CURVE SKETCHING

<u>Definition</u>. Let f be a continuous fn on an interval I.
We say that f is <u>concave up</u> on I when f is diff'ble on I and f' is increasing on I.
We say that f is <u>concave down</u> on I when f is diff'ble on I and f' is decreasing on I.

If f'' exists on I we can use it to determine the concavity of f as follows.

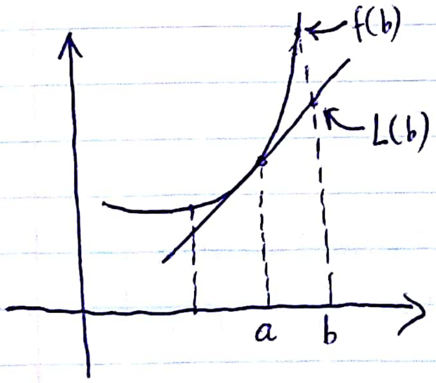
<u>Thm</u> (Concavity Test)
(a) If $f''(x) > 0$ for all $x \in I$, then the graph of f is concave up on I.
(b) If $f''(x) < 0$ for all $x \in I$, then the graph of f is concave down on I.

<u>Proof</u>. If $f''(x) > 0$ then we know that f' is increasing, so f is concave up on I. □

<u>Warning</u>. The converse is not true. For instance, $f(x) = x^4$ is concave up but $f''(0) = 0$.
It is useful to have a geometric picture of concavity.

<u>Thm</u> (Geometric meaning of concavity)
(a) f is concave up $\Leftrightarrow$ the graph of f <u>lies above</u> all its tangents on I.
(b) f is concave down $\Leftrightarrow$ the graph of f <u>lies below</u> all its tangents on I.

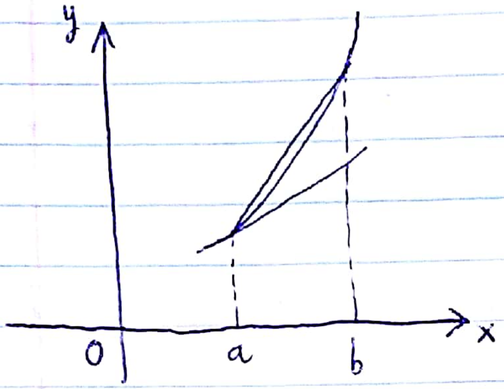
Here "lies above all its tangents on I" means the following:



for $a \in I$, let the tangent line to f at $(a, f(a))$ be
$$L(x) = f'(a)(x-a) + f(a).$$
Then for any $b \neq a$ (note that we don't assume $b > a$) we have
$$f(b) > L(b)$$
$$\cancel{\text{or}} \Rightarrow f(b) > f'(a)(b-a) + f(a)$$
$$\cancel{\text{or}} \Rightarrow \boxed{f'(a) < \frac{f(b)-f(a)}{b-a}}$$

<u>Proof</u>. We prove (a) and leave (b) as an exercise.
Note that this is an if and only if statement so we have to prove two directions.

(a) (⇒): We want to show f is concave up ⇒ its graph lies above its tangent lines.



Let $a \in I$ and $b \neq a$, then by the MVT there exists c between a and b s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Since f is concave up, we have f' is increasing. Assume that $a < c < b$. Then

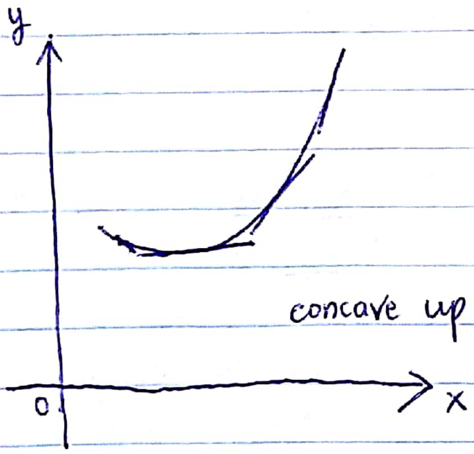
$$\frac{f(b) - f(a)}{b - a} = f'(c) > f'(a)$$

