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Given a free $\mathbb{Z}$-module $N \cong \mathbb{Z}^n$ and $M = \mathrm{Hom}(N, \mathbb{Z})$

let $\sigma \subset N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ be a strongly rational convex cone

$(\sigma \cap -\sigma = \{0\})$ $\quad \sigma = \{ r_1 v_1 + \cdots + r_s v_s \in N_{\mathbb{R}} \mid r_i \geq 0,\ v_i \in N \}$

$\rightsquigarrow \sigma^{\vee} = \{ u \in M_{\mathbb{R}},\ \langle u, v \rangle \geq 0 \ \ \forall v \in \sigma \}$

$S_\sigma := \sigma^{\vee} \cap M$ is a finitely generated semigroup

(Gorden's lemma : by σ is rational)

$\rightsquigarrow U_\sigma = \mathrm{spec}(\mathbb{C}[S_\sigma]) = \mathrm{spec}(\underbrace{\mathbb{C}[\sigma^{\vee} \cap M]}) = \mathrm{spec}(A_\sigma)$

affine variety $\qquad$ finited generated $\mathbb{C}$-Alg

General variety

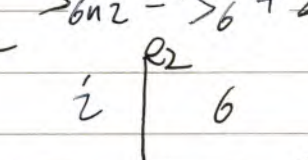
- a fan $\Delta \subseteq N_{\mathbb{R}}$ is a set of cones satisfying ① if $\sigma \in \Delta$, $\tau < \sigma$, then $\tau < \Delta$.
  ② if $\sigma, \sigma' \in \Delta$, then $\sigma \cap \sigma' \in \Delta$.

- Each cone $\sigma \in \Delta \rightsquigarrow$ affine toric variety U_σ
  We gluing U_σ as follows.:
  For cones $\sigma, \tau \in \Delta$, $\Rightarrow \sigma \cap \tau \in \Delta$ and $\sigma \cap \tau$ is face of both σ and τ.

$\xRightarrow{\text{lemma}}$ there is a $u \in S_\sigma$ with $\sigma \cap \tau = \sigma \cap u^{\perp}$

s.t $S_{\sigma \cap \tau} = S_\sigma + \mathbb{Z}_{\geq 0}(-u) = S_\tau + \mathbb{Z}_{\geq 0} u$

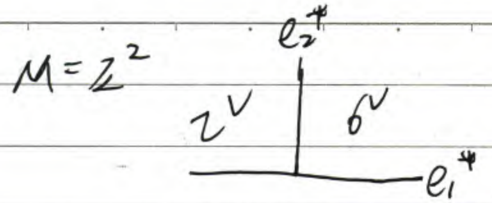
e.g. $N = \mathbb{Z}^2$



$\sigma = \mathrm{cone}(e_1, e_2)$

$\tau = \mathrm{cone}(-e_1, e_2)$

$\sigma \cap \tau = \mathrm{cone}(e_2)$



$\sigma^\vee = \mathrm{cone}(e_1^*, e_2^*)$

$\tau^\vee = \mathrm{cone}(-e_1^*, e_2^*)$

$(\sigma\cap\tau)^\vee = \mathrm{cone}(\pm e_1^*, e_2^*)$

$u = e_1^* \in S_\sigma \Rightarrow S_{\sigma\cap\tau} = S_\sigma + \mathbb{Z}_{\geq 0}(-u) = S_\tau + \mathbb{Z}_{\geq 0}u$

$\Rightarrow A_{\sigma\cap\tau} = \mathbb{C}[S_{\sigma\cap\tau}] = (A_\sigma)_{\chi^u} = (A_\tau)_{\chi^{-u}}$

$\Rightarrow U_{\sigma\cap\tau}$ is a principle open subvariety of U_σ and U_τ. $U_{\sigma\cap\tau} \hookrightarrow U_\sigma$

$U_{\sigma\cap\tau} \hookrightarrow U_\tau$

$\{U_i\}$ and $U_{ij} \subset U_i$

- $A_\sigma \xhookrightarrow{i} (A_\sigma)_{\chi^u} \xleftarrow[\cong]{\varphi_{ij}^*} (A_\tau)_{\chi^{-u}} \xhookleftarrow{i} A_\tau$

$U_i \hookleftarrow U_{ij} \xrightarrow[\cong]{\varphi_{ij}} U_{ji} \hookrightarrow U_j$

We need to confirm

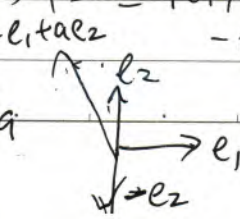
(1) $\varphi_{ji} = \varphi_{ij}^{-1}$ (2) $\varphi_{ij}(U_{ij}\cap U_{ik}) = U_{ji}\cap U_{jk}$ and $\varphi_{ik} = \varphi_{jk}\circ\varphi_{ij}$ on $\cancel{U_j}$ $U_{ij}\cup U_{ik}$

((1) $\varphi_{ij}^{*-1} = \varphi_{ji}^*$ (2) $\varphi_{ik}^* = \varphi_{ij}^*\circ\varphi_{jk}^*$ on $A_{i\cap j\cap k}$)

Example $\mathbb{C}^n$, $\mathbb{P}^n$,

$\mathrm{cone}(e_1,\cdots,e_n)$ $\Delta = \{\mathrm{cone}(S) \mid S \subseteq \{e_1,\cdots,e_n, -e_1-e_2-\cdots-e_n\}\}$

Hirzebruch surface F_a



$\Delta \rightsquigarrow X(\Delta)$

① ~~I~~

If $\tau < \sigma \Rightarrow A_\tau = (A_\sigma)_{\chi^u} \Rightarrow U_\tau \hookrightarrow U_\sigma$

For all $\sigma \in \Delta$, $\{0\} < \sigma$, $\{0\}^\vee \cap M = M$

$\Rightarrow U_{\{0\}} = \mathrm{spec}(\mathbb{C}[M]) \cong \mathbb{C}^n$

$\Rightarrow T_N := U_{\{0\}} \cong \mathbb{C}^n \hookrightarrow U_\sigma$

T_N is an dense open set of $X(\Delta)$

We also have a compatible action

$$\begin{tikzcd} X \times T_N \arrow[r] & X \\ T_N \times T_N \arrow[r] \arrow[u, hook] & T_N \arrow[u, hook] \end{tikzcd}$$

which is induced by

$$\begin{tikzcd} \mathbb{C}[S_\sigma] \otimes_{\mathbb{C}} \mathbb{C}[M] \arrow[d] & \mathbb{C}[S_\sigma] \arrow[l] \arrow[d] \\ \mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[M] & \mathbb{C}[M] \arrow[l] \end{tikzcd}$$

$$\chi^m \otimes \chi^m \leftarrow \chi^m$$

② $X(\Delta)$ is normal and seperated

Some properties of ~~varie~~ variety.

1. Smooth

$\sigma \subseteq N_{\mathbb{R}}$ strongly convex rational cone $\sigma = \mathrm{cone}(e_1, \cdots, e_k)$

① U_σ is nonsingular $\Leftrightarrow$ σ is generated by part of a basis for N, in which case $U_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$, $k = \dim(\sigma)$

② ~~Each Δ~~ $X(\Delta)$ is smooth $\Leftrightarrow$ each cone σ is smooth

2. finite quotient singularities

① U_σ has finite quotient singularities
$\Leftrightarrow$ σ is simplicial (i.e $\sigma = \text{cone}(v_1, \cdots, v_n)$)

② $X(\Delta)$ is an orbifold $\Leftrightarrow$ each cone σ is simplicial

3. $X(\Delta)$ is complete or compact in classical topology $\Leftrightarrow$ $|\Delta| = N_{\mathbb{R}}$

lemma: $\dim(\sigma) = N_{\mathbb{R}}$, then $\dim(T_{x_\sigma}(U_\sigma)) = |\mathcal{H}|$

prop: $N' \subseteq N$, $\Delta \leq N_{\mathbb{R}} = N'_{\mathbb{R}}$, Then
$\varphi : X(\Delta_{N'}) \to X(\Delta_N)$ induced by
$N' \hookrightarrow N$ presents $X(\Delta_N) = X(\Delta_{N'})/G$

$T_N = \text{Hom}(M, \mathbb{C}^*) \cong (\mathbb{C}^*)^n$
one-parameter group $\lambda_v : \mathbb{C}^* \to T_N$, $v = (a_1, \cdots, a_n) \in N$
λ_v is given by $t \mapsto (t^{a_1}, \cdots, t^{a_n})$
character $\chi^m : T_N \to \mathbb{C}^*$, $m = (m_1, \cdots, m_n) \in M$
$(t_1, \cdots, t_n) \mapsto t_1^{m_1} \cdots t_n^{m_n}$

Lemma: If v is not in any cone in Δ, then $\lim_{z \to 0} \lambda_v(z)$ does not exist in $X(\Delta)$

prop: Let $\Delta' \subseteq N'_{\mathbb{R}}$, $\Delta \subseteq N_{\mathbb{R}}$ and $\varphi : N' \to N$ is compatible with Δ' and Δ
(i.e. $\forall \sigma' \in \Delta'$, $\exists \sigma \in \Delta$ s.t $\varphi_{\mathbb{R}}(\sigma') \subset \sigma$)
Then The map $\varphi_* : X(\Delta') \to X(\Delta)$ is proper ~~i.e~~
iff $\varphi_{\mathbb{R}}^{-1}(|\Delta|) = |\Delta'|$

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Cor: If $\Delta \subseteq N_{\mathbb{R}}$, Δ' is a refinement of Δ, then there is a proper map $\varphi : X(\Delta') \to X(\Delta)$

$\Rightarrow$ ①

Application:

1. All complete nonsingular toric surfaces are obtained from $\mathbb{P}^2$ or $\mathbb{F}_a$ by a succession of blows-up
2. Resolusion of singularities.

Orbits.

For $\tau \in \Delta$, let N_τ be the sublattice of N generated by $\tau \cap N$ and $N(\tau) = N/N_\tau$, let $M(\tau) = \tau^\perp \cap M$,

there is a natural map $N(\tau) \times M(\tau) \to \mathbb{Z}$

$\Rightarrow$ Define $O_\tau = T_{N(\tau)} = \mathrm{Spec}(\mathbb{C}[M(\tau)])$

$\mathrm{Star}(\tau) = \{ \bar{\sigma} \subset N(\tau)_{\mathbb{R}} \mid \sigma > \tau \}$

($\mathrm{Star}(\tau)$ is a fan in $N(\tau)_{\mathbb{R}}$)

$V(\tau) = X(\mathrm{Star}(\tau))$ ($V(\tau)$ has an affine open covering $\{U_\sigma(\tau)\}$, $U_\sigma(\tau) \leftrightarrow \bar{\sigma} \subset \mathrm{Star}(\tau)$)

$\Rightarrow$ k-dim cone τ

$\leftrightarrow$ $(n-k)$-dim torus O_τ

$\leftrightarrow$ $(n-k)$-dim variety $V(\tau) = \overline{O_\tau}$

To embed $V(\tau)$ as a closed subvariety of $X(\Delta)$, we construct a closed embedding of $U_\sigma(\tau)$ in U_σ for each $\sigma > \tau$.

The embedding is

$U_\sigma(\tau) = \mathrm{Hom}_{sg}(\sigma^\vee \cap \tau^\perp \cap M, \mathbb{C})$

$\hookrightarrow \mathrm{Hom}_{sg}(\sigma^\vee \cap M, \mathbb{C}) = U_\sigma$

which corresponds to the surjection of rings $\mathbb{C}[\sigma^\vee \cap M] \twoheadrightarrow \mathbb{C}[\sigma^\vee \cap \tau^\perp \cap M]$

prop: ① O_τ is T_N-invariant

② $U_\sigma = \bigsqcup_{\tau < \sigma} O_\tau$

③ $V(\tau) = \bigsqcup_{\gamma > \tau} O_\gamma$, $X(\Delta) = \bigsqcup_{\gamma \in \Delta} O_\gamma$

④ $O_\tau = T_N \cdot x_\tau$

$x_\tau \in \mathrm{Hom}_{sg}(S_\sigma, \mathbb{C})$

$x_\tau : m \mapsto \begin{cases} 1 & S_\tau \cap \tau^\perp \\ 0 & S_\sigma \setminus \tau^\perp \end{cases}$

Example: $\mathbb{P}^2$

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Toric variety from polytopes

P is a lattice polytope in $M_{\mathbb{R}}$

i.e $P = \operatorname{conv}(u_1, \cdots, u_n)$, $u_i \in M$

Assume: P is n-dimension

For each $Q < P$, we define a cone

$$\sigma_Q = \{ v \in N_{\mathbb{R}} : \langle u, v \rangle \leq \langle u', v \rangle \text{ for all } u \in Q \text{ and } u' \in P \}$$

$\Rightarrow \{ \sigma_Q \mid Q < P \}$ is a fan in $N_{\mathbb{R}}$, denoted by Δ_P.

E.g. $P = \operatorname{conv}((\pm 1, \pm 1, \pm 1))$

Δ_P is generated by $(0,0,\pm 1)$

$(0, \pm 1, 0)$

$(\pm 1, 0, 0)$

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① $N, M \quad \Delta \subseteq N_{\mathbb{R}} \rightsquigarrow X(\Delta) \quad T_N \subseteq X$

$$T_N = \mathrm{spec}(\mathbb{C}[M])$$

$\forall \sigma \in \Delta \rightsquigarrow \sigma^{\vee} \subseteq M_{\mathbb{R}} \rightsquigarrow U_\sigma = \mathrm{spec}(\underbrace{\mathbb{C}[\sigma^{\vee} \cap M]}_{\overset{\Delta}{=} S_\sigma})$

② $T_N = \mathrm{Hom}(M, G_m) \cong (\mathbb{C}^*)^n$

One-parameter group $\lambda_v : G_m \to T_N \quad (v \in N)$
$t \mapsto (t^{a_1}, \cdots, t^{a_n})$

character $\chi^m : T_N \to G_m \quad (m \in M)$
$(t_1, \cdots, t_n) \mapsto t_1^{m_1} \cdots t_n^{m_n}$

③ Given $\tau \in \Delta$, $N_\tau = \tau \cap N \quad N(\tau) = N/N_\tau$

$M(\tau) = \tau^{\perp} \cap M$. $\quad \Delta \rightsquigarrow \mathrm{star}(\tau) = \{\sigma \mid \sigma > \tau\}$

Let $O_\tau \triangleq T_{N(\tau)} = \mathrm{spec}(\mathbb{C}[M(\tau)])$

$V(\tau) = X(\mathrm{Star}(\tau)) = \overline{O_\tau}$

$\tau \longleftrightarrow O_\tau \longleftrightarrow V(\tau)$

k-dimension $\quad (n-k)$-dimension torus

$$\begin{array}{ccc} V(\tau) & \hookrightarrow & X(\Delta) \\ \uparrow & & \uparrow \\ U_\tau(\tau) = O_\tau & \hookrightarrow & U_\tau \\ (\text{resp. } U_\sigma(\tau)) & & (\text{resp. } U_\sigma) \end{array}$$

- O_τ T_N invariant $\quad O(\tau) = T_N \cdot x_\sigma$
- $\bigsqcup_{\tau \in \Delta} O_\tau = X(\Delta)$

For a polytope $P \subseteq M_{\mathbb{R}} \rightsquigarrow X_{\Delta_P}$

Weil divisors

{the irreducible subvarieties of codim 1 that are T-stable} $\longleftrightarrow$ {edges of fans}

$D_i \longleftrightarrow \tau_i$

Cartier divisors

A Cartier divisor D is give by $\{(U_\alpha, f_\alpha)\}$
(U_α is an affine open covering of X.
f_α are rational function on U_α)
f_α / f_β are nonzero regular functions on $U_\alpha \cap U_\beta$

Thm 3.1.5. $X(\Delta)$ is a normal separated variety
Cor 3.1.8 Let X be a normal separated variety with torus T_N, then $\exists\, \Delta$ s.t $X \cong X(\Delta)$

The invertible sheaf $\mathcal{O}(D)$ associated D is the subsheaf of sheaf of the rational functions generated by f_α^{-1} on U_α.
The ideal sheaf $\mathcal{O}(-D)$ - - - - .
generated ~~~~ by f_α on U_α

Notation: $\sigma = \text{cone}(v_1, \cdots, v_n)$ (v_i is primitive)

① $\sigma(1) = \{\rho \mid \rho = \text{cone}(v_i)\}$

The T-Weil divisor

The irreducible subvarieties of codim 1 that are T-stable

$\Leftrightarrow$ The edges of fan

$D_\rho \Leftrightarrow \rho$ $\qquad D_\rho \Leftrightarrow \rho$

$D_\rho = V(\tau_i) = \overline{O_{\tau_i}}$ $\qquad D_\rho = V(\rho) = \overline{O_\rho}$

(Recall: k-dim cones $\sigma \in \Delta \Leftrightarrow (n-k)$-dim T_N-orbits in $X(\Delta)$)

The T-Weil divisors are the sums $\sum a_i D_i$ ($a_i \in \mathbb{Z}$)

(Q: Are all T-Weil divisors of the form $\sum a_i D_i$

② When X is normal, local rings are DVR

($\mathcal{O}_{X,D} = \{f/g \in K \mid f, g \in R, g \notin p\} = R_p$

$p = I(D)$

① $\mathcal{O}_{X,D}$ is a local ring with maximal ideal pR_p

② Since R is Noetherian and normal, so is R_p

③ The dimension of R_p is one; In fact, codim $D = 1 \Rightarrow$ no prime ideals strictly between $\{0\}$ and p.

$\Rightarrow$ the same is true for $\{0\}$ and m_p in R_p.

① + ② + ③ $\Rightarrow R_p$ is DVR $= \mathcal{O}_{X,D}$.

