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1 Contest

1.1 C++

```

1 #include <bits/stdc++.h>
2 using namespace std;
3
4 #ifdef LOCAL
5 #include "cp/debug.h"
6 #else
7 #define debug(...)
8 #endif
9
10 mt19937 rng(chrono::steady_clock::now().time_since_epoch().count());
11
12 int main() {
13     ios::sync_with_stdio(false); cin.tie(nullptr);
14     // freopen("input.txt", "r", stdin);
15     // freopen("output.txt", "w", stdout);
16
17     return 0;
18 }

```

1.2 Debug

```

1 #define debug(...) { string _s = #__VA_ARGS__; replace(begin(_s), end(_s),
2     ',', ' '); stringstream _ss(_s); istream_iterator<string> _it(_ss);
3     out_error(_it, __VA_ARGS__);}
4
5 void out_error(istream_iterator<string> it) { cerr << '\n'; }
6
7 template<typename T, typename ...Args>
8 void out_error(istream_iterator<string> it, T a, Args... args) {
9     cerr << " [" << *it << " = " << a << " ] ";
10    out_error(++it, args...);
11 }
12
13 template<typename T, typename G> ostream& operator<<(ostream &os, const
14    pair<T, G> &p) {
15    return os << "(" << p.first << ", " << p.second << ")";
16 }
17
18 template<class Con, class = decltype(begin(declval<Con>()))>
19 typename enable_if<!is_same<Con, string>::value, ostream&>::type
20 operator<<(ostream& os, const Con& container) {
21     os << "{";
22     for (auto it = container.begin(); it != container.end(); ++it)
23         os << (it == container.begin() ? "" : ", ") << *it;
24     return os << "}";
25 }

```

1.3 Java

```

1 import java.io.BufferedReader;
2 import java.util.StringTokenizer;
3 import java.io.IOException;

```

```

4 import java.io.InputStreamReader;
5 import java.io.PrintWriter;
6 import java.util.ArrayList;
7 import java.util.Arrays;
8 import java.util.Collections;
9 import java.util.Random;
10
11 public class Main {
12     public static void main(String[] args) {
13         FastScanner fs = new FastScanner();
14         PrintWriter out = new PrintWriter(System.out);
15         int n = fs.nextInt();
16         out.println(n);
17         out.close(); // don't forget this line.
18     }
19     static class FastScanner {
20         BufferedReader br;
21         StringTokenizer st;
22         public FastScanner() {
23             br = new BufferedReader(new InputStreamReader(System.in));
24             st = null;
25         }
26         public String next() {
27             while (st == null || st.hasMoreTokens() == false) {
28                 try {
29                     st = new StringTokenizer(br.readLine());
30                 }
31                 catch (IOException e) {
32                     throw new RuntimeException(e);
33                 }
34             }
35             return st.nextToken();
36         }
37
38         public int nextInt() {
39             return Integer.parseInt(next());
40         }
41
42         public long nextLong() {
43             return Long.parseLong(next());
44         }
45
46         public double nextDouble() {
47             return Double.parseDouble(next());
48         }
49     }
50 }

```

1.4 sublime-build

```

1 {
2     "cmd": ["g++", "-std=c++17", "-fmax-errors=5", "-DLOCAL", "-Wall",
3         "-Wextra", "-o", "${file_path}/${file_base_name}.out", "${file}"],
4     "file_regex": "^(..[^:]*):([0-9]+):?([0-9]+)??:? (.*)$",

```

```

4     "working_dir": "${file_path}",
5     "selector": "source.cpp, source.c++"
6 }

```

1.5 .bashrc

```

1 alias c++='g++ -std=c++2a -fmax-errors=5 -DLOCAL -Wall -Wextra -O2 -s'
2
3 #Stress-testing
4 function test {
5     SOL=$1
6     CHECKER=$2
7     for i in {1..100};
8     do
9         ./gen.out > in && $CHECKER < in > ans && $SOL < in > out && diff -Zb
10        out ans && echo "Test $i passed!!" || break;
11    done
12 }

```

2 Data structures

2.1 Sparse table

```

1 int st[MAXN][K + 1];
2 for (int i = 0; i < N; i++) {
3     st[i][0] = f(array[i]);
4 }
5 for (int j = 1; j <= K; j++) {
6     for (int i = 0; i + (1 << j) <= N; i++) {
7         st[i][j] = f(st[i][j - 1], st[i + (1 << (j - 1))][j - 1]);
8     }
9 }
10 // Range Minimum Queries.
11 int lg[MAXN + 1];
12 lg[1] = 0;
13 for (int i = 2; i <= MAXN; i++) {
14     lg[i] = lg[i / 2] + 1;
15 }
16 int j = lg[R - L + 1];
17 int minimum = min(st[L][j], st[R - (1 << j) + 1][j]);
18 // Range Sum Queries.
19 long long sum = 0;
20 for (int j = K; j >= 0; j--) {
21     if ((1 << j) <= R - L + 1) {
22         sum += st[L][j];
23         L += 1 << j;
24     }
25 }

```

2.2 Ordered set

```

1 #include <ext/pb_ds/assoc_container.hpp>
2 #include <ext/pb_ds/tree_policy.hpp>
3 using namespace __gnu_pbds;
4
5 template<typename key_type>

```

```

6 using set_t = tree<key_type, null_type, less<key_type>, rb_tree_tag,
7     tree_order_statistics_node_update>;
8
9 const int INF = 0x3f3f3f3f;
10 void example() {
11     vector<int> nums = {1, 2, 3, 5, 10};
12     set_t<int> st(nums.begin(), nums.end());
13
14     cout << *st.find_by_order(0) << '\n'; // 1
15     assert(st.find_by_order(-INF) == st.end());
16     assert(st.find_by_order(INF) == st.end());
17
18     cout << st.order_of_key(2) << '\n'; // 1
19     cout << st.order_of_key(4) << '\n'; // 3
20     cout << st.order_of_key(9) << '\n'; // 4
21     cout << st.order_of_key(-INF) << '\n'; // 0
22     cout << st.order_of_key(INF) << '\n'; // 5
23 }

```

2.3 Dsu

```

1 struct Dsu {
2     int n;
3     vector<int> par, sz;
4     Dsu(int _n) : n(_n) {
5         sz.resize(n, 1);
6         par.resize(n);
7         iota(par.begin(), par.end(), 0);
8     }
9     int find(int v) {
10         // finding leader/parent of set that contains the element v.
11         // with {path compression optimization}.
12         return (v == par[v] ? v : par[v] = find(par[v]));
13     }
14     bool same(int u, int v) {
15         return find(u) == find(v);
16     }
17     bool unite(int u, int v) {
18         u = find(u); v = find(v);
19         if (u == v) return false;
20         if (sz[u] < sz[v]) swap(u, v);
21         par[v] = u;
22         sz[u] += sz[v];
23         return true;
24     }
25     vector<vector<int>> groups() {
26         // returns the list of the "list of the vertices in a connected
27         // component".
28         vector<int> leader(n);
29         for (int i = 0; i < n; ++i) {
30             leader[i] = find(i);
31         }
32         vector<int> id(n, -1);
33         int count = 0;

```

```

33     for (int i = 0; i < n; ++i) {
34         if (id[leader[i]] == -1) {
35             id[leader[i]] = count++;
36         }
37     }
38     vector<vector<int>> result(count);
39     for (int i = 0; i < n; ++i) {
40         result[id[leader[i]]].push_back(i);
41     }
42     return result;
43 }
44 };

```

2.4 Segment tree

```

1  /**
2   * Description: A segment tree with range updates and sum queries that
3   * supports three types of operations:
4   * + Increase each value in range [l, r] by x (i.e. a[i] += x).
5   * + Set each value in range [l, r] to x (i.e. a[i] = x).
6   * + Determine the sum of values in range [l, r].
7   */
8  struct SegmentTree {
9      int n;
10     vector<long long> tree, lazy_add, lazy_set;
11     SegmentTree(int _n) : n(_n) {
12         int p = 1;
13         while (p < n) p *= 2;
14         tree.resize(p * 2);
15         lazy_add.resize(p * 2);
16         lazy_set.resize(p * 2);
17     }
18     long long merge(const long long &left, const long long &right) {
19         return left + right;
20     }
21     void build(int id, int l, int r, const vector<int> &arr) {
22         if (l == r) {
23             tree[id] += arr[l];
24             return;
25         }
26         int mid = (l + r) >> 1;
27         build(id * 2, l, mid, arr);
28         build(id * 2 + 1, mid + 1, r, arr);
29         tree[id] = merge(tree[id * 2], tree[id * 2 + 1]);
30     }
31     void push(int id, int l, int r) {
32         if (lazy_set[id] == 0 && lazy_add[id] == 0) return;
33         int mid = (l + r) >> 1;
34         for (int child : {id * 2, id * 2 + 1}) {
35             int range = (child == id * 2 ? mid - l + 1 : r - mid);
36             if (lazy_set[id] != 0) {
37                 lazy_add[child] = 0;
38                 lazy_set[child] = lazy_set[id];
39                 tree[child] = range * lazy_set[id];

```

```

39     }
40     lazy_add[child] += lazy_add[id];
41     tree[child] += range * lazy_add[id];
42 }
43 lazy_add[id] = lazy_set[id] = 0;
44 }
45 }
46 void update(int id, int l, int r, int u, int v, int amount, bool
47 set_value = false) {
48     if (r < u || l > v) return;
49     if (u <= l && r <= v) {
50         if (set_value) {
51             tree[id] = 1LL * amount * (r - l + 1);
52             lazy_set[id] = amount;
53             lazy_add[id] = 0; // clear all previous updates.
54         }
55         else {
56             tree[id] += 1LL * amount * (r - l + 1);
57             lazy_add[id] += amount;
58         }
59         return;
60     }
61     push(id, l, r);
62     int mid = (l + r) >> 1;
63     update(id * 2, l, mid, u, v, amount, set_value);
64     update(id * 2 + 1, mid + 1, r, u, v, amount, set_value);
65     tree[id] = merge(tree[id * 2], tree[id * 2 + 1]);
66 }
67 long long get(int id, int l, int r, int u, int v) {
68     if (r < u || l > v) return 0;
69     if (u <= l && r <= v) {
70         return tree[id];
71     }
72     push(id, l, r);
73     int mid = (l + r) >> 1;
74     long long left = get(id * 2, l, mid, u, v);
75     long long right = get(id * 2 + 1, mid + 1, r, u, v);
76     return merge(left, right);
77 }
78 };

```

2.5 Efficient segment tree

```

1  template<typename T> struct SegmentTree {
2      int n;
3      vector<T> tree;
4      SegmentTree(int _n) : n(_n), tree(2 * n) {}
5      T merge(const T &left, const T &right) {
6          return left + right;
7      }
8      template<typename G>
9      void build(const vector<G> &initial) {
10         assert((int) initial.size() == n);
11         for (int i = 0; i < n; ++i) {

