varphi : X(\Delta_{N'}) \to X(\Delta_N)$
induced by $N' \hookrightarrow N$ presents
$X(\Delta_N) = X(\Delta_{N'})/G$ $(G = N/N')$

pf: (3) $G = N/N' = \ker(T_{N'} \to T_N)$
$\Rightarrow G \curvearrowright X(\Delta_{N'})$ is compatible with $U_{\sigma,N'} \subseteq X_{\Delta_{N'}}$ $(\forall \sigma \in \Delta)$
$U_{\sigma,N'}/G \simeq U_{\sigma,N} \Rightarrow X(\Delta_{N'})/G \simeq X(\Delta_N)$

- We have $T_N \times U_6 \to U_6$ is an extension of $T_N \times T_N \to T_N$

Pf: $T_N \times T_N \to T_N$ is comes from

$$\mathbb{C}[M] \to \mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[M], \ \chi^m \mapsto \chi^m \otimes \chi^m$$

Then, we have following dual diagram

$$\begin{tikzcd}
\mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[M] & \mathbb{C}[M] \arrow[l] \arrow[dl, phantom, "\circlearrowright"] \\
\mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[S] \arrow[u, hook] & \mathbb{C}[S] \arrow[l] \arrow[u, hook]
\end{tikzcd}$$

(is given by $\chi^m \mapsto \chi^m \otimes \chi^m$)

$$\begin{tikzcd}
T \times T \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowright"] & T \arrow[d, hook] \\
T \times V \arrow[r] & V
\end{tikzcd}$$

(1) $\varphi : N' \to N$

$\varphi_{\mathbb{C}^*} : N' \otimes_{\mathbb{Z}} \mathbb{C}^* \to N' \otimes \mathbb{C}^*$

$\qquad\qquad \| \qquad\qquad\qquad \|$

$\qquad\qquad T_{N'} \qquad\qquad\quad T_N$

$n \otimes t \mapsto \varphi(n) \otimes t$

$M' = \mathrm{Hom}(T_{N'}, \mathbb{C}^*) \qquad M = \mathrm{Hom}(T_N, \mathbb{C}^*)$

$\tilde{\varphi} : M \to M'$

$$\begin{tikzcd}
T_{N'} \arrow[r] \arrow[dr] & T_N \arrow[d] \\
& \mathbb{C}^*
\end{tikzcd}$$

$$\left(\begin{array}{ll} M' = \mathrm{Hom}(N', \mathbb{Z}) & M = \mathrm{Hom}(N, \mathbb{Z}) \\ \varphi : N' \to N & \end{array} \right.$$

$$\begin{tikzcd} N' \arrow[r, "\varphi"] \arrow[dr, "m'"'] & N \arrow[d, "m"] \\ & \mathbb{Z} \end{tikzcd}$$

~~$\tilde{\varphi}(m') =$~~

$\tilde{\varphi}(m) = m \circ \varphi,$

$\tilde{\varphi} : M \to M'$

If $\varphi_R(\sigma') \subset \sigma$ $\quad S_{\sigma'} = \sigma'^{\vee} \cap M'$

$S_\sigma = \sigma^\vee \cap M$)

If $m \in S_\sigma \Rightarrow$ $m \in M$ $\langle m, u \rangle \geq 0 \ \forall u \in \sigma$

$\langle \tilde{\varphi}(m), u' \rangle$ $\quad u' \in \sigma'$ $\quad \exists n' \in \sigma' \cap N$ s.t $u' = r n$, $r > 0$

$= r \langle m, \varphi(n') \rangle$

Since $\varphi_R(\sigma') \subset \sigma \Rightarrow \varphi(n') \in \sigma \cap N$

so $\langle \tilde{\varphi}(m), u' \rangle \geq 0 \quad \forall u' \in \sigma'$

$\tilde{\varphi}|_{S_\sigma} \subset S_{\sigma'}$

$\Rightarrow \varphi^* : \mathbb{C}[S_\sigma] \to \mathbb{C}[S_{\sigma'}]$ $\left(\varphi^*(\chi^u) = \chi^{\tilde{\varphi}(u)} \right)$

$\Rightarrow \varphi_{\sigma'} : U_{\sigma'} \to U_\sigma$

(2) $\varphi^*_{\sigma_i}(\chi^u) = \chi^{\tilde{\varphi}(v_i - z_i m)}$

$\varphi^*_{\sigma_j}(\chi^u) = \chi^{\tilde{\varphi}(v_j - z_j(-m))}$

$u \in S_{\sigma_i \cap \sigma_j}$

$u = v_i - z_i m$

$= v_j - z_j(-m)$

Prop: Each ring $A_\sigma = \mathbb{C}[S_\sigma]$ is integrally closed

pf: $\sigma = \text{cone}(v_1, \cdots v_r)$

$\sigma^\vee = \cap \tau_i^\vee$, τ_i is the ray generated by v_i.

Then $\sigma^\vee \cap M = (\cap \tau_i^\vee) \cap M$

$= \cap (\tau_i^\vee \cap M)$

$$A_\sigma = \bigoplus_{u \in \sigma^\vee \cap M} \mathbb{C}\chi^u \quad A_{\tau_i} = \bigoplus_{u \in \tau_i^\vee \cap M} \mathbb{C}\chi^u$$

$\Rightarrow A_\sigma = \cap A_{\tau_i}$

Since $A_{\tau_i} \cong \mathbb{C}[X_1, X_2, X_2^{-1}, \cdots, X_n, X_n^{-1}]$, which is integrally closed.

then A_σ is integrally closed

R: integral domain

k: feild of fraction

If $r \in k$ is integral over R

$\Rightarrow r \in R$

(Any UID is integrally closed)

Lemma: Let R_α, $\alpha \in A$ be normal domin with the same field of fractions k.

Then $\cap_{\alpha \in A} R_\alpha$ is normal.

pf: $\forall v \in k \subseteq K$ is integral over $\cap_{\alpha \in A} R_\alpha$

$\Rightarrow v \in \cap_{\alpha \in A} R_\alpha$

§2.3 one parameter subgroup, limit points

- $Hom_{alg.gp}(G_m, G_m) = \mathbb{Z}$

- Give a lattice N, with dual M
  - the corresponding torus
    $$T_N = Hom(M, G_m) \cong (\mathbb{C}^*)^n$$
    $\forall t \in T_N$, $\forall u \in M$
    $$t(u) = \chi^u(t)$$
  - $\cancel{\#}$ $Hom(G_m, T_N) = Hom(\mathbb{Z}, N) = N$
    Take a basis for N
    $\forall (a_1, a_2, \cdots, a_n) \in N$
    $$G_m \longrightarrow T_N = (G_m)^n$$
    $$t \longmapsto (t^{a_1}, \cdots t^{a_n})$$
    $\Rightarrow$ Every one-parameter group $\lambda : G_m \to T_N$ is give by $v \in N$, denoted by λ_v
  - $Hom(T_N, G_m) = Hom(N, \mathbb{Z}) = M$
    Every character $\chi : T_N \to G_m$ is given by $u \in M$, denoted by χ^u
    $u = (m_1, \cdots, m_n) \in M$
    $$\chi^u : T_N \longrightarrow G_m$$
    $$(t_1, \cdots, t_n) \longrightarrow t_1^{m_1} t_2^{m_2} \cdots t_n^{m_n}$$
  - $Hom(T_N, G_m) \times Hom(G_m, T_N) \longrightarrow Hom(G_m, G_m)$
    $$(\chi^u, \lambda^v) \longmapsto \chi^u \circ \lambda^v : z \longmapsto z^{\langle u, v \rangle}$$

Recovering σ from the torus embedding $T_N \subset U_\sigma$.

prop: Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex cone and $u \in N$. Then

$$u \in \sigma \iff \lim_{t\to 0} \lambda_v(t) \text{ exists in } U_\sigma$$

Moreover, if $u \in \mathrm{Relint}(\sigma)$, then

$$\lim_{z\to 0} \lambda v(z) = X_{\sigma}$$

pf: (1) $\lim_{t\to 0} \lambda^u(t) \in U_\sigma$

$$\overset{\text{lemma1.}}{\iff} \lim_{t\to 0} \chi^m(\lambda^u(t)) \text{ exist in } \mathbb{C},\ \forall m \in S_\sigma$$

$$\iff \lim_{t\to 0} t^{\langle m,u\rangle} \text{ exists in } \mathbb{C}\ \forall m \in S_\sigma$$

$$\iff \langle m,u\rangle \geq 0 \quad \forall m \in \sigma^\vee \cap M$$

$$\iff u \in (\sigma^\vee)^\vee = \sigma$$

(2) $\lim_{t\to 0} \lambda^u(t) \in U_\sigma \rightsquigarrow x : \overset{\in S_\sigma}{m} \longmapsto \lim_{t\to 0} \chi^m(\lambda^u(t))$

$$= \lim_{t\to 0} t^{\langle m,u\rangle}$$

If $u \in \mathrm{Relint}(\sigma)$ $\langle m,u\rangle > 0 \quad \forall m \in S_\sigma \setminus \sigma^\perp$

$\langle m,u\rangle = 0 \quad \forall m \in S_\sigma \cap \sigma^\perp$

$$x : m \longmapsto \begin{cases} 1 & m \in S_\sigma \cap \sigma^\perp \\ 0 & m \in S_\sigma \setminus \sigma^\perp \end{cases}$$

lemma1: $\forall f : \mathbb{C}^* \to T_N$, Then

$\lim_{t\to 0} f(t)$ exists in U_σ

$$\iff \lim_{t\to 0} \chi^m(f(t)) \text{ exists in } \mathbb{C} \quad \forall m \in S_\sigma$$

"$\Rightarrow$" $\lim_{t\to 0} \chi^m(f(t)) = \lim_{t\to 0} f(t)(m)$ ($\exists x \in U_\sigma = \mathrm{Hom}(S_\sigma, \mathbb{C})$)

$\forall m \in S_\sigma \qquad = x(m) \in \mathbb{C}$

"$\Leftarrow$": $\forall m \in S_\sigma$

$\lim_{t\to 0} \chi^m(f(t)) \in \mathbb{C}$

$= \lim_{t\to 0} f(t)(m) \qquad = \lim_{t\to 0} \chi^m(f(t))$

$\chi: S_\sigma \longrightarrow \mathbb{C} \triangleq \chi(m) = \lim_{t\to 0} f(t)\, m$

Then $\chi = \lim_{t\to 0} f(t) \in \mathrm{Hom}(S_\sigma, \mathbb{C})$
$= U_\sigma$

For a general fan Δ.