$\Rightarrow$ Let $X = X(\Delta)$, each $\mathcal{O}_{X,D_\rho}$ induces a valuation: $v_{D_\rho}: \mathbb{C}(X)^* \to \mathbb{Z}$

Def: $f \in K^*$, we define the divisor of f

$(f) := \sum v_Y(f) \cdot Y$

A Cartier divisor D determines a Weil divisor $[D] = \sum_{\mathrm{cod}(V,X)=1} \mathrm{ord}_V(D) \cdot V$

For normal variety, $D \mapsto [D]$ embeds the group of Cartier divisors in the group of Weil divisors

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$\chi^u \in \mathrm{Hom}(T_N, \mathbb{C}^*)$, χ^u is a rational function on $X(\Delta)$

Lemma: Let $u \in M$, v be the first lattice point along an edge τ. Then

$V_{D_\tau}(\chi^u) = \mathrm{ord}_{D_\tau}(\mathrm{div}(\chi^u)) = \langle u, v \rangle$ and

V_{D_τ} is given by the corresponding $\mathcal{O}_{X,D_\tau}$

$[\mathrm{div}(\chi^u)] = \sum_i \langle u, v_i \rangle D_i$

pf: Since v is primitive, we can extend v to a basis $e_1 = v, e_2, \cdots, e_n$ of N.

$$\begin{tikzcd} V(\tau) \arrow[r, hook] & X(\Delta) \\ U_\tau(\tau) \arrow[r, hook] \arrow[u, hook] & U_\tau \arrow[u, hook] \end{tikzcd}$$

The affine open set $U_\tau = \mathrm{Spec}(\mathbb{C}[\tau^\vee \cap M])$
$= \mathrm{Spec}(\mathbb{C}[x_1, x_2^{\pm 1}, \cdots, x_n^{\pm 1}])$
$= \mathbb{C} \times (\mathbb{C}^*)^{n-1}$

$\mathcal{O}_{X(\Delta), D_\tau} = \mathcal{O}_{U_\tau, U_\tau \cap D_\tau} = \mathbb{C}[x_1, \cdots, x_n]_{(x_1)}$

$f \in \mathbb{C}(x_1, \cdots, x_n)^*$ $\quad v_{D_\tau}(f) = l$ when $f = x_1^l \frac{g}{h}$ $\quad g, h \in \mathbb{C}[x_1, \cdots, x_n] \setminus (x_1)$

$\Rightarrow V_{D_2}(\chi^u) = V_{D_2}(x_1^{\langle u,e_1\rangle} \cdot x_2^{\langle u,e_2\rangle} \cdots, x_n^{\langle u,e_n\rangle})$
$= \langle u, v\rangle$

Since $\chi^u \in Hom(T_N, \mathbb{C}^*)$ (i.e χ^u is nonzero on T_N)
and D_i are irreducible components of $X \setminus T_N$
$\Rightarrow div(\chi^u) = \sum_i V_{D_i}(\chi^u) D_i$
$= \sum_i \langle u, v_i\rangle D_i$

Thm: We have the exact sequence

$$M \to Div_{T_N}(X(\Delta)) \to Cl(X(\Delta)) \to 0$$

where $Div_{T_N}(X(\Delta)) = \bigoplus_\rho \mathbb{Z} D_\rho \subseteq Div(X(\Delta))$
and the first map is $m \mapsto div(\chi^m)$.
Moreover we have

$$0 \to M \to Div_{T_N}(X(\Delta)) \to Cl(X(\Delta)) \to 0$$

iff $\{u_\rho \mid \rho \in \Delta(1)\}$ spans $N_\mathbb{R}$.

pf: • Since D_ρ are the irreducible components of $X(\Delta) \setminus T_N$, we have an exact sequence

$$Div_{T_N}(X(\Delta)) = \bigoplus_\rho \mathbb{Z} D_\rho \to Cl(X(\Delta)) \to Cl(T_N) \to 0$$

Since $T_N = Spec(\mathbb{C}[M])$ and
$\mathbb{C}[M] \cong \mathbb{C}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$ is a UFD
then $Cl(T_N) = 0$
$\Rightarrow Div_{T_N}(X(\Delta)) \to Cl(X(\Delta))$ is surjective

• $M \to Div_{T_N}(X(\Delta)) \to Cl(X(\Delta))$ is zero.
Suppose $D \in Div_{T_N}(X(\Delta))$ maps to 0 in $Cl(X(\Delta))$
i.e. $D = div(f)$ for some $f \in \mathbb{C}(X(\Delta))^*$
Since $D = \sum a_i D_i \in Div_{T_N}(X(\Delta)) \Rightarrow$

then

$D = \mathrm{div}(f)$ restricts to 0 on T_N.

$\Rightarrow f \in \mathbb{C}[M]^*$ i.e $f = c\chi^m$, $c \in \mathbb{C}^*$, $m \in M$

$\Rightarrow D = \mathrm{div}(f) = \mathrm{div}(c\chi^m) = \mathrm{div}(\chi^m)$

- $m \in M \mapsto \mathrm{div}(\chi^m) = \sum_\rho \langle m, u_\rho \rangle D_\rho$

$M \to \mathrm{Div}_{T_N}(X(\Delta))$ is injective

$\Leftrightarrow \langle m, u_\rho \rangle = 0 \ \forall u_\rho \Rightarrow m = 0$

$\Leftrightarrow \{u_\rho\}$ spans $N_{\mathbb{R}}$. $\square$

Remark: ① The exactness at $Cl(X(\Delta))$ shows that $\forall$ Weil divisor $D \sim \sum a_\rho D_\rho$. ② The exactness at $\mathrm{Div}_{T_N}(X(\Delta))$ shows that all T_N-invariant principle divisors are of the form $\mathrm{div}(\chi^m)$

Example:

$M \cong \mathbb{Z}^n$, let $e_1^*, \cdots, e_n^*$ be the basis of M, and the ray generators of fan Δ is $u_1, \cdots, u_r \in N$.

Then the map $M \to \mathrm{Div}_{T_N}(X(\Delta))$ is the map $A: \mathbb{Z}^n \to \mathbb{Z}^r$ represented by the matrix

$$A = \begin{pmatrix} \langle e_1^*, u_1 \rangle, \cdots & \langle e_n^*, u_1 \rangle \\ \vdots & \\ \langle e_1^*, u_r \rangle, \cdots & \langle e_n^*, u_r \rangle \end{pmatrix}$$

Thus, $Cl(X(\Delta))$ ~~th~~ is the cokernel of A

Let $\sigma = \mathrm{cone}(de_1 - e_2, e_2) \subseteq \mathbb{R}^2$ ($d \in \mathbb{Z}_{>0}$)

$\Rightarrow \sigma^\vee = \mathrm{Cone}(e_1^*, e_1^* + de_2^*)$

$\Rightarrow U_\sigma = \mathrm{Spec}(\mathbb{C}[X, XY, XY^2, \cdots XY^d])$

$U_\sigma \cong$ rational normal cone $\hat{C}_d \subseteq \mathbb{C}^{d+1}$

$A = \begin{pmatrix} d & -1 \\ 0 & 1 \end{pmatrix} \Rightarrow Cl(\hat{C}_d) \simeq \mathbb{Z}/d\mathbb{Z}$

Let D_1, D_2 be divisors corresponding to rays ρ_1, ρ_2

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$Cl(\hat{C}_d)$ is generated by $[D_1]$ with $d[D_1]=0$

- The cokernel can be calculated by Smith normal form of A

- Cartier Divisors

Let $CDiv_{T_N}(X(\Delta))$ be the group of T_N-invariant Cartier divisor.

prop: $M \to CDiv_{T_N}(X(\Delta)) \to Pic(X(\Delta)) \to 0$ is exact.

Furthermore $0 \to M \to \cdots$ is exact $\iff \{u_\rho \mid \rho \in \Delta(1)\}$ spans $N_{\mathbb{R}}$

pf: • $\forall D$ cartier divisor is also a Weil divisor, and hence $\exists \sum_\rho a_\rho D_\rho$ s.t $D \sim \sum a_\rho D_\rho$. $(a_\rho \in \mathbb{Z})$

D is Cartier $\Rightarrow \sum a_\rho D_\rho$ is Cartier and T_N-invariant

Thus $CDiv_{T_N}(X(\Delta)) \to Pic(X(\Delta))$ is surjective

• ~~If $D \in CDiv_{T_N}(X(\Delta))$ maps to 0 in $Pic(X(\Delta))$, then $\exists m \in M$~~, s.t ~~$D = div(\chi^m)$~~

$div(\chi^m) \in CDiv_{T_N}(X(\Delta))$

How to describe $CDiv_{T_N}(X(\Delta))$?

prop: $\sigma \subseteq N_{\mathbb{R}}$ is a strongly convex cone, then

(a) Every T_N-invariant Cartier divisor on U_σ is the divisor of a charater.

(b) $Pic(U_\sigma) = 0$

pf: Let D be a T_N-invariant Cartier divisor, which is given by $\{(U_\alpha, f_\alpha)\}$,
The invertible sheaf $\mathcal{O}(D)$ is the subsheaf of the sheaf rational functions generated by $1/f_\alpha$ on U_α

If D be a Weil divisor on a normal variety X, then $\mathcal{O}_X(D)$ is a sheaf of $\mathcal{O}_X$-module defined by $U \mapsto \mathcal{O}_X(D)(U)$

$$= \{f \in \mathbb{C}(X)^* \mid (\mathrm{div}(f) + D)|_U \geq 0\} \cup \{0\}$$

Therefore if D is a Cartier divisor given by $\{(U_\alpha, f_\alpha)\}$, then $\mathcal{O}(D)$ is generated by $1/f_\alpha$ on U_α

Let $X = U_\sigma$, and $R = \mathbb{C}[\sigma^\vee \cap M]$

$I = \Gamma(X, \mathcal{O}(D))$

Claim: I is generated by χ^u for a unique $u \in \sigma^\vee \cap M$.

- pf: Since D is T-invariant, I is T-invariant.

Then $I = \bigoplus_{u \in A} \chi^u \quad A \subseteq M$

Since I is principle at X_σ

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another proof :

- Let $R = \mathbb{C}[\sigma^\vee \cap M]$. Suppose $D = \sum a_\rho D_\rho$ is an <u>effective</u> T_N-invariant Cartier divisor.

  Then $\Gamma(U_\sigma, \mathcal{O}(-D)) = \{ f \in K \mid f = 0 \text{ or } f \neq 0 \text{ and } \operatorname{div}(f) \geq D \}$

  is an ideal $I \subseteq R$

  Since D is T_N-invariant, then I is so and $I = \bigoplus_{\chi^m \in I} \mathbb{C} \cdot \chi^m = \bigoplus_{\operatorname{div}(\chi^m) \geq D} \mathbb{C} \cdot \chi^m$

  Since $x_\sigma = O(\sigma) \subseteq \overline{O(\rho)} = D_\rho \quad \forall \rho \in \sigma(1)$,

  $x_\sigma = O(\sigma) \subseteq \bigcap_\rho D_\rho$

  Since D is Cartier, it is local principle.

  Then D is principle in a neighborhood U of x_σ s.t $U = (U_\sigma)_h = \operatorname{Spec}(R_h)$ where $h \in R$ and $h(x_\sigma) \neq 0$

  Thus $D|_U = \operatorname{div}(f)|_U$ for some $f \in \mathbb{C}(U_\sigma)^*$

  Since D is effective, $f \in R_h$, and since h is invertible on U, we may assume $f \in R$. Then

$$\begin{aligned}\operatorname{div}(f) &= \sum_\rho V_{D_\rho}(f) D_\rho + \sum_{E \neq D_\rho} V_E(f) E \\ &\geq \sum_\rho V_{D_\rho}(f) D_\rho \quad (\text{since } f \in R) \\ &= D \quad (\text{since } x_\sigma \in U \cap D_\rho \text{ and } \operatorname{div}(f)|_U = D|_U)\end{aligned}$$

  Thus $f \in I$

Then, we can write $f = \sum_i a_i \chi^{m_i}$ with
$a_i \in \mathbb{C}^*$ and $\operatorname{div}(\chi^{m_i}) \geq D$
Restricing to U, this becomes
$\operatorname{div}(\chi^{m_i})|_U \geq \operatorname{div}(f)|_U$, which implies
that χ^{m_i}/f is a morphism on U. Then
$1 = \frac{\sum_i a_i \chi^{m_i}}{f} = \sum_i a_i \frac{\chi^{m_i}}{f}$ which imply that

~~pt~~ $\left(\frac{\chi^{m_i}}{f}\right)(\gamma_\sigma) \neq 0$ for some i Hence χ^{m_i}/f
is nonvanishing in some open V with
$\gamma_\sigma \in V \subseteq U$. It follows that
$$\operatorname{div}(\chi^{m_i})|_V = \operatorname{div}(f)|_V = D|_V$$
Since every $D_\rho \cap V \neq \phi$, and $\operatorname{div}(\chi^{m_i})$ and D have support contained in $\bigcup_\rho D_\rho$ (since $\chi^{m_i} \in \mathrm{Hom}(T_N, \mathbb{C}^*)$) we have
$$\operatorname{div}(\chi^{m_i}) = D$$

- Let D be an arbitary T_N-invariant Cartier divisor.
Since $\dim \sigma^\vee = \dim M_\mathbb{R}$, we can find
$m \in \sigma^\vee \cap M$ s.t. $\langle m, u_\rho \rangle > 0$ $\forall \rho \in \sigma(1)$
Thus $\operatorname{div}(\chi^m)$ is an effective divisor and
$D' = D + \operatorname{div}(\chi^{km}) \geq 0$ for sufficienty large k
The above implies that D' is
the divisor of a character, so that
the same is true for D.

Conclusion: A finitely generated abelian group is free $\Leftrightarrow$ torsion free.

Prop: Let $\Delta \subseteq N_{\mathbb{R}}$ and Δ contains a cone of dim n, then $\mathrm{Pic}(X(\Delta))$ is a free abelian group.

pf: It suffices to prove that if $D \in \mathrm{CDiv}_{T_N}(X(\Delta))$ and kD is the divisor of a character for some $k>0$, then the same is true for D

(Δ contains a cone of dim n

$\Rightarrow 0 \to M \to \mathrm{CDiv}_{T_N}(X(\Delta)) \to \mathrm{Pic}(X(\Delta)) \to 0$ exact

$\Rightarrow \mathrm{CDiv}_{T_N}(X(\Delta))/M \cong \mathrm{Pic}(X(\Delta))$ f.g.

$\Rightarrow$ It suffices to prove $\mathrm{CDiv}_{T_N}(X(\Delta))/M$ is torsion-free.)

To prove this, write $D = \sum_\rho a_\rho D_\rho$ and assume $kD = \mathrm{div}(\chi^m)$ $m \in M$.

Let σ have dimension n. D is Cartier

$\Rightarrow D|_{U_\sigma}$ is Cartier

~~$D|_{U_\sigma}$~~ $D|_{U_\sigma} = \sum_{\rho \in \Delta(1)} a_\rho D_\rho |_{U_\sigma} = \sum_{\rho \in \sigma(1)} a_\rho D_\rho$

$D|_{U_\sigma}$ is principal on U_σ by prop 1, so that $\exists m' \in M$ s.t. $D|_{U_\sigma} =$ ~~dim~~ $\mathrm{div}(\chi^{m'})|_{U_\sigma}$

This implies that ~~$k a_\rho$~~ $a_\rho = \langle m', u_\rho \rangle$ $\forall \rho \in \sigma(1)$

On the other hand,

$kD = \mathrm{div}(\chi^m) \Rightarrow k a_\rho = \langle m, u_\rho \rangle \ \forall \rho \in \Delta(1)$

$\Rightarrow \langle km', u_\rho \rangle = k a_\rho = \langle m, u_\rho \rangle \ \forall \rho \in \sigma(1)$

$\dim \sigma = n \Rightarrow km' = m$

$\Rightarrow D = \mathrm{div}(\chi^{m'})$

prop: TFAE:

(1) ~~Ee~~ Every Weil divisor on $X(\Delta)$ is Cartier

(2) $Pic(X(\Delta)) = Cl(X(\Delta))$

(3) $X(\Delta)$ is smooth

pf: (1) $\Leftrightarrow$ (2) (3) $\Rightarrow$ (1) ✓

(1) $\Rightarrow$ (3): Suppose every Weil divisor is Cartier. Let $\sigma \in \Delta$, ~~$Cl(X(\Delta))$~~

Since $Cl(X(\Delta)) \to Cl(U_\sigma)$ is surjective, it follows that every Weil divisors on U_σ is Cartier.

Since $Pic(U_\sigma) = 0$, then $m \mapsto div(\chi^m)$ induces a surjective map

$$M \xrightarrow{\Phi^*} Div_{T_N}(U_\sigma) = \bigoplus_{\rho \in \sigma(1)} \mathbb{Z} D_\rho$$

Writting $\sigma(1) = \{\rho_1, \cdots, \rho_s\}$, this map become $M \to \mathbb{Z}^s$

$$m \mapsto (\langle m, u_{\rho_1}\rangle, \cdots, \langle m, u_{\rho_s}\rangle)$$

Define $\Phi: \mathbb{Z}^s \to N$ by $\Phi(a_1, \cdots, a_s) = \sum_{i=1}^{s} a_i u_{\rho_i}$

The dual map $\Phi^*: M = Hom_{\mathbb{Z}}(N, \mathbb{Z}) \to Hom_{\mathbb{Z}}(\mathbb{Z}^s, \mathbb{Z}) = \mathbb{Z}^s$

is $m \mapsto (\langle m, u_{\rho_1}\rangle, \cdots, \langle m, u_{\rho_s}\rangle)$

Φ^* is surjective $\Rightarrow$ Φ is injective and $N/\Phi(\mathbb{Z}^s)$ is torsion-free

?

