Conformal Field Theory

知乎

杂货铺流形、

2024.6.25

version: 1.0

共形场论

Reference

1. David Simmons Duffin: TASI Lectures: 1602.07982,
   PHYS 229ab
2. Slava Rychkov: 1601.05000
3. Joao Penedones: 1608.04948 (AdS/CFT)
4. Agnese Bissi, Aninda Sinha, Xinan Zhou: 2202.08475 (2d CFT)

1. Introduction

QFT: A universal language

- Particle physics (SM & beyond)
- Formal HEP (string theory)
- Statistical physics
- Condensed matter
- Cosmology

Spacetime $\begin{cases}\text{Homogeneous}\\ \text{Isotropic}\end{cases}$

Poincare symmetry $\begin{cases}\text{Lorentz}\\ \text{Rotation}\\ \text{Translation}\end{cases}$ Isomorphic $SO(d-1,1)$ Rotation invariance $SO(d)$ theory

1 time

$d-1$ space

d space

These lectures

enlarged spacetime symmetry

Poincare $\longrightarrow$ Conformal symmetry $\begin{cases}\text{Dilatation} \quad x^\mu \to \lambda x^\mu\\ \text{Special conformal transformation}\end{cases}$ $\to$ conformal field theory

杂诗赋流形。

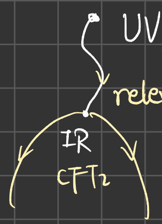
2024.6.25

In all known cases

Poincare + Dilatation $\longrightarrow$ Conformal $\leftarrow$ This is not a theorem

Where CFT arises?

1. QFT: Study of renormalization group flows



- A theory with a mass gap (Yang-Mills in 4d)
- A theory with massless particles (QED_4)
- A scale invariant theory with a continuous spectrum.
  $\to$ Described by a CFT.

2. very different microscopic descriptions (Many examples in statistical mechanism)

E.g. 3D Ising model

$$Z_{Ising} = \sum_{\{S_i\}} \exp\Big(-J \sum_{\langle ij \rangle} S_i S_j\Big) \qquad S_i = \pm 1$$

For special J $\to$ correlation length $= \infty$

CFT

Same as ϕ^4

$$S = \int d^dx \ \tfrac{1}{2}(\partial\phi)^2 + \tfrac{1}{2}m^2\phi^2 + \tfrac{1}{4!}g\phi^4 \qquad \text{critical universality}$$

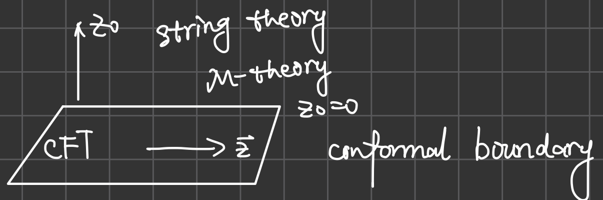
3. $\underbrace{\text{Anti de sitter}}_{\text{AdS}}$ / CFT correspondence

Statement

Gravitational theory in AdS $\longleftrightarrow$ Lower dimensional CFT

AdS: constant negative curvature

$$ds^2 = \frac{dz_0^2 + d\vec{z}^{\,2}}{z_0^2} \qquad z_0 > 0$$



What will be convered?

- Basic concept of CFT (conformal symmetry, operator product expansion, conformal blocks, etc)

& Technical details

- Nonperturbative: conformal bootstrap
- Holographic CFT $\leftrightarrow$ AdS/CFT

will not discuss 2D CFT

(conformal symmetry)$_{2D}$ → infinite dimensional Virasoro symmetry

2. Conformal symmetry

2.1 stress tensor & symmetries

Local QFT, exist conserved stress tensor → relate to global symmetry

$$\partial_\mu T^{\mu\nu}(x) = 0$$

Nother theorem: Exist ambiguites

"improvement terms" $T^{\mu\nu} \to T^{\mu\nu} + (\partial^\mu\partial^\nu - \delta^{\mu\nu}\partial^2)\Phi + \partial_\rho B^{\rho\mu\nu}$