```

```

12     tree[i + n] = initial[i];
13 }
14 for (int i = n - 1; i > 0; --i) {
15     tree[i] = merge(tree[i * 2], tree[i * 2 + 1]);
16 }
17 }
18 void modify(int i, int v) {
19     tree[i += n] = v;
20     for (i /= 2; i > 0; i /= 2) {
21         tree[i] = merge(tree[i * 2], tree[i * 2 + 1]);
22     }
23 }
24 T get_sum(int l, int r) {
25     // sum of elements from l to r - 1.
26     T ret{};
27     for (l += n, r += n; l < r; l /= 2, r /= 2) {
28         if (l & 1) ret = merge(ret, tree[l++]);
29         if (r & 1) ret = merge(ret, tree[--r]);
30     }
31     return ret;
32 }
33 };

```

2.6 Persistent lazy segment tree

```

1 struct Vertex {
2     int l, r;
3     long long val, lazy;
4     bool has_changed = false;
5     Vertex() {}
6     Vertex(int _l, int _r, long long _val, int _lazy = 0) : l(_l), r(_r),
7         val(_val), lazy(_lazy) {}
8 };
9 struct PerSegmentTree {
10     vector<Vertex> tree;
11     vector<int> root;
12     int build(const vector<int> &arr, int l, int r) {
13         if (l == r) {
14             tree.emplace_back(-1, -1, arr[l]);
15             return tree.size() - 1;
16         }
17         int mid = (l + r) / 2;
18         int left = build(arr, l, mid);
19         int right = build(arr, mid + 1, r);
20         tree.emplace_back(left, right, tree[left].val + tree[right].val);
21         return tree.size() - 1;
22     }
23     int add(int x, int l, int r, int u, int v, int amt) {
24         if (l > v || r < u) return x;
25         if (u <= l && r <= v) {
26             tree.emplace_back(tree[x].l, tree[x].r, tree[x].val + 1LL * amt *
27                 (r - l + 1), tree[x].lazy + amt);
28             tree.back().has_changed = true;
29             return tree.size() - 1;
30         }
31     }
32 };

```

```

28     }
29     int mid = (l + r) >> 1;
30     push(x, l, mid, r);
31     int left = add(tree[x].l, l, mid, u, v, amt);
32     int right = add(tree[x].r, mid + 1, r, u, v, amt);
33     tree.emplace_back(left, right, tree[left].val + tree[right].val, 0);
34     return tree.size() - 1;
35 }
36 long long get_sum(int x, int l, int r, int u, int v) {
37     if (r < u || l > v) return 0;
38     if (u <= l && r <= v) return tree[x].val;
39     int mid = (l + r) / 2;
40     push(x, l, mid, r);
41     return get_sum(tree[x].l, l, mid, u, v) + get_sum(tree[x].r, mid + 1,
42         r, u, v);
43 }
44 void push(int x, int l, int mid, int r) {
45     if (!tree[x].has_changed) return;
46     Vertex left = tree[tree[x].l];
47     Vertex right = tree[tree[x].r];
48     tree.emplace_back(left);
49     tree[x].l = tree.size() - 1;
50     tree.emplace_back(right);
51     tree[x].r = tree.size() - 1;
52
53     tree[tree[x].l].val += tree[x].lazy * (mid - l + 1);
54     tree[tree[x].l].lazy += tree[x].lazy;
55
56     tree[tree[x].r].val += tree[x].lazy * (r - mid);
57     tree[tree[x].r].lazy += tree[x].lazy;
58
59     tree[tree[x].l].has_changed = true;
60     tree[tree[x].r].has_changed = true;
61     tree[x].lazy = 0;
62     tree[x].has_changed = false;
63 };

```

2.7 Lichao tree

```

1 /**
2  * Description: A segment tree that allows insert a new line and query for
3  *             minimum value over all lines at point x.
4  * Usage: useful in convex hull trick.
5  */
6 const long long INF_LL = (long long) 4e18;
7
8 struct Line {
9     long long a, b;
10     Line(long long _a = 0, long long _b = INF_LL) : a(_a), b(_b) {}
11     long long operator()(long long x) {
12         return a * x + b;
13     }
14 };

```

```

14 };
15
16 struct SegmentTree { // min query
17     int n;
18     vector<Line> tree;
19     SegmentTree() {}
20     SegmentTree(int _n): n(1) {
21         while (n < _n) n *= 2;
22         tree.resize(n * 2);
23     }
24     void insert(int x, int l, int r, Line line) {
25         if (l == r) {
26             if (line(l) < tree[x](l)) tree[x] = line;
27             return;
28         }
29         int mid = (l + r) >> 1;
30         bool b_left = line(l) < tree[x](l);
31         bool b_mid = line(mid) < tree[x](mid);
32         if (b_mid) swap(tree[x], line);
33         if (b_left != b_mid) insert(x * 2, l, mid, line);
34         else insert(x * 2 + 1, mid + 1, r, line);
35     }
36     long long query(int x, int l, int r, int at) {
37         if (l == r) return tree[x](at);
38         int mid = (l + r) >> 1;
39         if (at <= mid) return min(tree[x](at), query(x * 2, l, mid, at));
40         else return min(tree[x](at), query(x * 2 + 1, mid + 1, r, at));
41     }
42 };

```

2.8 Old driver tree (Chtholly tree)

```

1 /**
2  * Description: An optimized brute-force approach to deal with problem that
3  * has operation of setting an interval to the same number.
4  * Note: caution TLE, only works when input is random
5  */
6 struct ODT {
7     map<int, long long> tree;
8     using It = map<int, long long>::iterator;
9
10     It split(int x) {
11         It it = tree.upper_bound(x);
12         assert(it != tree.begin());
13         --it;
14         if (it->first == x) return it;
15         return tree.emplace(x, it->second).first;
16     }
17
18     void add(int l, int r, int amt) {
19         It it_l = split(l);
20         It it_r = split(r + 1);
21         while (it_l != it_r) {
22             it_l->second += amt;

```

```

22         ++it_l;
23     }
24 }
25
26 void set(int l, int r, int v) {
27     It it_l = split(l);
28     It it_r = split(r + 1);
29     while (it_l != it_r) {
30         tree.erase(it_l++);
31     }
32     tree[l] = v;
33 }
34
35 long long kth_smallest(int l, int r, int k) {
36     // return the k-th smallest value in range [l..r]
37     vector<pair<long long, int>> values; // pair(value, count)
38     It it_l = split(l);
39     It it_r = split(r + 1);
40     while (it_l != it_r) {
41         It prev = it_l++;
42         values.emplace_back(prev->second, it_l->first - prev->first);
43     }
44     sort(values.begin(), values.end());
45     for (auto [value, cnt] : values) {
46         if (k <= cnt) return value;
47         k -= cnt;
48     }
49     return -1;
50 }
51
52 int powmod(long long a, long long n, int mod) {}
53 int sum_of_xth_power(int l, int r, int x, int mod) {
54     It it_l = split(l);
55     It it_r = split(r + 1);
56     int res = 0;
57     while (it_l != it_r) {
58         It prev = it_l++;
59         res = (res + 1LL * (it_l->first - prev->first) *
60             powmod(prev->second, x, mod)) % mod;
61     }
62     return res;
63 }
64 };

```

2.9 Disjoint sparse table

```

1 /**
2  * Description: range query on a static array.
3  * Time: O(1) per query.
4  * Tested: stress-test.
5  */
6 const int MOD = (int) 1e9 + 7;
7 struct DisjointSparseTable { // product queries.
8     int n, h;
9     vector<vector<int>> dst;

```

```

10     vector<int> lg;
11     DisjointSparseTable(int _n) : n(_n) {
12         h = 1; // in case n = 1: h = 0 !!.
13         int p = 1;
14         while (p < n) p *= 2, h++;
15         lg.resize(p); lg[1] = 0;
16         for (int i = 2; i < p; ++i) {
17             lg[i] = 1 + lg[i / 2];
18         }
19         dst.resize(h, vector<int>(n));
20     }
21     void build(const vector<int> &A) {
22         for (int lv = 0; lv < h; ++lv) {
23             int len = (1 << lv);
24             for (int k = 0; k < n; k += len * 2) {
25                 int mid = min(k + len, n);
26                 dst[lv][mid - 1] = A[mid - 1] % MOD;
27                 for (int i = mid - 2; i >= k; --i) {
28                     dst[lv][i] = 1LL * A[i] * dst[lv][i + 1] % MOD;
29                 }
30                 if (mid == n) break;
31                 dst[lv][mid] = A[mid] % MOD;
32                 for (int i = mid + 1; i < min(mid + len, n); ++i) {
33                     dst[lv][i] = 1LL * A[i] * dst[lv][i - 1] % MOD;
34                 }
35             }
36         }
37     }
38     int get(int l, int r) {
39         if (l == r) {
40             return dst[0][l];
41         }
42         int i = lg[l ^ r];
43         return 1LL * dst[i][l] * dst[i][r] % MOD;
44     }
45 };

```

2.10 Fenwick tree

```

1  /**
2   * Description: range update and range sum query.
3   */
4
5  using tree_type = long long;
6  struct FenwickTree {
7      int n;
8      vector<tree_type> fenw_coeff, fenw;
9      FenwickTree() {}
10     FenwickTree(int _n) : n(_n) {
11         fenw_coeff.assign(n, 0); // fenwick tree with coefficient (n - i).
12         fenw.assign(n, 0); // normal fenwick tree.
13     }
14     template<typename G>
15     void build(const vector<G> &A) {

```

```

16     assert((int) A.size() == n);
17     vector<int> diff(n);
18     diff[0] = A[0];
19     for (int i = 1; i < n; ++i) {
20         diff[i] = A[i] - A[i - 1];
21     }
22     fenw_coeff[0] = (long long) diff[0] * n;
23     fenw[0] = diff[0];
24     for (int i = 1; i < n; ++i) {
25         fenw_coeff[i] = fenw_coeff[i - 1] + (long long) diff[i] * (n - i);
26         fenw[i] = fenw[i - 1] + diff[i];
27     }
28     for (int i = n - 1; i >= 0; --i) {
29         int j = (i & (i + 1)) - 1;
30         if (j >= 0) {
31             fenw_coeff[i] -= fenw_coeff[j];
32             fenw[i] -= fenw[j];
33         }
34     }
35 }
36 void add(vector<tree_type> &fenw, int i, tree_type val) {
37     while (i < n) {
38         fenw[i] += val;
39         i |= (i + 1);
40     }
41 }
42 tree_type __prefix_sum(vector<tree_type> &fenw, int i) {
43     tree_type res{};
44     while (i >= 0) {
45         res += fenw[i];
46         i = (i & (i + 1)) - 1;
47     }
48     return res;
49 }
50 tree_type prefix_sum(int i) {
51     return __prefix_sum(fenw_coeff, i) - __prefix_sum(fenw, i) * (n - i - 1);
52 }
53 void range_add(int l, int r, tree_type val) {
54     add(fenw_coeff, l, (n - 1) * val);
55     add(fenw_coeff, r + 1, (n - r - 1) * (-val));
56     add(fenw, l, val);
57     add(fenw, r + 1, -val);
58 }
59 tree_type range_sum(int l, int r) {
60     return prefix_sum(r) - prefix_sum(l - 1);
61 }
62 };