$\forall \sigma \in \Delta$, $\chi_\sigma: S_\sigma \longrightarrow \mathbb{C}$

$$u \longmapsto \begin{cases} 1 & u \in \sigma^\perp \\ 0 & \text{otherwise} \end{cases}$$

If $\tau < \sigma$, $S_\sigma \subseteq S_\tau$ $\quad U_\tau \subseteq U_\sigma$, $\chi_\tau \in U_\sigma \underset{\mathrm{Hom}(S_\sigma,\mathbb{C})}{\shortparallel}$

- $\chi_\tau|_{S_\sigma}: S_\sigma \longrightarrow \mathbb{C}$

$$u \longmapsto \begin{cases} 1 & u \in \tau^\perp \cap S_\sigma \\ 0 & \text{otherwise} \end{cases}$$

~~Since~~ $\chi_\tau|_{S_\sigma}$ is well defined since $\tau^\perp \cap \sigma^\vee$ is a face of $\sigma^\vee$

- $\chi_\sigma \notin U_\tau$

Example: (1) Consider $V = V(xy - zw)$

$\sigma^\vee = \mathrm{Cone}(e_1^*, e_2^*, e_3^*, e_1^* + e_2^* - e_3^*)$

(2) $V = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M])$

~~Hom(Lm, TN)~~ $\mathbb{C}[\sigma^\vee \cap M]$

$= \mathbb{C}[\chi^{e_1^*}, \chi^{e_2^*}, \chi^{e_3^*}, \chi^{e_1^*+e_2^*-e_3^*}]$

$= \mathbb{C}[X, Y, Z, W] / (XY - ZW)$

$\mathbb{C}[M] = \mathbb{C}[X^{\pm 1}, Y^{\pm 1}, Z^{\pm 1}]$

$= (\mathbb{C}[\sigma^\vee \cap M])_{(XY)}$ x

$\mathbb{C}[\sigma^\vee \cap M] \hookrightarrow \mathbb{C}[M]$

$V = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M]) \longleftarrow \mathrm{Spec}(\mathbb{C}[M])$

Take $u = (a, b, c) \in \mathbb{Z}^3 \cong N \cong (\mathbb{C}^*)^3$

$\cancel{\lim_{t \to 0} (X^u, y}$ $\cancel{\lim_{t \to 0} \lambda^u(t)}$

$\cancel{= \lim_{t \to 0} (t^a, t^b, t^c) \in \mathbb{C}^3}$

$\lambda_u(t) = (t^a, t^b, t^c)$

$\cancel{\chi^{e_1^*} = t^a}$ $\chi^{e_1^*}(\lambda_u(t)) = t^a$

$\chi^{e_2^*}(\lambda_u(t)) = t^b$

$\chi^{e_3^*}(\lambda_u(t)) = t^c$

$\chi^{e_1^*+e_2^*-e_3^*}(\lambda_u(t)) = t^{a+b-c}$

$\lim_{t \to 0} t^{\langle m, n \rangle}$ exists in $\mathbb{C}$

$\Rightarrow a, b, c \geq 0.\ a + b - c \geq 0$

$\Rightarrow \sigma = \mathrm{cone}(e_1, e_2, e_1 + e_3, e_2 + e_3)$

$(\mathbb{C}^*)^3 \hookrightarrow U_6$

$(t_1, t_2, t_3) \longmapsto (t_1, t_2, t_3, t_1 t_2 t_3^{-1})$

~~$\lambda_u(t)$~~ $\lambda_u(t) = (t^a, t^b, t^c, t^{a+b-c})$

$\lim_{t\to 0} \lambda_u(t) \in V \iff a, b, c \geq 0$
$a+b \geq c$

$\Rightarrow 6 = \mathrm{Cone}(e_1, e_2, e_1+e_3, e_2+e_3)$

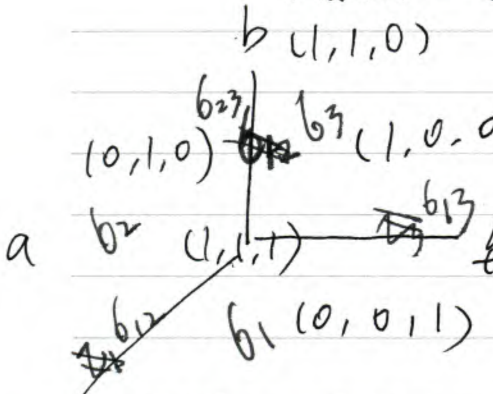
Example:

Consider $\mathbb{P}^2$

$T_N = (\mathbb{C}^*)^2 \subseteq \mathbb{P}^2 = \{(1, s, t) \mid s, t \neq 0\}$

$\forall u = (a,b) \in N = \mathbb{Z}^2$

$\lambda_u(t) = (1, t^a, t^b)$ ~~u x~~



$\lim_{t\to 0} \lambda_u(t) = (1, 0, 1)$

$\lim_{t\to 0} \lambda_u(t) = (0, 1, 1)$

$\shortparallel$

X_6

T_N-orbits in $\mathbb{P}^2$

$T_N \times \mathbb{P}^2 \longrightarrow \mathbb{P}^2$

$(1, s, t) \times (x_0 : x_1 : x_2) \longmapsto (x_0 : s x_1 : t x_2)$

Date

$O_1 = \{(x_0 : x_1 : x_2) \mid x_0, x_1, x_2 \neq 0\} = O_{301}$

$= \cancel{\gamma_{301}}\ T_N \cdot \gamma_{301}$

$= T_N \cdot (1,1,1)$

$O_2 = \{(x_0 : x_1 : x_2) \mid x_0 = 0,\ x_1, x_2 \neq 0\}$

$= O_{612} = T_N \cdot X_{612}$

$O_3 = \{(x_0 : x_1 : x_2) \mid x_0, x_2 \neq 0,\ x_1 = 0\} = O_{613} = T_N \cdot X_{613}$

$O_4 = \{(x_0 : x_1 : x_2) \mid x_0, x_1 \neq 0,\ x_2 = 0\} = O_{623} = T_N \cdot X_{623}$

$O_5 = \{(x_0 : x_1 : x_2) \mid x_0 = x_1 = 0,\ x_2 \neq 0\} = O_{61} = T_N \cdot X_{61}$

$O_6 = \{ \quad \mid x_0 = x_2 = 0,\ x_1 \neq 0\} = O_{62} = T_N \cdot X_{62}$

$O_7 = \{ \quad \mid x_1 = x_2 = 0,\ x_0 \neq 0\} = O_{63} = T_N \cdot X_{63}$

$X_{301} = (1,1,1)$?

$\gamma_{301} \in \mathrm{Hom}(M, \mathbb{C})$

$\gamma_{301} : m \longmapsto 1 \quad \forall m \in M$

$X_{301} \in \mathrm{Specm}(\mathbb{C}[M]) = (1,1,1)$

The torus orbit

Lemma : $O(\sigma) \triangleq T_N \cdot x_\sigma$

$= \{ x : S_\sigma \to \mathbb{C} \mid x(m) \neq 0 \Leftrightarrow m \in \sigma^\perp \cap M \}$

$\cong \mathrm{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*) = T_{N(\sigma)}$

$N(\sigma) = N/N_\sigma$

pf: " $T_N \cdot x_\sigma \subseteq \{ x : S_\sigma \to \mathbb{C} \mid \cdots \}$ "

显然

" $T_N \cdot x_\sigma \supseteq \{ x : S_\sigma \to \mathbb{C} \mid \cdots \}$ "

① $\{ x : S_\sigma \to \mathbb{C} \mid x(m) \neq 0 \Leftrightarrow m \in \sigma^\perp \cap M \}$

$\cong \mathrm{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*)$

② $0 \to N_\sigma \to N \to N(\sigma) \to 0$

$\xRightarrow{\otimes_\mathbb{Z} \mathbb{C}^*}$

$$\begin{array}{ccc} T_N & \to & T_{N(\sigma)} \to 0 \\ \| & & \| \\ \mathrm{Hom}_\mathbb{Z}(M, \mathbb{C}^*) & & \mathrm{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*) \end{array}$$

prop: Give a fan Δ and $X(\Delta)$

(a) $\{ \text{cones in } \Delta \} \Longleftrightarrow \{ T_N\text{-orbits in } X(\Delta) \}$

$\sigma \Longleftrightarrow O(\sigma)$

(b) $n = \dim N_\mathbb{R}$, then $\dim O(\sigma) = n - \dim \sigma$

(c) $U_\sigma = \bigcup_{\tau < \sigma} O(\tau)$

(d) $\tau < \sigma \Longleftrightarrow O(\sigma) \subseteq \overline{O(\tau)}$ and