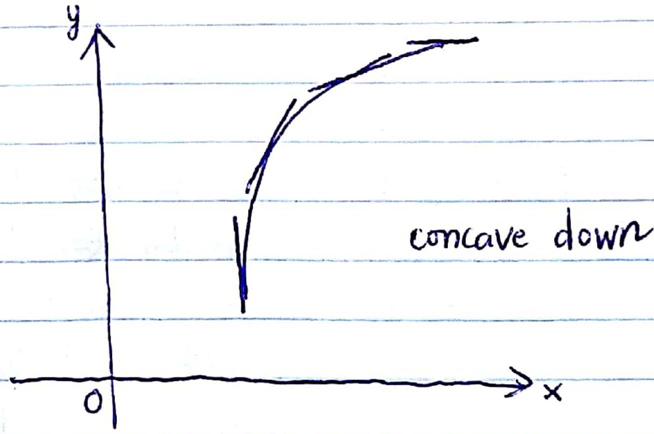
$$\Rightarrow \quad f(b) - f(a) > f'(a)(b-a)$$

$$\Rightarrow \quad f(b) > f(a) + f'(a)(b-a).$$

(⇐): if the graph of f lies above all its tangent lines, then f is concave up. You are required to prove this in Problem Set 5.

□



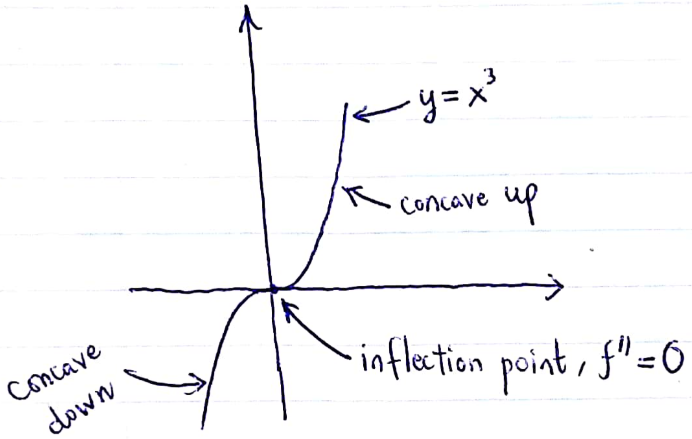


<u>Mnemonic</u>:

$f'' > 0$ (positive) ⇒ up (happy) ⇒ ☺ (smiley) ⇒ graph lies above tangents.
$f'' < 0$ (negative) ⇒ down (sad) ⇒ ☹ (frowney) ⇒ graph lies below tangents.

> <u>Definition</u>. A point P on a curve $y = f(x)$ is called an <u>inflection point</u> if f is continuous there and the curve changes from concave up to concave down ~~at P~~ or vice versa at P.

Fact: At an inflection point, $f'' = 0$ or undefined.
(This may not be so easy to prove as you think it is)



$$f(x) = \begin{cases} x^2, & x \geq 0, \\ -x^2, & x < 0 \end{cases}$$

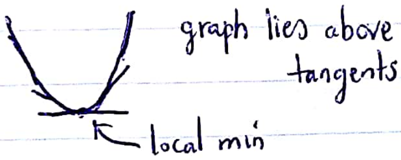
then
$$f'(x) = \begin{cases} 2x, & x \geq 0 \\ -2x, & x < 0 \end{cases}$$

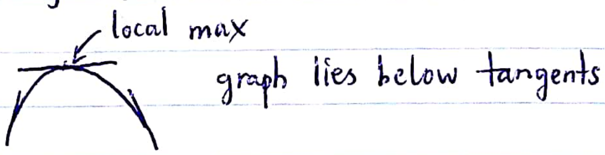
$$f''(x) = \begin{cases} 2, & x > 0 \\ -2, & x < 0 \end{cases}$$

In this case 0 is an inflection point and $f''(0)$ D.N.E.

> **Thm** (The second derivative test)
> Suppose $f'(a) = 0$. If $f''(a) > 0$, then f has a local min at a; if $f''(a) < 0$, then f has a local max at a.

This theorem is quite obvious from the geometric intuition. If $f''(a) > 0$, then the graph of f lies above its tangents, so f has a local min at a





Proof. We have

$$f''(a) = \lim_{h \to 0} \frac{f'(a+h) - f'(a)}{h} \overset{\text{since } f'(a)=0}{=} \lim_{h \to 0} \frac{f'(a+h)}{h}.$$

Suppose $f''(a) > 0$, then for sufficiently small h we have $\frac{f'(a+h)}{h} > 0$. This means that

$$f'(a+h) > 0 \quad \text{for sufficiently small } h > 0$$
$$f'(a+h) < 0 \quad \text{for sufficiently small } h < 0.$$

So we see that ~~f~~ f increases on the right of a and f decreases on the left of a, so f has a local min at a. The case when $f''(a) < 0$ is similar. $\square$

Warning: At a local extremum, f'' can still be zero. For instance, look at $f(x) = x^4$.