```

2.11 Fenwick tree 2D

```

1  /**
2   * Description: range update and range sum query on a 2D array.
3   */

```

```

4 using tree_type = long long;
5 struct FenwickTree2D {
6     int n, m;
7     vector<vector<tree_type>> fenw[4];
8     FenwickTree2D(int _n, int _m) : n(_n), m(_m) {
9         for (int i = 0; i < 4; i++) {
10             fenw[i].resize(n, vector<tree_type>(m));
11         }
12     }
13     void add(int u, int v, tree_type val) {
14         for (int i = u; i < n; i |= (i + 1)) {
15             for (int j = v; j < m; j |= (j + 1)) {
16                 fenw[0][i][j] += val;
17                 fenw[1][i][j] += (u + 1) * val;
18                 fenw[2][i][j] += (v + 1) * val;
19                 fenw[3][i][j] += (u + 1) * (v + 1) * val;
20             }
21         }
22     }
23     void range_add(int r, int c, int rr, int cc, tree_type val) { // [r, rr]
24         x [c, cc].
25         add(r, c, val);
26         add(r, cc + 1, -val);
27         add(rr + 1, c, -val);
28         add(rr + 1, cc + 1, val);
29     }
30     tree_type prefix_sum(int u, int v) {
31         tree_type res{};
32         for (int i = u; i >= 0; i = (i & (i + 1)) - 1) {
33             for (int j = v; j >= 0; j = (j & (j + 1)) - 1) {
34                 res += (u + 2) * (v + 2) * fenw[0][i][j];
35                 res -= (v + 2) * fenw[1][i][j];
36                 res -= (u + 2) * fenw[2][i][j];
37                 res += fenw[3][i][j];
38             }
39         }
40         return res;
41     }
42     tree_type range_sum(int r, int c, int rr, int cc) { // [r, rr] x [c, cc].
43         return prefix_sum(rr, cc) - prefix_sum(r - 1, cc) - prefix_sum(rr, c
44         - 1) + prefix_sum(r - 1, c - 1);
45     }
46 };

```

2.12 Implicit treap

```

1 struct Node {
2     int val, prior, cnt;
3     bool rev;
4     Node *left, *right;
5     Node() {}
6     Node(int _val) : val(_val), prior(rng()), cnt(1), rev(false),
7     left(nullptr), right(nullptr) {}
8 };

```

```

8 // Binary search tree + min-heap.
9 struct Treap {
10     Node *root;
11     Treap() : root(nullptr) {}
12     int get_cnt(Node *n) { return n ? n->cnt : 0; }
13     void upd_cnt(Node *&n) {
14         if (n) n->cnt = get_cnt(n->left) + get_cnt(n->right) + 1;
15     }
16     void push_rev(Node *treap) {
17         if (!treap || !treap->rev) return;
18         treap->rev = false;
19         swap(treap->left, treap->right);
20         if (treap->left) treap->left->rev ^= true;
21         if (treap->right) treap->right->rev ^= true;
22     }
23     pair<Node*, Node*> split(Node *treap, int x, int smaller = 0) {
24         if (!treap) return {};
25         push_rev(treap);
26         int idx = smaller + get_cnt(treap->left); // implicit val.
27         if (idx <= x) {
28             auto pr = split(treap->right, x, idx + 1);
29             treap->right = pr.first;
30             upd_cnt(treap);
31             return {treap, pr.second};
32         }
33         else {
34             auto pl = split(treap->left, x, smaller);
35             treap->left = pl.second;
36             upd_cnt(treap);
37             return {pl.first, treap};
38         }
39     }
40     Node* merge(Node *l, Node *r) {
41         push_rev(l); push_rev(r);
42         if (!l || !r) return (l ? l : r);
43         if (l->prior < r->prior) {
44             l->right = merge(l->right, r);
45             upd_cnt(l);
46             return l;
47         }
48         else {
49             r->left = merge(l, r->left);
50             upd_cnt(r);
51             return r;
52         }
53     }
54     void insert(int pos, int val) {
55         if (!root) {
56             root = new Node(val);
57             return;
58         }
59         Node *l, *m, *r;
60         m = new Node(val);

```



```

61     tie(l, r) = split(root, pos - 1);
62     root = merge(l, merge(m, r));
63 }
64 void erase(int pos_l, int pos_r) {
65     Node *l, *m, *r;
66     tie(l, r) = split(root, pos_l - 1);
67     tie(m, r) = split(r, pos_r - pos_l);
68     root = merge(l, r);
69 }
70 void reverse(int pos_l, int pos_r) {
71     Node *l, *m, *r;
72     tie(l, r) = split(root, pos_l - 1);
73     tie(m, r) = split(r, pos_r - pos_l);
74     m->rev ^= true;
75     root = merge(l, merge(m, r));
76 }
77 int query(int pos_l, int pos_r);
78 // returns answer for corresponding types of query.
79 void inorder(Node *n) {
80     if (!n) return;
81     push_rev(n);
82     inorder(n->left);
83     cout << n->val << ' ';
84     inorder(n->right);
85 }
86 void print() {
87     inorder(root);
88     cout << '\n';
89 }
90 };

```

2.13 Line container

```

1 /**
2  * Source: kactl
3  * Description: container that allow you can add lines in form 'ax + b' and
4  * query maximum value at 'x'.
5  */
6 using num_t = int;
7 struct Line {
8     num_t a, b; // ax + b
9     mutable num_t x; // x-intersect with the next line in the hull
10    bool operator<(const Line &other) const {
11        return a < other.a;
12    }
13    bool operator<(num_t other_x) const {
14        return x < other_x;
15    }
16 };
17 struct LineContainer : multiset<Line, less<>> { // max-query
18     // for doubles, use INF = 1 / 0.0
19     static const num_t INF = numeric_limits<num_t>::max();
20 };

```

```

21 num_t floor_div(num_t a, num_t b) {
22     return a / b - ((a ^ b) < 0 && a % b != 0);
23 }
24 bool isect(iterator u, iterator v) {
25     if (v == end()) {
26         u->x = INF;
27         return false;
28     }
29     if (u->a == v->a) u->x = (u->b > v->b ? INF : -INF);
30     else u->x = floor_div(v->b - u->b, u->a - v->a);
31     return u->x >= v->x;
32 }
33 void add(num_t a, num_t b) {
34     auto z = insert({a, b, 0}), y = z++, x = y;
35     while (isect(y, z)) z = erase(z);
36     if (x != begin() && isect(--x, y)) {
37         y = erase(y);
38         isect(x, y);
39     }
40     while ((y = x) != begin() && (--x)->x >= y->x) {
41         isect(x, erase(y));
42     }
43 }
44 num_t query(num_t x) {
45     assert(!empty());
46     auto it = *lower_bound(x);
47     return it.a * x + it.b;
48 }
49 };

```

3 Mathematics

3.1 Trigonometry

3.1.1 Sum - difference identities

$$\sin(u \pm v) = \sin(u) \cos(v) \pm \cos(u) \sin(v)$$

$$\cos(u \pm v) = \cos(u) \cos(v) \mp \sin(u) \sin(v)$$

$$\tan(u \pm v) = \frac{\tan(u) \pm \tan(v)}{1 \mp \tan(u) \tan(v)}$$

3.1.2 Sum to product identities

$$\cos(u) + \cos(v) = 2 \cos\left(\frac{u+v}{2}\right) \cos\left(\frac{u-v}{2}\right)$$

$$\cos(u) - \cos(v) = -2 \sin\left(\frac{u+v}{2}\right) \sin\left(\frac{u-v}{2}\right)$$

$$\sin(u) + \sin(v) = 2 \sin\left(\frac{u+v}{2}\right) \cos\left(\frac{u-v}{2}\right)$$

$$\sin(u) - \sin(v) = 2 \cos\left(\frac{u+v}{2}\right) \sin\left(\frac{u-v}{2}\right)$$

3.1.3 Product identities

$$\cos(u) \cos(v) = \frac{1}{2} [\cos(u+v) + \cos(u-v)]$$

$$\sin(u) \sin(v) = -\frac{1}{2} [\cos(u+v) - \cos(u-v)]$$

$$\sin(u) \cos(v) = \frac{1}{2} [\sin(u+v) + \sin(u-v)]$$

3.1.4 Double - triple angle identities

$$\sin(2u) = 2 \sin(u) \cos(u)$$

$$\cos(2u) = 2 \cos^2(u) - 1 = 1 - 2 \sin^2(u)$$

$$\tan(2u) = \frac{2 \tan(u)}{1 - \tan^2(u)}$$

$$\sin(3u) = 3 \sin(u) - 4 \sin^3(u)$$

$$\cos(3u) = 4 \cos^3(u) - 3 \cos(u)$$

$$\tan(3u) = \frac{3 \tan(u) - \tan^3(u)}{1 - 3 \tan^2(u)}$$

3.2 Sums

$$c^a + c^{a+1} + \dots + c^b = \frac{c^{b+1} - c^a}{c - 1}, c \neq 1$$

$$c + 2c^2 + \dots + nc^n = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c-1)^2}, c \neq 1$$

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \left(\frac{n(n+1)}{2} \right)^2$$

$$1^4 + 2^4 + 3^4 + \dots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

$$1^5 + 2^5 + 3^5 + \dots + n^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$$

$$1^6 + 2^6 + 3^6 + \dots + n^6 = \frac{n(n+1)(2n+1)(3n^4+6n^3-3n+1)}{42}$$

$$1^7 + 2^7 + 3^7 + \dots + n^7 = \frac{n^2(n+1)^2(3n^4+6n^3-n^2-4n+2)}{24}$$

4 String

4.1 Prefix function

```

1 /**
2  * Description: The prefix function of a string 's' is defined as an array pi
3  *   of length n,
4  *   where pi[i] is the length of the longest proper prefix of the substring
5  *   s[0..i] which is also a suffix of this substring.
6  * Time complexity: O(|S|).
7  */
8 vector<int> prefix_function(const string &s) {
9     int n = (int) s.length();
10    vector<int> pi(n);
11    pi[0] = 0;
12    for (int i = 1; i < n; ++i) {
13        int j = pi[i - 1]; // try length pi[i - 1] + 1.
14        while (j > 0 && s[j] != s[i]) {
15            j = pi[j - 1];
16        }
17        if (s[j] == s[i]) {
18            pi[i] = j + 1;
19        }
20    }
21    return pi;
22 }
```