$\overline{O(\tau)} = \bigcup_{\tau < \sigma} O(\sigma)$

↓

both the classical and Zariski closure

pf: (a) Let O be a T_N-orbit in $X(\Delta)$
Since $X(\Delta)$ is covered by U_σ,
and U_σ is T_N-invariant and
$U_{\sigma_i} \cap U_{\sigma_j} = U_{\sigma_i \cap \sigma_j}$
Then $\exists$ minimal cone $\sigma \in \Delta$
with $O \subseteq U_\sigma$.
Claim: $O = O(\sigma)$
Let $\gamma \in O$, and $S_\gamma = \{ m \in \overset{\sigma^\vee \cap M}{\overset{\shortparallel}{S_\sigma}} \mid \gamma(m) \neq 0 \}$
~~S_γ is a face of~~
then $\exists$ a face of $\sigma^\vee$ s.t $S_\gamma \underset{= F \cap M}{=}$
~~Then $\exists$~~
Since faces of $\sigma^\vee$ are all of the form
$\sigma^\vee \cap \tau^\perp$ for some $\tau < \sigma$,
then $\exists \tau < \sigma$ s.t $S_\gamma = \sigma^\vee \cap \tau^\perp \cap M$

Date

Let N_τ to be the sublattice of N generated by $\tau \cap N$, and
$$N(\tau) = N/N_\tau \quad M(\tau) = \tau^\perp \cap M$$
Then $N(\tau) \times M(\tau) \to \mathbb{Z}$ is induces by $N \times M \to \mathbb{Z}$

Let $O_\tau \triangleq T_{N(\tau)} = \mathrm{Hom}(M(\tau), \mathbb{C}^*)$
$$= \mathrm{spec}(\mathbb{C}[M(\tau)]) = N(\tau) \otimes_{\mathbb{Z}} \mathbb{C}^*$$
~~is a torus of~~
$\dim(O_\tau) = \dim(T_{N(\tau)}) = n-k$ where $k = \dim(\tau)$

The star of τ
$$\triangleq \mathrm{Star}(\tau) = \{\bar{\sigma} \subset N_\mathbb{R}/(N_\tau)_\mathbb{R} = N(\tau)_\mathbb{R} \mid \overset{\Delta \ni}{\sigma} > \tau\}$$
$\downarrow$
is a fan in $N(\tau)$

Let $V(\tau) = X(\mathrm{Star}(\tau))$ a $(n-k)$-d toric variety.

Then
$O_\tau = T_{N(\tau)} \subset V(\tau)$ corresponds to the cone $\{0\} = \bar{\tau}$

and $V(\tau) = \bigcup\limits_{\sigma > \tau} U_\sigma(\tau)$ is an affine open covering

$$U_\sigma(\tau) = \mathrm{spec}(\mathbb{C}[\bar{\sigma}^\vee \cap M(\tau)])$$
$$= \mathrm{spec}(\mathbb{C}[\sigma^\vee \cap \tau^\perp \cap M])$$

$$U_\tau(\tau) = O_\tau$$

classical topology?

A closed embedding of $U_\sigma(\tau)$ in U_σ is given by

$$\begin{tikzcd} U_\sigma(\tau) = \mathrm{Hom}_{sg}(\sigma^\vee \cap \tau^\perp \cap M, \mathbb{C}) \arrow[r, hook] \arrow[r, phantom, shift right=3ex, "\text{extension by zero}"] & \mathrm{Hom}_{sg}(\sigma^\vee \cap M, \mathbb{C}) = U_\sigma \end{tikzcd}$$

($\sigma^\vee \cap \tau^\perp$ is a face of $\sigma^\vee$ implies the extension by zero is well-defined)

The corresponding surjection of rings

$$\mathbb{C}[\sigma^\vee \cap M] \twoheadrightarrow \mathbb{C}[\sigma^\vee \cap \tau^\perp \cap M]$$

is projection

- The embedding map is compatible:
  if $\tau < \sigma$, $\sigma < \sigma'$

$$\begin{tikzcd} \mathrm{Hom}_{sg}(\sigma^\vee \cap \tau^\perp \cap M, \mathbb{C}) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Hom}_{sg}(\sigma'^\vee \cap \tau^\perp \cap M, \mathbb{C}) \arrow[d] \\ \mathrm{Hom}_{sg}(\sigma^\vee \cap M, \mathbb{C}) \arrow[r, hook] & \mathrm{Hom}_{sg}(\sigma'^\vee \cap M, \mathbb{C}) \end{tikzcd}$$

$$\Downarrow$$

$$\begin{tikzcd} U_\sigma(\tau) \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowright"] & U_{\sigma'}(\tau) \arrow[d, hook] \\ U_\sigma \arrow[r, hook] & U_{\sigma'} \end{tikzcd}$$

Then we have a closed embedding

$$V(\tau) \hookrightarrow X(\Delta)$$

If $\tau < \tau'$, we have closed embedding
$V(\tau') \hookrightarrow V(\tau)$
regard $V(\tau)$ as a toric variety

For $\sigma \in \mathrm{Star}(\tau')$ $U_\sigma(\tau') = \mathrm{Hom}_{sg}(\sigma^\vee \cap \tau'^\perp \cap M, \mathbb{C})$
$$\hookrightarrow \mathrm{Hom}_{sg}(\sigma^\vee \cap \tau^\perp \cap M, \mathbb{C})$$
$$= U_\sigma(\tau)$$
gives ~~an~~ a closed embedding $U_\sigma(\tau') \hookrightarrow U_\sigma(\tau)$

Apply preceding constructions we
have $V(\tau') \hookrightarrow V(\tau)$

prop: (a) $U_\sigma = \bigsqcup_{\tau < \sigma} O_\tau$

(o) $O_\tau = T_N \cdot x_\tau$
pf: $T_N \cdot x_\tau = \{ x : S_\sigma \to \mathbb{C}$
$x(m) \neq 0 \Leftrightarrow m \in \sigma^\vee \cap M \}$

(b) $V(\tau) = \bigsqcup_{\gamma > \tau} O_\gamma$

(c) $O_\tau = V(\tau) \setminus \bigcup_{\gamma \supsetneq \tau} V(\gamma)$

pf: (a) "$\supseteq$" $O_\tau = U_\tau(\tau) \hookrightarrow U_\tau \subseteq U_\sigma$

"$\subseteq$" $\forall x \in U_\sigma$, $x : \sigma^\vee \cap M \to \mathbb{C}$

$\exists \tau < \sigma$, s.t. $x^{-1}(\mathbb{C}^*) = \sigma^\vee \cap \tau^\perp \cap M$

since, the sum of two elements of $\sigma^\vee$ cannot be in $x^{-1}(\mathbb{C}^*)$ unless both are in $x^{-1}(\mathbb{C}^*)$

So x correspond to a point of O_τ

(c) ~~Working in $V(\tau)$, $\tau = \{0\}$~~
~~we need to prove $T_N = X(\Delta) \setminus \bigcup_{\gamma \neq \{0\}} V(\gamma)$~~

$X(\Delta) \setminus \bigcup_{\gamma \neq \{0\}} V(\gamma)$

$= X(\Delta) \setminus \bigcup_{\gamma \neq \{0\}} \bigcup_{\sigma > \gamma} U_\sigma(\gamma)$

$= \bigcap_{\substack{\gamma \neq \{0\} \\ \sigma > \gamma}} X(\Delta) \setminus U_\sigma(\gamma)$

$\ni O_\tau$

(b) $V(\tau) = O_\tau \cup \left(\bigcup_{\gamma \gneq \tau} V(\gamma) \right)$

$\cancel{\bigcup_{\gamma \gneq \tau} V(\gamma)}$

$V(\gamma) = O_\gamma \cup$

(a) $V(\tau) = \underline{\bigcup_{\sigma > \tau} U_\sigma(\tau)}$

$= \bigcup_{\sigma > \tau} O_\sigma = \coprod_{\sigma > \tau} O_\sigma$

(c) $V(\gamma) = \coprod_{\sigma > \gamma} O_\sigma$

$\bigcup_{\gamma \gneq \tau} V(\gamma) = \coprod_{\sigma \gneq \tau} O_\sigma$

$V(\tau) \setminus \bigcup_{\gamma \gneq \tau} V(\gamma) = O_\tau$

Working $U_\sigma(\tau)$

From (a) $U_\sigma(\tau) = \coprod_{\sigma > \gamma > \tau} \widetilde{O}_\gamma$

claim: $\widetilde{O}_\gamma = O_\gamma$

$O_\gamma = \mathrm{spec}(\mathbb{C}[\gamma^\perp \cap M])$

In $N(\tau)$, $\bar{\gamma} \subset N_R / (N_\tau)_R \subset N(\tau)_R$

$\bar{\gamma} \subset N_R / (N_\tau)_R \subset N(\tau)_R$

$\widetilde{N(\bar{\gamma})} = N(\tau) / N_{\bar{\gamma}}$

$\widetilde{M(\bar{\gamma})} = \tau^\perp \cap M \cap \bar{\gamma}^\perp$

$= \tau^\perp \cap M \cap \gamma^\perp$

$= \gamma^\perp \cap M$

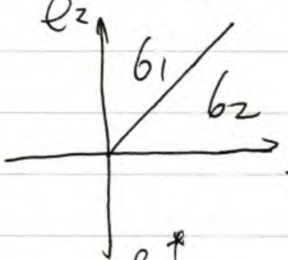
$\widetilde{O}_\gamma = \mathrm{spec}(\mathbb{C}[\widetilde{M(\bar{\gamma})}])$

$= \mathrm{spec}(\mathbb{C}[\gamma^\perp \cap M])$

Construct the blowup.