$$(B^{\rho\mu\nu} = -B^{\mu\rho\nu})$$

Couple to $g_{\mu\nu}(x)$

$T^{\mu\nu}$: reaction to $\delta g_{\mu\nu}$, $T^{\mu\nu}(x) = -\frac{2}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}(x)}$ ✓

correlation functions:

$$\langle O_1(x_1)\cdots O_n(x_n)\rangle_g = \frac{1}{Z[g]}\int [d\phi]\, e^{-S[\phi,g]}\, O_1(x_1)\cdots O_n(x_n)$$

$Z[g]$ is the partition function: $Z[g] = \int [d\phi]\, e^{-S[\phi,g]}$

$g \to g + \delta g$,

$$\boxed{\begin{aligned}&\langle O_1(x_1)\cdots O_n(x_n)\rangle_{g+\delta g} - \langle O_1(x_1)\cdots O_n(x_n)\rangle_g = \frac{1}{2}\int dx\,\sqrt{g}\,\delta g_{\mu\nu}(x)\\ &\quad\times\left[\langle T^{\mu\nu}(x)\, O_1(x_1)\cdots O_n(x_n)\rangle_g - \langle T^{\mu\nu}(x)\rangle_g\langle O_1(x_1)\cdots O_n(x_n)\rangle_g\right] + O(\delta g^2)\end{aligned}}$$

coordinate transformation (around flat space $g_{\mu\nu}(x) = \eta_{\mu\nu}$)

$x^\mu \to \tilde{x}^\mu = x^\mu + \epsilon^\mu(x)$, $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = g_{\mu\nu} - \partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\mu$

$$\boxed{\langle O_1(\tilde{x}_1)\cdots O_n(\tilde{x}_n)\rangle_{\tilde{g}} = \langle O_1(x_1)\cdots O_n(x_n)\rangle_g}$$

$$\sum_{i=1}^{n}\epsilon^\mu(x_i)\frac{\partial}{\partial x_i^\mu}\langle O_1(x_1)\cdots O_n(x_n)\rangle = -\int dx\,\epsilon_\nu(x)\langle \partial_\mu T^{\mu\nu}(x)\, O_1(x_1)\cdots O_n(x_n)\rangle$$

Alternatively — Ward Identity

$$\partial_\mu\langle T^{\mu\nu}(x)\, O_1(x_1)\cdots O_n(x_n)\rangle = -\underbrace{\sum_{i=1}^{n}\delta(x-x_i)\frac{\partial}{\partial x_i^\nu}\langle O_1(x_1)\cdots O_n(x_n)\rangle}_{\text{contact term}}$$

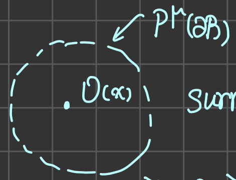
2.1.2 Topological operators & symmetries

$$P^{\nu}(\Sigma) \equiv -\int_{\Sigma} dS_{\mu} T^{\mu\nu}(x)$$

Σ: codimension 1 surface

Independent of deformations of Σ as long as Σ does not cross operators

$P^{\mu}(\Sigma)$: Topological surface operator $\longleftrightarrow$ symmetry



surronding $O(x)$ with P^{μ} is equivalent to taking a derivative.

$\int$

$\partial^{\mu} O(x)$

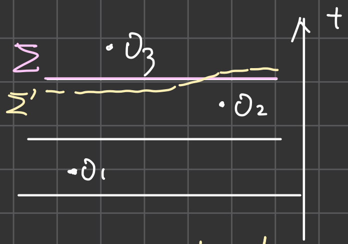
$\partial B = \Sigma$, $\quad O(x)$

$\hookrightarrow$ ball

$$\langle P^{\mu}(\Sigma)\, O(x) \cdots \rangle = \partial^{\mu} \langle O(x) \cdots \rangle = \langle P^{\mu}(\Sigma')\, O(x) \cdots \rangle$$