4.2 Counting occurrences of each prefix

```

1 #include "prefix_function.h"
2 vector<int> count_occurrences(const string &s) {
3     vector<int> pi = prefix_function(s);
4     int n = (int) s.size();
5     vector<int> ans(n + 1);
6     for (int i = 0; i < n; ++i) {
7         ans[pi[i]]++;
8     }
9     for (int i = n - 1; i > 0; --i) {
10        ans[pi[i - 1]] += ans[i];
11    }
12    for (int i = 0; i <= n; ++i) {
13        ans[i]++;
14    }
15    return ans;
16    // Input: ABACABA
17    // Output: 4 2 2 1 1 1 1
18 }
```

4.3 Knuth–Morris–Pratt algorithm

```

1 /**
2  * Searching for a substring in a string.
3  * Time complexity: O(N + M).
4  */
5
6 #include "prefix_function.h"
7
8 vector<int> KMP(const string &text, const string &pattern) {
9     int n = (int) text.length();
```

```

10     int m = (int) pattern.length();
11     string s = pattern + '$' + text;
12     vector<int> pi = prefix_function(s);
13     vector<int> indices;
14     for (int i = 0; i < (int) s.length(); ++i) {
15         if (pi[i] == m) {
16             indices.push_back(i - 2 * m);
17         }
18     }
19     return indices;
20 }

```

4.4 Suffix array

```

1 struct SuffixArray {
2     string s;
3     int n, lim;
4     vector<int> sa, lcp, rank;
5     SuffixArray(const string &s, int _lim = 256) : s(_s), n(s.length() + 1),
6         lim(_lim), sa(n), lcp(n), rank(n) {
7         s += '$';
8         build();
9         kasai();
10        sa.erase(sa.begin());
11        lcp.erase(lcp.begin());
12        s.pop_back();
13    }
14    void build() {
15        vector<int> nrank(n), norder(n), cnt(max(n, lim));
16        for (int i = 0; i < n; ++i) {
17            sa[i] = i; rank[i] = s[i];
18        }
19        for (int k = 0, rank_cnt = 0; rank_cnt < n - 1; k = max(1, k * 2),
20            lim = rank_cnt + 1) {
21            // counting sort.
22            for (int i = 0; i < n; ++i) norder[i] = (sa[i] - k + n) % n;
23            for (int i = 0; i < n; ++i) cnt[rank[i]]++;
24            for (int i = 1; i < lim; ++i) cnt[i] += cnt[i - 1];
25            for (int i = n - 1; i >= 0; --i) sa[--cnt[rank[norder[i]]]] =
26                norder[i];
27            rank[sa[0]] = rank_cnt = 0;
28            for (int i = 1; i < n; ++i) {
29                int u = sa[i], v = sa[i - 1];
30                int nu = u + k, nv = v + k;
31                if (nu >= n) nu -= n;
32                if (nv >= n) nv -= n;
33                if (rank[u] != rank[v] || rank[nu] != rank[nv]) ++rank_cnt;
34                nrank[sa[i]] = rank_cnt;
35            }
36            for (int i = 0; i < rank_cnt + 1; ++i) cnt[i] = 0;
37            rank.swap(nrank);
38        }
39    }
40    void kasai() {

```

```

38     for (int i = 0; i < n; ++i) rank[sa[i]] = i;
39     for (int i = 0, k = 0; i < n - 1; ++i, k = max(0, k - 1)) {
40         int j = sa[rank[i] - 1];
41         while (s[i + k] == s[j + k]) k++;
42         lcp[rank[i]] = k;
43     }
44     // Note: lcp[i] = longest common prefix(sa[i - 1], sa[i]).
45 }
46 };

```

4.5 Suffix array slow

```

1 vector<int> suffix_array_slow(string &s) {
2     s += '$';
3     int n = (int) s.size();
4     vector<int> order(n), rank(n);
5     for (int i = 0; i < n; ++i) {
6         order[i] = i; rank[i] = s[i];
7     }
8     for (int k = 0; k < n; k = max(1, k * 2)) {
9         stable_sort(order.begin(), order.end(), [&](int i, int j) {
10             return make_pair(rank[i], rank[(i + k) % n]) < make_pair(rank[j],
11                 rank[(j + k) % n]);
12         });
13         vector<int> nrank(n);
14         for (int i = 0, cnt = 0; i < n; ++i) {
15             if (i > 0 && rank[order[i]] != rank[order[i - 1]]) ++cnt;
16             else if (i > 0 && rank[(order[i] + k) % n] != rank[(order[i - 1]
17                 + k) % n]) ++cnt;
18             nrank[order[i]] = cnt;
19         }
20         rank.swap(nrank);
21     }
22     s.pop_back(); order.erase(order.begin());
23     return order;
24 }
25 // Time complexity: O(N * log(N)^2).

```

4.6 Manacher's algorithm

```

1 /**
2  * Description: for each position, computes d[0][i] = half length of
3  * longest palindrome centered on i (rounded up), d[1][i] = half length of
4  * longest palindrome centered on i and i - 1.
5  * Time complexity: O(N).
6  * Tested: https://judge.yosupo.jp/problem/enumerate\_palindromes,
7  * stress-tested.
8  */
9 array<vector<int>, 2> manacher(const string &s) {
10     int n = (int) s.size();
11     array<vector<int>, 2> d;
12     for (int z = 0; z < 2; ++z) {
13         d[z].resize(n);
14         int l = 0, r = 0;
15         for (int i = 0; i < n; ++i) {

```

```

15         int mirror = l + r - i + z;
16         d[z][i] = (i > r ? 0 : min(d[z][mirror], r - i));
17         int L = i - d[z][i] - z, R = i + d[z][i];
18         while (L >= 0 && R < n && s[L] == s[R]) {
19             d[z][i]++; L--; R++;
20         }
21         if (R > r) {
22             l = L; r = R;
23         }
24     }
25 }
26 return d;
27 }

```

4.7 Trie

```

1 struct Trie {
2     const static int ALPHABET = 26;
3     const static char minChar = 'a';
4     struct Vertex {
5         int next[ALPHABET];
6         bool leaf;
7         Vertex() {
8             leaf = false;
9             fill(next, next + ALPHABET, -1);
10        }
11    };
12    vector<Vertex> trie;
13    Trie() { trie.emplace_back(); }
14
15    void insert(const string &s) {
16        int i = 0;
17        for (const char &ch : s) {
18            int j = ch - minChar;
19            if (trie[i].next[j] == -1) {
20                trie[i].next[j] = trie.size();
21                trie.emplace_back();
22            }
23            i = trie[i].next[j];
24        }
25        trie[i].leaf = true;
26    }
27    bool find(const string &s) {
28        int i = 0;
29        for (const char &ch : s) {
30            int j = ch - minChar;
31            if (trie[i].next[j] == -1) {
32                return false;
33            }
34            i = trie[i].next[j];
35        }
36        return (trie[i].leaf ? true : false);
37    }
38 };

```

4.8 Hashing

```

1 struct Hash61 {
2     static const uint64_t MOD = (1LL << 61) - 1;
3     static uint64_t BASE;
4     static vector<uint64_t> pw;
5     uint64_t addmod(uint64_t a, uint64_t b) const {
6         a += b;
7         if (a >= MOD) a -= MOD;
8         return a;
9     }
10    uint64_t submod(uint64_t a, uint64_t b) const {
11        a += MOD - b;
12        if (a >= MOD) a -= MOD;
13        return a;
14    }
15    uint64_t mulmod(uint64_t a, uint64_t b) const {
16        uint64_t low1 = (uint32_t) a, high1 = (a >> 32);
17        uint64_t low2 = (uint32_t) b, high2 = (b >> 32);
18
19        uint64_t low = low1 * low2;
20        uint64_t mid = low1 * high2 + low2 * high1;
21        uint64_t high = high1 * high2;
22
23        uint64_t ret = (low & MOD) + (low >> 61) + (high << 3) + (mid >> 29)
24        + (mid << 35 >> 3) + 1;
25        // ret %= MOD;
26        ret = (ret >> 61) + (ret & MOD);
27        ret = (ret >> 61) + (ret & MOD);
28        return ret - 1;
29    }
30    void ensure_pw(int m) {
31        int sz = (int) pw.size();
32        if (sz >= m) return;
33        pw.resize(m);
34        for (int i = sz; i < m; ++i) {
35            pw[i] = mulmod(pw[i - 1], BASE);
36        }
37    }
38
39    vector<uint64_t> pref;
40    int n;
41    template<typename T> Hash61(const T &s) { // strings or arrays.
42        n = (int) s.size();
43        ensure_pw(n);
44        pref.resize(n + 1);
45        pref[0] = 0;
46        for (int i = 0; i < n; ++i) {
47            pref[i + 1] = addmod(mulmod(pref[i], BASE), s[i]);
48        }
49    }
50    inline uint64_t operator()(const int from, const int to) const {
51        assert(0 <= from && from <= to && to < n);
52        // pref[to + 1] - pref[from] * pw[to - from + 1]

```

```

52         return submod(pref[to + 1], mulmod(pref[from], pw[to - from + 1]));
53     }
54 };
55 rng((unsigned int) chrono::steady_clock::now().time_since_epoch().count());
56 uint64_t Hash61::BASE = (MOD >> 2) + rng() % (MOD >> 1);
57 vector<uint64_t> Hash61::pw = vector<uint64_t>(1, 1);

```

4.9 Minimum rotation

```

1  /**
2   * Author: Stjepan Glavina
3   * License: Unlicense
4   * Source: https://github.com/stjepang/snippets/blob/master/min_rotation.cpp
5   * Description: Finds the lexicographically smallest rotation of a string.
6   * Time: O(N)
7   * Usage:
8   *   rotate(v.begin(), v.begin()+minRotation(v), v.end());
9   * Status: Stress-tested
10  */
11 #pragma once
12
13 int minRotation(string s) {
14     int a = 0, N = (int) s.size(); s += s;
15     rep(b, 0, N) rep(k, 0, N) {
16         if (a + k == b || s[a + k] < s[b + k]) {b += max(0, k - 1); break;}
17         if (s[a + k] > s[b + k]) {a = b; break;}
18     }
19     return a;
20 }

```

5 Numerical

5.1 Fast Fourier transform

```

1  const double PI = acos(-1);
2  using Comp = complex<double>;
3  int reverse_bit(int n, int lg) {
4      int res = 0;
5      for (int i = 0; i < lg; ++i) {
6          if (n & (1 << i)) {
7              res |= (1 << (lg - i - 1));
8          }
9      }
10     return res;
11 }
12 void fft(vector<Comp> &a, bool invert = false) {
13     int n = (int) a.size();
14     int lg = 0;
15     while (1 << (lg) < n) ++lg;
16     for (int i = 0; i < n; ++i) {
17         int rev_i = reverse_bit(i, lg);
18         if (i < rev_i) swap(a[i], a[rev_i]);
19     }
20     for (int len = 2; len <= n; len *= 2) {
21         double angle = 2 * PI / len * (invert ? -1 : 1);

```

```

22         Comp w_base(cos(angle), sin(angle));
23         for (int i = 0; i < n; i += len) {
24             Comp w(1);
25             for (int j = i; j < i + len / 2; ++j) {
26                 Comp u = a[j], v = a[j + len / 2];
27                 a[j] = u + w * v;
28                 a[j + len / 2] = u - w * v;
29                 w *= w_base;
30             }
31         }
32     }
33     if (invert) for (int i = 0; i < n; ++i) a[i] /= n;
34 }
35 vector<int> mult(vector<int> &a, vector<int> &b) {
36     vector<Comp> A(a.begin(), a.end()), B(b.begin(), b.end());
37     int n = (int) a.size(), m = (int) b.size(), p = 1;
38     while (p < n + m) p *= 2;
39     A.resize(p), B.resize(p);
40     fft(A, false);
41     fft(B, false);
42     for (int i = 0; i < p; ++i) {
43         A[i] *= B[i];
44     }
45     fft(A, true);
46     vector<int> res(n + m - 1);
47     for (int i = 0; i < n + m - 1; ++i) {
48         res[i] = (int) round(A[i].real());
49     }
50     return res;
51 }

```

6 Number Theory

6.1 Euler's totient function

- Euler's totient function, also known as **phi-function** $\phi(n)$ counts the number of integers between 1 and n inclusive, that are **coprime to** n .