$\Rightarrow u_{\rho_1}, \cdots, u_{\rho_s}$ can be extended to a basis of N

$\Rightarrow \forall \sigma$ is smooth

$\Rightarrow X(\Delta)$ is smooth

$0 \to \mathbb{Z}^s \xrightarrow{\Phi} N \to N/\Phi(\mathbb{Z}^s) \to 0$

$0 \to Hom_{\mathbb{Z}}(N/\Phi(\mathbb{Z}^s), \mathbb{Z}) \to M \to \mathbb{Z}^s \to 0$

$Ext^1_{\mathbb{Z}}(N/\Phi(\mathbb{Z}^s), \mathbb{Z}) = 0$

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prop: T.F.A.E

(1) Every Weil divisor is $\mathbb{Q}$-Cartier

(2) $X(\Delta)$ is simplicial (i.e Δ is simplicial)

pf: (2) $\Rightarrow$ (1)

Δ is simplicial $\Rightarrow$ $X(\Delta)$ is finite singularities

$\Rightarrow$ $X(\Delta)$ is $\mathbb{Q}$-Cartier.

(1) $\Rightarrow$ (2) Suppose ~~Eve~~ every weil divisor is $\mathbb{Q}$-Cartier, it follows that on U_σ is so.

Since $\mathrm{Pic}(U_\sigma) = 0$, then

$$0 \to M \to \mathrm{Div}_T(U_\sigma) \to \mathrm{Cl}(U_\sigma) \to 0$$
$$m \mapsto \mathrm{div}(\chi^m)$$

U_σ is simplicial $\Leftrightarrow$ Φ is injective and $M/\Phi(\mathbb{Z}^s)$

Exercise: $\sigma = \mathrm{cone}(\underbrace{2e_1 - e_2}_{v_1}, \underbrace{-e_1 + 2e_2}_{v_2})$ and v_1, v_2 correspond the divisors D_1 and D_2.

Then $a_1 D_1 + a_2 D_2$ is a Cartier divisor

$\Leftrightarrow \exists u \in M$, s.t $\mathrm{div}(\chi^u) = a_1 D_1 + a_2 D_2$

$\Leftrightarrow \exists u\ \langle u, v_1 \rangle = a_1,\ \langle u, v_2 \rangle = a_2$

$\Leftrightarrow \exists u = (p, q)\quad 2p - q = a_1$

$\qquad -p + 2q = a_2$

$\Leftrightarrow a_1 \equiv a_2 \mod 3$

Exercise: Show that for each irreducible Weil divisor D_ν on an affine toric variety, there is an effective Cartier divisor that contains D_ν with multiplicity one

Exercise: Let $D = \sum a_\rho D_\rho$ ~~is~~ be a Weil divisor

Then (1) D is Cartier

(2) $\Leftrightarrow$ for each cone σ $\exists\, u(\sigma) \in M$ s.t. $\langle u(\sigma), v_\rho\rangle = -a_\rho$ $\forall \rho \in \sigma(1)$

(3) $\Leftrightarrow$ for each maximal cone σ $\cdots$

(~~ma~~ means that σ are not proper subsets of another cone in Δ.

Remark: the minus signs are ~~not~~ related to the minus signs ~~of~~ in the expression of polytope

$P = \{v \in E : \langle u_1, v\rangle \geq -a_1, \cdots \langle u_r, v\rangle \geq -a_r$

pf: $(2) \Rightarrow (3)$ ✓

$(3) \Rightarrow (2)$ $\forall$ cone $\sigma \in \Delta$ is a face of some maximal cone σ_0.

Thus if m_σ works for σ_0, m_σ also works for σ.

$(1) \Leftrightarrow (2)$ $D|_{U_\sigma} = \sum_{\rho \in \sigma(1)} a_\rho D_\rho$

D is Cartier

$\Leftrightarrow D|_{U_\sigma}$ is Cartier

$\Leftrightarrow D|_{U_\sigma} = \operatorname{div}(\chi^{m})$

$= \sum_{\rho \in \sigma(1)} \langle m, v_\rho\rangle D_\rho = \sum_{\rho \in \sigma(1)} a_\rho D_\rho$

$\Leftrightarrow \exists\, u(\sigma) = -m \in M$ s.t. $\langle u(\sigma), v_\rho\rangle = -a_\rho$

Remark: ① $u(\sigma)$ is unique modulo $M(\sigma) = \sigma^\perp \cap M$

(Since if $\langle u(\sigma), v_\rho \rangle = \langle u'(\sigma), v_\rho \rangle = -a_\rho \ \forall \rho \in \sigma(1)$
then $\langle u(\sigma) - u'(\sigma), v_\rho \rangle = 0 \ \forall \rho \in \sigma(1)$
Hence $u(\sigma) - u'(\sigma) \in \sigma^\perp \cap M$)

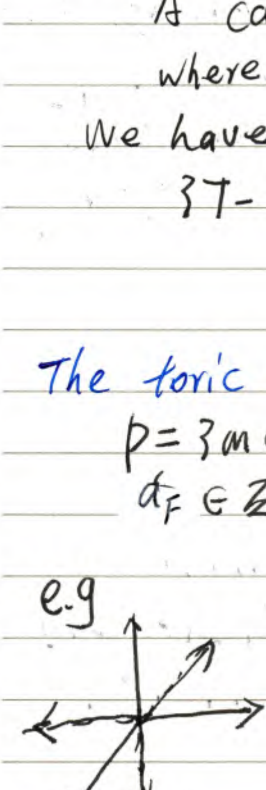
② If $\tau < \sigma$, then the canonical map $M/M(\sigma) \to M/M(\tau)$ sends $u(\sigma)$ to $u(\tau)$

According to Exercise
A cartier divisor is given by $\{U_{\sigma_i}, \chi^{-u(\sigma_i)}\}$
where $\{\sigma_i\}$ are maximal cone
We have
$$\{T\text{-Cartier divisors}\} = \ker\Big(\bigoplus_i M/M(\sigma_i) \to \bigoplus_{i<j} M/M(\sigma_i \cap \sigma_j)\Big)$$

The toric variety of a polytope

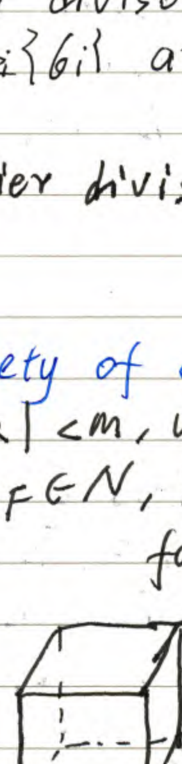
$P = \{m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle \geq -a_F$ for all facets of $P\}$
$a_F \in \mathbb{Z}$, $u_F \in N$, inward-pointing facet normal and minimal

e.g



$N_{\mathbb{R}}$

$u_F \in \{(\pm 1, 0, 0)$
$(0, \pm 1, 0)$
$(0, 0, \pm 1)\}$



$M_{\mathbb{R}}$

$a_F = 1$

$\Delta_P = \{\sigma_Q \mid Q < P\}$

$\sigma_Q = \mathrm{Cone}(u_F \mid F$ contains $Q\}$

$\dim(\Delta_P) = n$

$\dim(Q) = k \iff \dim(\sigma_Q) = n-k$

$\{\text{facets of } P\} \longleftrightarrow \{\rho_F = \text{cone}(u_F)\}$
$\longleftrightarrow \{\text{prime divisor } D_F\}$

Let $D_P = \sum_F a_F D_F$, which tells us a_F

prop: D_P is a Cartier divisor on $X(\Delta_P)$
and $D_P \not\sim 0$

pf: ① $\{\text{vertices of } P\} \longleftrightarrow \{\text{maximal cones } \sigma_v\}$
$\sigma_v = \text{cone}(u_F \mid v < F)$
$\forall F \ni v$, $\langle v, u_F \rangle = -a_F$
and $v \in M \Rightarrow D$ is Cartier.
② If $D_P \sim 0 \Rightarrow \exists\, m \in M$ s.t $\langle m, u_F \rangle = a_F$
which is impossible. □

Remark: The data of D_P is $\{v \mid v \text{ is the vertex of } P\}$

Exercise P64 - P65

1. The example of $X(\Delta)$, which is complete but not projective.

The fan Δ is generated by
$u_1 = (1,2,3) \cup \{(\pm1, \pm1, \pm1)\} \setminus \{(1,1,1)\}$
we can compute all piecewise linear and integral functions on $|\Delta|$ is linear.
$\Rightarrow$ that is $\text{Pic}(X(\Delta)) = 0$
But for $\forall$ projective X_P, we have a $D_P \not\sim 0$.
$\Rightarrow X(\Delta)$ is not projective

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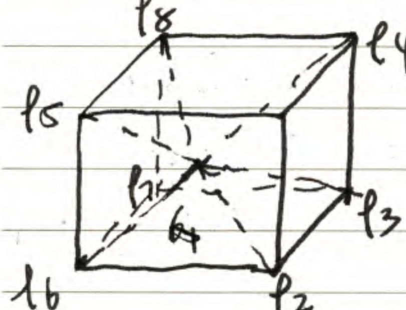
$\mathrm{Pic}(X(\Delta)) = 0$

Indeed, let $D = \sum' a_\rho D_\rho$ be a Cartier divisor.
and $\psi_D(v)$ be its supporting function.
Suppose $\psi_D(v) = \langle u_i, v \rangle$ for each $\sigma_i \in \Delta(n)$



$\sigma_1 = \mathrm{cone}(\rho_1, \rho_2, \rho_5, \rho_6)$

σ_1 (front)	$\rho_1, \rho_2, \rho_5, \rho_6$
σ_2 (back)	$\rho_3, \rho_4, \rho_7, \rho_8$
σ_3 (left)	$\rho_5, \rho_6, \rho_7, \rho_8$
σ_4 (right)	$\rho_1, \rho_2, \rho_3, \rho_4$
σ_5 (up)	$\rho_1, \rho_4, \rho_5, \rho_8$
σ_6 (down)	$\rho_2, \rho_3, \rho_6, \rho_7$.

$\langle u_1, \rho_1 \rangle = \langle u_4, \rho_1 \rangle = \langle u_5, \rho_1 \rangle = -a_1$

$\langle u_1, \rho_2 \rangle = \langle u_4, \rho_2 \rangle = \langle u_6, \rho_2 \rangle = -a_2$

$\langle u_1, \rho_5 \rangle = \langle u_3, \rho_5 \rangle = \langle u_5, \rho_5 \rangle = -a_5$

$\langle u_1, \rho_6 \rangle = \langle u_3, \rho_6 \rangle = \langle u_6, \rho_6 \rangle = -a_6$

Let $\varphi(v) = \psi_D(v) - \langle u_1, v \rangle$

Then $\varphi(\rho_1) = \varphi(\rho_2) = \varphi(\rho_5) = \varphi(\rho_6) = 0$

$\rho_2 + \rho_7 = \rho_3 + \rho_6 \Rightarrow \varphi(\rho_2) + \varphi(\rho_7) = \varphi(\rho_3) + \varphi(\rho_6)$

$\varphi(\rho_7) = \varphi(\rho_3)$

Similarly $\varphi(\rho_7) = \varphi(\rho_8)$, $\cancel{\varphi}$

$\varphi(\rho_3) + \cancel{\varphi(\rho_8)} = \cancel{\varphi(\rho_7)} + \varphi(\rho_4)$

$\Rightarrow \varphi(\rho_3) = \varphi(\rho_4) = \varphi(\rho_7) = \varphi(\rho_8)$

σ_5: $2\rho_1 + 3\rho_8 = 4\rho_5 + 5\rho_4$

$\Rightarrow \cancel{2\varphi(\rho_4)}\ 2\varphi(\rho_4) = 0$

σ_4: $2\rho_1 + 4\rho_3 = 3\rho_2 + 5\rho_4 \Rightarrow 2\varphi(\rho_4) = 0$

Supporting function.

Given a T-Cartier divisor D and its data $\{u(\sigma) \in M/M(\sigma)\}$

— Define a continuous piecewise linear function
$\psi_D(v) = \langle u(\sigma), v \rangle$ for $v \in \sigma$.

(ψ_D is well defined: if $\tau < \sigma$, $\forall v \in \tau$
$\langle u(\tau), v \rangle = \langle u(\sigma), v \rangle$)

If $[D] = \sum a_i D_i$, then $\psi_D(v_i) = -a_i$, where v_i are the minimal generators of the rays of fan
(i.e $[D] = \sum -\psi_D(v_i) D_i$)

properties:

① $\psi_{D+E} = \psi_D + \psi_E$

② $\psi_{mD} = m\psi_D$

③ $\psi_{\mathrm{div}\chi^u}(v) = \langle -u, v \rangle$

④ if $D \sim E$, $\psi_D - \psi_E = u$ for some $u \in M$

— $D = \sum a_i D_i$ ~~T-Cartier divisor~~

Define $P_D = \{u \in M_{\mathbb{R}} : \langle u, v_i \rangle \geqslant -a_i \ \forall i\}$

Note 2. $= \{u \in M_{\mathbb{R}} : u \geqslant \psi_D \text{ on } |\Delta|\}$

$P_D \subseteq M_{\mathbb{R}}$ a rational convex polyhedron

If D is Cartier $\not\Rightarrow$ P_D is a lattice polytope.

Lemma: $\Gamma(X, \mathcal{O}(D)) = \bigoplus_{u \in P_D \cap M} \mathbb{C} \cdot \chi^u$

pf: $\Gamma(U_\sigma, \mathcal{O}(D)) = \bigoplus_{\mathrm{div}(\chi^u) + D|_{U_\sigma} \geqslant 0} \mathbb{C} \cdot \chi^u$

$= \bigoplus_{\substack{\langle u, v_i \rangle \geqslant -a_i \\ \forall v_i \in \sigma}} \mathbb{C} \cdot \chi^u = \bigoplus_{u \in P_D(\sigma) \cap M} \mathbb{C} \cdot \chi^u$

$P_D(\sigma) = \{u \in M_{\mathbb{R}} : \langle u, v_i \rangle \geqslant -a_i \ \forall v_i \in \sigma\}$

$$\Rightarrow \Gamma(X, \mathcal{O}(D)) = \bigcap_\sigma \Gamma(U_\sigma, \mathcal{O}(D))$$
$$= \bigoplus_{u \in \bigcap_\sigma P_D(\sigma) \cap M} \mathbb{C} \cdot \chi^u$$
$$= \bigoplus_{u \in P_D \cap M} \mathbb{C} \cdot \chi^u \qquad \square$$

For a polytope $P \subseteq M_\mathbb{R}$, we have
$$\Gamma(X_P, \mathcal{O}(D_P)) = \bigoplus_{u \in P \cap M} \mathbb{C} \cdot \chi^u$$

Ex: Show that (i) $P_{mD} = mP_D$ (ii) $P_{D+\mathrm{div}(\chi^u)} = P_D - u$
(iii) $P_D + P_E \subset P_{D+E}$

pf: (i) ✓

(ii) $m \in P_{D+\mathrm{div}(\chi^u)}$
$\Leftrightarrow \langle m, v_i \rangle \geq -(a_i + \langle u, v_i \rangle) \quad \forall v_i$
$\Leftrightarrow \langle m+u, v_i \rangle \geq -a_i \quad \forall v_i$
$\Leftrightarrow m + u \in P_D$
$\Leftrightarrow m \in P_D - u$

$D = \sum a_i D_i \quad E = \sum b_i D_i$

(iii) if $u_1 \in P_D$, $u_2 \in P_E$, then
$\langle u_1, v_i \rangle \geq -a_i$
$\langle u_2, v_i \rangle \geq -b_i$
$\Rightarrow \langle u_1 + u_2, v_i \rangle \geq -(a_i + b_i) \quad \forall i$
$\Rightarrow u_1 + u_2 \in P_{D+E}$

~~if $u \in P_{D+E}$, then $\langle u, v_i \rangle \geq -(a_i+b_i)$~~

When $|\Delta| = N_\mathbb{R}$, $X(\Delta)$ is complete
$\Rightarrow$ cohomology group of a coherent sheaf are finite dimensional on any complete variety
that is. P_D is bounded.

For polytope X_P

$$\Gamma(X_P, \mathcal{O}(D_P)) = \bigoplus_{m \in P \cap M} \mathbb{C} \cdot \chi^m$$

$$\Gamma(X_P, \mathcal{O}(kD_P)) = \bigoplus_{m \in (kP) \cap M} \mathbb{C} \cdot \chi^m$$

kP is very ample $\Rightarrow$ $X_{(kP) \cap M}$ is the toric variety X_P

so the characters χ^m that realize X_P as a projective variety come from global sections of $\mathcal{O}_{X_P}(kD_P)$

prop: If the cones in Δ span $N_{\mathbb{R}}$ (i.e $|\Delta| = N_{\mathbb{R}}$), then $\Gamma(X, \mathcal{O}(D))$ is finite dimensional.
In particular, $P_D \cap M$ is finite.

pf: If P_D were unbounded, then $\exists$ a sequence of vectors ~~...~~ u_i in P_D, and positive numbers t_i converging to zero, s.t $t_i u_i$ converges to some nonzero vector u in $M_{\mathbb{R}}$

Since $\langle u_i, v_\rho \rangle \geq -a_\rho$ $\forall \rho \in \Delta(1)$, then

$$\langle u, v_j \rangle = \lim_{n \to \infty} \langle t_i u_i, v_j \rangle \geq \lim_{n \to \infty} -t_i a_\rho$$
$$= 0 \quad \forall \rho \in \Delta(1)$$

Since $|\Delta| = N_{\mathbb{R}}$, $u = 0$, a contradiction.

prop: Assume all maximal cones in Δ are n-dim. Let D be a Cartier divisor on $X(\Delta)$. Then $\mathcal{O}(D)$ is generated by its sections if and only if $\psi(D)$ is convex

pf: $\mathcal{O}(D)$ is generated by its sections

$\Longleftrightarrow$ for each cone σ, $\exists\, u(\sigma) \in M$ s.t. $\chi^{u(\sigma)} \in \Gamma(X, \mathcal{O}(D))$

Note 1 and $\chi^{u(\sigma)}$ generates $\Gamma(U_\sigma, \mathcal{O}(D))$.