E.g. Sketch the curve $y = x^4 - 4x^3$.

Soln. Let $f(x) = x^4 - 4x^3$. Domain: $\mathbb{R}$

We have

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$
$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

Now $f'(x) = 0 \Leftrightarrow x = 0$ or $x = 3$. So we have two critical points.

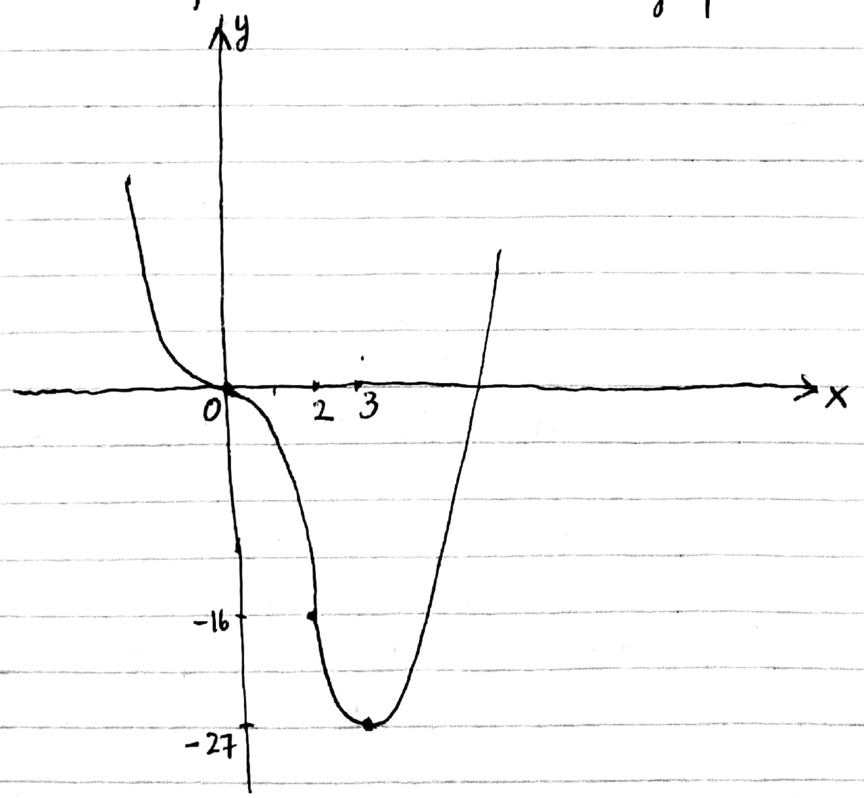
x	$-\infty$		0		3		$+\infty$
$x-3$		$-$		$-$	0	$+$	
$f'(x)$		$-$	0	$-$	0	$+$	
$f(x)$	$+\infty$	$\searrow$			$f(3)$	$\nearrow$	$+\infty$

So $f(3) = -27$ is a local minimum. We have

$$f''(x) = 0 \Leftrightarrow x = 0 \text{ or } x = 2$$

x	$-\infty$		0		2		$+\infty$
$f''(x)$		$+$	0	$-$	0	$+$	
$f(x)$		concave up	infl point	concave down	infl point	concave up	

With these info we can sketch the graph



E.g. Sketch the graph of the function $f(x) = x^{2/3}(6-x)^{1/3}$.

Soln. Domain: $\mathbb{R}$

$\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = +\infty$.