Path integral v.s. Hamiltonian Formalism

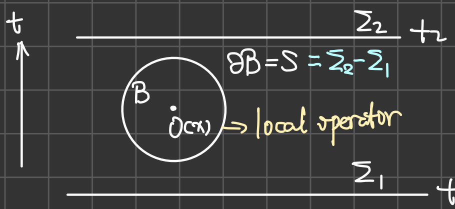
$\hookrightarrow$ choose a foliate spacetime



$$\langle O_1(x_1)\, O_2(x_2) \cdots O_n(x_n) \rangle = \langle 0 | T\{ \hat{O}_1(\vec{x}_1, t_1) \cdots \hat{O}_n(\vec{x}_n, t_n) \} | 0 \rangle$$

$\hookrightarrow$ time ordering

In QFT, having a topological codimension-1 operator is the same as having a symmetry



$S = \Sigma_2 - \Sigma_1$

$(t_2 > t > t_1)$

$$\text{LHS} = \langle (P^{\mu}(\Sigma_2) - P^{\mu}(\Sigma_1))\, O(x) \cdots \rangle$$

$$= \langle 0 | T\{ [\hat{P}, \hat{O}(x)] \cdots \} | 0 \rangle$$

$\hookrightarrow$ momentum charge

$P^{\mu}(\Sigma)$ can be moved freely in time as long as it doesn't cross any operator insections.

$$= \langle P^{\mu}(S)\, O(x) \cdots \rangle \quad \text{Ward identity}$$

$\hookrightarrow$ operator identity $\quad [\hat{P}^{\mu}, \hat{O}(x)] = \partial^{\mu} \hat{O}(x)$

$$= \partial^{\mu} \langle O(x) \cdots \rangle = \partial^{\mu} \langle 0 | T\{ \hat{O}(x) \cdots \} | 0 \rangle$$

$$O(x) = e^{x \cdot P}\, O(0)\, e^{-x \cdot P}$$

More general ϵ^{μ}, $\quad \epsilon(x) = \epsilon^{\mu}(x)\, \partial_{\mu}$ vector field

Conserved current $\quad J^{\mu}_{\epsilon}(x) = \epsilon_{\nu}(x) T^{\mu\nu}(x)$, $\quad$ Charge $\quad Q_{\epsilon}(\Sigma) = -\int_{\Sigma} dS_{\mu}\, \epsilon_{\nu}(x) T^{\mu\nu}(x)$

$$\partial_{\mu} J^{\mu}_{\epsilon}(x) = 0 = \partial_{\mu}(\epsilon_{\nu} T^{\mu\nu}) = (\partial_{\mu}\epsilon_{\nu}) T^{\mu\nu} + \overset{0}{\cancel{\epsilon_{\nu}\partial_{\mu} T^{\mu\nu}}} = \frac{1}{2}(\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu}) T^{\mu\nu}$$

$\rightarrow \boxed{\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = 0}$ Killing equation

Solutions (In flat space):

corresponding charges are momentum $P_{\mu} = Q_{p_{\mu}}$, angular momentum $M_{\mu\nu} = Q_{m_{\mu\nu}}$.

$$\left.\begin{array}{ll} \text{translations} & \boxed{\begin{array}{l} p_{\mu} = \partial_{\mu}, \\ m_{\mu\nu} = x_{\nu}\partial_{\mu} - x_{\mu}\partial_{\nu} \end{array}} \\ \text{rotations} & \end{array}\right\} \text{generates flat space isometries}$$

$J^\mu_\epsilon(x) = \epsilon_\nu(x) T^{\mu\nu}(x)$

In CFT: $T^\mu_\mu(x) = 0$ (Around flat space), traceless

$\delta g_{\mu\nu} = w(x) g_{\mu\nu}(x)$, local rescaling

$0 = \partial_\mu J^\mu_\epsilon = \frac{1}{2}(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) T^{\mu\nu}$

$\hookrightarrow$ A more general equation (because $T^\mu_\mu = 0$)

$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = c(x) \delta_{\mu\nu}$, relax the killing equation in flat space.