Note 1

$\Longleftrightarrow \exists\, u(\sigma) \in P_D$ i.e. $\langle u(\sigma), v_\rho \rangle \geq -a_\rho \quad \forall \rho \in \Delta(1)$

and $\langle u(\sigma), v_\rho \rangle = -a_\rho \quad \forall \rho \in \sigma(1)$

since $\psi_D(v_\rho) = -a_\rho \quad \forall \rho \in \sigma(1)$

$\Longleftrightarrow \psi_D$ is convex. $\square$

Prop: If $\mathcal{O}(D)$ is generated by its sections, and all maximal cones of the fan are n-dim, then

$$\psi_D(v) = \min_{u \in P_D \cap M} \langle u, v \rangle = \min \langle u_i, v \rangle$$

where u_i are the vertices of P_D.

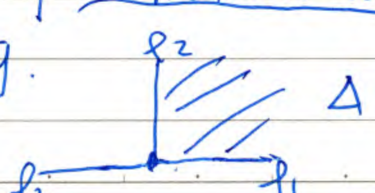
pf: The first equality holds $u(\sigma) \in P_D \cap M \quad \forall \sigma \in \Delta$.

~~The second~~

(P_D is unbounded and $X(\Delta)$ is not complete)

Q: If $\exists$ maximal cone $\sigma \in \Delta$, $\dim(\sigma) \neq n$, what happens?

e.g.



$X(\Delta) = \mathbb{P}^1 \times \mathbb{C} \setminus \{(\infty, 0)\}$

$D_{\rho_1} \cong \mathbb{C} \quad D_{\rho_3} \cong \mathbb{C} \quad D_{\rho_2} \cong \mathbb{P}^1$

Let $D=\sum_{i=1}^{3} a_i D_i$

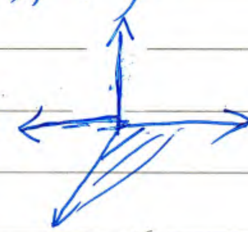
$$\left.\begin{array}{l}\langle u(\sigma), (1,0)\rangle = -a_1 \\ \langle u(\sigma), (0,1)\rangle = -a_2\end{array}\right\} \Rightarrow u(\sigma) = (-a_1, -a_2)$$

$\langle u(\rho_1), (1,0)\rangle = -a_1 \Rightarrow u(\rho_1) = (-a_1, *)$

$\langle u(\rho_2), (0,1)\rangle = -a_2 \Rightarrow u(\rho_2) = (*, -a_2)$

$\langle u(\rho_3), (-1,0)\rangle = -a_3 \Rightarrow u(\rho_3) = (a_3, *)$

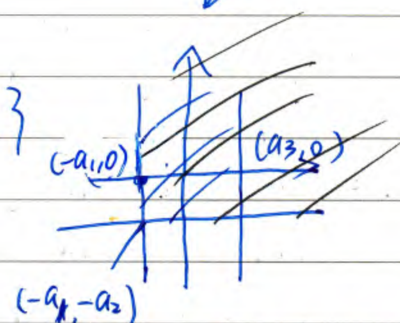
$$\psi_D(v) = \begin{cases} \langle(-a_1,-a_2), v\rangle & \forall v \in \sigma \\ \langle(a_3, *), v\rangle & \forall v \in \rho_3 \end{cases}$$



$$P_D = \left\{ u \;\middle|\; \begin{array}{l} \langle u, (1,0)\rangle \geq -a_1 \\ \langle u, (0,1)\rangle \geq -a_2 \\ \langle u, (-1,0)\rangle \geq -a_3 \end{array} \right\}$$

$P_D(\sigma) =$

$P_D(\rho_3) =$



$a_1 \geq 0 \quad a_3 \geq 0$

$a_2 \geq 0$

Note 1

$D = \sum a_\rho D_\rho$ is Cartier

Prop: If $\Sigma_{max} = \Sigma(n)$, then TFAE

(a) D has no basepoints, i.e. $\mathcal{O}(D)$ is generated by global sections

(b) $u(\sigma) \in P_D$ for all $\sigma \in \Delta(n)$

pf: "$\Rightarrow$" Suppose D is generated by global sections and take $\sigma \in \Delta(n)$. there is a fixed point $x_\sigma \in \bigcap_{\rho \in \sigma(1)} D_\rho$

Then $\exists$ a global section s s.t. $s(x_\sigma) \neq 0$, which amounts to $x_\sigma \notin \text{supp}(\text{div}_0(s))$

Assume that s is given by $\chi^m \otimes$, $m \in P_D \cap M$

Then $\text{div}_0(s) = D + \text{div}(\chi^m)$

$= \sum_\rho (a_\rho + \langle m, u_\rho \rangle) D_\rho$

$x_\sigma \notin \text{supp}(\text{div}_0(s))$ yet lies in D_ρ.

This forces $a_\rho + \langle m, u_\rho \rangle = 0$ $\forall \rho \in \sigma(1)$.

Since $\dim(\sigma) = n$, then $m = m_\sigma \in P_D$

"$\Leftarrow$" Suppose $u(\sigma) \in P_D$, $\forall \sigma \in \Delta(n)$

Then $\chi^{u(\sigma)}$ gives a global section s whose divisor of zeros is

$\text{div}_0(s) = D + \text{div}(\chi^{u(\sigma)})$

$= \sum_{\rho \notin \sigma(1)} (a_\rho + \langle u(\sigma), v_\rho \rangle) D_\rho$,

which implies that the support of $\text{div}_0(s)$ misses U_σ.

Hence s is nonvanishing on U_σ.

Then we are done since U_σ cover $X(\Delta)$

Note 2.

~~If~~ If $D = \sum a_\rho D_\rho$ be a Cartier divisor, then

$P_D = \{u \in M_{\mathbb{R}} \mid u \geq \psi_D \text{ on } |\Delta|\}$

pf: "$\supseteq$" Suppose $\langle u, v\rangle \geq \psi_D(v)$ $\forall v \in |\Delta|$

Then $\psi_D(v_\rho) = a_\rho \leq \langle u, v_\rho\rangle$,

$\Rightarrow u \in P_D$

"$\subseteq$" Take $u \in P_D$, and $v \in |\Delta|$

$\exists\, \sigma \in \Delta$ s.t $v = \sum_{\rho \in \sigma(1)} \lambda_\rho v_\rho$, $\lambda_\rho \geq 0$.

Then $\langle u, v\rangle = \sum \lambda_\rho \langle u, v_\rho\rangle \geq \sum \lambda_\rho \cancel{a_\rho} (-a_\rho)$

$= \sum \lambda_\rho \psi_D(v_\rho)$

$= \psi_D(v)$

Cor: If $X(\Delta)$ is complete, then TFAE

(1) $\mathcal{O}(D)$ is generated by its sections

(2) $u(\sigma) \in P_D$

(3) ψ_D is convex

(4) $P_D = \mathrm{Conv}(u(\sigma) \mid \sigma \in \Delta(n))$

i.e $u(\sigma)$ is the set of vertices of P_D

(5) $\psi_D(v) = \min_{u \in P_D \cap M} \langle u, v\rangle$

pf: We only need to prove $(2) \Rightarrow (4)$

Suppose m is a vertex of P_D and

$H_{v,a}$ is a supporting hyperplane of m

$\{m \in M_{\mathbb{R}} \mid \langle m, v\rangle = a\}$

i.e $H_{v,a} \cap P_D = \{m\}$ and $\forall u \in P_D$ $\langle u, v\rangle \geq a$

Since $|\Delta| = N_{\mathbb{R}}$, $\exists\, \sigma \in \Delta(n)$ s.t $v \in \sigma$

Then $\psi_D(v) = \langle u(\sigma), v\rangle = \min_{u \in P_D} \langle u, v\rangle = \langle m, v\rangle$

Hence $u(\sigma) = m$

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Ex: If $\mathcal{O}(D)$ and $\mathcal{O}(E)$ are generated by their sections, show that $P_{D+E} = P_D + P_E$

pf: We only need to prove $P_{D+E} \subset P_D + P_E$

~~If $X(\Delta)$ is complete, then P_{D+E}, P_D, P_E is boundary.~~ Let $D = \sum a_\rho D_\rho$ $E = \sum b_\rho D_\rho$

$\forall u \in P_{D+E}$, we have $\langle u, v_\rho \rangle \geq -(a_\rho + b_\rho)$ $\forall \rho \in \Delta(1)$

Since $\mathcal{O}(D)$ and $\mathcal{O}(E)$ are generated by their sections, then $\exists m_\rho \in P_D$, $m_\rho' \in P_E$

s.t $\langle m_\rho, v_\rho \rangle = -a_\rho$ $\langle m_\rho', v_\rho \rangle = -b_\rho$

Let $u_1 = \frac{1}{2}(u - m_\rho - m_\rho') + m_\rho$

$u_2 = \frac{1}{2}(u - m_\rho - m_\rho') + m_\rho'$, then $u = u_1 + u_2$

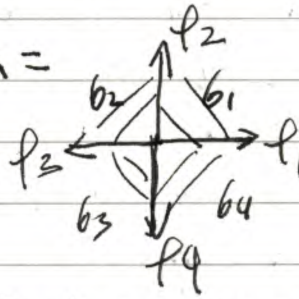
Since $\langle u - m_\rho - m_\rho', v_\rho \rangle \geq 0$, then

$\langle u_1, v_\rho \rangle \geq -a_\rho$, $\langle u_2, v_\rho \rangle \geq -b_\rho$

i.e $u_1 \in P_D$, $u_2 \in P_E$.

$\Rightarrow P_{D+E} \subset P_D + P_E$ $\square$

Example. $\Delta =$



$X(\Delta) = \mathbb{P}^1 \times \mathbb{P}^1$

$P_{D_1} = \{u \mid \langle u, (1,0) \rangle \geq -1, \langle u, (0,1) \rangle \geq 0$

$\langle u, (0,-1) \rangle \geq 0$, $\langle u, (-1,0) \rangle \geq 0$

$=$

$u(\sigma_1) = u(\sigma_4) = (-1, 0)$

$u(\sigma_2) = u(\sigma_3) = (0, 0)$

Line bundles and maps to projective space

Fix a finite-dimensional subspace $W \subseteq \Gamma(X, \mathcal{L})$ with no basepoints and let $W^\vee = \mathrm{Hom}_{\mathbb{C}}(W, \mathbb{C})$. The projective space $\mathbb{P}(W^\vee) = (W^\vee \setminus \{0\}) / \mathbb{C}^*$

We define a map $\phi_{\mathcal{L},W} : X \to \mathbb{P}(W^\vee)$ as follows.

Fix $p \in X$ and pick a nonzero element $v_p \in \pi^{-1}(p) \simeq \mathbb{C}$ where $\pi : V_{\mathcal{L}} \to X$.

For each $s \in W$, there is $\lambda_s \in \mathbb{C}$ s.t. $s(p) = \lambda_s v_p$. Then the map $l_p(s) = \lambda_s$ is linear and nonzero since W has no basepoint. Thus $l_p \in W^\vee$, and since v_p is unique up to an element of $\mathbb{C}^*$, the same is true for l_p. Then

$$\phi_{\mathcal{L},W}(p) = l_p \text{ defines } \phi_{\mathcal{L},W} : X \to \mathbb{P}(W^\vee)$$

Lemma: The map $\phi_{\mathcal{L},W} : X \to \mathbb{P}(W^\vee)$ is a morphism

pf: Let $s_0, \cdots, s_m$ be a basis of W and let $U_i = \{p \in X \mid s_i(p) \neq 0\}$.

$\Rightarrow U_i$ cover X since W has no basepoints.

Furthermore, the natural map

$$U_i \times \mathbb{C} \to \pi^{-1}(U_i) \quad (p, \lambda) \mapsto (p, \lambda s_i(p))$$

is an isomorphism.

Since all sections of $U_i \times \mathbb{C} \to \mathbb{C}$ are of the form $p \mapsto (p, h(p))$ for $h \in \mathcal{O}_X(U_i)$, it follows that we can write

$$s_j|_{U_i} = h_{ij} s_i|_{U_i} \quad h_{ij} \in \mathcal{O}_X(U_i).$$

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The definition of $\phi_{\mathcal{L},W}$ uses a nonzero vector $v_p \in \pi^{-1}(p)$. Over U_i, we can use $0 \neq s_i(p) \in \pi^{-1}(p)$. Then $s_j(p) = h_{ij}(p)\, s_i(p)$ implies ~~$l_p(s_j(p))$~~ $l_p(s_j) = h_{ij}(p)$.
Since $l \mapsto (l(s_1), \cdots, l(s_m))$ gives an isomorphism $\mathbb{P}(W^\vee) \simeq \mathbb{P}^m$, the $\phi_{\mathcal{L},W}|_{U_i}$ can be written

$$U_i \longrightarrow \mathbb{P}^m \quad p \mapsto (h_{i0}(p), \cdots h_{im}(p))$$

which is a morphism since $h_{ii} = 1$

$\square$

Furthermore, when $\mathcal{L} = \mathcal{O}_X(D)$, we can think of the global sections s_i as rational functions g_i s.t $D + \mathrm{div}(g_i) \geq 0$
Then $\phi_{\mathcal{L},W}$ can be written

$$X \longrightarrow \mathbb{P}^m, \ p \longmapsto (g_0(p), \cdots, g_m(p)) \in \mathbb{P}^m \quad (*)$$

Since $g_i(p)$ may be ~~undif~~ undefined, this need explanation. The local data $\{(U_j, f_j)\}$ of D implies that $f_j g_0, \cdots f_j g_m \in \mathcal{O}_X(U_j)$
Then $(*)$ means that $\phi_{\mathcal{L},W}|_{U_j}$ is

$$U_j \longrightarrow \mathbb{P}^m \quad p \mapsto (f_j g_0(p), \cdots, f_j g_m(p)) \in \mathbb{P}^m$$

This is a morphism on U_j since the global sections corresponding to $g_0, \cdots, g_m$ have no base points.

Def: Let D be a Cartier divisor on a complete normal variety X. Then $W = \Gamma(X, \mathcal{O}_X(D))$ is finite-dimensional.

(a) The divisor D and the line bundle $\mathcal{O}_X(D)$ are very ample when D has no basepoints and $\varphi_D : X \to \mathbb{P}(W^\vee)$ is a closed embedding.

(b) D and $\mathcal{O}_X(D)$ are ample when kD is very ample for some integer $k > 0$.

$\varphi_D : X(\Delta) \to \mathbb{P}^{r-1}$, $x \mapsto (\chi^{u_1}(x) : \cdots : \chi^{u_r}(x))$

$r = \mathrm{Card}(P_D \cap M) \overset{?}{=} |P_D \cap M|$

- Strictly convex

$\psi_D(v) = \langle u(\sigma), v \rangle$ if $v \in \sigma$

ψ_D is strictly convex

$\Leftrightarrow \langle u(\sigma), v \rangle = \psi_D(v) \Leftrightarrow v \in \sigma$ and ψ_D is convex

$\Leftrightarrow \psi_D(v) < \langle u(\sigma), v \rangle$ for all $v \in |\Delta| \setminus \sigma$ and $\sigma \in \Delta(n)$

$\Leftrightarrow \psi_D$ is convex and $m_\sigma \neq m_{\sigma'}$ when $\sigma \neq \sigma'$ in $\Delta(n)$

$\Leftrightarrow \psi_D(u+v) > \psi_D(u) + \psi_D(v)$ for all u, v not in the same cone.

A nonzero global section of $\mathcal{O}_X(D)$ have two expression ① $f \in \mathbb{C}(X)^*$ satisfying $D + \mathrm{div}(f) \geq 0$
② $s: X \to V_L$ ~~whose~~ s.t $\pi \circ s$ is the identity on X

Lemma: If $|\Delta| = N_{\mathbb{R}}$, D is very ample iff φ_D is strictly convex and for every $\sigma \in \Delta(n)$, S_σ is generated by $\{u - u(\sigma) : u \in P_D \cap M\}$

pf: "$\Leftarrow$" Let $\sigma \in \Delta(n)$, ~~Then $u(\sigma) \in P_D \cap M$~~
Since φ_D is convex, then $u(\sigma) \in P_D \cap M$.
$\chi^{u(\sigma)} \in \Gamma(X, \mathcal{O}(D))$

$\{(U_i, f_i)\}$ is the data of D. Then over U_i the section s looks like $\{p, s_i(p)\}$ for $s_i = f_i \cdot f \in \mathcal{O}_X(U_i)$.
Then $\mathrm{div}(s_i)|_{U_i} = \mathrm{div}(f_i f)|_{U_i} = D + \mathrm{div}(f)|_{U_i}$
$\Rightarrow$ the divisor of zeros of s is
$\mathrm{div}_0(s) = D + \mathrm{div}(f)$
i.e $\{p \mid s(p) = 0\} = \mathrm{Supp}(\mathrm{div}_0(s))$

Let s is the global section corresponding to $\chi^{u(\sigma)}$, and ~~$\mathrm{div}(s)_0$~~
$\mathrm{div}(s)_0 = D + \mathrm{div}(\chi^{u(\sigma)})$
Since φ_D is strictly convex, i.e
$\langle u(\sigma), v_\rho \rangle = -a_\rho \quad \rho \in \sigma(1)$
$\langle u(\sigma), v_\rho \rangle > -a_\rho \quad \rho \in \Delta(1) \setminus \sigma(1)$
then the support of $\mathrm{div}(s)_0$ consists of those divisors D_ρ ($\rho \in \Delta(1) \setminus \sigma(1)$) missing the affine open U_σ.
It follows that U_σ is the nonvanishing locus of $\chi^{u(\sigma)}$.