We have

$$f'(x) = \frac{2}{3}x^{-1/3}(6-x)^{1/3} - \frac{1}{3}x^{2/3}(6-x)^{-2/3} = \frac{2(6-x)^{1/3}}{3x^{1/3}} - \frac{x^{2/3}}{3(6-x)^{2/3}}$$

$$= \frac{4-x}{x^{1/3}(6-x)^{2/3}}$$

$$f''(x) = \frac{-x^{1/3}(6-x)^{2/3} - (4-x)\left(\frac{1}{3}x^{-2/3}(6-x)^{2/3} - \frac{2}{3}x^{1/3}(6-x)^{-1/3}\right)}{x^{2/3}(6-x)^{4/3}}$$

$$= \frac{-8}{x^{4/3}(6-x)^{5/3}}.$$

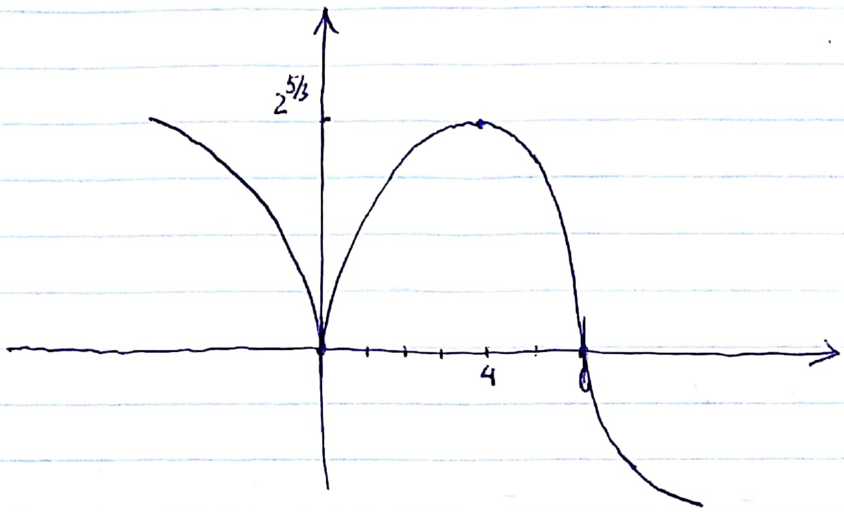
When $x=4$, $f'(x)=0$, when $x=0$ or 6, $f'(x)$ D.N.E.

x	$-\infty$		0		4		6		$+\infty$
$4-x$		$+$	|	$+$	0	$-$	|	$-$	
$x^{1/3}$		$-$	|	$+$	|	$+$	|	$+$	
$f'(x)$		$-$	∞	$+$	0	$-$	∞	$-$	
$f(x)$	$+\infty$	$\searrow$	0	$\nearrow$	$2^{5/3}$	$\searrow$	0	$\searrow$	$-\infty$

So f decreases on $(-\infty, 0] \cup [4, +\infty)$ and increases on $[0,4]$. It has a local min at 0 and a local max at 4.

x	$-\infty$		0		6		$+\infty$
$(6-x)^{5/3}$		$+$	|	$+$	0	$-$	
$f''(x)$		$-$	∞	$-$	∞	$+$	
$f(x)$		concave down			infl. point	concave up	

Now we can sketch the graph.



SUMMARY OF CURVE SKETCHING.

Generally to sketch a curve we pay attention to the following features:

1. <u>Domain</u> : where the fn is defined
2. <u>Intercepts</u> : find x-intercepts and y-intercepts
3. <u>Symmetry</u>: even, odd, periodic fn
4. <u>Asymptotes</u>: horizontal asymptotes, vertical asymptotes, slant asymptotes.
5. <u>Intervals of increase/decrease</u> : look at the sign of f'
6. <u>Local max / local min</u> : find the crit pts of f'.
7. <u>Concavity and points of inflection</u> : investigate f''.

<u>Example</u>. Sketch the curve $y = \dfrac{2x^2}{x^2-1}$

<u>Soln</u>. Domain: $D = \mathbb{R} \setminus \{\pm 1\} = (-\infty,-1) \cup (-1,1) \cup (1,+\infty)$.

Intercepts: x-intercept: 0 , y-intercept: 0.

Since $f(-x) = f(x)$, f is even, so it is symmetric about the y-axis.

$\lim\limits_{x\to\pm\infty} \dfrac{2x^2}{x^2-1} = 2 \Rightarrow$ horizontal asymptote: $y = 2$.

$\lim\limits_{x\to 1^+} \dfrac{2x^2}{x^2-1} = \infty$, $\lim\limits_{x\to 1^-} \dfrac{2x^2}{x^2-1} = -\infty \Rightarrow$ vertical asymptotes: $x = \pm 1$

$$f'(x) = \frac{4x(x^2-1) - 2x(2x^2)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

So we have one critical point $x = 0$.