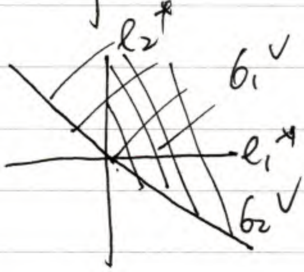
- Let $\sigma_1 = \text{cone}(e_1+e_2, e_2)$

  $\sigma_2 = \text{cone}(e_1, e_1+e_2)$



$$\sigma_1^\vee = \text{cone}(e_1^*, -e_1^*+e_2^*)$$

$$\sigma_2^\vee = \text{cone}(e_1^*-e_2^*, e_2^*)$$



$$S_2 = S_{\sigma_1} + (-(e_2^*-e_1^*))$$

$$= S_{\sigma_2} + (-(e_1^*-e_2^*))$$

$$\mathbb{C}[S_{\sigma_1}] = \mathbb{C}[X, X^{-1}Y]$$

$$\mathbb{C}[S_{\sigma_2}] = \mathbb{C}[XY^{-1}, Y]$$

$$= \mathbb{C}[S_2] = \mathbb{C}[S_{\sigma_1}]_{X^{-1}Y} = \mathbb{C}[S_{\sigma_2}]_{XY^{-1}}$$

$\mathbb{C}[X^{-1}Y, \cancel{XY}, XY^{-1}]$
X

$$\begin{array}{ccc|ccc}
\mathbb{C}[S_{\sigma_1}]_{X^{-1}Y} & & \mathbb{C}[S_2] & & & \mathbb{C}[S_{\sigma_2}]_{XY^{-1}} \\
X & \longmapsto & X & XY^{-1} & \longleftarrow & XY^{-1} \\
X^{-1}Y & \longmapsto & X^{-1}Y & Y & \longleftarrow & Y \\
(w,z) & \longmapsto & (u,v) = (wz, \tfrac{1}{z}) & & &
\end{array}$$

- Consider $\mathbb{P}^1 \times \mathbb{C}^2$ with coordinates $(x_0 : x_1)$ on $\mathbb{P}^1$ and (x, y) on $\mathbb{C}^2$

  $W = V(x_0 y - x_1 x) \subseteq \mathbb{P}^1 \times \mathbb{C}^2$ is the blowup of $\mathbb{C}^2$ at the origin $Bl_0(\mathbb{C}^2)$

- $\mathbb{P}^1 \times \mathbb{C}^2$ is covered by

  $U_0 \times \mathbb{C}^2 = \text{Spec}(\mathbb{C}[x_1/x_0, x, y])$

  $U_1 \times \mathbb{C}^2 = \text{Spec}(\mathbb{C}[x_0/x_1, x, y])$

  W is covered by

  $W_0 = V(y - (x_1/x_0)x) \subseteq U_0 \times \mathbb{C}^2$

$W_1 = V((x_0/x_1)y - x) \subseteq U_1 \times \mathbb{C}^2$

For W_0

$\mathbb{C}[x_1/x_0, x, y]/(y-(x_1/x_0)x) \cong \mathbb{C}[x, x_1/x_0]$

via $y \mapsto (x_1/x_0)x$

For W_1

$\mathbb{C}[x_0/x_1, x, y]/(x-(x_0/x_1)y) \cong \mathbb{C}[x_0/x_1, y]$

via $x \mapsto (x_0/x_1)y$

Glue W_0 and W_1 by

$u = wz, \; v = \frac{1}{z}$

$X(\Delta) \cong W = Bl_0(\mathbb{C}^2)$

- Let $N = \mathbb{Z}^n$ with standard basis $e_1, \cdots, e_2$

set $e_0 = e_1 + \cdots + e_n$

$\Delta = \{ \sigma \mid \sigma$ genereted by all subsets of $\{e_0, e_1, \cdots e_n\}$ except $\{e_1, \cdots, e_n\} \}$

$\Rightarrow X(\Delta) \cong Bl_0(\mathbb{C}^n)$ ($W = Bl_0(\mathbb{C}^2) \subseteq \mathbb{P}^1 \times \mathbb{C}^2$

$W \setminus \mathbb{P}^1 \times \{0\} \cong \mathbb{C}^2 \setminus \{0\}$

- ~~Show how to construct the~~

Refinement of fans and blowup

$\Sigma \subseteq N_{\mathbb{R}}$, Σ' is a refinement of Σ

$(|\Sigma'| = |\Sigma|)$

Then the toric morphism $\phi : X_{\Sigma'} \to X_\Sigma$ is proper

Date

Def: Σ a fan in $N_{\mathbb{R}}$, $\sigma = \mathrm{Cone}(u_1, \cdots, u_n)$ be a smooth cone in Σ,
Let $u_0 = u_1 + \cdots + u_n$ and
$\Sigma'(\sigma)$ be the set of all cones generated by subsets of $\{u_0, \cdots, u_n\}$ not containing $\{u_1, \cdots, u_n\}$
Then $\Sigma^*(\sigma) = (\Sigma \setminus \{\sigma\}) \cup \Sigma'(\sigma)$
is called the star subdivision of Σ along σ.

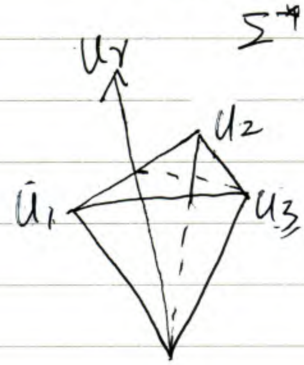
prop: $\phi: X_{\Sigma^*(\sigma)} \to X_\Sigma$ makes $X_{\Sigma^*(\sigma)}$ the blowup of X_Σ at the distinguished point x_σ
(Denoted by $Bl_{\gamma_0}(X_\Sigma)$)

pf: We only need to consider $\Sigma = \{$all faces of $\sigma\}$
$\Rightarrow X_\Sigma = U_\sigma \cong \mathbb{C}^2$
and γ_σ is the unique fixed point of torus action (i.e γ_σ is the origin)
We only need to check
$$\phi: X_{\Sigma'(\sigma)} \to U_\sigma \cong \mathbb{C}^n$$
is the blowup of $\mathbb{C}^n$ at the origin

Def: Σ a fan in $N_{\mathbb{R}}$, $\tau \in \Sigma$ has the property that all cones $\sigma \succ \tau$ are smooth. Let $u_\tau = \sum_{\rho \in \tau(1)} u_\rho$ and for each $\sigma \succ \tau$ set $\Sigma_\sigma^*(\tau) = \{ \mathrm{Cone}(A) \mid A \subseteq \{u_\tau\} \cup \sigma(1), \tau(1) \nsubseteq A \}$

Then the star subdivision of Σ relative to τ is

$$\Sigma^*(\tau) = \{ \sigma \in \Sigma \mid \sigma \not\succeq \tau \} \cup \bigcup_{\tau \preceq \sigma} \Sigma_\sigma^*(\tau)$$



eg. $\tau = \mathrm{cone}(u_1, u_2)$

$u_\tau = u_1 + u_2$

$\Sigma_\sigma^*(\tau) = \{ \mathrm{cone}(u_1, u_\tau, u_3)$
$\mathrm{cone}(u_2, u_\tau, u_3) \}$

prop: $X_{\Sigma^*(\tau)}$ is the blowup $Bl_{V(\tau)}(X_\Sigma)$ of X_Σ along the orbit closure $V(\tau)$

§2.4. compactness and properness

prop: $X(\Delta)$ is compact if and only if ~~its~~ $|\Delta| = N_{\mathbb{R}}$

Notion: a complex variety is compact in classical topology when it is complete (proper) as an algebraic ~~of~~ variety.

Notion: proper map.

$f: X \to Y$ is proper if the preimage of compact set in Y is compact in X

Remark: X is compact $\Longleftrightarrow$ the constant map $X \to Y = \{\text{a single point}\}$ is proper

$\varphi: N' \to N$ is compatible

prop: The map $\varphi_*: X(\Delta') \to X(\Delta)$ is proper if and only if $\varphi^{-1}(|\Delta|) = |\Delta'|$

prop.: Let X be a locally compact Haustorff topological space

X is compat $\Longleftrightarrow \forall Z$, $\pi_Z: X \times Z \to Z$ is closed (i.e. $\pi_Z(W) \subseteq Z$ is closed, $\forall$ closed $W \subseteq X \times Z$)

Def: A variety X is complete if $\forall$ variety Z the projection map $\pi_Z: X \times Z \to Z$ is a closed mapping in the Zariski topology

Remark: Any projective variety is complete

Def: (proper map): above

prop: $f: X \to Y$ be a continuous mapping of locally compact first countable Hausdorff spaces. Then TFAE:

(a) f is proper

(b) f is a closed mapping, i.e. $f(W) \subseteq Y$ is closed, $\forall$ closed subset $W \subseteq X$ and all fibers $f^{-1}(y)$, $y \in Y$ are compact

(c) $\forall x_k \in X$, s.t. $f(x_k) \in Y$ converges in Y has a subsequence x_{k_t} that converges in X

prop: $f: X \to Y$ be a continuous map between locally compact Hausdorff space.