$\delta^{\mu\nu}$: $2(\partial\cdot\epsilon) = cd \Rightarrow c = \frac{2}{d}(\partial\cdot\epsilon) \Rightarrow \boxed{(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = \frac{2}{d}(\partial\cdot\epsilon)\delta_{\mu\nu}}$ conformal killing equation

$(*)$

Solving conformal killing equation

Act with $\partial^\mu \partial^\nu$ on $(*)$ $\Rightarrow (2 - \frac{2}{d}) \partial^2 \partial\cdot\epsilon = 0$

Assume $d \neq 1 \Rightarrow \partial^2 \partial\cdot\epsilon = 0$

Act on $(*)$ $\partial^\nu \partial_\rho \Rightarrow \partial_\mu \partial_\rho (\partial\cdot\epsilon) + \partial^2 \partial_\rho \epsilon_\mu = \frac{2}{d} \partial_\rho \partial_\mu (\partial\cdot\epsilon)$

Symmetrize in μ, ρ $\Rightarrow 2\partial_\mu \partial_\rho (\partial\cdot\epsilon) + \partial^2 \cancelto{0}{(\partial_\rho \epsilon_\mu + \partial_\mu \epsilon_\rho)} - \frac{4}{d} \partial_\rho \partial_\mu (\partial\cdot\epsilon) = 0$

$\hookrightarrow \frac{2}{d}(\partial\cdot\epsilon)\delta_{\mu\rho}$

$\hookrightarrow (1 - \frac{2}{d}) \partial_\rho \partial_\mu (\partial\cdot\epsilon) = 0$.

If $d > 2 \Rightarrow \partial_\rho \partial_\mu (\partial\cdot\epsilon) = 0$

Act w $\partial_\alpha \partial_\beta$ on $(*)$

$RHS = \frac{2}{d} \partial_\alpha \partial_\beta (\partial\cdot\epsilon) \delta_{\mu\nu} = 0 \Rightarrow \partial_\alpha \partial_\beta \partial_\mu \epsilon_\nu + \partial_\alpha \partial_\beta \partial_\nu \epsilon_\mu = 0$

$\partial_\alpha \partial_\mu \partial_\beta \epsilon_\nu + \partial_\alpha \partial_\mu \partial_\nu \epsilon_\beta = 0$.

$\partial_\alpha \partial_\nu \partial_\mu \epsilon_\beta + \partial_\alpha \partial_\nu \partial_\beta \epsilon_\nu = 0$.

$\Rightarrow \partial_\alpha \partial_\beta \partial_\mu \epsilon_\nu = 0$

ϵ_μ is at most quadratic.

Two additional types of solutions in $\mathbb{R}^d$.

$\boxed{d = x^\mu \partial_\mu}$ (Dilatations) $\rightarrow D = Q_d$

$\boxed{k_\mu = x_\mu (x\cdot\partial) - x^2 \partial_\mu}$ (special conformal transformations) $\rightarrow K_\mu = Q_{k_\mu}$

$d=2$ $\{x, y\} = \{z, \bar{z}\}$, $\epsilon = \epsilon^\nu \partial_z - \epsilon^{\bar{\nu}} \partial \bar{E}$

Conformal killing equation: $\partial_z \epsilon^{\bar{z}} = 0$, $\partial_{\bar{z}} \epsilon^z = 0$

$\rightarrow \epsilon^z(z, \bar{z}) = \epsilon^z$

$\epsilon^{\bar{z}}(z, \bar{z}) = \epsilon^{\bar{z}}$

2.2 Conformal algebra

The charges give a representation of the conformal algebra.

$[D, P_\mu] = P_\mu$, $[D, K_\mu] = -K_\mu$, $[K_\mu, P_\nu] = 2\delta_{\mu\nu}D - 2M_{\mu\nu}$.