- Properties:

- Divisor sum property: $\sum_{d|n} \phi(d) = n$.

- $\phi(n)$ is a **prime number** when $n = 3, 4, 6$.

- If p is a prime number, then $\phi(p) = p - 1$.

- If p is a prime number and $k \geq 1$, then $\phi(p^k) = p^k - p^{k-1}$.

- If a and b are **coprime**, then $\phi(ab) = \phi(a) \cdot \phi(b)$.

- In general, for **not coprime** a and b , with $d = \gcd(a, b)$ this equation holds:

$$\phi(ab) = \phi(a) \cdot \phi(b) \cdot \frac{d}{\phi(d)}.$$

– With $n = p_1^{k_1} \cdot p_2^{k_2} \cdots p_m^{k_m}$:

$$\begin{aligned}\phi(n) &= \phi(p_1^{k_1}) \cdot \phi(p_2^{k_2}) \cdots \phi(p_m^{k_m}) \\ &= n \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_m}\right)\end{aligned}$$

• Application in Euler's theorem:

– If $\gcd(a, M) = 1$, then:

$$a^{\phi(M)} \equiv 1 \pmod{M} \Rightarrow a^n \equiv a^{n \bmod \phi(M)} \pmod{M}$$

– In general, for arbitrary a, M and $n \geq \log_2 M$:

$$a^n \equiv a^{\phi(M) + [n \bmod \phi(M)]} \pmod{M}$$

6.2 Mobius function

• For a positive integer $n = p_1^{k_1} \cdot p_2^{k_2} \cdots p_m^{k_m}$:

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } \exists k_i > 1 \\ (-1)^m & \text{otherwise} \end{cases}$$

• Properties:

$$\sum_{d|n} \mu(d) = [n = 1].$$

– If a and b are **coprime**, then $\mu(ab) = \mu(a) \cdot \mu(b)$.

– Mobius inversion: let f and g be arithmetic functions:

$$g(n) = \sum_{d|n} f(d) \Leftrightarrow f(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) g(d)$$

6.3 Primes

Approximating the number of primes up to n :

n	$\pi(n)$	$\frac{n}{\ln n - 1}$
100 ($1e^2$)	25	28
500 ($5e^2$)	95	96
1000 ($1e^3$)	168	169
5000 ($5e^3$)	669	665
10000 ($1e^4$)	1229	1218
50000 ($5e^4$)	5133	5092
100000 ($1e^5$)	9592	9512
500000 ($5e^5$)	41538	41246
1000000 ($1e^6$)	78498	78030
5000000 ($5e^6$)	348513	346622

($\pi(n)$ = the number of primes less than or equal to n , $\frac{n}{\ln n - 1}$ is used to approximate $\pi(n)$).

6.4 Wilson's theorem

A positive integer n is a prime if and only if:

$$(n - 1)! \equiv n - 1 \pmod{n}$$

6.5 Zeckendorf's theorem

The Zeckendorf's theorem states that every positive integer n can be represented uniquely as a sum of one or more distinct non-consecutive Fibonacci numbers. For example:

$$64 = 55 + 8 + 1$$

$$85 = 55 + 21 + 8 + 1$$

```
1 vector<int> zeckendofth_theorem(int n) {
2     vector<int> fibs = {1, 1};
3     int sz = 2;
4     while (fibs.back() <= n) {
5         fibs.push_back(fibs[sz - 1] + fibs[sz - 2]);
6         sz++;
7     }
8     fibs.pop_back();
9     vector<int> nums;
10    int p = sz - 1;
11    while (n > 0) {
12        if (n >= fibs[p]) {
13            nums.push_back(fibs[p]);
14            n -= fibs[p];
15        }
16        p--;
17    }
18    return nums;
19 }
```

6.6 Bitwise operation

- $a + b = (a \oplus b) + 2(a \& b)$
- $a | b = (a \oplus b) + (a \& b)$
- $a \& (b \oplus c) = (a \& b) \oplus (a \& c)$
- $a | (b \& c) = (a | b) \& (a | c)$
- $a \& (b | c) = (a \& b) | (a \& c)$
- $a | (a \& b) = a$
- $a \& (a | b) = a$
- $n = 2^k \Leftrightarrow (n \& (n - 1)) = 0$
- $-a = \sim a + 1$
- $4i \oplus (4i + 1) \oplus (4i + 2) \oplus (4i + 3) = 0$
- Iterating over all subsets of a set and iterating over all submasks of a mask:

```

1 int n;
2 void mask_example() {
3     for (int mask = 0; mask < (1 << n); ++mask) {
4         for (int i = 0; i < n; ++i) {
5             if (mask & (1 << i)) {
6                 // do something...
7             }
8         }
9         // Time complexity: O(n * 2^n).
10    }
11    for (int mask = 0; mask < (1 << n); ++mask) {
12        for (int submask = mask; ; submask = (submask - 1) & mask) {
13            // do something...
14            if (submask == 0) break;
15        }
16        // Time complexity: O(3^n).
17    }
18 }

```

6.7 Pollard's rho algorithm

```

1 using num_t = long long;
2 const int PRIME_MAX = (int) 4e4; // for handle numbers <= 1e9.
3 const int LIMIT = (int) 1e9;
4 vector<int> primes;
5 int small_primes[] = {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 73, 113,
6     193, 311, 313, 407521, 299210837};
7 void linear_sieve(int n);
8 num_t mulmod(num_t a, num_t b, num_t mod);
9 num_t powmod(num_t a, num_t n, num_t mod);
10 bool miller_rabin(num_t a, num_t d, int s, num_t mod) {
11     num_t x = powmod(a, d, mod);
12     if (x == mod - 1 || x == 1) {
13         return true;
14     }
15     for (int i = 0; i < s - 1; ++i) {
16         x = mulmod(x, x, mod);
17         if (x == mod - 1) return true;
18     }
19     return false;
20 }
21 bool is_prime(num_t n, int tests = 10) {
22     if (n < 4) return (n > 1);

```

```

22     num_t d = n - 1;
23     int s = 0;
24     while (d % 2 == 0) { d >>= 1; s++; }
25     for (int i = 0; i < tests; ++i) {
26         int a = small_primes[i];
27         if (n == a) return true;
28         if (n % a == 0 || !miller_rabin(a, d, s, n)) return false;
29     }
30     return true;
31 }
32 num_t f(num_t x, int c, num_t mod) { // f(x) = (x^2 + c) % mod.
33     x = mulmod(x, x, mod);
34     x += c;
35     if (x >= mod) x -= mod;
36     return x;
37 }
38 num_t pollard_rho(num_t n, int c) {
39     // algorithm to find a random divisor of 'n'.
40     // using random function: f(x) = (x^2 + c) % n.
41     num_t x = 2, y = x, d;
42     long long p = 1;
43     int dist = 0;
44     while (true) {
45         y = f(y, c, n);
46         dist++;
47         d = __gcd(llabs(x - y), n);
48         if (d > 1) break;
49         if (dist == p) { dist = 0; p *= 2; x = y; }
50     }
51     return d;
52 }
53 void factorize(int n, vector<num_t> &factors);
54 void llfactorize(num_t n, vector<num_t> &factors) {
55     if (n < 2) return;
56     if (is_prime(n)) {
57         factors.emplace_back(n);
58         return;
59     }
60     if (n < LIMIT) {
61         factorize(n, factors);
62         return;
63     }
64     num_t d = n;
65     for (int c = 2; d == n; c++) {
66         d = pollard_rho(n, c);
67     }
68     llfactorize(d, factors);
69     llfactorize(n / d, factors);
70 }
71 vector<num_t> gen_divisors(vector<pair<num_t, int>> &factors) {
72     vector<num_t> divisors = {1};
73     for (auto &x : factors) {
74         int sz = (int) divisors.size();

```

```

75     for (int i = 0; i < sz; ++i) {
76         num_t cur = divisors[i];
77         for (int j = 0; j < x.second; ++j) {
78             cur *= x.first;
79             divisors.push_back(cur);
80         }
81     }
82 }
83 return divisors; // this array is NOT sorted yet.
84 }

```

6.8 Segment divisor sieve

```

1  const int MAXN = (int) 1e6; // R - L + 1 <= N.
2  int divisor_count[MAXN + 3];
3  void segment_divisor_sieve(long long L, long long R) {
4      for (long long i = 1; i <= (long long) sqrt(R); ++i) {
5          long long start1 = ((L + i - 1) / i) * i;
6          long long start2 = i * i;
7          long long j = max(start1, start2);
8          if (j == start2) {
9              divisor_count[j - L] += 1;
10             j += i;
11         }
12         for (; j <= R; j += i) {
13             divisor_count[j - L] += 2;
14         }
15     }
16 }

```

6.9 Linear sieve

```

1  /**
2   * Description: Finding primes and computing value for multiplicative
3   *              function in O(N)
4   */
5  const int N = (int) 1e6;
6  bool is_prime[N + 1];
7  int spf[N + 1]; // smallest prime factor
8  int phi[N + 1]; // euler's totient function
9  int mu[N + 1]; // mobius function
10 int func[N + 1]; // a multiplicative function, f(p^k) = k
11 int cnt[N + 1]; // cnt[i] = the power of the smallest prime factor of i
12 int pw[N + 1]; // pw[i] = p^cnt[i] where p is the smallest prime factor of i
13 vector<int> primes;
14
15 void sieve(int n = N) {
16     spf[0] = spf[1] = -1;
17     phi[1] = mu[1] = func[1] = 1;
18     for (int x = 2; x <= n; ++x) {
19         if (spf[x] == 0) {
20             primes.push_back(x);
21             is_prime[x] = true;
22             phi[x] = x - 1;