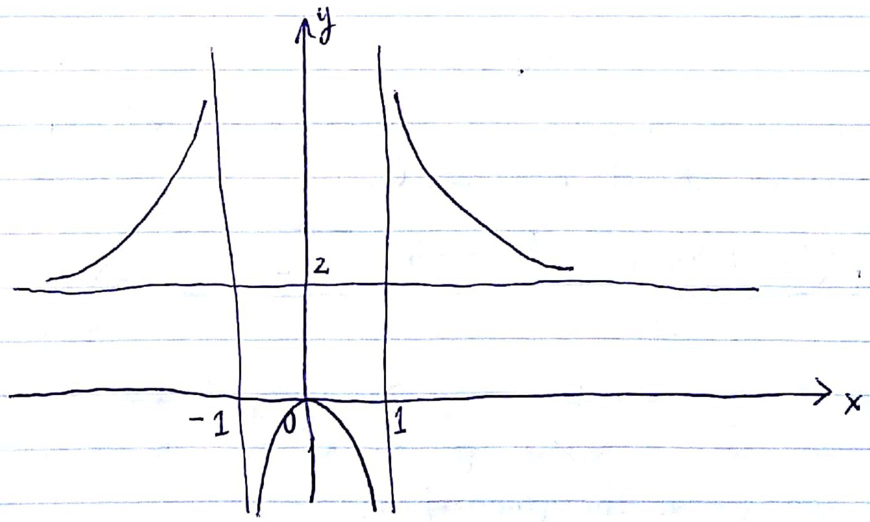
x	$-\infty$		-1		0		1		$+\infty$
$f'(x)$		+	∞	+	0	-	∞	-	
$f(x)$	2	↗	$+\infty$ ‖ $-\infty$	↗	0	↘	$-\infty$ ‖ $+\infty$	↘	2

$$f''(x) = \frac{12x^2 + 4}{(x^2-1)^3}$$

Here $f''(x)$ is undefined at ± 1, but since $\pm 1 \notin$ domain, they are not inflection points

x	$-\infty$		-1		1		$+\infty$
$f''(x)$		+	∞	-	∞	+	
$f(x)$		concave up	‖	concave down	‖	concave up	

Now we can sketch the graph



Example. Sketch the graph of $f(x) = \dfrac{x^2}{\sqrt{x+1}}$

Soln. Domain: $(-1, \infty)$

Intercept: x-intercept 0, y-intercept 0.

$$\lim_{x\to\infty} \frac{x^2}{\sqrt{x+1}} = \infty \Rightarrow \text{no horizontal asymptote}$$

$$\lim_{x\to\infty} \frac{x^2}{\sqrt{x+1}} = \infty \Rightarrow \text{vertical asymptote}: x = -1.$$

We have

$$f'(x) = \frac{2x\sqrt{x+1} - \frac{x^2}{2\sqrt{x+1}}}{x+1} = \frac{3x^2+4x}{2(x+1)^{3/2}} = \frac{x(3x+4)}{2(x+1)^{3/2}}$$

Note that $f'(x) = 0$ when $x = 0$ ($x = -\frac{4}{3}$ does not lie in the domain)

So we have only one critical point $x = 0$.

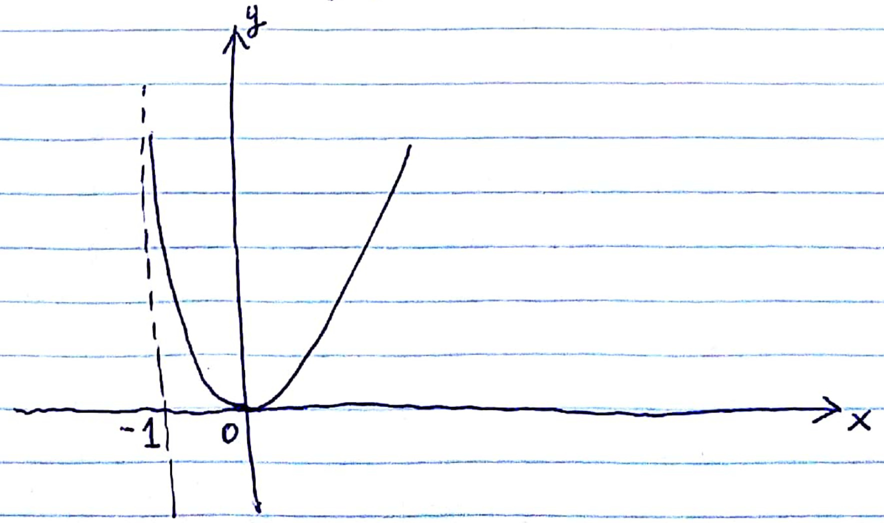
x	-1		0		∞
$f'(x)$		$-$	0	$+$	
$f(x)$	∞	$\searrow$	0	$\nearrow$	∞

