Prove transitive closure is transitive. (Huanyu Li)

For each $(x, y), (y, z) \in \bigcup_{n \in Z^+} R^n$.

there exists a i and a j satisfy that $(x, y) \in R^i$ and $(y, z) \in R^j$

Because $\bigcup_{n \in Z^+} R^n = R \cup R^2 \cup R^3 \cup \cdots \cup R^i \cup \cdots \cup R^j \cup \cdots \cup R^n$

The we can get $R^i \circ R^j = \{(x, z)\}$ based on the composition definition.

since $R^{n+1} = R^n \circ R$ $(n \in N)$

Then we can get

$$
\begin{aligned}
R^{i+j} &= R^{i+j-1} \circ R \\
&= R^{i+j-2} \circ R \circ R \\
&\ \ \vdots \\
&= R^{i+j-(i+j-1)} \circ \underbrace{R \circ \cdots \circ R \circ R}_{i+j-1} \\
&= \underbrace{R \circ R \circ \cdots \circ R \circ R}_{i+j}
\end{aligned}
$$

With the similar step. we can get

$$R^i = \underbrace{R \circ R \circ \cdots \circ R \circ R}_{i} \qquad R^j = \underbrace{R \circ R \circ \cdots \circ R \circ R}_{j}$$

Then. $R^i \circ R^j = \underbrace{R \circ R \circ \cdots \circ R \circ R}_{i+j} = R^{i+j}$

$i+j \in Z^+$ $\therefore (x, z) \in \bigcup_{n \in Z^+} R^n$

No we can conclude. that for each $(x, y), (y, z) \in \bigcup_{n \in Z^+} R^n$ it has $(x, z) \in \bigcup_{n \in Z^+} R^n$ so R^+ is transitive.