Need: D is ample $\Rightarrow X \to \mathbb{P}^r$ is finite

$$\varphi_D : X(\Delta) \to \mathbb{P}^{r-1}$$
$$x \mapsto (\chi^{u_1}(x) : \cdots : \chi^{u_r}(x))$$
$$\uparrow$$
$$(T_{u_1} : \cdots : T_{u_r})$$

~~Hence the in~~

Hence the inverse image by φ_D of the set $\mathbb{C}^{r-1} \subset \mathbb{P}^{r-1}$ where $T_{u(\sigma)} \neq 0$ is the open set U_σ.

Then the restriction $U_\sigma \to \mathbb{C}^{r-1}$ is then given by the function $\chi^{u-u(\sigma)}$, $u \in P_D \cap M$.

Since S_σ is generated by $\{u - u(\sigma), u \in P_D \cap M\}$ then the corresponding map of rings is surjective $(\mathbb{C}[\chi^{u_1-u(\sigma)}, \dots, \widehat{\chi^{u(\sigma)-u(\sigma)}}, \chi^{u_r-u(\sigma)}] \twoheadrightarrow \mathbb{C}[S_\sigma])$

Hence the mapping is a closed embedding $U_\sigma \hookrightarrow \mathbb{C}^{r-1} \subset \mathbb{P}^{r-1}$

"$\Leftarrow$" ① D is b.f $\Rightarrow$ $u(\sigma) \in P \cap M$ and $\{u(\sigma)\}$ are the vertices of P.

$\Rightarrow \langle u(\sigma), v_\rho \rangle \geq -a_\rho \quad \forall \rho \in \Delta(1)$

Hence the inverse image containing U_σ. Since φ_D is an embedding, then the ~~ima~~ inverse image equals to U_σ.

Hence $\langle u(\sigma), v_\rho \rangle > -a_\rho \quad \forall \rho \in \Delta(1) \setminus \sigma(1)$

② S_σ is generated by $\{u - u(\sigma) ; u \in P_D \cap M\}$ follows form the surjection of ~~ring~~ map of rings

prop: Let $|\Delta| = N_\mathbb{R}$, a T-Cartier divisor D is ample, ~~the~~ iff ψ_D is strictly convex.

pf: "$\Rightarrow$" Since $\psi_{mD} = m\psi_D$, then strict convexity of ψ_{mD} implies that ψ_D is strict convex

"$\Leftarrow$" ψ_D is strictly convex

$\Rightarrow \forall m$, $\psi_{mD} = m\psi_D$ is strictly convex

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Then we only ~~need~~ need to prove

$\exists\, m>0$ s.t S_σ is generated by

$\{u - m u(\sigma) \mid u \in P_{mD} \cap M\}$

$(P_{mD} = \{u \in M \mid \langle u, v_\rho\rangle \geq -m a_\rho \ \forall \rho \in \Delta(1)\})$

Since ψ_D is strictly convex,

$\langle u(\sigma), v_\rho\rangle > -a_\rho \quad \forall \rho \in \Delta(1) \setminus \sigma(1)$

$\Rightarrow \langle u(\sigma), v_\rho\rangle + a_\rho > 0, \ \langle u(\sigma), v_\rho\rangle + a_\rho = 0$
$\rho \in \sigma(1)$

Take $v \in S_\sigma$, $\langle v, v_\rho\rangle \geq 0 \quad \forall \rho \in \sigma(1)$

$\langle v, v_\rho\rangle$

$\exists\, m \gg 0$, s.t $\langle v + m u(\sigma), v_\rho\rangle + m a_\rho$

$= \langle v, v_\rho\rangle + m(\langle u_\sigma, v_\rho\rangle + a_\rho)$

$\geq 0 \quad \forall \rho \in \Delta(1)$

$\Rightarrow v + m u(\sigma) \in P_{mD}$.

prop: Any complete toric variety can be dominated birationally by a projective toric variety.

pf: (i.e. A complete fan Δ has a refinement Δ' s.t $X(\Delta')$ is projective)

Let Δ' be obtained from refining Δ using $\bigcup_{\gamma \in \Delta(n-1)} \mathrm{Span}(\gamma)$

Then $\bigcup_{\gamma' \in \Delta'(n-1)} \gamma' = \bigcup_{\gamma \in \Delta(n-1)} \mathrm{Span}(\gamma)$

Choosing $m_\gamma \in M$ s.t $\mathrm{Span}(\gamma)$

$= \{u \in N_{\mathbb{R}} \mid \langle m_\gamma, u\rangle = 0\}$

and define the map $\varphi: N_{\mathbb{R}} \to \mathbb{R}$ by

$\varphi^* \mathcal{O}_{\mathbb{P}^r}$

$$\varphi(u) = - \sum_{\gamma \in \Delta(n-1)} |\langle m_\gamma, u \rangle|$$

Claim: $\varphi(u)$ is an integral, piecewise linear and stricly convex on $|\Delta'|$

Then $D = - \sum_{\rho \in \Delta'(1)} \varphi(a_\rho) D_\rho$

(1) φ takes integer values on N ✓

(2) φ is convex by the triangle inequality. ✓

(3) φ is piecewise linear on Δ'

Fix $\gamma \in \Delta(n-1)$ and note that each cone σ' is contained in one of the closed half-spaces bounded by $\mathrm{Span}(\gamma)$.

This implies $\forall u \in \sigma$ $u \mapsto |\langle m_\gamma, u \rangle|$ is linear for each $\gamma \in \Delta(n-1)$. Hence φ is linear on σ.

(4) φ is strictly convex.

Let $\gamma' = \sigma_1' \cap \sigma_2'$ is a wall of Δ'

Then $\gamma' \subseteq \mathrm{Span}(\gamma_0)$ $\gamma_0 \in \Delta(n-1)$

Then $\varphi|_{\sigma_1'}(u) = -\langle m_{\gamma_0}, u \rangle - \underbrace{\sum_{\gamma \neq \gamma_0, \gamma \in \Delta(n-1)} |\langle m_\gamma, u \rangle|}$

$\varphi|_{\sigma_2'}(u) = \langle m_{\gamma_0}, u \rangle - \underbrace{\sum_{\gamma \neq \gamma_0, \gamma \in \Delta(n-1)} |\langle m_\gamma, u \rangle|}$

$\Downarrow$

is linear on $\sigma_1' \cup \sigma_2'$.

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Cor: Any two-dimensional complete toric variety is projective, and any ample divisor on such a variety is very ample.

pf: $\varphi(u) = -\sum_{\gamma\in\Delta(1)} |\langle m_\gamma, u\rangle|$

§ Cohomology of line bundles

1. The toric Cech complex

Choose the open cover $\mathcal{U} = \{U_\sigma\}_{\sigma \in \Delta_{max}}$

We write these as σ_i and order them.

Given a T-Cartier divisor $D = \sum a_\rho D_\rho$,

the Cech complex is given by

$$C^p = \bigoplus_{(i_0, \cdots, i_p)} H^0(U_{\sigma_{i_0}} \cap \cdots U_{\sigma_{i_p}}, \mathcal{O}_X(D))$$

(Recall: $H^0(U_\sigma, \mathcal{O}_X(D)) = \bigoplus_{m \in P_D(\sigma) \cap M} \mathbb{C} \cdot \chi^m$)

$$P_D(\sigma) = \{ m \in M_\mathbb{R} \mid \langle m, v_\rho \rangle \geq -a_\rho \ \forall \rho \in \sigma(1) \}$$

$$H^0(X, \mathcal{O}_X(D)) = \bigoplus_{m \in P_D \cap M} \mathbb{C} \cdot \chi^m$$

$$P_D = \{ m \in M_\mathbb{R} \mid \langle m, v_\rho \rangle \geq -a_\rho \ \forall \rho \in \Delta(1) \}$$

Then we can write

$$H^0(U_\sigma, \mathcal{O}_X(D)) = \bigoplus_{m \in M} H^0(U_\sigma, \mathcal{O}_X(D))_m$$

$$H^0(U_\sigma, \mathcal{O}_X(D))_m = \begin{cases} \mathbb{C} \cdot \chi^m & m \in P_D(\sigma) \\ 0 & \text{otherwise} \end{cases}$$

$$H^0(X, \mathcal{O}_X(D)) = \bigoplus_{m \in M} H^0(X, \mathcal{O}_X(D))_m$$

$$H^0(X, \mathcal{O}_X(D))_m = \begin{cases} \mathbb{C} \cdot \chi^m & m \in P_D \\ 0 & \text{otherwise} \end{cases}$$

$H^0(X, \mathcal{O}_X(D))$ and $H^0(U_\sigma, \mathcal{O}_X(D))$ are graded module and this induces a grading of Cech complex

(The Čech complex preserves grading)

$$C^{p-1} \xrightarrow{d^{p-1}} C^p$$

$$\bigoplus_{m\in M}\bigoplus_{(i_0,\cdots,i_{p-1})} H^0(\quad)_m \;\Vert\qquad \overset{\Vert}{\bigoplus_{(i_0,\cdots,i_p)}} \bigoplus_{m\in M} H^0(U_{\sigma_{i_0}} \cap \cdots \cap U_{\sigma_{i_p}}, \mathcal{O}(D))_m$$

$$= \bigoplus_{m\in M} \bigoplus_{(i_0,\cdots i_p)} H^0(U_{\sigma_{i_0}} \cap \cdots \cap U_{\sigma_{i_p}}, \mathcal{O}(D))_m$$

Fix $m\in M$,

$$d^{p-1}(\alpha)(i_0,\cdots,i_p) = \sum_{k=0}^{p} (-1)^k \alpha(i_0,\cdots,\hat{i}_k,\cdots,i_p)\Big|_{U_{i_0}\cap\cdots U_{i_p}}$$

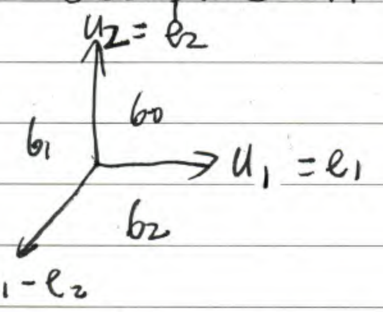
$$\alpha(i_0,\cdots \hat{i}_k,\cdots,i_p)$$
$$\in H^0(U_{\sigma_{i_0}} \cap \cdots \hat{\cap}\, U_{\sigma_{i_k}} \cdots \cap U_{\sigma_{i_p}}, \mathcal{O}_X(D))_m$$
$$= \begin{cases} \mathbb{C}\cdot \chi^m & \text{if } m \in P_D(\sigma_{i_0} \cap \cdots \hat{\sigma}_{i_k} \cap \cdots \cap \sigma_{i_p} \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow d^{p-1}(\alpha)(i_0,\cdots,i_p) \in H^0(U_{\sigma_{i_0}} \cap \cdots \cap U_{\sigma_{i_p}}, \mathcal{O}_X(D))_m$$

(Since $H^p(X, \mathcal{O}_X(D)) = \check{H}^p(\mathcal{U}, \mathcal{O}_X(D))$,
we obtain a natural decomposition
$$H^p(X, \mathcal{O}_X(D)) = \bigoplus_{m\in M} H^p(X, \mathcal{O}_X(D))_m \quad)$$

Example (Compute $H^p(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(a))$ $a>0$)

Let



$$\mathcal{O}_{\mathbb{P}^2}(a) = \mathcal{O}_{\mathbb{P}^2}(aD_0)$$

The Čech complex is

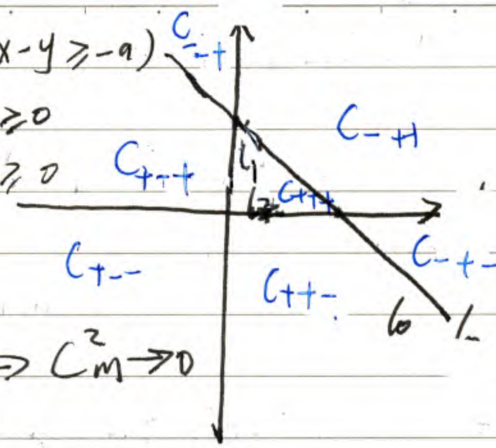
$$0 \to C^0 \xrightarrow[d^1]{\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}} C^1 \xrightarrow[d^2]{(1\ -1\ 1)} C^2 \to 0$$

$$C^0 = \bigoplus_{i=0}^{2} H^0(U_i, \mathcal{O}_{\mathbb{P}^2}(aD_0)), \quad C^1 = \bigoplus_{i<j} H^0(U_{ij}, \mathcal{O}_{\mathbb{P}^2}(aD_0)), \quad C^2 = H^0(U_{012}, \mathcal{O}_{\mathbb{P}^2}(aD_0))$$

$\langle m, u_0 \rangle \geq -a \quad (-x-y \geq -a)$

$\langle m, u_1 \rangle \geq 0 \quad x \geq 0$

$\langle m, u_2 \rangle \geq 0 \quad y \geq 0$



Consider

$$0 \to C_m^0 \to C_m^1 \to C_m^2 \to 0$$

For C_m^0

$$H^0(U_0, \mathcal{O}_{\mathbb{P}^2}(a))_m \neq 0 \iff m \in (C_{+++} \cup C_{-++}) \cap M$$

$$H^0(U_1, \mathcal{O}_{\mathbb{P}^2}(a))_m \neq 0 \iff m \in (C_{+++} \cup C_{+-+}) \cap M$$

$$H^0(U_2, \mathcal{O}_{\mathbb{P}^2}(a))_m \neq 0 \iff m \in (C_{+++} \cup C_{++-}) \cap M$$

$m \in C_{+++} \quad \dim C_m^0 = 3$

$m \in C_{-++} \cup C_{+-+} \cup C_{++-} \quad \dim C_m^0 = 1$

$m \in C_{+--} \cup C_{-+-} \cup C_{--+} \quad \dim C_m^0 = 0$

① $m \in C_{+++}$

$$\dim(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(a))_m) = 1$$

② $m \in$ other

$$\dim(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(a))_m) = 0$$

$\Rightarrow \dim(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(a)) = |a\Delta_2 \cap M|$

For C_m^1

$$H^0(U_{01}, \mathcal{O}_{\mathbb{P}^2}(a))_m \neq 0 \iff m \in (C_{+++} \cup C_{+-+} \cup C_{-++} \cup C_{--+})_m$$

$$H^0(U_{02}, \mathcal{O}_{\mathbb{P}^2}(a))_m \neq 0 \iff m \in (C_{+++} \cup C_{++-} \cup \cancel{C_{+-+}}\ C_{-++} \cup C_{-+-}) \cap M$$

$$\cancel{H^0(U_{12}, \mathcal{O}_{\mathbb{P}^2}(a))_m \neq 0 \iff m \in (C_{+++} \cup C_{+-+} \cup C_{+--} \cup C_{++-}) \cap M}$$

For C_m^2

$H^0(U_{012}, \mathcal{O}_{\mathbb{P}^2}(a))_m \neq 0 \iff m \in M$

	$\dim C_m^0$	$\dim(\ker d^1)$	$\dim(\mathrm{Im}\, d^1)$
$m \in C_{+++}$	3	1	2
		$\dim(\ker d^2)$	$\dim(\mathrm{Im}\, d^2)$
		2	1

$\Rightarrow \dim(H^0(\mathbb{P}^2, \mathcal{O}_X(D))_m) = 1$

$\dim(H^1(\mathbb{P}^2, \mathcal{O}_X(D))_m) = \dim(H^2(\mathbb{P}^2, \mathcal{O}_X(D))_m) = 0$

	$\dim C_m^0$ / $\dim C_m^1$ / $\dim C_m^2$	$\dim(\ker d^1)$ / $\dim(\ker d^2)$	$\dim(\mathrm{Im}\, d^1)$ / $\dim(\mathrm{Im}\, d^2)$
~~$m \in C_{+++}$~~			
$m \in C_{-++}$	1	0	1
$\cup C_{+-+}$	2	1	1
$\cup C_{++-}$	1		
	0	0	0
$m \in C_{+--}$	0	0	0
$\cup C_{-+-}$	1	0	1
$\cup C_{--+}$	1		
	0	0	0

$\dim(H^0(\mathbb{P}^2, \mathcal{O}_X(D))_m)$
$= \dim \ker d^1$

$\dim(H^1(\mathbb{P}^2, \mathcal{O}_X(D))_m)$
$= \dim(\ker d^2)$
$- \dim(\mathrm{Im}\, d^1)$

$\dim(H^2(\mathbb{P}^2, \mathcal{O}_X(D))_m)$
$= \dim(C_m^2)$
$- \dim(\mathrm{Im}\, d^2)$

If $a \geq 0$ $\dim(H^p(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(a)))$

$$= \begin{cases} |a\Delta_2 \cap M| = \binom{a+2}{2} & p=0 \\ 0 & p>0 \end{cases}$$

If $a<0$ $\dim(H^p(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(a)))$

$$= \begin{cases} 0 & p \neq 2 \\ |\mathrm{Int}(a\Delta_2) \cap M| = \binom{-a-1}{2} & p=2 \end{cases}$$

$Z(u) = \{ v \in |\Delta| : \langle u, v \rangle \geq \varphi(v) \}$

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② $H^p(X, \mathcal{O}(D))_u = H^p_{Z(u)}(|\Delta|)$

$$\begin{aligned} &\wr\downarrow \\ &C^p_M \\ &\| \\ \bigoplus_{\sigma_{i_0}, \cdots, \sigma_{i_p}} &H^0(U_{\sigma_{i_0}} \cap \cdots \cap U_{\sigma_{i_p}}, \mathcal{O}(D))_u \\ &\| \\ \bigoplus_{\sigma_{i_0}, \cdots, \sigma_{i_p}} &H^0_{Z(u) \cap \sigma_{i_0} \cap \cdots \cap \sigma_{i_p}}(\sigma_{i_0} \cap \cdots \cap \sigma_{i_p}) \end{aligned}$$