So f decreases on $(-1, 0]$ and increases on $[0, \infty)$ and has a local min (in fact an absolute min) at 0.

$$f''(x) = \frac{1}{2}\,\frac{(6x+4)(x+1)^{3/2} - \frac{3}{2}(3x^2+4x)\sqrt{x+1}}{(x+1)^3} = \frac{3x^2+8x+8}{4(x+1)^{5/2}}.$$

Notice that the numerator is always >0 (why?) and the denominator is >0 since $x > -1$. So $f''(x) > 0$ for all $x > -1$. So f is concave up on $(-1, \infty)$

Now we can sketch the graph of f.



Example. Sketch the graph of $f(x) = \frac{\cos x}{2+\sin x}$.

Soln. Domain: $\mathbb{R}$ because $2+\sin x > 0 \ \forall x \in \mathbb{R}$.

Intercepts: $f(x) = 0 \Leftrightarrow \cos x = 0 \Leftrightarrow x = \frac{\pi}{2} + n\pi, \ n \in \mathbb{Z}$

$f(0) = 1/2$

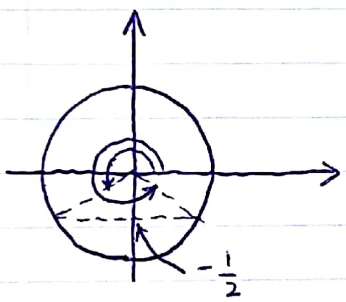
f is periodic with period 2π, so it suffices to restrict the domain to $0 \leq x \leq 2\pi$.

Asymptotes: none

We have

$$f'(x) = \frac{-\sin x(2+\sin x) - \cos x \cos x}{(2+\sin x)^2} = \frac{-2\sin x - 1}{(2+\sin x)^2} = -\frac{2\sin x + 1}{(2+\sin x)^2}$$

For $x \in [0, 2\pi]$, we see that $f'(x) = 0 \Leftrightarrow \sin x = \frac{-1}{2} \Leftrightarrow x = \frac{7\pi}{6}$ or $x = \frac{11\pi}{6}$.



x	0		$\frac{7\pi}{6}$		$\frac{11\pi}{6}$		2π
$f'(x)$	$-\frac{1}{4}$	$-$	0	$+$	0	$-$	$-\frac{1}{4}$
$f(x)$		$\searrow$	$f(\frac{7\pi}{6})$	$\nearrow$	$f(\frac{11\pi}{6})$	$\searrow$	

So f decreases on $[0, \frac{7\pi}{6}] \cup [\frac{11\pi}{6}, 2\pi]$ and increases on $[\frac{7\pi}{6}, \frac{11\pi}{6}]$. It has a local min at $\frac{7\pi}{6}$ and a local max at $\frac{11\pi}{6}$.

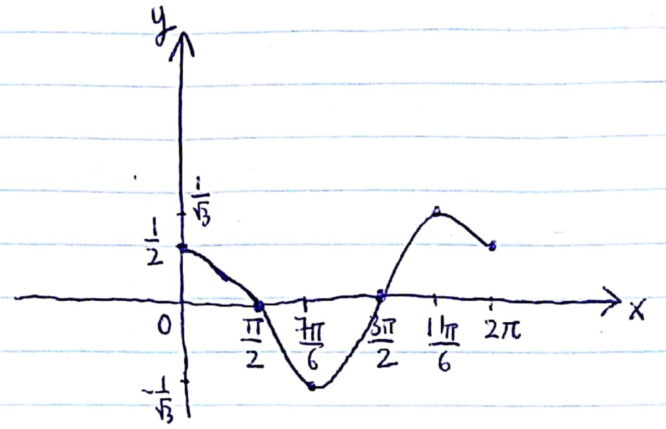
$$f''(x) = -\frac{2\cos x(2+\sin x)^2 - 2(2+\sin x)\cos x(2\sin x + 1)}{(2+\sin x)^4} = -\frac{2\cos x(1-\sin x)}{(2+\sin x)^3}$$