$[M_{\mu\nu}, P_\alpha] = \delta_{\nu\alpha}P_\mu - \delta_{\mu\alpha}P_\nu$, $[M_{\mu\nu}, K_\alpha] = \delta_{\nu\alpha}K_\mu - \delta_{\mu\alpha}K_\nu$.

$[M_{\alpha\beta}, M_{\mu\nu}] = \delta_{\beta\mu}M_{\alpha\nu} + \delta_{\alpha\nu}M_{\beta\mu} - \delta_{\alpha\mu}M_{\beta\nu} - \delta_{\beta\nu}M_{\alpha\mu}$

All other commutators vanish.

W.R.T.: M, P K are vectors

W.R.T: D, P is raising operator, K is lowering operator.

Conformal algebra is isomorphic to $SO(1, d+1)$

$\underbrace{\mathbb{R}^{1,d+1}}_{\text{Embedding space}}$

$L_{\mu\nu} = M_{\mu\nu}$

$L_{-1,0} = D$

$L_{0,\mu} = \frac{1}{2}(P_\mu + K_\mu)$

$L_{-1,\mu} = \frac{1}{2}(P_\mu - K_\mu)$

$L_{ab} = -L_{ba}$ satisfy commutation relations of $SO(1, d+1)$

Conformal transformation is non-linear

Rotation in $\mathbb{R}^{1,d+1}$ is linear

$\mathbb{R}^{1,d+1}$ linearizes the action of conformal symmetry

3. Local operators

3.1 Primaries and descendants

Classify operators w.r.t. conserved charges.

Start with Poincare subalgebra

Rotation invariance $\longrightarrow$ operators at origin transform in irreducible representations of the rotation group $SO(d)$

$[M_{\mu\nu}, O^a(0)] = (S_{\mu\nu})_b{}^a O^b(0)$

$$M_{\mu\nu} O(x) = M_{\mu\nu} e^{x\cdot P} O(0)$$

$$= e^{x\cdot P}(e^{-x\cdot P} M_{\mu\nu} e^{x\cdot P}) O(0)$$

BCH formula

$e^A B e^{-A} = e^{[A,\cdot]}B$

$= B + [A, B] + \frac{1}{2!}[A, [A, B]] + \cdots$

$[P_\mu, O(x)] = \partial_\mu O(x)$

$$= e^{x\cdot P}(-x_\mu P_\nu + x_\nu P_\mu + M_{\mu\nu}) O(0)$$

$$= (x_\mu \partial_\nu - x_\nu \partial_\mu + S_{\mu\nu}) e^{x\cdot P} O(0)$$

$$= (-x_\mu \partial_\nu + x_\nu \partial_\mu + S_{\mu\nu}) O(x)$$

P ✓
M ✓

D: Scale invariant theory $[D, O(x)] = \Delta O(0)$
$\hookrightarrow$ conformal dimension

Diagonalize D

$$D K_\mu O(0) = (\underbrace{[D, K_\mu]}_{-K_\mu} + K_\mu D) O(0) = (\Delta - 1) K_\mu O(0)$$

lowers dimension

In physical theories, Δ can not be indefinitely lowered.

Exist $[K_\mu, O(0)] = 0$

Such operators are called primary operators from which we can construct descendant operators.

$P_{\mu_1} \cdots P_{\mu_n} O(0)$

conformal dimension $= \Delta + n$

$[K_\mu, O(x)] = (K_\mu + 2\Delta x_\mu - 2x^\nu S_{\mu\nu}) O(x)$ Exercise

Q_ϵ can be put into a single formula

$$[Q_\epsilon, O(x)] = (\epsilon \cdot \partial + \frac{\Delta}{d}(\partial \cdot \epsilon) - \frac{1}{2}(\partial^\mu \epsilon^\nu) S_{\mu\nu}) O(x)$$

$$\epsilon^\mu: x^\mu \to x'^\mu = x^\mu + \epsilon^\mu, \quad \frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu + \partial_\nu \epsilon^\mu(x) = \underbrace{(1 + \frac{1}{d}(\partial \cdot \epsilon))}_{\text{rescaling}} \underbrace{(\delta^\mu_\nu + \frac{1}{2}(\partial_\nu \epsilon^\mu - \partial^\mu \epsilon_\nu))}_{SO(d) \text{ rotation}}$$