(a) f is proper

$\Leftrightarrow$ (b) f is universally closed, i.e.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow{\scriptstyle \pi_X} & \circlearrowleft & \uparrow{\scriptstyle g} \text{ continuous mapping} \\ X \times_Y Z & \xrightarrow[\pi_Z]{} & Z \end{array}$$

the projection π_Z defined by the commutative diagram is a closed mapping

Def: ~~prop~~: A morphism of varieties $\phi: X \to Y$ is proper if it is universally closed i.e $\forall$ variety Z and morphism $\psi: Z \to Y$ the projection π_Z is a closed mapping in the zariski topology

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{\pi_X} & X \\ \downarrow{\scriptstyle \pi_Z} & & \downarrow{\scriptstyle \phi} \\ Z & \xrightarrow{\psi} & Y \end{array}$$

Remark: a variety X is complete $\Longleftrightarrow$ $\phi: X \to \{pt\}$ is proper

pf: "$\Rightarrow$" X complete $\Rightarrow$ $X \times_Y Z \xrightarrow{\pi_Z} Z$ is a closed mapping

"$\Leftarrow$" ✓

Furthermore, if X is complete $\Rightarrow \pi_Z : X \times Z \to Z$ is proper, $\forall Z$

proof of prop:

"$\Rightarrow$" $|\Delta'| \subseteq \varphi^{-1}(|\Delta|)$ ✓

We only need to prove $\varphi^{-1}(|\Delta|) \subseteq |\Delta'|$

Suppose there exist $v \in |\Delta|$, $v = \varphi(v')$ but $v' \notin |\Delta'|$.

Then $\lim\limits_{z \to 0} \varphi_*(\lambda_{v'}(z)) = \lim\limits_{z\to 0} \lambda_v(z)$ ~~has a limit~~ exists in $X(\Delta)$

but $\lim\limits_{z \to 0} \lambda_{v'}(z)$ doesn't exist, which contradicts to properness.

"$\Leftarrow$" (valuative criterion of properness)

A morphism $f: X \to Y$ of varieties (or a separated morphism of schemes of finite type) is proper $\Longleftrightarrow$ $\forall$ DVR R, with quotient field K, the commutative diagram can be filled by

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{Spec}(R) \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

When X is irreducible, one may assume the image of $\mathrm{Spec}(k) \to X$ is in a given open subset U of X.

Apply this criterion

$$\begin{tikzcd} \mathrm{Spec}(k) \arrow[r] \arrow[d, hook] & X(\Delta') \arrow[d, "\varphi_*"] \\ \mathrm{Spec}(R) \arrow[r] & X(\Delta) \end{tikzcd}$$

Assume the image of $\mathrm{Spec}(k) \to X(\Delta')$ is in $T_{N'}$ and consider each affine piece U_σ of $X(\Delta)$

we have the diagram

$$\begin{tikzcd} k & \mathbb{C}[M'] \arrow[l] \\ R \arrow[u, hook] & \mathbb{C}[S_\sigma] \arrow[l] \arrow[u] \end{tikzcd}$$

To fill this diagram, we only need to find σ' and a map: $\sigma' \to \sigma$ to fill the following diagram

$$\begin{tikzcd} k & \mathbb{C}[M'] \arrow[l] & \mathbb{C}[S_{\sigma'}] \arrow[l, phantom, "\supset"] \arrow[ddll, dashed] \\ & & \\ R \arrow[uu, hook] & \mathbb{C}[S_\sigma] \arrow[l] \arrow[uu] \arrow[uur, dashed] & \end{tikzcd}$$

(generating set of k)

Let $\alpha: M' \to k^*$ is given by $\mathrm{Spec}(k) \to T_{N'}$

Then $\forall \in \sigma^\vee \cap M$, $\mathrm{ord} \circ \alpha \circ \varphi^*(m) \geq 0$

$\Rightarrow \varphi(\mathrm{ord} \circ \alpha) = \mathrm{ord} \circ \alpha \circ \varphi^* \in (\sigma^\vee)^\vee = \sigma$

Since ~~$\forall$~~ $|\Delta'| = \varphi^{-1}(|\Delta|)$, $\exists\, \sigma' \in \Delta'$, s.t. $\mathrm{ord} \circ \alpha \in \sigma'$ and $\varphi(\sigma') < \sigma$

Then $\mathrm{ord} \circ \alpha \in (\sigma'^\vee)^\vee$, so we fill the diagram

Date

Jacobi racial of a ring

- In commutative case

$$J(R) = \bigcap_{m \in \mathrm{SpecmR}} m$$

- In Noncommutative case, R a ring with unity

$J(R) = \{ r \in R \mid rM = 0$ where M is simple R-module$\}$

- Motivation
- Example: local ring $\overset{(R,p)}{\Rightarrow}$ $J(R) = p$

Artin ring$_R$ $\Rightarrow$

local ring

- a commutative local ring ofen aries as the result of the localization of a ring at a prime ideal
- R is a local ring if it has any one of following properties

(a) R has a unique maximal left (or right) ideal

(b) $1 \neq 0$ and the sum of two non-unit in R is a non-unit

(i.e the set of non-units in R forms a (proper) ideal)

(c) $1 \neq 0$ and if x or $1-x$ is a unit ($\forall x \in R$)

(i.e there do not exist two coprime proper ideal $\nexists I_i, I_2 \subsetneq R$ s.t $R = I_i + I_2$)

- Example: discrete valuation ring and not a field (which is a local principle ideal domain)

Valuation ring & discrete valuation ring

- Def: Valuation ring D
  the following are equivalent
  (a) $\forall$ non-zero $x \in K$, at least one x or $x^{-1} \in D$ (K: the field of fraction)
  (b) The ideals of D are totally ordered by inclusion
  (c) There is a totally ordered abelian group T and a valuation $v: K \to T \cup \{\infty\}$ with $D = \{x \in K,\ v(x) \geq 0\}$
- Def: Discrete ... if $T \cong \mathbb{Z}$
  Example of DVR: $F[[X]]$ (F field)

Date

Conclusion:

① U_σ is smooth, and $U_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$

$\iff \sigma = \text{cone}(e_1, \cdots, e_k)$

$e_1, \cdots, e_k$ is part of a basis of N

(We say σ is smooth)

(Δ is smooth $\iff$ each σ is smooth)

(or X_Δ)

② U_σ has finite quotient singularities

$\iff \sigma$ (or Δ) is simplicial

③ X_Δ is complete $\iff |\Delta| = N_{\mathbb{R}}$

(Δ is complete)

Toric resolution of singularities

let X be a normal toric surface and

X_{sing} is the finite set of singular points of X

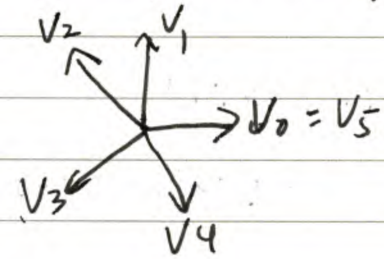
Def: A proper morphism: $\varphi: Y \to X$ is a resolution of singularities of X if Y is smooth surface and

$$Y \setminus \varphi^{-1}(X_{sing}) \simeq X \setminus X_{sing}$$

Nonsingular surface

Consider 2-d -nonsingular complete toric varieties. $X(\Delta)$

$N = \mathbb{Z}^2$, $\Delta \subset N_{\mathbb{R}}$ is given by a sequence of lattice points $v_0, v_1, \cdots, v_d = v_0$



- complete $\sim$ $|\Delta| = N_{\mathbb{R}}$
- nonsigular $\sim$ $\forall\, 0 \le i \le d-1$ v_i and v_{i+1} are a basis for $N \cong \mathbb{R}^2$

$$\Rightarrow \begin{pmatrix} 0 & 1 \\ -1 & a_1 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \text{i.e } v_2 = -v_0 + a_1 v_1$$

$$\Rightarrow a_i v_i = v_{i-1} + v_{i+1} \quad \forall\, 1 \le i \le d,\ a_i \in \mathbb{Z}$$

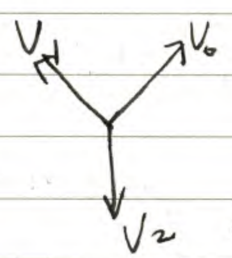
$$\begin{pmatrix} 0 & 1 \\ -1 & a_d \end{pmatrix} \begin{pmatrix} v_{d-1} \\ \underset{v_0}{v_d} \end{pmatrix} = \begin{pmatrix} v_d \\ v_1 \end{pmatrix} \quad v_1 + v_{d-1} = a_d v_d$$

- If $d = 3$ $\begin{pmatrix} 0 & 1 \\ -1 & a_3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & a_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & a_1 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}$

$\Rightarrow a_1 = a_2 = a_3 = -1$

$\Rightarrow v_0 + v_2 + v_1 = 0$

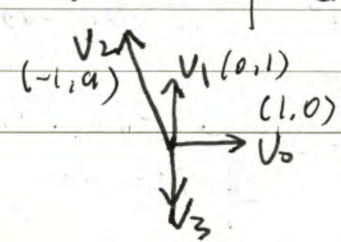
$\Rightarrow X(\Delta) = \mathbb{P}^2$



- If $d = 4$ $a_1 = a_3 = 0$ $a_2 = -a_4 \neq 0$

or $a_2 = a_4 = 0$ $a_1 = -a_3 \neq 0$

$a_1 = a$



$\Rightarrow X(\Delta) = F_a$ $\begin{cases} a v_1 = v_0 + v_2 \\ v_1 + v_3 = 0 \\ -a v_3 = v_2 + v_4 \end{cases}$

Claim: If $d \geq 5$, there must be some j, $1 \leq j \leq d$, s.t. ~~V_{j-1} and~~

$$V_j = V_{j-1} + V_{j+1}$$

① for $\forall i, j$, V_j and V_{j+1} cannot be ~~are~~ arranged with

$$V_j \in \text{Relint}(\text{cone}(V_{i+1}, -V_i))$$

and $V_{j+1} \in \text{Relint}(\text{cone}(-V_i, -V_{i+1}))$

Pf: Suppose

$$\begin{pmatrix} V_j \\ V_{j+1} \end{pmatrix} = \begin{pmatrix} -a & b \\ -c & -d \end{pmatrix} \begin{pmatrix} V_i \\ V_{i+1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & a_j \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & a_{j-1} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -1 & a_{i+1} \end{pmatrix} \begin{pmatrix} V_i \\ V_{i+1} \end{pmatrix}$$

where $a, b, c, d \in \mathbb{Z}_{>0}$

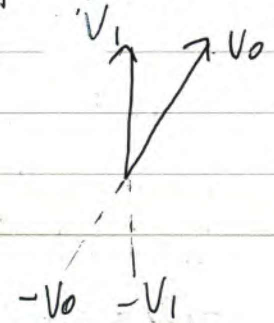
But $\det \begin{pmatrix} -a & b \\ -c & -d \end{pmatrix}$

$$= ad + cb = 1$$

contradict!