```

```

23         mu[x] = -1;
24         func[x] = 1;
25         cnt[x] = 1;
26         pw[x] = x;
27     }
28     for (int p : primes) {
29         if (p > spf[x] || x * p > n) break;
30         spf[x * p] = p;
31         if (p == spf[x]) {
32             phi[x * p] = phi[x] * p;
33             mu[x * p] = 0;
34             func[x * p] = func[x / pw[x]] * (cnt[x] + 1);
35             cnt[x * p] = cnt[x] + 1;
36             pw[x * p] = pw[x] * p;
37         }
38         else {
39             phi[x * p] = phi[x] * phi[p];
40             mu[x * p] = mu[x] * mu[p]; // or -mu[x]
41             func[x * p] = func[x] * func[p];
42             cnt[x * p] = 1;
43             pw[x * p] = p;
44         }
45     }
46 }
47 }

```

6.10 Bitset sieve

```

1  /**
2   * Description: sieve of eratosthenes for large n (up to 1e9).
3   * Time and space (tested on codeforces):
4   * + For n = 1e8: ~200 ms, 6 MB.
5   * + For n = 1e9: ~4000 ms, 60 MB.
6   */
7  const int N = (int) 1e8;
8  bitset<N / 2 + 1> isPrime;
9  void sieve(int n = N) {
10     isPrime.flip();
11     isPrime[0] = false;
12     for (int i = 3; i <= (int) sqrt(n); i += 2) {
13         if (isPrime[i >> 1]) {
14             for (int j = i * i; j <= n; j += 2 * i) {
15                 isPrime[j >> 1] = false;
16             }
17         }
18     }
19 }
20 void example(int n) {
21     sieve(n);
22     int primeCnt = (n >= 2);
23     for (int i = 3; i <= n; i += 2) {
24         if (isPrime[i >> 1]) {
25             primeCnt++;
26         }

```



```

27     }
28     cout << primeCnt << '\n';
29 }

```

6.11 Block sieve

```

1  /**
2   * Description: very fast sieve of eratosthenes for large n (up to 1e9).
3   * Source: kactl.
4   * Time and space (tested on codeforces):
5   *   + For n = 1e8: ~160 ms, 60 MB.
6   *   + For n = 1e9: ~1600 ms, 505 MB.
7   * Need to check memory limit.
8   */
9  const int N = (int) 1e8;
10 bitset<N + 1> is_prime;
11 vector<int> fast_sieve() {
12     const int S = (int) sqrt(N), R = N / 2;
13     vector<int> primes = {2};
14     vector<bool> sieve(S + 1, true);
15     vector<array<int, 2>> cp;
16     for (int i = 3; i <= S; i += 2) {
17         if (sieve[i]) {
18             cp.push_back({i, i * i / 2});
19             for (int j = i * i; j <= S; j += 2 * i) {
20                 sieve[j] = false;
21             }
22         }
23     }
24     for (int L = 1; L <= R; L += S) {
25         array<bool, S> block{};
26         for (auto &[p, idx] : cp) {
27             for (; idx < S + L; idx += p) block[idx - L] = true;
28         }
29         for (int i = 0; i < min(S, R - L); ++i) {
30             if (!block[i]) primes.push_back((L + i) * 2 + 1);
31         }
32     }
33     for (int p : primes) is_prime[p] = true;
34     return primes;
35 }

```

6.12 Combinatorics

6.12.1 Catalan numbers

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{n!(n+1)!}$$

$$C_{n+1} = \sum_{i=0}^n C_i C_{n-i}, \quad C_0 = 1, \quad C_n = \frac{4n-2}{n+1} C_{n-1}$$

- The first 12 Catalan numbers ($n = 0, 1, 2, \dots, 11$):

$$C_n = 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786$$

- Applications of Catalan numbers:

- difference binary search trees with n vertices from 1 to n .
- rooted binary trees with $n + 1$ leaves (vertices are not numbered).
- correct bracket sequence of length $2 * n$.
- permutation $[n]$ with no 3-term increasing subsequence (i.e. doesn't exist $i < j < k$ for which $a[i] < a[j] < a[k]$).
- ways a convex polygon of $n + 2$ sides can split into triangles by connecting vertices.

6.12.2 Fibonacci numbers

$$F_n = \begin{cases} 0, & \text{if } n = 0 \\ 1, & \text{if } n = 1 \\ F_{n-1} + F_{n-2}, & \text{otherwise} \end{cases}$$

- The first 20 Fibonacci numbers ($n = 0, 1, 2, \dots, 19$):

$$F_n = 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181$$

- Properties:

$$\left. \begin{aligned} F_{2n+1} &= F_n^2 + F_{n+1}^2 \\ F_{2n} &= F_{n-1} \cdot F_n + F_n \cdot F_{n+1} \\ F_{n+1} \cdot F_{n-1} - F_n^2 &= (-1)^n \end{aligned} \right| \begin{aligned} n \mid m &\iff F_n \mid F_m \\ (F_n, F_m) &= F_{(n,m)} \end{aligned}$$

6.12.3 Stirling numbers of the second kind

Partitions of n distinct elements into exactly k non-empty groups.

$$S(n, k) = S(n-1, k-1) + kS(n-1, k)$$

$$S(n, 1) = S(n, n) = 1$$

$$S(n, k) = \frac{1}{k!} \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} i^n$$

6.12.4 Derangements

Permutation of the elements of a set, such that no element appears in its original position (no fixed point). Recursive formulas:

$$D(n) = (n-1)[D(n-1) + D(n-2)] = nD(n-1) + (-1)^n$$

7 Geometry

7.1 Fundamentals

7.1.1 Point

```

1 #pragma once
2
3 const double PI = acos(-1);
4 const double EPS = 1e-9;
5 typedef double ftype;
6 struct Point {
7     ftype x, y;
8     Point(ftype _x = 0, ftype _y = 0): x(_x), y(_y) {}
9     Point& operator+=(const Point& other) {
10         x += other.x; y += other.y; return *this;
11     }
12     Point& operator-=(const Point& other) {
13         x -= other.x; y -= other.y; return *this;
14     }
15     Point& operator*=(ftype t) {
16         x *= t; y *= t; return *this;
17     }
18     Point& operator/=(ftype t) {
19         x /= t; y /= t; return *this;
20     }
21     Point operator+(const Point& other) const {
22         return Point(*this) += other;
23     }
24     Point operator-(const Point& other) const {
25         return Point(*this) -= other;
26     }
27     Point operator*(ftype t) const {
28         return Point(*this) *= t;
29     }
30     Point operator/(ftype t) const {
31         return Point(*this) /= t;
32     }
33     Point rotate(double angle) const {
34         return Point(x * cos(angle) - y * sin(angle), x * sin(angle) + y *
35             cos(angle));
36     }
37     friend istream& operator>>(istream &in, Point &t);
38     friend ostream& operator<<(ostream &out, const Point& t);
39     bool operator<(const Point& other) const {
40         if (fabs(x - other.x) < EPS)
41             return y < other.y;
42         return x < other.x;
43     };
44
45     istream& operator>>(istream &in, Point &t) {
46         in >> t.x >> t.y;
47         return in;
48     }

```

```

49     ostream& operator<<(ostream &out, const Point& t) {
50         out << t.x << ' ' << t.y;
51         return out;
52     }
53
54     ftype dot(Point a, Point b) {return a.x * b.x + a.y * b.y;}
55     ftype norm(Point a) {return dot(a, a);}
56     ftype abs(Point a) {return sqrt(norm(a));}
57     ftype angle(Point a, Point b) {return acos(dot(a, b) / (abs(a) * abs(b)));}
58     ftype proj(Point a, Point b) {return dot(a, b) / abs(b);}
59     ftype cross(Point a, Point b) {return a.x * b.y - a.y * b.x;}
60     bool ccw(Point a, Point b, Point c) {return cross(b - a, c - a) > EPS;}
61     bool collinear(Point a, Point b, Point c) {return fabs(cross(b - a, c - a)) <
62         EPS;}
63     Point intersect(Point a1, Point d1, Point a2, Point d2) {
64         double t = cross(a2 - a1, d2) / cross(d1, d2);
65         return a1 + d1 * t;
66     }

```

7.1.2 Line

```

1 #include "point.h"
2
3 struct Line {
4     double a, b, c;
5     Line(double _a = 0, double _b = 0, double _c = 0): a(_a), b(_b), c(_c) {}
6     friend ostream & operator<<(ostream& out, const Line& l);
7 };
8 ostream & operator<<(ostream& out, const Line& l) {
9     out << l.a << ' ' << l.b << ' ' << l.c;
10    return out;
11 }
12 void PointsToLine(const Point& p1, const Point& p2, Line& l) {
13     if (fabs(p1.x - p2.x) < EPS)
14         l = {1.0, 0.0, -p1.x};
15     else {
16         l.a = - (double)(p1.y - p2.y) / (p1.x - p2.x);
17         l.b = 1.0;
18         l.c = - l.a * p1.x - l.b * p1.y;
19     }
20 }
21 void PointsSlopeToLine(const Point& p, double m, Line& l) {
22     l.a = -m;
23     l.b = 1;
24     l.c = -l.a * p.x - l.b * p.y;
25 }
26 bool areParallel(const Line& l1, const Line& l2) {
27     return fabs(l1.a - l2.a) < EPS && fabs(l1.b - l2.b) < EPS;
28 }
29 bool areSame(const Line& l1, const Line& l2) {
30     return areParallel(l1, l2) && fabs(l1.c - l2.c) < EPS;
31 }
32 bool areIntersect(Line l1, Line l2, Point& p) {
33     if (areParallel(l1, l2)) return false;
34     p.x = - (l1.c * l2.b - l1.b * l2.c) / (l1.a * l2.b - l1.b * l2.a);

```

```

35     if (fabs(l1.b) > EPS) p.y = - (l1.c + l1.a * p.x);
36     else p.y = - (l2.c + l2.a * p.x);
37     return 1;
38 }
39 double distToLine(Point p, Point a, Point b, Point& c) {
40     double t = dot(p - a, b - a) / norm(b - a);
41     c = a + (b - a) * t;
42     return abs(c - p);
43 }
44 double distToSegment(Point p, Point a, Point b, Point& c) {
45     double t = dot(p - a, b - a) / norm(b - a);
46     if (t > 1.0)
47         c = Point(b.x, b.y);
48     else if (t < 0.0)
49         c = Point(a.x, a.y);
50     else
51         c = a + (b - a) * t;
52     return abs(c - p);
53 }
54 bool intersectTwoSegment(Point a, Point b, Point c, Point d) {
55     ftype ABxAC = cross(b - a, c - a);
56     ftype ABxAD = cross(b - a, d - a);
57     ftype CDxCA = cross(d - c, a - c);
58     ftype CDxCB = cross(d - c, b - c);
59     if (ABxAC == 0 || ABxAD == 0 || CDxCA == 0 || CDxCB == 0) {
60         if (ABxAC == 0 && dot(a - c, b - c) <= 0) return true;
61         if (ABxAD == 0 && dot(a - d, b - d) <= 0) return true;
62         if (CDxCA == 0 && dot(c - a, d - a) <= 0) return true;
63         if (CDxCB == 0 && dot(c - b, d - b) <= 0) return true;
64         return false;
65     }
66     return (ABxAC * ABxAD < 0 && CDxCA * CDxCB < 0);
67 }
68 void perpendicular(Line l1, Point p, Line& l2) {
69     if (fabs(l1.a) < EPS)
70         l2 = {1.0, 0.0, -p.x};
71     else {
72         l2.a = -l1.b / l1.a;
73         l2.b = 1.0;
74         l2.c = -l2.a * p.x - l2.b * p.y;
75     }
76 }