$H^0(U_{\sigma_{i_0}} \cap \cdots \cap U_{\sigma_{i_p}}, \mathcal{O}(D))_u \neq 0$

$\Leftrightarrow u \in P_D(\sigma_{i_0} \cap \cdots \cap \sigma_{i_p}) \cap M \Leftrightarrow$

$\Leftrightarrow \forall v \in |\sigma_{i_0} \cap \cdots \cap \sigma_{i_p}|,\ v \in Z(u) \Leftrightarrow H^0(|\sigma_{i_0} \cap \cdots \cap \sigma_{i_p}| \setminus Z(u)) = 0$

$\Leftrightarrow H^0_{Z(u) \cap \sigma_{i_0} \cap \cdots \cap \sigma_{i_p}}(\sigma_{i_0} \cap \cdots \cap \sigma_{i_p}) = \mathbb{C}$

We want to prove

$$H^p_{Z(u)}(|\Delta|, \mathbb{C}) = H^i(C^\bullet(\{\sigma_i\}, \mathbb{C}))$$

$$\| $$

$$H^p(|\Delta|, |\Delta| \setminus Z(u); \mathbb{C})$$

Where $C^\bullet(\{\sigma_i\}, \mathbb{C})$ is the "Cech" complex whose p^{th} term is

$$C^p(\{\sigma_i\}, \mathbb{C}) = \bigoplus_{\sigma_{i_0}, \cdots, \sigma_{i_p}} H^0_{Z(u) \cap \sigma_{i_0} \cap \cdots \cap \sigma_{i_p}}(\sigma_{i_0} \cap \cdots \cap \sigma_{i_p}, \mathbb{C})$$

Let $E_1^{p,q} = \bigoplus_{\sigma_{i_0}, \cdots, \sigma_{i_p}} H^q_{Z(u) \cap \sigma_{i_0} \cap \cdots \cap \sigma_{i_p}}(\sigma_{i_0} \cap \cdots \cap \sigma_{i_p}, \mathbb{C})$

$$\Rightarrow H^{p+q}_{Z(u)}(|\Delta|, \mathbb{C})$$

Since $\cancel{E_1^{p+q} =}\ H^q_{Z(u) \cap |\tau|}(|\tau|) = 0$ for $q > 0$ and $\tau \in \Delta$

then $E_1^{p,q} = 0$ for all $q > 0$

Hence $E_2^{p,q} = \begin{cases} H^p(C^\bullet\{\sigma_i\}, \mathbb{C}) & q = 0 \\ 0 & q > 0 \end{cases}$

and $E_2^{p,0} \cong H^p(C^\bullet\{\sigma_i\}, \mathbb{C}) \cong H^p_{Z(u)}(|\Delta|, \mathbb{C})$

Lemma: Let $\mathcal{C} = \{C_i\}$ be a locally finite closed cover of X and $\mathcal{F}$ be a sheaf on X. Then there is a spectral sequence

$$E_1^{p,q} = \bigoplus_{(i_0,\cdots,i_p)} H^q(C_{i_0} \cap \cdots \cap C_{i_p}, \mathcal{F})$$

$$\Rightarrow H^{p+q}(X, \mathcal{F})$$

where $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ is induced by inclusion

Relative

Cohomomolgy: we have an exact sequence

$$0 \to H^0(|\Delta|, |\Delta| \setminus Z(u)) \to H^0(|\Delta|) \to H^0(|\Delta| \setminus Z(u))$$
$$\to H^1(|\Delta|, |\Delta| \setminus Z(u)) \to H^1(|\Delta|) \to H^1(|\Delta| \setminus Z(u))$$
$$\to \cdots.$$

$|\Delta|$ is contractible

$\Rightarrow H^0(|\Delta|) = \mathbb{C} \qquad H^i(|\Delta|) = 0 \quad i > 0$

$\Rightarrow H^0(|\Delta|, |\Delta| \setminus Z(u)) \simeq H^0(|\Delta|)$

$H^p(|\Delta| \setminus Z(u)) \simeq H^{p+1}(|\Delta|, |\Delta| \setminus Z(u)) \quad p \neq 0$

$H^0(\cancel{\sigma}\, \sigma, \sigma \setminus Z(u)) \simeq H^0(|\sigma|)$

$H^p(\sigma \setminus Z(u)) \simeq H^{p+1}(|\sigma|, \cancel{\sigma}\,\sigma \setminus Z(u))$

Date

Background: Sheaf cohomology.

1. Sheaf and cohomology

~~Let $f^\bullet: 0 \to \mathcal{F} \to f^0 \xrightarrow{d^0} f^1 \xrightarrow{d^1} \cdots$~~

~~be an injective resolution of~~

We uses an exact sequence of sheaves

$$0 \to \mathcal{F} \to f^0 \xrightarrow{d^0} f^1 \xrightarrow{d^1} \cdots$$

where $f^0, f^1, \cdots$ are injective

(f is injective if given α and injection β

$\exists \theta$ s.t
$$\begin{tikzcd} & f \\ 0 \arrow[r] & \mathcal{H} \arrow[u, "\alpha"] \arrow[r, "\beta"] & \mathcal{G} \arrow[ul, dashed, "\theta"'] \end{tikzcd}$$
)

We say $f^\bullet$ is an injective resolution of $\mathcal{F}$

$\Rightarrow$ Then we have a complex

$$0 \xrightarrow{d^{-1}} \Gamma(X, f^0) \xrightarrow{d^0} \Gamma(X, f^1) \xrightarrow{d^1} \Gamma(X, f^2) \to \cdots$$

Then $H^p(X, \mathcal{F}) := H^p(\Gamma(X, f^\bullet))$

$$= \ker d^p / \operatorname{Im} d^{p-1}$$

Remark: One can prove that injective resolution always exist and that two diffirent resolutions of $\mathcal{F}$ give the same sheaf cohomology group.

By the defintion, we have

① $H^0(X, \mathcal{F}) = \ker d^0$
$$\cong \Gamma(X, \mathcal{F})$$

② $\mathcal{F} \xrightarrow{\alpha} \mathcal{G}$ induces $H^i(X, \mathcal{F}) \to H^i(X, \mathcal{G})$

Indeed, given α and $\mathcal{F} \to \mathcal{A}^\bullet$, $\mathcal{G} \to \mathcal{B}^\bullet$,

$\exists \alpha^p$ s.t
$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d, "\alpha"] & \mathcal{A}^0 \arrow[r, "d^0"] \arrow[d, "\alpha^0"] & \mathcal{A}^1 \arrow[r] \arrow[d, "\alpha^1"] & \cdots \\ \mathcal{G} \arrow[r] & \mathcal{B}^0 \arrow[r, "d^1"] & \mathcal{B}^1 \arrow[r] & \cdots \end{tikzcd}$$
commutes

Then α^p induces $H^p(X, \mathcal{F}) \to H^p(X, \mathcal{G})$

③ A short exact sequnce of sheaves

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

gives ~~an~~ a long exact sequence

$$0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^0(X, \mathcal{H})$$
$$\xrightarrow{\delta_0} H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G}) \to H^1(X, \mathcal{H})$$
$$\to \cdots H^p(X, \mathcal{F}) \to H^p(X, \mathcal{G}) \to H^p(X, \mathcal{H})$$
$$\to \cdots$$

In fact, we can show that

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{A}^\bullet & \to & \mathcal{B}^\bullet & \to & \mathcal{C}^\bullet & \to & 0 \\ & & \uparrow & & \uparrow & & & & \\ 0 & \to & \mathcal{F} & \to & \mathcal{G} & \to & \mathcal{H} & \to & 0 \end{array}$$

is an exact sequence of injective resolutions

and this gives an exact sequence of

complex $0 \to \Gamma(X, \mathcal{A}^\bullet) \to \Gamma(X, \mathcal{B}^\bullet)$
$\to \Gamma(X, \mathcal{C}^\bullet) \to 0$

Then we have desired long exact sequence

2. Cech complex

$$\hat{C}^\bullet(\mathcal{U}, \mathcal{F}) \rightsquigarrow \hat{H}^p(\mathcal{U}, \mathcal{F})$$

Notice: $\hat{H}^0(\mathcal{U}, \mathcal{F}) = H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$

However, $\hat{H}^p(\mathcal{U}, \mathcal{F})$ need not equal $H^p(X, \mathcal{F})$ for $p > 0$

3. When $\hat{H}^p(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F})$

Thm (Serre vanishing for affine variety

$\mathcal{F}$ quassi-coherent sheaf, U affine

Then $H^p(U, \mathcal{F}) = 0 \quad \forall p > 0$.

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By Serre vanishing Thm, we have

Thm: Let $\mathcal{U} = \{U_i\}$ be an affine open cover of X and $\mathcal{F}$ be a quasi-coherent sheaf. Then $\check{H}^p(\mathcal{U}, \mathcal{F}) \simeq H^p(X, \mathcal{F})$.

- Thm (Serre vanishing for projective varieties)
  Let $\mathcal{L}$ be an ample line bundle on ~~X~~ projective varieties X. Then $\forall$ coherent $\mathcal{F}$, ~~the~~ we have $H^p(X, \mathcal{F} \otimes_X \mathcal{L}^{\otimes l}) = 0$
  $$\forall p > 0, l > 0$$

- Higher direct image
  - Given a morphism $f: X \to Y$ of varieties and an $\mathcal{O}_X$-module sheaf on X. Then we have $f_* \mathcal{F}$ is an $\mathcal{O}_Y$-module sheaf
  - By definition of $f_* \mathcal{F}$,
    $$H^0(Y, f_* \mathcal{F}) = H^0(X, \mathcal{F})$$
    and there are homomorphism
    $$\underline{H^p(Y, f_* \mathcal{F}) \to H^p(X, \mathcal{F})} \, ?$$

- $R^p f_* \mathcal{F}$ is the sheaf on Y associated to the presheaf defined by
  $$U \mapsto H^p(f^{-1}(U), \mathcal{F})$$

prop: Suppose $\mathcal{F}$ is quasicoherent. Then
(1) $R^p f_* \mathcal{F}$ are quasicoherent
(2) $U \subseteq Y$ affine, then $R^p f_* \mathcal{F}|_U$ is the sheaf associated to $H^p(f^{-1}(U), \mathcal{F})$

- Spectral sequence

1. Leray ~~se~~ spectral sequence

$$E_2^{p,q} = H^p(Y, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F})$$

Furthermore

$H^p(Y, f_* \mathcal{F}) \to H^p(X, \mathcal{F})$ is the edge homomorphism

$$E_2^{p,0} \to H^p(X, \mathcal{F})$$

Prop: $f: X \to Y$, $\mathcal{F}$ is a quasi-coherent sheaf on X s.t. $R^q f_* \mathcal{F} = 0 \ \forall q > 0$

i.e. $E_2^{p,q} = \begin{cases} H^p(Y, f_* \mathcal{F}) & q = 0 \\ 0 & q > 0 \end{cases}$

Then $H^p(Y, f_* \mathcal{F}) \simeq H^p(X, \mathcal{F})$

2. Covering spectral sequence

$$E_1^{p,q} = \bigoplus_{(i_0, \cdots i_p)} H^q(U_{i_0} \cap \cdots \cap U_{i_q}) \Rightarrow H^{p+q}(X, \mathcal{F})$$

proof of "$\hat{H}^p(\mathcal{U}, \mathcal{F}) \simeq H^p(X, \mathcal{F})$"

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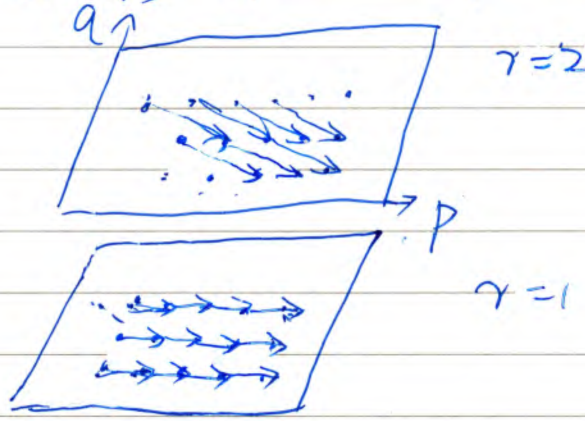
Spectral sequence

Def: (cohomology) spectral sequence is a collection of abelian groups $E_r^{p,q}$ and homomorphism $d_r^{p,q}$ satisfy

(1) In the rth sheet,

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,\, q-r+1}$$

satisfy $d_r^{p+r,\, q-r+1} \circ d_r^{p,q} = 0$



(2) In the $(r+1)$st sheet,

$$E_{r+1}^{p,q} = \ker(d_r^{p,q}) / \operatorname{Im}(d_r^{p-r,\, q+r-1})$$

We will always suppose $E_r^{p,q} = 0$ when $p<0$ or $q<0$ and the minimum value of r is 1 or 2

E_1 spectral sequence $\quad$ E_2 spectral sequence

- In the first quadrant spectral sequence for fixed p, q, $0 \longrightarrow E_r^{p,q} \longrightarrow 0$ for $r \gg 0$

It follows that, $E_r^{p,q} = E_{r+1}^{p,q} = E_{r+2}^{p,q} = \cdots$

The common value is defined to be $E_\infty^{p,q}$

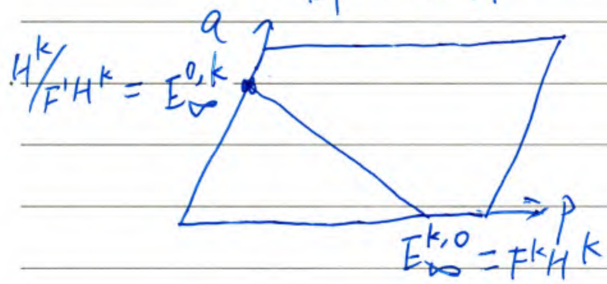
Def: A first quadrant spectral sequence $(E_r^{p,q}, d_r^{p,q})$ converges to a sequence of abelian group H^k, $k \geq 0$, if there is a fibration

$$0 = F^{k+1}H^k \subseteq F^kH^k \subseteq F^{k-1}H^k \subseteq \cdots \subseteq F^0H^k = H^k$$

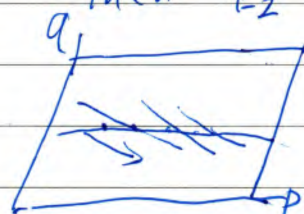
s.t $E_\infty^{p,q} = F^pH^{p+q} / F^{p+1}H^{p+q}$

For E_1 or E_2 spectral sequence, we write

$$E_1^{p,q} \Rightarrow H^{p+q} \quad E_2^{p,q} \Rightarrow H^{p+q}$$



prop: Suppose $E_2^{p,q} \Rightarrow H^{p+q}$ with $E_2^{p,q} = 0$ for all $q > 0$. Then $E_2^{p,0} \simeq H^p$ $\forall p \geq 0$



$E_2^{p,q}$ ($E_2^{p,q} = 0$ i.e $d_2^{p,q} = 0$ $\forall p, q$

Def: We say that a spectral sequence degenarates at E_r sheet if $d_s^{p,q} = 0$ $\forall p, q$, $s \geq r$ (i.e $E_r^{p,q} = E_\infty^{p,q}$)

Note that knowing $E_\infty^{p,q}$ for all p, q is not quite enough to determine H^{p+q}. Then we have following prop.

prop. Let $E_{r_0}^{p,q} \Rightarrow H^{p+q}$. If $E_\infty^{p,q} = 0$ for all $p+q = k$ except $p, q = p_0, q_0$, then $H^k \simeq E_\infty^{p_0, q_0}$

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Edge Homomorphism

For first quadrant E_2 spectral sequence,

$E_r^{p,0} \xrightarrow{d^*} \cdots$ ~~d*~~ all $d^* = 0$

So for each p, there is

$E_2^{p,0} \to E_3^{p,0} \to \cdots \to E_\infty^{p,0} = F^pH^p \subseteq H^p$

Def: The map $E_2^{p,0} \to H^p$ is called an edge ~~mor~~ homomorphism

prop: $E_2^{p,q} \Rightarrow H^{p+q}$ with $E_2^{p,q} = 0$ for all $q > 0$

Then $E_2^{p,0} \to H^p$ is an isomorphism

- Cohen–Macaulay varieties
  A local ring is Cohen-Macaulay if
  $\text{depth}(R) = \dim(R)$
  (Elements $f_1, \cdots, f_r \in m$ form a regular sequence if f_i is not a zero divisor in $R/(f_1, \cdots, f_{i-1})$ for all i
  The depth of R is the maximal length of a regular sequence)

- Singular cohomology.
  - A continuous map $f: Z \to W$ induces
    $f^*: H^p(W, R) \to H^p(Z, R)$
    s.t if f, g are homtopic maps, then $f^* = g^*$
  - If $i: A \hookrightarrow Z$ is a deformation retract, then $i^*: H^p(Z, R) \to H^p(A, R)$ is an isomorphism
  - If Z is ~~contradic~~ contractible, then
    $$H^p(Z, R) = \begin{cases} R & p = 0 \\ 0 & \text{otherwise} \end{cases}$$
  - For $n > 1$, $H^p(S^{n-1}, R) = \begin{cases} R & p = 0, n-1 \\ 0 & \text{otherwise} \end{cases}$

- Relative cohomology

we have a exact sequence

$0 \to H^0 C$

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Prop: Let Δ' be a refinement of Δ, giving a birational proper map $f: \underset{X'}{X(\Delta')} \to \underset{X}{X(\Delta)}$

Then $f_*(\mathcal{O}_{X'}) = \mathcal{O}_X$ and
$R^i f_*(\mathcal{O}_{X'}) = 0 \quad \forall i > 0$

In particular, taking X' to be a resolution of singularities, then this says that X has rational singularities.

pf: ① $f_*(\mathcal{O}_{X'}) = \mathcal{O}_X$ is a general fact since X is normal and f is birational.
Moreover, there is a homomorphism of sheaves $\mathcal{O}_X \to f_*(\mathcal{O}_{X'})$
It suffices to show that this map is an isomorphism on each $U_\sigma \subseteq X$, $\sigma \in \Delta$.
Let $X = U_\sigma$, $|X'| = \tilde{\varphi}_{\mathbb{R}}^{-1}(\sigma) = |\sigma|$
Hence $H^0(X, \mathcal{O}_X) = H^0(X', \mathcal{O}_{X'}) = \bigoplus_{m \in \sigma^\vee \cap M} \mathbb{C} \cdot X^m$

② By proposition, $R^i f_*(\mathcal{O}_{X'})|_{U_\sigma}$ is the sheaf associated to $H^i(f^{-1}(U_\sigma), \mathcal{O}_{X'})$
$f^{-1}(U_\sigma)$ is toric variety whose fan $|X'| = |\sigma|$
Since $H^i(X', \mathcal{O}_{X'}) = 0 \quad \forall i > 0$
$\Rightarrow$ ~~R^i~~ $R^i f_*(\mathcal{O}_{X'}) = 0$

Remark: Let $\mathcal{L}$ be a line bundle on X.