② If $d \geq 4$, there must exist $V_j = -V_i$ for some i, j.

Pf:



Consider $V_i \in \text{Relint}(\text{cone}(V_1, -V_1))$

If $V_i \in \text{Relint}(\text{cone}(V_1, -V_0))$

$\Rightarrow V_{i+1} = -V_0$

If $V_i \in \text{Relint}(\text{cone}(-V_0, -V_1))$

$\Rightarrow V_{i+1} = -V_1$ If $V_i = -V_0$ done!

Date

③ Suppose $V_i = -V_0$, $i \geq 3$, then there exists $0 < j < i$ s.t. $V_j = V_{j-1} + V_{j+1}$

pf: Let $V_j = -b_j V_0 + b_j' V_1$ ($1 \leq j \leq i$)

($b_j, b_j' \geq 0$, $b_j + b_j' \neq 0$)

Let $c_j = b_j + b_j'$ $\cancel{c_j \geq 0}$ $c_j \in \mathbb{Z}_{>0}$

$c_1 = 1$ $c_i = 1$ $c_2 \geq 2$

$\exists j$ s.t. $c_j \geq c_{j-1}$, $c_j > c_{j+1}$

Then $a_j v_j = V_{j-1} + V_{j+1}$

$\Rightarrow (b_{j+1}' + b_{j-1}' - a_j b_j') v_1 = (b_{j+1} + b_{j-1} - a_j b_j) v_0$

$\Rightarrow b_{j+1}' + b_{j-1}' - a_j b_j' + b_{j+1} + b_{j-1} - a_j b_j =$

$= c_{j+1} + c_{j-1} - a_j c_j = 0$

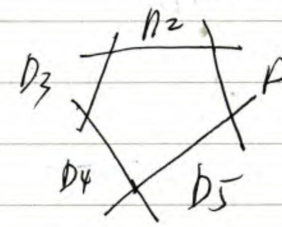
$\Rightarrow c_{j+1} - c_j + (c_{j-1} - c_j) = (a_j - 2) c_j$

$\Rightarrow a_j - 2 < 0 \Rightarrow a_j = 1$

prop: All complete nonsingular toric surface are obtaind from $\mathbb{P}^2$ or $\mathbb{F}_a$ by a succession of blow-ups at T_N-fixed points

- Each V_i determines a curve

$$D_i = \underset{\substack{\| \\ R_{\geq 0} V_i}}{V(\tau_i)} \cong \mathbb{P}^1 \text{ and } (D_i \cdot D_i) = -a_i$$



§2.6 Date

Resolution of singularities

Lemma: $\sigma \subseteq N_{\mathbb{R}} \cong \mathbb{R}^2$. Then there exists a basis e_1, e_2 for N s.t

$$\sigma = \text{Cone}(e_2, me_1 - ke_2)$$

where $0 \le k < m$, $\gcd(m,k) = 1$

pf: Let $\sigma = \text{cone}(u_1, u_2)$, where u_i are primitive

Then we can take $e_2 = u_1$.

Let e_1', e_2 be the basis for N,

then $u_2 = me_1' + le_2$ for some $m > 0$

Then there exist $s, k \in \mathbb{Z}$ s.t

$$l = sm - k, \quad \underline{0 \le k < m}$$

Let $e_1 = e_1' + se_2$, e_1, e_2 is also a basis for N

and $u_2 = m(e_1 - se_2) + le_2$

$= me_1 - ke_2$

Since u_2 is primitive $\Rightarrow \gcd(m,k) = 1$

Hirzebruch-Jung Continued fractions

Given $\sigma = \text{cone}(e_2, me_1 - ke_2) \subseteq N_{\mathbb{R}} \cong \mathbb{R}^2$

① refine σ to $\sigma' = \text{Cone}(e_2, e_1) \sim$ smooth

$\sigma'' = \text{Cone}(e_1, me_1 - ke_2)$

(If $k=1$, σ'' smooth

If $k > 1$, σ'' not smooth)

② $m = a_1 k - k_1 \quad 0 \le k_1 < k \;,\; a_1 \geq 2$ (since $m>k$)

$k = a_2 k_1 - k_2 \quad 0 \le k_2 < k_1 \;,\; a_2 \geq 2$

$k_1 = a_3 k_2 - k_3$

$\vdots$

$k_{r-2} = a_r k_{r-1} + \underbrace{k_r}_{=0}$ since $k > k_1 > k_2 \cdots$ $\exists\, r$, s.t $k_r = 0$

$$\Rightarrow \frac{m}{k} = a_1 - \frac{k_1}{k} = a_1 - \frac{1}{a_2 - \frac{k_2}{k_1}} = \cdots = a_1 - \frac{1}{a_2 - \frac{1}{\cdots - \frac{1}{a_r}}}$$

Denote: $\frac{m}{k} = [[a_1, a_2, \dots, a_r]] \quad (a_i \geq 2)$

~~③ Let $m>k>0$, $\gcd(m,k)=1$, and~~
~~$\frac{m}{k} = [[a_1, a_2, \dots, a_r]]$~~
~~Let $p_0 = 1$, $p_1 = a_1$ $p_i = a_i p_{i-1} - p_{i-2}$~~
~~$Q_0 = 0$ $Q_1 = 1$, $Q_i = a_i Q_{i-1} - P_{i-2}$~~
$(2 \le i \le r)$
~~claim:~~

③ We get a refinement of σ''

Let $m>k>0$, $\gcd(m,k)=1$, $\frac{m}{k} = [[a_1, \dots, a_r]]$

Let $p_0 = 1$, $p_1 = a_1$, $p_i = a_i p_{i-1} - p_{i-2}$

$Q_0 = 0$ $Q_1 = 1$, $Q_i = a_i Q_{i-1} - Q_{i-2}$ $(2 \le i \le r)$

$u_i = p_{i-1} e_1 - Q_{i-1} e_2 \quad 1 \le i \le r+1$, $u_0 = e_2$

$\sigma_i = \mathrm{cone}(u_{i-1}, u_i) \quad 1 \le i \le r+1$

Claim: $\{\sigma_i\}$ is a refinement of σ''

~~Each U_{σ_i} is smooth~~

④ We get $\phi: X_\Delta \to U_\sigma$ is a resolusion of singularities

$\frac{n}{k} = [[a_1, a_2, a_3, a_4, a_5, a_6]]$
$\;\;11$

Date

Example: $17/11 = [[2, 3, 2, 2, 2, 2]]$

$17 = 2 \times 11 - 5$

$11 = 3 \times 5 - 4$

$5 = 2 \times 4 - 3$

$4 = 2 \times 3 - 2$

$3 = 2 \times 2 - 1$

$2 = 2 \times 1 - 0$

	P_i	Q_i	u_i	$\det(b_i)$
$i=0$	1	0		$= \begin{vmatrix} P_{i-2}, & -Q_{i-2} \\ P_{i-1}, & -Q_{i-1} \end{vmatrix}$
$i=1$	a_1	1	e_1	
2	$a_2 a_1 - 1$	a_2	$a_1 e_1 - e_2$	
3			$(a_2 a_1 - 1)e_1 - a_1 e_2$	
4				
5				
6				

(a) P_i, Q_i are increasing sequences $\Rightarrow P_i, Q_i \geq 0$

pf: $P_i - P_{i-1} = (a_i - 1) P_{i-1} - P_{i-2}$

$\geq P_{i-1} - P_{i-2} \quad (a_i \geq 2)$

By induction, $P_i - P_{i-1} \geq 0$

$\Rightarrow P_i$ is increasing

(b) $P_{i-1} Q_i - P_i Q_{i-1} = 1 \quad \forall 1 \leq i \leq r$

pf: $i=1$ ✓

Suppose the results holds for $i \leq s$

Let $i = s+1$ $\quad P_{i-1} Q_i - P_i Q_{i-1}$

$= P_s (a_{s+1} Q_s - Q_{s-1}) - (a_{s+1} P_s - P_{s-1}) Q_s$

$= P_{s-1} Q_s - P_s Q_{s-1} = 1$

(c) $P_{i-1}Q_i - P_iQ_{i-1} > 0 \quad \forall 1 \le i \le r$

$$\Rightarrow \frac{P_{i-1}}{Q_{i-1}} > \frac{P_i}{Q_i} \qquad \forall 2 \le i \le r$$

$$\Rightarrow \frac{P_r}{Q_r} < \frac{P_{r-1}}{Q_{r-1}} < \cdots < \frac{P_1}{Q_1}$$