```

7.1.3 Circle

```

1  #include "point.h"
2
3  int insideCircle(const Point& p, const Point& center, ftype r) {
4      ftype d = norm(p - center);
5      ftype rSq = r * r;
6      return fabs(d - rSq) < EPS ? 0 : (d - rSq >= EPS ? 1 : -1);
7  }
8  bool circle2PointsR(const Point& p1, const Point& p2, ftype r, Point& c) {
9      double h = r * r - norm(p1 - p2) / 4.0;
10     if (fabs(h) < 0) return false;

```

```

11     h = sqrt(h);
12     Point perp = (p2 - p1).rotate(PI / 2.0);
13     Point m = (p1 + p2) / 2.0;
14     c = m + perp * (h / abs(perp));
15     return true;
16 }

```

7.1.4 Triangle

```

1  #include "point.h"
2  #include "line.h"
3
4  double areaTriangle(double ab, double bc, double ca) {
5      double p = (ab + bc + ca) / 2;
6      return sqrt(p) * sqrt(p - ab) * sqrt(p - bc) * sqrt(p - ca);
7  }
8  double rInCircle(double ab, double bc, double ca) {
9      double p = (ab + bc + ca) / 2;
10     return areaTriangle(ab, bc, ca) / p;
11 }
12 double rInCircle(Point a, Point b, Point c) {
13     return rInCircle(abs(a - b), abs(b - c), abs(c - a));
14 }
15 bool inCircle(Point p1, Point p2, Point p3, Point &ctr, double &r) {
16     r = rInCircle(p1, p2, p3);
17     if (fabs(r) < EPS) return false;
18     Line l1, l2;
19     double ratio = abs(p2 - p1) / abs(p3 - p1);
20     Point p = p2 + (p3 - p2) * (ratio / (1 + ratio));
21     PointsToLine(p1, p, l1);
22     ratio = abs(p1 - p2) / abs(p2 - p3);
23     p = p1 + (p3 - p1) * (ratio / (1 + ratio));
24     PointsToLine(p2, p, l2);
25     areIntersect(l1, l2, ctr);
26     return true;
27 }
28 double rCircumCircle(double ab, double bc, double ca) {
29     return ab * bc * ca / (4.0 * areaTriangle(ab, bc, ca));
30 }
31 double rCircumCircle(Point a, Point b, Point c) {
32     return rCircumCircle(abs(b - a), abs(c - b), abs(a - c));
33 }

```

7.1.5 Convex hull

```

1  #include "point.h"
2
3  vector<Point> CH_Andrew(vector<Point> &Pts) { // overall O(n log n)
4      int n = Pts.size(), k = 0;
5      vector<Point> H(2 * n);
6      sort(Pts.begin(), Pts.end());
7      for (int i = 0; i < n; ++i) {
8          while ((k >= 2) && !ccw(H[k - 2], H[k - 1], Pts[i])) --k;
9          H[k++] = Pts[i];
10     }
11     for (int i = n - 2, t = k + 1; i >= 0; --i) {

```

```

12     while ((k >= t) && !ccw(H[k - 2], H[k - 1], Pts[i])) --k;
13     H[k++] = Pts[i];
14 }
15 H.resize(k);
16 return H;
17 }

```

7.1.6 Polygon

```

1 #include "point.h"
2
3 double perimeter(const vector<Point> &P) {
4     double ans = 0.0;
5     for (int i = 0; i < (int)P.size() - 1; ++i)
6         ans += abs(P[i] - P[i + 1]);
7     return ans;
8 }
9 double area(const vector<Point> &P) {
10    double ans = 0.0;
11    for (int i = 0; i < (int)P.size() - 1; ++i)
12        ans += (P[i].x * P[i + 1].y - P[i + 1].x * P[i].y);
13    return fabs(ans) / 2.0;
14 }
15 bool isConvex(const vector<Point> &P) {
16    int n = (int)P.size();
17    if (n <= 3) return false;
18    bool firstTurn = ccw(P[0], P[1], P[2]);
19    for (int i = 1; i < n - 1; ++i)
20        if (ccw(P[i], P[i + 1], P[(i + 2) % n]) != firstTurn)
21            return false;
22    return true;
23 }
24 int insidePolygon(Point pt, const vector<Point> &P) {
25    int n = (int)P.size();
26    if (n <= 3) return -1;
27    bool on_polygon = false;
28    for (int i = 0; i < n - 1; ++i)
29        if (fabs(abs(P[i] - pt) + abs(pt - P[i + 1]) - abs(P[i] - P[i + 1]))
30            < EPS)
31            on_polygon = true;
32    if (on_polygon) return 0;
33    double sum = 0.0;
34    for (int i = 0; i < n - 1; ++i) {
35        if (ccw(pt, P[i], P[i + 1]))
36            sum += angle(P[i] - pt, P[i + 1] - pt);
37        else
38            sum -= angle(P[i] - pt, P[i + 1] - pt);
39    }
40    return fabs(sum) > PI ? 1 : -1;

```

7.2 Minimum enclosing circle

```

1 /**
2  * Description: computes the minimum Circle that encloses all the given
3  * Points.

```

```

3  */
4
5 #include "point.h"
6 // #include "circle.h"
7 // TODO:
8
9 Point center_from(double bx, double by, double cx, double cy) {
10     double B = bx * bx + by * by, C = cx * cx + cy * cy, D = bx * cy - by *
11     cx;
12     return Point((cy * B - by * C) / (2 * D), (bx * C - cx * B) / (2 * D));
13 }
14 Circle Circle_from(Point A, Point B, Point C) {
15     Point I = center_from(B.x - A.x, B.y - A.y, C.x - A.x, C.y - A.y);
16     return Circle(I + A, abs(I));
17 }
18
19 const int N = 100005;
20 int n, x[N], y[N];
21 Point a[N];
22
23 Circle emo_welzl(int n, vector<Point> T) {
24     if (T.size() == 3 || n == 0) {
25         if (T.size() == 0) return Circle(Point(0, 0), -1);
26         if (T.size() == 1) return Circle(T[0], 0);
27         if (T.size() == 2) return Circle((T[0] + T[1]) / 2, abs(T[0] - T[1])
28         / 2);
29         return Circle_from(T[0], T[1], T[2]);
30     }
31     random_shuffle(a + 1, a + n + 1);
32     Circle Result = emo_welzl(0, T);
33     for (int i = 1; i <= n; i++)
34         if (abs(Result.x - a[i]) > Result.y + 1e-9) {
35             T.push_back(a[i]);
36             Result = emo_welzl(i - 1, T);
37             T.pop_back();
38         }
39     return Result;

```

8 Linear algebra

8.1 Gauss elimination

```

1 const double EPS = 1e-9;
2 const int INF = 2; // it doesn't actually have to be infinity or a big number
3 int gauss (vector < vector<double> > a, vector<double> & ans) {
4     int n = (int) a.size();
5     int m = (int) a[0].size() - 1;
6     vector<int> where (m, -1);
7     for (int col=0, row=0; col<m && row<n; ++col) {
8         int sel = row;
9         for (int i=row; i<n; ++i) {
10            if (abs (a[i][col]) > abs (a[sel][col])) sel = i;

```

```

11     }
12     if (abs (a[sel][col]) < EPS) continue;
13     for (int i=col; i<=m; ++i) {
14         swap (a[sel][i], a[row][i]);
15     }
16     where[col] = row;
17
18     for (int i=0; i<n; ++i) {
19         if (i != row) {
20             double c = a[i][col] / a[row][col];
21             for (int j=col; j<=m; ++j) {
22                 a[i][j] -= a[row][j] * c;
23             }
24         }
25     }
26     ++row;
27 }
28 ans.assign (m, 0);
29 for (int i=0; i<m; ++i) {
30     if (where[i] != -1) {
31         ans[i] = a[where[i]][m] / a[where[i]][i];
32     }
33 }
34 for (int i=0; i<n; ++i) {
35     double sum = 0;
36     for (int j=0; j<m; ++j) {
37         sum += ans[j] * a[i][j];
38     }
39     if (abs (sum - a[i][m]) > EPS) return 0;
40 }
41 for (int i=0; i<m; ++i) {
42     if (where[i] == -1) return INF;
43 }
44 return 1;
45 }

```

9 Graph

9.1 Bellman-Ford algorithm

```

1 /**
2  * Description: single source shortest path in a weighted (negative or
3  *             positive) directed graph.
4  * Time: O(N * M).
5  * Tested: https://open.kattis.com/problems/shortestpath3
6  */
7 const int64_t INF = (int64_t) 2e18;
8 struct Edge {
9     int u, v; // u -> v
10    int64_t w;
11    Edge() {}
12    Edge(int _u, int _v, int64_t _w) : u(_u), v(_v), w(_w) {}
13 };
14 int n;

```

```

14 vector<Edge> edges;
15 vector<int64_t> bellmanFord(int s) {
16     // dist[stating] = 0.
17     // dist[u] = +INF, if u is unreachable.
18     // dist[u] = -INF, if there is a negative cycle on the path from s to u.
19     // -INF < dist[u] < +INF, otherwise.
20     vector<int64_t> dist(n, INF);
21     dist[s] = 0;
22     for (int i = 0; i < n - 1; ++i) {
23         bool any = false;
24         for (auto [u, v, w] : edges) {
25             if (dist[u] != INF && dist[v] > w + dist[u]) {
26                 dist[v] = w + dist[u];
27                 any = true;
28             }
29         }
30         if (!any) break;
31     }
32     // handle negative cycles
33     for (int i = 0; i < n - 1; ++i) {
34         for (auto [u, v, w] : edges) {
35             if (dist[u] != INF && dist[v] > w + dist[u]) {
36                 dist[v] = -INF;
37             }
38         }
39     }
40     return dist;
41 }

```

9.2 Articulation point and Bridge

```

1 /**
2  * Description: finding articulation points and bridges in a simple
3  *             undirected graph.
4  * Tested: https://oj.vnoi.info/problem/graph_
5  */
6 const int N = (int) 1e5;
7 vector<int> g[N];
8 int num[N], low[N], dfs_timer;
9 bool joint[N];
10 vector<pair<int, int>> bridges;
11 void dfs(int u, int prev) {
12     low[u] = num[u] = ++dfs_timer;
13     int child = 0;
14     for (int v : g[u]) {
15         if (v == prev) continue;
16         if (num[v]) low[u] = min(low[u], num[v]);
17         else {
18             dfs(v, u);
19             low[u] = min(low[u], low[v]);
20             child++;
21             if (low[v] >= num[v]) {
22                 bridges.emplace_back(u, v);
23             }
24         }
25     }
26     joint[u] = (child > 1);
27 }