Similarly, we have

(1) $f_* f^* \mathcal{L} \simeq \mathcal{L}$

(2) $R^i f_* f^* \mathcal{L} = 0 \quad \forall i > 0$

Pf: (1) $f_* f^* \mathcal{L} \simeq f_* \mathcal{O}_{X'} \otimes_{\mathcal{O}_X} \mathcal{L} \simeq \mathcal{L}$

(2) $\mathcal{L}|_{U_6} \simeq \mathcal{O}_X|_{U_6}$, so

$$R^i f_* f^* \mathcal{L}|_{U_6} \simeq R^i f_* f^* \mathcal{O}_X|_{U_6}$$
$$\simeq R^i f_* \mathcal{O}_{X'}|_{U_6} = 0$$

Cor: $HP(X, f^* \mathcal{L}) = HP(X', \mathcal{L})$

Example: $\mathbb{P}^2$,

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prop: The toric varieties are all Cohen-Macauley
(i.e each of local rings has depth n
i.e contains a regular sequence of n)
(since, in general, we have depth $(R) \leq \dim(R)$)

pf: Let $\dim(\sigma) = n$, set $A = A_\sigma$, and
$$I = \bigoplus_{u \in \mathrm{Int}(\sigma^\vee) \cap M} \mathbb{C} \cdot \chi^u$$

$$C_{n-k} = \bigoplus_{\substack{\dim(\tau)=k \\ \tau < \sigma}} \mathbb{C}[\sigma^\vee \cap \tau^\perp \cap M]$$

rank ~~dim~~ $(\sigma^\vee \cap \tau^\perp \cap M) = n-k$

$$C_{n-(k-1)} \xrightarrow{\delta} C_{n-k}$$

$$\delta = \bigoplus_{\substack{\tau \in \sigma(k-1) \\ \gamma \in \sigma(k)}} \delta_{\tau,\gamma}$$

If $\tau \not< \gamma$, $\delta_{\tau,\gamma} = 0$

If $\tau < \gamma$, ~~$\delta_{\tau,\gamma}$~~

$\sigma^\vee \cap \tau^\perp \cap M \supseteq \sigma^\vee \cap \gamma^\perp \cap M$

$\delta_{\tau,\gamma} = $ sign $\circ$ projection $\mathbb{C}[\sigma^\vee \cap \tau^\perp \cap M] \to \mathbb{C}[\sigma^\vee \cap \gamma^\perp \cap M]$

Then we have an exact sequence
$$0 \to A/I \to C_{n-1} \to C_{n-2} \to \cdots C_1 \to C_0 \to 0$$

(exactness at A/I
$$A/I = \bigoplus_{u \in \sigma(1)^\perp} \mathbb{C} \cdot \chi^u$$
exactness at C_{n-k}: let

Then by induction, ~~C_{n-k}~~ depth $(C_{n-k}) = n-k$

$\Rightarrow A/I = n-1$

If I is principal, then A has depth n

i.e. $\exists\, u_0 \in \mathrm{Int}(\sigma^\vee) \cap M$ s.t. $\forall\, u \in \mathrm{Int}(\sigma^\vee) \cap M$

$u - u_0 \in \sigma^\vee \cap M$

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The fundamental group

1. $T_N \simeq (\mathbb{C}^*)^n \simeq (\mathbb{R}_{>0})^n \times (S^1)^n$

$\Rightarrow^{(1)} \pi_1(T_N) \simeq \pi_1(\mathbb{R}_{>0})^n \times \pi_1(S^1)^n$
$\simeq \mathbb{Z}^n$

$\Rightarrow (2)$ There is a deformation retract
$i: (S^1)^n \hookrightarrow T_N$.
(i.e $(S^1)^n$ is a deformation retract of T_N)

(Remark: A more intrinsic way to understand $\pi_1(T_N) \simeq \mathbb{Z}^n$

$\forall u \in N \rightsquigarrow \lambda_u: \mathbb{C}^* \to T_N$
$t \mapsto (t^{u_1}, \cdots, t^{u_n})$

Restricting λ_u to $S^1 \subseteq \mathbb{C}^*$ gives a closed path in T_N.

Then we can show that
$N \to \pi_1(T_N)$ is an isomorphism.
$u \mapsto [\sigma_u]$
?

2. complete toric variety is simple connected.

Thm: Let X be a normal variety and $i: U \hookrightarrow X$ be the inclusion of an open subvariety. Then $i_*: \pi_1(U) \to \pi_1(X)$ is surjective.

pf: Any suitable nice topological space X has a simply connected, universal covering space $\bar{X} \xrightarrow{p} X$

Then the conclusion follows from the following lemma.

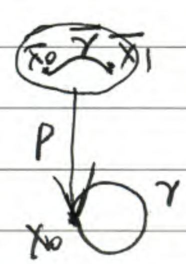
Lemma 1: If X is a normal variety and $i: U \hookrightarrow X$ is the inclusion of an open subvariety U, then $U \times_X \overline{X}$ is path-connected.

Lemma 2: Let X, Z be topological spaces and assume X has a universal covering space $p: \overline{X} \to X$.
Let $f: Z \to X$ be continuous.
If $Z \times_X \overline{X}$ is path-connected, then $f_*: \pi_1(Z) \to \pi_1(X)$ is surjective.

pf of lemma 2: Let $[\gamma] \in \pi_1(X)$.
To prove $\exists \lambda$ s.t. $f_*([\lambda]) = [\gamma]$
i.e. $f \circ \lambda \simeq \gamma$

$$\begin{tikzcd} \overline{X} \arrow[d] & \\ X & Z \arrow[l] \end{tikzcd}$$

Pick a base point $x_0 \in X$ with $x_0 = f(z_0)$
Let $\overline{x}_0 \in p^{-1}(x_0)$ in $\overline{X}$, and lift γ to $\overline{\gamma}$ in $\overline{X}$ starting from $\overline{x}_0$
The final point of $\overline{\gamma}$ will be some $\overline{x}_1 \in p^{-1}(x_0)$



Hence $(z_0, \overline{x}_0), (z_0, \overline{x}_1) \in Z \times_X \overline{X} = \{(z, \overline{x}) \in Z \times \overline{X}, f(z) = p(\overline{x})\}$
Since $Z \times_X \overline{X}$ is path-connected,
$\exists\ \delta$ s.t. $\delta(0) = (z_0, \overline{x}_0)$, $\delta(1) = (z_0, \overline{x}_1)$
Let $p_1: Z \times \overline{X} \to Z$ $p_2: Z \times \overline{X} \to \overline{X}$.
Since $\overline{X}$ is simply-connected, then $p_2 \circ \delta \simeq \overline{\gamma}$ in $\overline{X}$
Hence $p \circ p_2 \circ \delta \simeq \gamma$ in X
Moreover $p_1 \circ \delta = \lambda$ is a loop in Z
s.t. $f \circ \lambda = p \circ p_2 \circ \delta$. Hence $f_*([\lambda]) = [\gamma]$.

prop: Let X be a normal variety and $x \in X$. Then there is a basis $\{V_\alpha \mid \alpha \in A\}$ of open neighborhoods of x in X s.t. $V_\alpha \setminus (V_\alpha \cap \mathrm{Sing}(X))$ is connected

i.e the normal variety is locally irreducible, $\forall \alpha$

Lemma 1

pf: Let $U = X \setminus Y$, where Y is Zariski closed in X

Since X is locally irreducible, so is $\bar{X}$

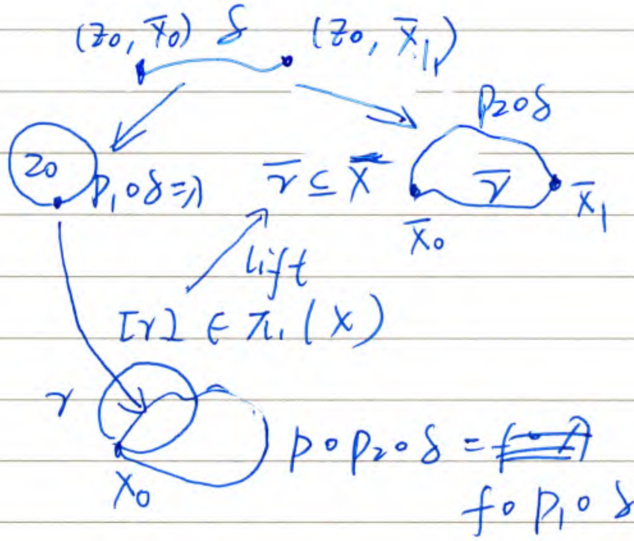
Hence $\bar{X} \setminus p^{-1}(Y) = p^{-1}(X \setminus Y) = p^{-1}(U)$ is connected and path-connected

Then $U \times_X \bar{X} = \{(u, \bar{x}) \mid p(\bar{x}) = u\}$ is the graph of p restricted to $p^{-1}(U)$

Since p is continuous, then ~~path~~ $U \times_X \bar{X}$ is path-connected

Lemma 2.

$$\begin{tikzcd} & Z \times_X \bar{X} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ Z \arrow[dr, "f"'] & & \bar{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$



Covering space

Def: Let X be a topological space. A covering of X is a continuous map

$\pi: \bar{X} \to X$.

s.t $\forall x \in X$, $\exists$ open nghb $U_x \ni x$ and a discrete space D_x s.t $\pi^{-1}(U_x) = \coprod_{d \in D_x} V_d$

and $\pi|_{V_d}: V_d \to U_x$ is a homeomorphism for every $d \in D_x$

(The open set V_d are uniquely determined up to homeomorphism if U_x is connected)

Remark (1): If X is connected, it can be shown that π is surjective and the cardinality of D_x is the same for all $x \in X$

(2) $\pi^{-1}(x)$ is discrete $\forall x \in X$

properties — local homeomorphism

i.e π is continuous and $\forall e \in E$

$\exists_{e \in} V \subset E$ s.t $\pi|_V: V \to \pi(V)$ is a homeomorphism

It follows that

- If X is a connected mainfold, then $\exists$ a covering $\pi: \bar{X} \to X$ is a connected and simple connected mainfold.

lifting property

Date

prop: Let Δ be a fan that contains an n-dim cone. Then $X(\Delta)$ is simply connected

pf: $i: T_N \hookrightarrow X(\Delta)$ gives $i_*: \pi_1(T_N) \twoheadrightarrow \pi_1(X(\Delta))$

Then $\pi_1(X(\Delta)) = \pi_1(T_N)/\ker(i_*) \simeq N/\ker(i_*)$

~~Next~~, Then $\ker(i_*) = ?$

~~To complete the proof, we first discuss the affine case:~~

~~prop: (a) $O(\sigma)$ is a deformation retract of U_σ~~

~~(b) $\pi_1(U_\sigma) \simeq \pi_1(T_{N(\sigma)}) \simeq N(\sigma) \simeq \mathbb{Z}^{n-k}$~~

~~where $N(\sigma) = N/N_\sigma$, $\dim(\sigma) = k$~~

Let $\sigma \in \Delta$, $\dim(\sigma) = n$. Take $v \in \sigma \cap N$

Then $\lim_{z \to 0} \lambda_v(z) = x \in U_\sigma$.

Then we can extend λ_v to a map from $\mathbb{C}$ to U_σ by

$$\lambda_{v,t}(z) = \begin{cases} \lambda_v(tz) & z \in S^1,\ 0 < t \leq 1 \\ x & t = 0 \end{cases}$$

So if $v \in \sigma \cap N$, then $i_*[\sigma_v] = 0$

i.e $v \in \ker(i_*)$

Since $\sigma \cap N$ generates N as a group, and $\ker(i_*)$ is a subgroup of N, $\ker(i_*) = N$. Hence $\pi_1(X(\Delta)) = 0$

Cor. If ~~σ is~~ $\dim(\sigma) = k$, then $\pi_1(U_\sigma) \cong \mathbb{Z}^{n-k}$

pf: ~~$\cong$~~ $U_\sigma = U_{\sigma'} \times (\mathbb{C}^*)^{n-k}$

Hence $\pi_1(U_\sigma) \cong \pi_1((\mathbb{C}^*)^{n-k}) \cong \mathbb{Z}^{n-k}$ 12

Remark: More intrinsically, let N_σ be the sublattice generated by $\sigma \cap N$ and $N(\sigma) = N/N_\sigma$

Then $N = N_\sigma \oplus N(\sigma)$ $\quad \sigma = \sigma' \oplus \{0\}$

$M = M' \oplus M''$ and $S_\sigma = \underset{(\sigma')^\vee \cap M'}{\underset{\parallel}{S_{\sigma'}}} \oplus M''$

i.e $N_\sigma \to N \to N(\sigma)$

$\sigma' \to \sigma \to \{0\}$

determine a fibler bundle

$U_{\sigma'} \to U_\sigma \to T_{N(\sigma)}$

that splits : $U_\sigma \cong U_{\sigma'} \times T_{N(\sigma)}$

Then $\pi_1(U_\sigma) \cong \pi_1(T_{N(\sigma)}) \cong \mathbb{Z}^{n-k}$

prop : Let $\dim(\sigma) = k$, then $O_\sigma \hookrightarrow U_\sigma$ is a deformation retract

In particular, if $\dim(\sigma) = n$, $x_\sigma = O_\sigma \hookrightarrow U_\sigma$ is a deformation retract.

pf : We want to define a homotopy

$H : U_\sigma \times [0,1] \to U_\sigma$

between $\gamma : U_\sigma \to O_\sigma$ and id_{U_σ}

Choose $v \in \text{Reint}(\sigma) \cap N$

Regarding the points of U_σ as $x \in Hom_{sg}(S_\sigma, \mathbb{C})$

Then $H(x,t)(u) := \begin{cases} (x_\sigma \cdot x)(u) & t = 0 \\ t^{\langle u,v \rangle} x(u) & 0 < t \leq 1 \end{cases}$

① $t = 1$ $\quad H(x,1)(u) = x(u)$

i.e $H(x,1) = id_{U_\sigma}$

⓪ $\forall (x,t)$ $\quad H(x,t) \in Hom_{sg}(S_\sigma, \mathbb{C})$

$\left(x_\sigma = \begin{cases} 1 & \text{if } u \in \sigma^\perp \cap M \\ 0 & \text{otherwise} \end{cases} \right)$

② $O_\sigma = \{ \gamma : S_\sigma \to \mathbb{C} \mid \gamma(m) \neq 0 \Leftrightarrow m \in \sigma^\perp \cap M \}$

If $x \in O_\sigma$ $\quad H(x,t) \in O_\sigma$ $\quad \forall\, 0 \leq t \leq 1$

③ $t^{\langle u,v \rangle} = \lambda_v(t)(u)$, $\quad \lim_{t \to 0} \lambda_v(t) = x_\sigma$ $\quad \lim_{t \to 0} t^{\langle u,v \rangle} = x_\sigma(u)$

prop: Let Δ be a fan in $N_{\mathbb{R}}$ and N' be the sublattice generated by $|\Delta| \cap N$.
Then $\pi_1(X(\Delta)) \simeq N/N'$

Van Kampen's Theorem.
Let ~~it~~ $X = U_1 \cup U_2$ and U_1, U_2 be path-connected
Suppose $U_1 \cap U_2$ is path connected and nonempty, and let $x_0 \in U_1 \cap U_2$ be a base point.
Then we have

$$\begin{tikzcd}
 & \pi_1(U_1) \arrow[d, dashed] \arrow[drr, bend left, "j_1"] & & \\
\pi_1(U_1 \cap U_2) \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & \pi_1(U_1) *_{\pi_1(U_1 \cap U_2)} \pi_1(U_2) \arrow[rr, dashed, "k"] & & \pi_1(X) \\
 & \pi_1(U_2) \arrow[u, dashed] \arrow[urr, bend right, "j_2"'] & &
\end{tikzcd}$$

The natural morphism k is an isomorphism

pf: We use induction on the number of cones in Δ.
① If Δ consists of a single cone $\{0\}$, then the result holds.
② Assume the result ~~ha~~ holds for all fans with $k-1$ cones or fewer and let Δ contain k cones
Pick a maximal cone $\sigma \in \Delta$, and
$\Delta' = \Delta \setminus \{\sigma\}$.
Then $X(\Delta) = X(\Delta') \cup U_\sigma$ and
$X(\Delta') \cap U_\sigma = X(\Delta'')$ where
Δ'' consists all the proper faces of σ

Using Van Kampen's Theorem, we have

$$\begin{tikzcd} & \pi_1(X(\Delta')) \arrow[dr, dashed] \arrow[drr, bend left, "j_{1*}"] & & \\ \pi_1(X(\Delta'')) \arrow[ur, "i_{1*}"] \arrow[dr, "i_{2*}"'] & & \pi_1(X(\Delta')) *_{\pi_1(X(\Delta''))} \pi_1(U_6) \arrow[r, dashed, "\simeq"] & \pi_1(X(\Delta)) \\ & \pi_1(U_6) \arrow[ur, dashed] \arrow[urr, bend right, "j_{2*}"'] & & \end{tikzcd}$$

i.e $\pi_1(X(\Delta)) \simeq \pi_1(X(\Delta')) * \pi_1(U_6) / G$

G is the subgroup generated by

$(j_1 i_1)_*([w]) (j_2 i_2)_*[w])^{-1}$

?