(d) $\frac{P_r}{Q_r} = \frac{m}{k}$, $\frac{P_i}{Q_i} = [[a_1, a_2, \cdots, a_i]] \; \forall 1 \le i \le r$

pf: $$\frac{P_i}{Q_i} = \frac{a_iP_{i-1} - P_{i-2}}{a_iQ_{i-1} - Q_{i-2}}$$

$$= \frac{P_{i-1} - \frac{1}{[[a_i]]}P_{i-2}}{Q_{i-1} - \frac{1}{[[a_i]]}Q_{i-2}} = \frac{a_{i-1}P_{i-2} - P_{i-3} - \frac{1}{[[a_i]]}P_{i-2}}{a_{i-1}Q_{i-2} - Q_{i-3} - \frac{1}{[[a_i]]}Q_{i-2}}$$

$$= \frac{[[a_i, a_{i-1}]]P_{i-2} - P_{i-3}}{[[a_i, a_{i-1}]]Q_{i-2} - Q_{i-3}} = \frac{P_{i-2} - \frac{1}{[[a_{i-1}, a_i]]}P_{i-3}}{Q_{i-2} - \frac{1}{[[a_{i-1}, a_i]]}Q_{i-3}}$$

$$= \cdots$$

$$= \frac{P_1 - \frac{1}{[[a_2, \cdots, a_i]]}P_0}{Q_1 - \frac{1}{[[a_2, \cdots, a_i]]}Q_0}$$

$$= a_1 - \frac{1}{[[a_2, \cdots, a_i]]} = [[a_1, \cdots, a_i]]$$

④ $\phi: X_\Delta \to U_\sigma$ is a resolution al singularities
ϕ: is proper and birational, X_Δ is smooth

$U_\sigma \setminus U_{sing} = U_\sigma \setminus \{x_\sigma\} \cong X_\Delta \setminus \phi^{-1}(x_\sigma)$

and $\phi^{-1}(x_\sigma) = V(\tau_1) \cup V(\tau_2) \cup \cdots \cup V(\tau_r)$

where $\tau_i = \text{cone}(u_i)$

Denote $E_i = V(\tau_i)$, which is call exceptiolal divisor

$E = \bigcup E_i$

prop: u_i satisfy $u_{i-1}+u_{i+1}=a_i u_i \quad 1\le i\le r$

pf: $u_{i-1}+u_{i+1}$

$$=(p_{i-2}e_1-Q_{i-2}e_2)+(p_ie_1-Q_ie_2)$$
$$=(p_{i-2}+p_i)e_1-(Q_{i-2}+Q_i)e_2$$
$$=a_ip_{i-1}e_1-a_iQ_{i-1}e_2$$
$$=a_iu_i$$

prop: $\forall i$, $E_i\cong\mathbb{P}^1$, and the self-intersection numbers $(E_i\cdot E_i)=-a_i$

$$\underset{i\neq j}{(E_i\cdot E_j)}=\begin{cases}1 & \text{if } |i-j|=1\\ 0 & \text{other}\end{cases}$$

prop: (a) $\{u_0,u_1,\cdots,u_{r+1}\}$ is the Hilbert basis of $\sigma\cap N$

(b) $S=\{u_0,u_1,\cdots,u_{r+1}\}$ is the set of lattice point on the bounded edges of $\Theta_\sigma=\mathrm{Cone}(\sigma\cap N\setminus\{0\})$

pf: (a) ① Each pair u_{i-1},u_i is a basis of N, then $\sigma_i\cap N$ is generated by u_{i-1},u_i

Then $\sigma\cap N$ is generated by $S=\{u_0,\cdots,u_{r+1}\}$

② Claim: all the u_i are irreducible elements of $\sigma\cap N$

It is clear for $u_0=e_2$, $u_{r+1}=de_1-ke_2$

If $1\le i\le r$ and u_i is not irreducible then $u_i = P_{i-1} e_i - Q_{i-1} e_2$

$$= \sum_{j \neq i} c_j u_j = \left(\sum_{j\neq i} c_j P_{j-1}\right) e_1 - \left(\sum_{j \neq i} c_j Q_{j-1}\right) e_2$$

where $c_j \in \mathbb{Z}_{\geq 0}$

$\Rightarrow \quad P_{i-1} = \sum_{j \neq i} c_j P_{j-1} \quad Q_{i-1} = \sum_{j\neq i} c_j Q_{j-1}$

Since P_i, Q_i are increasing sequence, then $c_j = 0, \forall j > i$

i.e. $u_i = \sum_{j<i} c_j u_j$

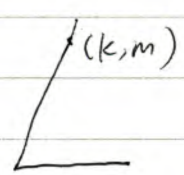
But the slopes of u_i are strictly decreasing, this is impossible $\square$

- Consider the Hilbert basis of $\sigma^\vee \cap M$ $\mathcal{H}(\sigma^\vee \cap M)$

Recall: $|\mathcal{H}| = \dim(T_{x_\sigma}(U_\sigma))$

Let $\sigma = \mathrm{cone}(e_2, me_1 - ke_2)$

Then $\sigma^\vee = \mathrm{cone}(e_1^*, ke_1^* + me_2^*)$



$$\begin{pmatrix} 1 & 0 \\ k & m \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ k-m & m \end{pmatrix}$$

$\therefore \sigma^\vee = \mathrm{cone}(e_2'^*, me_1'^* - (m-k) e_2'^*)$

~~If~~ $\cancel{\frac{m}{k} = [[a_1, \dots, a_r]]}$

How to get the Hirzebruch-Jung continued fraction $\frac{m}{m-k}$

The resolution of singularities for $X(\Delta)$

1. The singular locus

prop: $X(\Delta)_{sing} = \bigcup_{\sigma \text{ not smooth}} V(\sigma)$

$$X(\Delta) \setminus X(\Delta)_{sing} = \bigcup_{\sigma \text{ smooth}} U_\sigma$$

pf: Observation:

① σ is smooth, then so is every face of σ

② σ is not smooth, then so is every cone of Σ containing σ

From ①, $\underbrace{\bigcup_{\sigma \text{ smooth}}}_{\text{the smooth cone of } \Sigma \text{ forms a fan}} U_\sigma$ is smooth

From ②, we have $X(\Delta) \setminus \bigcup_{\sigma \text{ smooth}} U_\sigma$

$$= \bigcup_{\sigma \text{ not smooth}} O(\sigma)$$

$$= \bigcup_{\sigma \text{ not smooth}} V(\sigma)$$

Then, we only need to prove each point of $O(\sigma)$ is singular in $X(\Delta)$, where σ is not smooth

Let $N_\sigma = \operatorname{Span}(\sigma) \cap N$, and pick $N_2 \leq N$ s.t $N = N_\sigma \oplus N_2$, we have

$$U_{\sigma,N} \simeq U_{\sigma,N_\sigma} \times T_{N_2}$$

Since $\dim \sigma = \operatorname{rank} N_\sigma$, then

$U_{\sigma,N_\sigma} \supseteq O_{N_\sigma}(\sigma) = x_\sigma$ is the fixed point of the action of T_{N_σ} and x_σ is singular in U_{σ,N_σ}

② Since $O(\sigma) = O_N(\sigma) \simeq O_{N_\sigma}(\sigma) \times T_{N_2}$
$= X_\sigma \times T_{N_2}$

then every point of $O(\sigma)$ is singular in $X(\Delta)$

Star subdivisions

Given $\Delta \subseteq N_{\mathbb{R}}$ and a primitive element $v \in |\Delta| \cap N$

let $\Delta^*(v)$ be the set of the following cones

(a) σ, where $v \notin \sigma \in \Delta$

(b) $\mathrm{Cone}(v, \tau)$, where $v \in \sigma$ and $v \notin \tau < \sigma$

Step 1: Simplicialization.

prop: Every fan Δ has a refinement Δ' such that:

(a) Δ' is obtained from Δ by a sequence of star subdivision

(b) Δ' is simplicial

(c) $\Delta'(1) = \Delta(1)$

pf: ?

