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1.3.

To prove preorder $(A, \leq)$ and let $x \equiv y$ iff $x \leq y$ and $y \leq x$ is an equivalence relation.

we need to prove it is reflexive, symmetric and transitive.

Prove:

① for each $x \in A$. we know that

$x \leq x$ and $x \leq x$ then we get $x \equiv x$

so $(A, \overset{\leq}{\equiv})$ is reflexive.

② for each $(x, y) \in (A, \overset{\leq}{\equiv})$

since $x \equiv y$ we can get $x \leq y$ and $y \leq x$.

as well as, it's obvious $y \equiv x$. $\therefore (y, x) \in (A, \equiv)$

so $(A, \overset{\leq}{\equiv})$ is symmetric.

③ for each pair $(x, y), (y, z) \in (A, \overset{\leq}{\equiv})$

we know that.

$x \equiv y$ and $y \equiv z$

$\therefore x \leq y$ and $y \leq x$

$y \leq z$ and $z \leq y$

we can get $x \leq z$ and $z \leq x$

$\therefore x \equiv z$, $(x, z) \in (A, \overset{\leq}{\equiv})$

$\therefore (A, \overset{\leq}{\equiv})$ is an equivalence relation.

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relation B: To prove $[x] \leq_= [y]$ iff $x \leq y$ is a partial order.

we need to prove it is reflexive, antisymmetric and transitive.

Prove: for each $x \in B$ $x \leq x$ exists.

so $[x] \leq_= [x]$

this relation is reflexive.

for each pairs $(x, y), (y, x) \in B$.

it's $[x] \leq_= [y] \Rightarrow x \leq y$

$[y] \leq_= [x] \Rightarrow y \leq x$

$\therefore x = y$

$\therefore$ This relation is antisymmetric.

for each pairs $(x, y), (y, z) \in B$.

$[x] \leq_= [y] \Rightarrow x \leq y$

$[y] \leq_= [z] \Rightarrow y \leq z$

$\therefore y \leq z$.

$\therefore [y] \leq_= [z]$

$\therefore (y, z) \in B$

$\therefore$ This relation is transitive. As a result, it is a poset

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To prove $[x] \leq_{\equiv} [y]$ iff $x \leq y$ is well defined. (which is not ambiguous). We need to make sure $[x] = [x']$ and $[y] = [y']$ then $[x] \leq_{\equiv} [y]$ iff $[x'] \leq_{\equiv} [y']$

Prove: $([x] \leq_{\equiv} [y] \Rightarrow [x'] \leq [y'])$.

Based on the definition. we know that

$$[x] \leq_{\equiv} [y] \Rightarrow \underbrace{x \leq y}_{②}$$

let set A stand for equivalence class of x and x'

$[x] = [x'] = A$.

let set B stand for equivalence class of y and y'

for each $a \in A$ and $b \in B$:

Based on the relation $(A \leq_{\equiv})$

$(x, a) \in (A, \leq_{\equiv}) \Rightarrow x \equiv a \Rightarrow x \leq a$ and $\underbrace{a \leq x}_{①}$

$(y, b) \in (A, \leq_{\equiv}) \Rightarrow y \equiv b \Rightarrow \underbrace{y \leq b}_{③}$ and $b \leq y$

Based on ①, ②, ③ we get $a \leq x \leq y \leq b$ $\therefore \underbrace{a \leq b}_{④}$.

Based on previous prove, we know that $(A, \leq_{\equiv})$ is reflexive.

Which means $(x', x') \in (A, \leq_{\equiv})$ $(y', y') \in (A, \leq_{\equiv})$

$\therefore x' \in [x'] = A$ $y' \in [y'] = B$.

We already get that for each $a \in A$. and $b \in B$.

$a \leq B$ $\therefore x' \leq y'$

Based on definition $[x] \leq_{\equiv} [y]$ iff $x \leq y$.

we get $[x'] \leq_{\equiv} [y']$ } So $[x] \leq_{\equiv} [y] \Rightarrow [x'] \leq [y']$ is true
it's obvious, $[x'] \leq [y'] \Rightarrow [x] \leq_{\equiv} [y]$ is true