$i_{1*}([w]) (i_2[w])^{-1}$

By the induction hypothesis,

$\pi_1(X(\Delta'')) \simeq N / N_{\Delta''}$

$\pi_1(X(\Delta')) \simeq N / N_{\Delta'}$

$\pi_1(U_6) \simeq N / N_6$

Hence $\pi_1(X(\Delta)) \simeq \left(N/N_{\Delta'} \oplus N/N_6 \right) / G$

$$\simeq N / (N_{\Delta'} + N_6)$$

$$= N / N_\Delta$$

$$\begin{tikzcd} & N/N_{\Delta'} \\ N/N_{\Delta''} \arrow[ur] \arrow[dr, two heads] & \\ & N/N_6 \end{tikzcd}$$

$\square$.

$$\begin{tikzcd} & \tilde{n} \\ n \arrow[ur, mapsto] \arrow[dr, mapsto] & \\ & \bar{n} \end{tikzcd}$$

$\tilde{n}_1 \cdots \tilde{n}_m \bar{n}_1 \cdots \bar{n}_k / (\tilde{n}_1 \cdot \bar{n}_1^{-1} \cdots \tilde{n}_L \cdot \bar{n}_L^{-1})$

Remark: Using general Van-Kampen's Theorem

$$\pi_1(X(\Delta)) = \pi_1(\cup U_6)$$

$$= \varinjlim \pi_1(U_6)$$

$$\overset{?}{=} \varinjlim N/N_6 \overset{?}{=} N/\Sigma N_6 = N/N'$$

The homomorphism $j_\alpha : \pi_1(A_\alpha) \to \pi_1(X)$ induced by $A_\alpha \hookrightarrow X$ can extend to a homomorphism $\Phi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$

Van-Kampen Theorem

If $X = \bigcup A_\alpha$, A_α path-connected and containing the basepoint $x_0 \in X$, and if each $A_\alpha \cap A_\beta$ is path-connected, then the homomorphism $\Phi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$ is surjective.

If in addition, each $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then the kernel Φ is the normal subgroup N generated by all elements of the form $i_{\alpha\beta}(w) i_{\beta\alpha}(w)^{-1}$ $w \in \pi_1(A_\alpha \cap A_\beta)$

Remark:

- $j_\alpha : \pi_1(A_\alpha) \to \pi_1(X)$ extend to a homormorphism $\Phi : *_1(A_\alpha) \to \pi_1(X)$

  The Van-kampen theorem says Φ is very often surjective

- The kernel of Φ.

  We have

$$\begin{tikzcd}
 & \pi_1(A_\alpha) \arrow[dr, "j_\alpha"] & \\
\pi_1(A_\alpha \cap A_\beta) \arrow[ur, "i_{\alpha\beta}"] \arrow[dr, "i_{\beta\alpha}"'] & & \pi_1(X) \\
 & \pi_1(A_\beta) \arrow[ur, "j_\beta"'] &
\end{tikzcd}$$

Naturally, $j_\alpha i_{\alpha\beta} = j_\beta i_{\beta\alpha}$ since both compositions ~~Φ~~ are induced by $A_\alpha \cap A_\beta \hookrightarrow X$

So the kernel of Φ consist all the elements of the form $i_{\alpha\beta}(w) i_{\beta\alpha}(w)^{-1}$ for $w \in \pi_1(A_\alpha \cap A_\beta)$

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prop: There is a canonical isomorphism
$H^i(U_\sigma; \mathbb{Z}) \cong \Lambda^i(M(\sigma))$

pf: $T_{N(\sigma)} = O_\sigma$ is a deformation retract of U_σ
Hence $H^i(U_\sigma, \mathbb{Z}) \simeq H^i(T_{N(\sigma)}, \mathbb{Z}) \simeq \Lambda^i(M(\sigma))$
$(H^\bullet((S^1)^n, R) \simeq \Lambda^\bullet R^n$ (R is a ring))
$H^\bullet((\mathbb{C}^*)^n, R) \simeq H^\bullet((S^1)^n, R) \simeq \Lambda^\bullet R^n$
More canonically $T_N = N \otimes \mathbb{C}^*$, then
$H^\bullet(T_N, R) \simeq \Lambda^\bullet M \otimes_{\mathbb{Z}} R$?

- The cohomology ~~for~~ of $X(\Delta)$
There is a spectral sequence
$$E_1^{p,q} = \bigoplus_{i_0 < \cdots < i_p} H^q(U_{i_0} \cap \cdots \cap U_{i_p}) \Rightarrow H^{p+q}(X)$$
$$\overset{\text{prop}}{\Longrightarrow} E_1^{p,q} = \bigoplus_{i_0 < \cdots < i_p} \Lambda^q M(\sigma_{i_0} \cap \cdots \cap \sigma_{i_p}) \Rightarrow H^{p+q}(X)$$

Then $\chi(X(\Delta)) = ?$

Observation:

① $E_1^{p,0} = \bigoplus_{i_0 < \cdots < i_p} \mathbb{Z}$

$$\begin{array}{ccccc} E_1^{0,0} & \to & E_1^{1,0} & \to & E_1^{2,0} \cdots \\ \| & & & & \\ \bigoplus_i \mathbb{Z}_{\sigma_i} & \to & \bigoplus_{i<j} \mathbb{Z}_{\sigma_i \cap \sigma_j} & \to & \bigoplus_{i<j<k} \mathbb{Z}_{\sigma_i \cap \sigma_j \cap \sigma_k} \end{array}$$
→ Koszul complex

$\Rightarrow E_2^{0,0} = \mathbb{Z} \quad E_2^{p,0} = 0 \quad p > 0$
?

② Assume all maximal cones in Δ are n-dim

$\Rightarrow E_1^{0,q} = 0 \quad q \geq 1$

E_1 sheet

$$\begin{array}{llll}
0 & \cdot & & \\
0 & \cdot & & \\
0 & \to E_1^{1,1} & \to E_1^{2,1} & \\
E_1^{0,0} & \to E_1^{1,0} & \to E_1^{2,0} & \to \cdots
\end{array}$$

E_2 sheet

$$\begin{tikzcd}
0 & \cdot & & & \\
0 & \cdot & E_2^{1,2} \arrow[rrd] & E_2^{2,2} & \\
0 & \cdot & E_2^{1,1} \arrow[rrd] & E_2^{2,1} & E_2^{3,1} \\
\mathbb{Z} & \cdot & 0 & 0 & 0
\end{tikzcd}$$

$\Rightarrow E_r^{2,0} = 0 \quad r \geq 2 \qquad E_r^{0,2} = 0 \quad r \geq 0$

$E_2^{1,1} = E_\infty^{1,1} \simeq H^2(X_\Sigma, \mathbb{Z})$

$E_2^{1,1} = \ker(E_1^{1,1} \to E_1^{2,1})$

$= \ker\left(\bigoplus_{i<j} M(\sigma_i \cap \sigma_j) \to \bigoplus_{i<j<k} M(\sigma_i \cap \sigma_j \cap \sigma_k)\right)$

Hence, we get the commutative diagram

$\exists y$ s.t $g_1(y) = f_1(x)$

$$\begin{tikzcd}
& & & 0 \arrow[d] & 0 \arrow[d] \\
\text{exact} & 0 \arrow[r] & H^2(X(\Delta), \mathbb{Z}) \arrow[r, "g_1"] & \bigoplus_{i<j} M(\sigma_i \cap \sigma_j) \arrow[r, "g_2"] \arrow[d] & \bigoplus_{i<j<k} M(\sigma_i \cap \sigma_j \cap \sigma_k) \arrow[d] \\
\text{exact, } & M \arrow[r] & \bigoplus_i M \arrow[r, "f_1"] \arrow[dr, "\psi"'] & \bigoplus_{i<j} M \arrow[r, "f_2"] \arrow[d] & \bigoplus_{i<j<k} M \arrow[d] \\
& & & \bigoplus_{i<j} M/M(\sigma_i \cap \sigma_j) \arrow[r] \arrow[d] & \bigoplus_{i<j<k} M/M(\sigma_i \cap \sigma_j \cap \sigma_k) \arrow[d] \\
& & & 0 & 0
\end{tikzcd}$$

since it is the Koszul complex

$x \in \ker \psi$

$f_1(x)$

$f_1(x) \in \ker g_2$

$\mapsto 0$

Then by diagram chase we have an exact sequence

$$M \to \ker(\psi) \xrightarrow{\delta} H^2(X(\Delta), \mathbb{Z}) \to 0$$

On the other hand, since $M(\sigma_i) = 0 \ \forall i$

$$\ker(\psi) = \ker\left(\bigoplus_i M/M(\sigma_i) \to \bigoplus_{i<j} M/M(\sigma_i \cap \sigma_j)\right)$$

$$\simeq CDiv_{T_N}(X(\Delta))$$

Hence $H^2(X(\Delta, \mathbb{Z}) \simeq Pic(X(\Delta))$

Exactness at $\ker(\psi)$:

$\forall \ \delta(x) = 0 \Rightarrow x \in \ker(f_1) \quad x \in Im(M \to \ker\psi)$

Exactness at $H^2(X(\Delta), \mathbb{Z})$

$\forall y \in H^2(X(\Delta), \mathbb{Z}) \quad g_1(y) \in \ker(g_2)$

$\Rightarrow g_1(y) \in \ker(f_2)$

$\Rightarrow \exists \ x \in \bigoplus_i M \quad s.t \quad f_1(x) = g_1(y)$

$\Rightarrow \psi(x) = 0 \quad$ i.e $x \in \ker\psi$

$\square$.

- Euler characteristic

$$\chi(X(\Delta)) \overset{?}{=} \sum (-1)^{p+q} \operatorname{rank} E_1^{p,q}$$

$$= \sum_p \sum_q (-1)^{p+q}$$

Date . .

Exterior algebra

Koszul complex

Given a ring R and $f_1, \cdots, f_l \in R$, define

$$d^p : \Lambda^p R^l \longrightarrow \Lambda^{p+1} R^l$$

by $d^p(\alpha) = (\sum_{i=1}^{l} f_i e_i) \wedge \alpha$, where $e_1, \cdots, e_l$ are the standard basis of R^l.

Setting $K^p = \Lambda^p R^l$, we get the complex

$$K^\bullet : K^0 \xrightarrow{d^0} K^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{l-2}} K^{l-1} \xrightarrow{d^{l-1}} K^l \longrightarrow 0$$

and for an R-mod M, we get

$$K^\bullet(f_1, \cdots, f_l; M) = K^\bullet \otimes_R M$$

$$K^\bullet : K^0 \xrightarrow{d^0} K^1 \xrightarrow{d^1} K^2 \longrightarrow \cdots$$

$$R \longrightarrow R^l \longrightarrow$$

$$1 \longmapsto \sum_{i=1}^{l} f_i e_i$$

~~$(\sum f_i e_i) \wedge$~~

(1) Given $\varphi: R^l \to R$, show that there are $s^p: K^p \to K^{p-1}$ s.t. $s^1 = \varphi$ and

$$s^{p+q}(\alpha \wedge \beta) = s^p(\alpha) \wedge \beta + (-1)^p \alpha \wedge s^q(\beta)$$

$\cancel{(s^{p+1}(\alpha\wedge\beta) = s^p(\alpha)\wedge\beta + (-1)^{p+1}}\ \alpha \in K_p,\ \beta \in K_q$

(2) $\forall \alpha \in K^p$, s^p satisfy

$$d^{p-1}(s^p(\alpha)) + s^{p+1}(d^p(\alpha)) = \varphi\left(\sum_{i=1}^{l} f e_i\right)\alpha$$

(3) Assume that $(f_1, \cdots, f_l) = R$. Prove that $K^\cdot(f_1, \cdots, f_l; M)$ is exact $\forall$ R-mod M.

pf: $\cancel{\alpha = g_1 e_1 + \cdots + g_l e_l}$

$\cancel{s^1: K^1 \to K^0}$

(1) $s^p: K^p \to K^{p-1}$ $\quad s^p(\alpha_1 \wedge \alpha_2 \cdots \wedge \alpha^p)$

$$= \sum_{i=1}^{p} (-1)^{i+1} \varphi(\alpha_i)\, \alpha_1 \wedge \cdots \wedge \hat{\alpha}_i \wedge \cdots \wedge \alpha^p$$

$s^p(\alpha) \wedge \beta + (-1)^p \alpha \wedge s^q(\beta)$

$$= \sum_{i=1}^{p} (-1)^{i-1} \varphi(\alpha_i)\, \alpha_1 \wedge \cdots \wedge \hat{\alpha}_i \wedge \cdots \wedge \alpha_p \wedge \beta_1 \cdots \wedge \beta_q$$

$$+ \sum_{i=1}^{q} (-1)^p (-1)^{i-1} \varphi(\beta_i)\, \underbrace{\beta_1 \wedge \cdots \hat{\wedge}}_{\alpha_1 \wedge \cdots \wedge \alpha_p} \beta_i \cdots \wedge \beta_q$$

$$= s^{p+q}(\alpha \wedge \beta)$$

(2) $d^{p-1}(s^p(\alpha)) + s^{p+1}(d^p(\alpha))$ $\quad \alpha = \alpha_1 \wedge \cdots \wedge \alpha_p$

$$= d^{p-1}\left(\sum_{i=1}^{p} (-1)^{i+1} \varphi(\alpha_i)\, \alpha_1 \wedge \cdots \wedge \hat{\alpha}_i \cdots \wedge \alpha_p\right)$$

$$+ s^{p+1}\left(\left(\sum_{i=1}^{l} f_i e_i\right) \wedge \alpha_1 \wedge \cdots \wedge \alpha_p\right)$$

$$= \sum_{i=1}^{p} (-1)^{i+1} \varphi(\alpha_i) \left(\sum_{i=1}^{l} f_i e_i \wedge \alpha_1 \wedge \cdots \hat{\alpha}_i \wedge \alpha_p\right)$$

$$+ \varphi\left(\sum_{i=1}^{l} f_i e_i\right) \alpha_1 \wedge \cdots \wedge \alpha_p$$

$$+ \sum_{i=1}^{p} (-1)^i \left(\left(\sum_{i=1}^{l} f_i e_i\right) \wedge \alpha_1 \wedge \cdots \hat{\alpha}_i \cdots \wedge \alpha_p\right)$$

$$= \varphi\left(\sum_{i=1}^{l} f_i e_i\right) \alpha$$

Since $(f_1,\cdots,f_l)=R \Rightarrow \sum_{i=1}^{l} g_i f_i = 1$

(3) Let $\varphi(e_i)=g_i$

Then $d^{p-1}(s^p(\alpha)) + s^{p+1}(d^p(\alpha)) = \alpha \quad \forall \alpha \in K^p$

$$K^{p+1} \underset{d^p}{\overset{s^{p+1}}{\rightleftarrows}} K^p \underset{d^{p-1}}{\overset{s^p}{\rightleftarrows}} K^{p-1}$$

$\alpha \in \ker d^p \Rightarrow d^p(\alpha)=0 \Rightarrow s^{p+1}(d^p(\alpha))=0$

$\Rightarrow \alpha = d^{p-1}(s^p(\alpha)) \Rightarrow \alpha \in \cancel{\ker}\ \mathrm{Im}\, d^{p-1}$

Date

The action of T_N on $\mathbb{C}[M]$

$\forall t \in T_N$, $f \in \mathbb{C}[M]$, then $t \cdot f \in \mathbb{C}[M]$ is defined by $p \mapsto f(t^{-1}p)$ for $p \in T_N$

Lemma: Let $A \subseteq \mathbb{C}[M]$ be a subspace stable under the action of T_N. Then

$$A = \bigoplus_{\chi^m \in A} \mathbb{C} \cdot \chi^m$$

pf: Let $A' = \bigoplus_{\chi^m \in A} \mathbb{C} \cdot \chi^m$ and note that $A' \subseteq A$.

For the opposite inclusion, $\forall f \neq 0 \in A \subsetneq \mathbb{C}[M]$

$f = \sum_{m \in \mathcal{B}} c_m \chi^m$ where $\mathcal{B} \subseteq M$ is finite, $c_m \neq 0$

Let $B = \text{span}\{\chi^m \mid m \in \mathcal{B}\}$. Then $f \in B \cap A$

B is stable, since $t \cdot \chi^m = \chi^m(t^{-1}) \chi^m$

Hence $B \cap A$ is stable.

According to following lemma, $B \cap A$ is spanned by simultaneous eigenvectors of T_N, i.e, spanned by character.

Thus $f \in B \cap A \Rightarrow \chi^m \in A$ $\forall m \in \mathcal{B}$. Hence $f \in A'$

Lemma: (The linear maps $w \mapsto t \cdot w$ are diagonalizable and can be simultaneously diagonalized)

Given $m \in M$, $W_m = \{w \in W \mid t \cdot w = \chi^m(t) w \ \forall t \in T\}$

We have $W = \bigoplus_{m \in M} W_m$

Prop: X is smooth $\Rightarrow Cl(X) = 0$

pf: If X is smooth, then $\mathcal{O}_{X,p}$ is a regular local ring for all $p \in X$