Date

Step 2: The multiplicity of a simplicial cones

$N_\sigma = \operatorname{Span}(\sigma) \cap N \qquad \sigma = \operatorname{cone}(v_1, \cdots, v_k)$

$\operatorname{mult}(\sigma) = [N_\sigma : \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_k]$

Lemma: If σ is simplicial, then

$$\operatorname{mult}(\sigma) = |\Pi(\sigma) \cap N| = |\{\sum_i^k t_i v_i \mid 0 \le t_i < 1\} \cap N| = \det(v_1, \cdots, v_k)$$

~~The~~ Thm: Δ is a simplicial fan, there is a refinment Δ' such that

(1) Δ' is smooth

(2) Δ' is obtained from Δ by a sequence of star subdivision

pf: If $\operatorname{mult}(\Delta) = \max_{\sigma \in \Delta} \operatorname{mult}(\sigma) > 1$

let $\sigma_0 \in \Delta$ have maximal multiplicity

Since $\operatorname{mult}(\sigma_0) > 1 \Rightarrow \exists v \in \Pi(\sigma_0) \cap N \setminus \{0\}$

Let $v = t_1 v_1 + \cdots + t_d v_d \quad 0 < t_i < 1$

where $v_1, \cdots, v_d$ is the generators of the minimal face τ_0 of σ_0 containing v

$$\Delta^*(v) = \bigcup_{v \notin \sigma} \sigma \cup \Big(\bigcup_{\substack{v \in \sigma \\ v \notin \tau \in \sigma}} \operatorname{cone}(v, \tau) \Big)$$

Claim: $\operatorname{mult}(\operatorname{cone}(v, \tau)) < \operatorname{mult}(\sigma)$

Since $v \in \sigma \Rightarrow \sigma = \operatorname{cone}(v_1, \cdots v_d, v_{d+1}, \cdots v_s)$

Since $v \notin \tau \Rightarrow$ there exists v_i $(1 \le i \le d) \notin \tau$

Then $\mathrm{mult}(\mathrm{cone}(v, \tau))$

$$\le \mathrm{mult}(\mathrm{cone}(v, u_1, \cdots, \hat{u}_i, \cdots, u_d, u_{d+1}, \cdots u_s))$$
$$= \det((v, u_1, \cdots, \hat{u}_i, \cdots, u_d, u_{d+1}, \cdots, u_s))$$
$$= \det((u_1, \cdots, t_i u_i, \cdots, u_d, u_{d+1}, \cdots, u_s))$$
$$= t_i \det(u_1, \cdots, u_s)$$
$$= t_i \,\mathrm{mult}(\sigma) < \mathrm{mult}(\sigma)$$

$\square$

Exceptional locus

The resolution $\phi: \Delta' \to \Delta$, $\mathrm{Exc}(\phi) = \phi^{-1}(X(\Delta)_{sing})$

$X(\Delta)_{sing} = V(\sigma_1) \cup \cdots \cup V(\sigma_s)$

$\sigma_1, \cdots, \sigma_s$ are the minimal non-smooth cones of Δ.

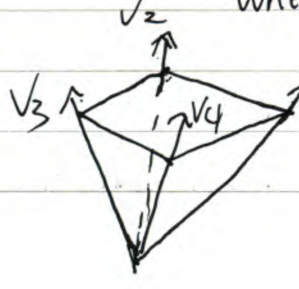
$\phi^{-1}(V(\sigma)) = V(\sigma_1') \cup \cdots \cup V(\sigma_r')$

where $\sigma_1', \cdots, \sigma_r'$ are the minimal cones of Δ' that meet the relative interior of σ

Example: $\sigma = \mathrm{cone}(v_1, v_2, v_3, v_4)$

where v_1, v_2, v_3, v_4 generate N and

$v_1 + v_3 = v_2 + v_4$



$$\begin{tikzcd}
 & X(\Delta_3) \arrow[dl] \arrow[dr] \arrow[dd, "\varphi_3"] & \\
X(\Delta_1) \arrow[dr, "\varphi_1"'] & & X(\Delta_2) \arrow[dl, "\varphi_2"] \\
 & X(\Delta) &
\end{tikzcd}$$

Date

$\Delta_1 = \Delta^*(v) \quad (v = v_1)$

$\Delta_2 = \Delta^*(v) \quad (v = v_2)$

$\Delta_3 = \Delta^*(v) \quad (v = v_1 + v_3 = v_2 + v_4)$

$\Rightarrow$ $\Delta_1, \Delta_2, \Delta_3$ are all smooth

$X(\Delta)_{sing} = \{x_6\}$

$\varphi_1^{-1}(x_6) = \mathbb{P}^1 = \varphi_2^{-1}(x_6)$

$\varphi_3^{-1}(x_6) = \mathbb{P}^1 \times \mathbb{P}^1$

The minimal resolution is not unique when dimension ≥ 3

Note:

1. If $\sigma^\vee$ is strongly convex
$\Rightarrow \mathcal{H}$ is unique

2. $G \curvearrowright A'$ $\quad A = (A')^G \hookrightarrow A'$

$$\begin{array}{c} G \curvearrowright \operatorname{spec}(A') \\ \downarrow \\ \operatorname{spec}(A) \end{array} \qquad \operatorname{spec}(A) = \operatorname{spec}(A')/G$$

① geometric quotient
If G is a finite group
$\Rightarrow \operatorname{spec}(A)$ is a geometric

② GIT quotient

eg: $\mathbb{C}^* \times \mathbb{C}^2 \to \mathbb{C}^2$

$\lambda \in \mathbb{C}^* \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$

- 3 classes of orbit
  ① $(0,0)$
  ② $(x,0)$ or $(0,y)$
  ③ $xy = a$ $(a \neq 0)$
- $\mathbb{C}[XY] = \mathbb{C}[X,Y]^G \subseteq \mathbb{C}[X,Y]$

$$\begin{array}{c} \operatorname{spec}(\mathbb{C}[X,Y]) \doteq \mathbb{C}^2 \\ \downarrow \\ \operatorname{spec}(\mathbb{C}[XY]) = \mathbb{C} \end{array}$$

3. $M = \mathbb{Z}S_\sigma$ (Ref Cox Thm 1.2.18)

Thm: Let $\sigma \subseteq N_{\mathbb{R}} \simeq \mathbb{R}^n$ be a rational cone with semigroup $S_\sigma = \sigma^\vee \cap M$. Then

$$U_\sigma = \operatorname{Spec}(\mathbb{C}[S_\sigma])$$

is an affine toric variety.

Furthermore

$$\dim U_\sigma = n \iff \text{the torus of } U_\sigma \text{ is } T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^*$$
$$\iff \sigma \text{ is strongly convex}$$

pf: U_σ's torus has character lattice $\mathbb{Z}S_\sigma \subseteq M$.

$$\mathbb{Z}S_\sigma = S_\sigma - S_\sigma$$
$$= \{m_1 - m_2 \mid m_1, m_2 \in S_\sigma\}$$

Suppose $km \in \mathbb{Z}S_\sigma$ for some $k > 1$ and $m \in M$

Then $km = m_1 - m_2$, $m_1, m_2 \in S_\sigma = \sigma^\vee \cap M$

$$\Rightarrow m + m_2 = \tfrac{1}{k}m_1 + \tfrac{k-1}{k}m_2 \in \sigma^\vee$$

$$\Rightarrow m = \underbrace{(m + m_2)}_{\substack{m \in M,\, m_2 \in M \\ m + m_2 \in \sigma^\vee \\ m + m_2 \in \sigma^\vee \cap M}} - m_2 \in \mathbb{Z}S_\sigma$$

Thus $M / \mathbb{Z}S_\sigma$ is torsion-free

(Ref Cox prop 1.2.23

prop: Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational cone of maximal dimension and let $S_\sigma = \sigma^\vee \cap M$. Then

$$\mathcal{H} = \{ m \in S_\sigma \mid m \text{ is irreducible} \}$$

has the following properties

(a) $\mathcal{H}$ is finite and generates S_σ

(b) $\mathcal{H}$ contains the ray generators of the egde of $\sigma^\vee$

(c) $\mathcal{H}$ is the minimal generating set of S_σ

Remark: The proposition holds for any affine semigroup S satisfying $S \cap (-S) = \{0\}$

① finitely generated

② commutative

③ can be embedded in a lattice M

4. We have

$$0 \to \underset{0}{\underset{\parallel}{\mathrm{Hom}(N/N', \mathbb{Z})}} \to M \to M' \to \mathrm{Ext}^1(N/N', \mathbb{Z}) \to 0$$

Then we have exact

$$0 \to M \otimes \mathbb{Q} \to M' \otimes \mathbb{Q} \to \mathrm{Ext}^1(N/N', \mathbb{Z}) \otimes \mathbb{Q} \to 0$$

$\mathrm{Ext}^1(N/N', \mathbb{Z})$ is a $\mathbb{Z}$-module and ~~has~~ is a finite group

$\Rightarrow \mathrm{Ext}^1(N/N', \mathbb{Z}) \otimes \mathbb{Q} = 0$

$\Rightarrow M_{\mathbb{Q}} = M_{\mathbb{Q}'}$

$Ext^1(\quad)$

M, N R-modules

$\xi \in Ext^1_R(M,N)$ $\quad r \in R$

$r\cdot\xi \in Ext^1_R(M,N)$

$$\begin{tikzcd}
\xi: 0 \arrow[r] & N \arrow[r, "\psi"] & L \arrow[r, "\pi"] & M \arrow[r] & 0 \\
r\cdot\xi: 0 \arrow[r] & N \arrow[r] \arrow[u, "\varphi"] & L' \arrow[r, "p_2"] \arrow[u, "p_1"] & M \arrow[r] \arrow[u, "\cdot r"] & 0
\end{tikzcd}$$

$L' \triangleq L' \subseteq L \oplus M$

$\quad = \{(l,m) \mid \pi(l) := r(m)\}$

- $\varphi = id$?

  $Im(\psi) = ker(\pi)$

  $ker(p_2) = ?$

  $ker(p_2) = \{(l,m) \mid \pi(l) = 0\}$

  $= Im(\psi)$

- If $r\cdot m = 0$ $\forall m \in M$

  $L' = M \oplus N \rightsquigarrow r\cdot\xi = 0$

- $Ext^1(\mathbb{Z}/a\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/a\mathbb{Z}$.

- $r\cdot m = 0$ $\forall m \in M$

  $M \otimes_{\mathbb{Z}} \mathbb{Q} = 0$

  $m \otimes_{\mathbb{Z}} q = r\cdot m \otimes \frac{q}{r}$

  $= 0 \otimes \frac{q}{r}$

$0 \otimes \frac{q}{r} + x \otimes y \implies 0 \otimes \frac{q}{r} = 0$

$= 0 \otimes y + x \otimes y$

$= x \otimes y$