Because $1 - \sin x \geq 0$ and $2+\sin x > 0$ the sign of $f''(x)$ only depends on $\cos x$. So we get

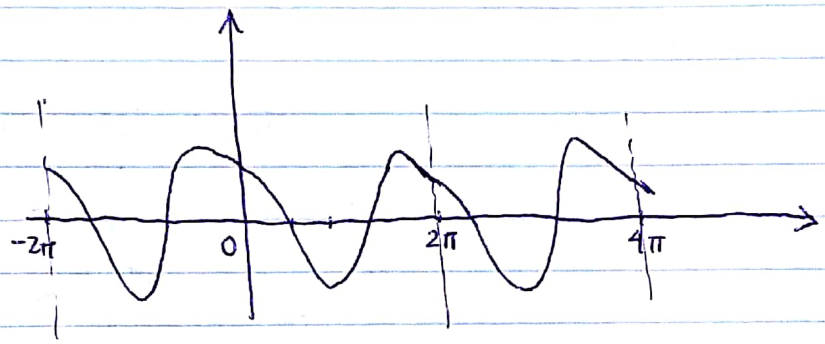
x	0		$\frac{\pi}{2}$		$\frac{3\pi}{2}$		2π
$f''(x)$		$-$	0	$+$	0	$-$	
$f(x)$		concave down	infl point	concave up	infl point	concave down	

Now we can sketch the graph of f. Notice the special points

$f(0) = 1/2$, $f(2\pi) = 1/2$, $f(\frac{\pi}{2}) = 0$, $f(\frac{3\pi}{2}) = 0$, $f(\frac{7\pi}{6}) = \frac{-1}{\sqrt{3}}$, $f(\frac{11\pi}{6}) = \frac{1}{\sqrt{3}}$.



to get the full graph of f we extend the above graph periodically.



Slant Asymptotes:

Definition. The line $y = mx + b$ is called a slant asymptote of f if

$$\lim_{x\to\infty}\left[f(x) - (mx+b)\right] = 0.$$

When $m = 0$ we obtain a horizontal asymptote.

E.g. Sketch the graph of $f(x) = \dfrac{x^3}{x^2+1}$.

Domain: $D = \mathbb{R}$.

intercept: x-intercept: 0, y-intercept: 0.

Since $f(-x) = -f(x)$, f is odd $\Rightarrow$ its graph is symmetric about the origin

We have

$$\frac{x^3}{x^2+1} = \frac{x(x^2+1) - x}{x^2+1} = x - \frac{x}{x^2+1}$$

Now

$$\lim_{x\to\infty}\left[f(x) - x\right] = \lim_{x\to\infty}\left(x - \frac{x}{x^2+1} - x\right) = \lim_{x\to\infty}\left(\frac{-x}{x^2+1}\right) = 0.$$

So the line $y = x$ is a slant asymptote.

We have

$$\cancel{\text{tho}}\ f'(x) = \frac{3x^2(x^2+1) - 2x\,x^3}{(x^2+1)^2} = \frac{x^4+3x^2}{(x^2+1)^2} = \frac{x^2(x^2+3)}{(x^2+1)^2}$$

We only have one critical point $x = 0$, however since $f'(x) \geq 0 \ \forall x \in \mathbb{R}$ it is neither a local max nor local min, and f is increasing on $\mathbb{R}$.

$$f''(x) = \frac{(4x^3+6x)(x^2+1)^2 - 2(x^2+1)2x(x^4+3x^2)}{(x^2+1)^2}$$

$$= \frac{2x(3-x^2)}{(x^2+1)^3}.$$

We have $f''(x) = 0 \Leftrightarrow x = 0$ or $x = \pm\sqrt{3}$.

x	$-\infty$		$-\sqrt{3}$		0		$\sqrt{3}$		∞
x		$-$	|	$-$	0	$+$	|	$+$	
$3-x^2$		$-$	0	$+$	|	$+$	0	$-$	
$f''(x)$		$+$	0	$-$	0	$+$	0	$-$	
$f(x)$		con. up	infl. point	con. down	infl. point	con. up	infl. point	con. down	

Now we can sketch the graph. Note that $f(0)=0$, $f(\sqrt{3}) = \frac{3\sqrt{3}}{4}$, $f(-\sqrt{3}) = \frac{-3\sqrt{3}}{4}$