```

```

23         if (u != prev && low[v] >= num[u]) joint[u] = true;
24     }
25 }
26 if (u == prev && child > 1) joint[u] = true;
27 }
28
29 int solve() {
30     int n, m;
31     cin >> n >> m;
32     for (int i = 0; i < m; ++i) {
33         int u, v;
34         cin >> u >> v;
35         u--; v--;
36         g[u].push_back(v);
37         g[v].push_back(u);
38     }
39     for (int i = 0; i < n; ++i) {
40         if (!num[i]) dfs(i, i);
41     }
42     return 0;
43 }

```

9.3 Strongly connected components

```

1 /**
2  * Description: Tarjan's algorithm finds strongly connected components
3  * in a directed graph. If vertices u and v belong to the same component,
4  * then scc_id[u] == scc_id[v].
5  * Tested: https://judge.yosupo.jp/problem/scc
6  */
7 const int N = (int) 5e5;
8 vector<int> g[N], st;
9 int low[N], num[N], dfs_timer, scc_id[N], scc;
10 bool used[N];
11 void Tarjan(int u) {
12     low[u] = num[u] = ++dfs_timer;
13     st.push_back(u);
14     for (int v : g[u]) {
15         if (used[v]) continue;
16         if (num[v] == 0) {
17             Tarjan(v);
18             low[u] = min(low[u], low[v]);
19         }
20         else {
21             low[u] = min(low[u], num[v]);
22         }
23     }
24     if (low[u] == num[u]) {
25         int v;
26         do {
27             v = st.back(); st.pop_back();
28             debug(u, v)
29             used[v] = true;
30             scc_id[v] = scc;

```

```

31     } while (v != u);
32     scc++;
33 }
34 }

```

9.4 Topo sort

```

1 /**
2  * Description: A topological sort of a directed acyclic graph
3  * is a linear ordering of its vertices such that for every directed edge
4  * from vertex u to vertex v, u comes before v in the ordering.
5  * Note: If there are cycles, the returned list will have size smaller than n
6  * (i.e, topo.size() < n).
7  * Tested: https://judge.yosupo.jp/problem/scc
8  */
9 vector<int> topo_sort(const vector<vector<int>> &g) {
10     int n = (int) g.size();
11     vector<int> indeg(n);
12     for (int u = 0; u < n; ++u) {
13         for (int v : g[u]) indeg[v]++;
14     }
15     queue<int> q; // Note: use min-heap to get the smallest lexicographical
16     // order.
17     for (int u = 0; u < n; ++u) {
18         if (indeg[u] == 0) q.emplace(u);
19     }
20     vector<int> topo;
21     while (!q.empty()) {
22         int u = q.front(); q.pop();
23         topo.emplace_back(u);
24         for (int v : g[u]) {
25             if (--indeg[v] == 0) q.emplace(v);
26         }
27     }
28     return topo;
29 }

```

9.5 K-th smallest shortest path

```

1 /** Description: Finding the k-th smallest shortest path from vertex s to
2  * vertex t,
3  * each vertex can be visited more than once.
4  */
5 using adj_list = vector<vector<pair<int, int>>>;
6 vector<long long> k_smallest(const adj_list &g, int k, int s, int t) {
7     int n = (int) g.size();
8     vector<long long> ans;
9     vector<int> cnt(n);
10     using pli = pair<long long, int>;
11     priority_queue<pli, vector<pli>, greater<pli>> pq;
12     pq.emplace(0, s);
13     while (!pq.empty() && cnt[t] < k) {
14         int u = pq.top().second;
15         long long d = pq.top().first;
16         pq.pop();

```

```

16         if (cnt[u] == k) continue;
17         cnt[u]++;
18         if (u == t) {
19             ans.push_back(d);
20         }
21         for (auto [v, cost] : g[u]) {
22             pq.emplace(d + cost, v);
23         }
24     }
25     assert(k == (int) ans.size());
26     return ans;
27 }

```

9.6 Eulerian path

9.6.1 Directed graph

```

1 /**
2  * Hierholzer's algorithm.
3  * Description: An Eulerian path in a directed graph is a path that visits
4  * all edges exactly once.
5  * An Eulerian cycle is a Eulerian path that is a cycle.
6  * Time complexity: O(|E|).
7  */
8 vector<int> find_path_directed(const vector<vector<int>> &g, int s) {
9     int n = (int) g.size();
10    vector<int> stack, cur_edge(n), vertices;
11    stack.push_back(s);
12    while (!stack.empty()) {
13        int u = stack.back();
14        stack.pop_back();
15        while (cur_edge[u] < (int) g[u].size()) {
16            stack.push_back(u);
17            u = g[u][cur_edge[u]++];
18        }
19        vertices.push_back(u);
20    }
21    reverse(vertices.begin(), vertices.end());
22    return vertices;
23 }

```

9.6.2 Undirected graph

```

1 /**
2  * Hierholzer's algorithm.
3  * Description: An Eulerian path in a undirected graph is a path that visits
4  * all edges exactly once.
5  * An Eulerian cycle is a Eulerian path that is a cycle.
6  * Time complexity: O(|E|).
7  */
8 struct Edge {
9     int to;
10    list<Edge>::iterator reverse_edge;
11    Edge(int _to) : to(_to) {}
12 };
13 vector<int> vertices;

```

```

13 void find_path(vector<list<Edge>> &g, int u) {
14     while (!g[u].empty()) {
15         int v = g[u].front().to;
16         g[v].erase(g[u].front().reverse_edge);
17         g[u].pop_front();
18         find_path(g, v);
19     }
20     vertices.emplace_back(u); // reversion list.
21 }
22 void add_edge(vector<list<Edge>> &g, int u, int v) {
23     g[u].emplace_front(v);
24     g[v].emplace_front(u);
25     g[u].front().reverse_edge = g[v].begin();
26     g[v].front().reverse_edge = g[u].begin();
27 }

```

10 Misc.

10.1 Ternary search

```

1 const double eps = 1e-9;
2 double f(double x) { return 42; }
3 double ternary_search_max(double l, double r) {
4     // find x0 such that: f(x0) > f(x), \all x: l <= x <= r.
5     while (r - l > eps) {
6         double mid1 = l + (r - l) / 3;
7         double mid2 = r - (r - l) / 3;
8         if (f(mid1) < f(mid2)) l = mid1;
9         else r = mid2;
10    }
11    return l;
12 }
13 double ternary_search_min(double l, double r) {
14     // find x0 such that: f(x0) < f(x), \all x: l <= x <= r.
15     while (r - l > eps) {
16         double mid1 = l + (r - l) / 3;
17         double mid2 = r - (r - l) / 3;
18         if (f(mid1) > f(mid2)) l = mid1;
19         else r = mid2;
20    }
21    return l;
22 }

```

10.2 Ternary search 2

```

1 /**
2  * Source: https://github.com/icpcvn/icpcvn.github.io/blob/master/2017/
3  * national/solutions/E/RR_ternary.cpp
4  */
5
6 #include <bits/stdc++.h>
7 using namespace std;
8 #include "../geometry/point.h"
9
10 Point A, B, C;

```

```

11 double f(Point P) {
12     return abs(P - A) + abs(P - B) + abs(P - C);
13 }
14 double f(double x, double& y) {
15     double l = -2000, r = 2000;
16     for (int i = 0; i < 500; ++i) {
17         double y1 = (l * 2 + r) / 3.0;
18         double y2 = (l + r * 2) / 3.0;
19
20         double f1 = f(Point(x, y1));
21         double f2 = f(Point(x, y2));
22
23         if (f1 < f2) r = y2;
24         else l = y1;
25     }
26     y = (l + r) / 2.0;
27     return f(Point(x, y));
28 }
29 int32_t main() {
30     ios::sync_with_stdio(0);
31     cin.tie(0);
32     cout << fixed << setprecision(9) << boolalpha;
33     cin >> A >> B >> C;
34
35     double l = -2000, r = 2000;
36     for (int i = 0; i < 500; ++i) {
37         double x1 = (l * 2 + r) / 3.0;
38         double x2 = (l + r * 2) / 3.0;
39
40         double y;
41         double f1 = f(x1, y);
42         double f2 = f(x2, y);
43
44         if (f1 < f2) r = x2;
45         else l = x1;
46     }
47     double y;
48     f(l, y);
49     cout << l << ' ' << y << '\n';
50     return 0;
51 }

```

10.3 Dutch flag national problem

```

1 void dutch_flag_national(vector<int> &arr) {
2     // All elements that are LESS than pivot are moved to the LEFT.
3     // All elements that are GREATER than pivot are moved to the RIGHT.
4     // E.g. [1, 2, 0, 0, 2, 2, 1], pivot = 1 -> [0, 0, 1, 1, 2, 2, 2].
5     int n = (int) arr.size();
6     int i = 0, j = 0, k = n - 1;
7     int pivot = 1;
8     // 0...i...j...k...n
9     while (j <= k) {
10         if (arr[j] < pivot) {

```

```

11         swap(arr[i], arr[j]);
12         i++;
13         j++;
14     }
15     else if (arr[j] > pivot) {
16         swap(arr[j], arr[k]);
17         k--;
18     }
19     else {
20         j++;
21     }
22 }
23 // 0 <= index <= i - 1: arr[index] < mid.
24 // i <= index <= k: arr[index] = mid.
25 // k + 1 <= index < sz: arr[index] > mid.
26 }

```

10.4 Matrix

```

1 using matrix_type = int;
2 const int MOD = (int) 1e9 + 7;
3 struct Matrix {
4     static const matrix_type INF = numeric_limits<matrix_type>::max();
5     int N, M;
6     vector<vector<matrix_type>> mat;
7
8     Matrix(int _N, int _M, matrix_type v = 0) : N(_N), M(_M) {
9         mat.assign(N, vector<matrix_type>(M, v));
10    }
11    static Matrix identity(int n) { // return identity matrix.
12        Matrix I(n, n);
13        for (int i = 0; i < n; ++i) {
14            I[i][i] = 1;
15        }
16        return I;
17    }
18
19    vector<matrix_type>& operator[](int r) { return mat[r]; }
20    const vector<matrix_type>& operator[](int r) const { return mat[r]; }
21
22    Matrix& operator*=(const Matrix &other) {
23        assert(M == other.N); // [N x M] [other.N x other.M]
24        Matrix res(N, other.M);
25        for (int r = 0; r < N; ++r) {
26            for (int c = 0; c < other.M; ++c) {
27                long long square_mod = (long long) MOD * MOD;
28                long long sum = 0;
29                for (int g = 0; g < M; ++g) {
30                    sum += (long long) mat[r][g] * other[g][c];
31                    if (sum >= square_mod) sum -= square_mod;
32                }
33                res[r][c] = sum % MOD;
34            }
35        }

```



```
36         mat.swap(res.mat); return *this;
37     }
38 };
```