Exponentiate

$$\frac{\partial x'^\mu}{\partial x^\nu} = \Omega(x') R^\mu_\nu(x') \qquad R^T \cdot R = I_{d \times d}$$

$$\delta_{\mu\nu} \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} = \Omega^2(x') \delta_{\alpha\beta}$$

$\Rightarrow U = e^{Q_\epsilon}$ finite transformation

$$U O^a(x) U^{-1} = \Omega(x')^\Delta \underbrace{\rho^a{}_b (R(x')^{-1})}_{\text{Scalar } \rho(R) = 1} O^b(x')$$

3.2 Conformal correlators

Ward identities

$\langle O_1(x_1) O_2(x_2) \cdots O_n(x_n) \rangle$

$= \langle U O_1(x_1) U^{-1} \cdots U O_n(x_n) U^{-1} \rangle$ scalar O_i, $\rho = 1$

$= \Omega(x_1')^{\Delta_1} \cdots \Omega(x_n')^{\Delta_n} \langle O(x_1') \cdots O(x_n') \rangle$

Constraints from Ward identity

2-point case:

$$\langle O_{\Delta_1}(x_1) O_{\Delta_2}(x_2) \rangle = f(|x_1 - x_2|)$$

scale transformation $\quad U O_\Delta(x) U^{-1} = \lambda^{\Delta} O_\Delta(\lambda x)$

$$\langle O_1(x_1) O_2(x_2)\rangle = \lambda^{\Delta_1+\Delta_2} \langle O_1(\lambda x_1) O_2(\lambda x_2)\rangle$$

$$f(|x_1-x_2|) = \lambda^{\Delta_1+\Delta_2} f(\lambda|x_1-x_2|)$$

$$\hookrightarrow f(|x_1-x_2|) = \frac{a}{|x_1-x_2|^{\Delta_1+\Delta_2}}$$

$K_\mu : I \circ P_\mu \circ I \qquad$ Exercise

I : conformal inversion , $x^\mu \longrightarrow -\frac{x^\mu}{x^2}$

$$O_\Delta(x) = (x^2)^{-\Delta} O_\Delta(-x^\mu/x^2), \quad (x_1'-x_2')^2 = \frac{(x_1-x_2)^2}{x_1^2 x_2^2}$$

$$\langle O_{\Delta_1}(x_1) O_{\Delta_2}(x_2)\rangle = (x_1^2)^{-\Delta_1}(x_2^2)^{-\Delta_2}\langle O_{\Delta_1}(x_1') O_{\Delta_2}(x_2')\rangle$$

$$LHS = \frac{a}{|x_1-x_2|^{\Delta_1+\Delta_2}}$$

$$RHS = (x_1^2)^{-\Delta_1}(x_2^2)^{-\Delta_2}\frac{a}{|x_1'-x_2'|^{\Delta_1+\Delta_2}} = \frac{a}{|x_1-x_2|^{\Delta_1+\Delta_2}} |x_1^2|^{-\Delta_1+\frac{\Delta_1+\Delta_2}{2}} |x_2^2|^{-\Delta_2+\frac{\Delta_1+\Delta_2}{2}} \to 1$$

Only okay if $\Delta_1 = \Delta_2$

$$O_i \longrightarrow O_i/\sqrt{a}$$

$$\langle O_{\Delta_1}(x_1) O_{\Delta_2}(x_2)\rangle = \frac{1}{(x_1-x_2)^{2\Delta_1}}\delta_{\Delta_1\Delta_2}$$

3-point case:

$$\langle O_1(x_1) O_2(x_2) O_3(x_3)\rangle$$

P, M ✓ $\quad K = I\circ P \circ I, \quad I: x^\mu \to x'^\mu = \frac{x^\mu}{x^2}, \quad (x_1'-x_2')^2 = \frac{(x_1-x_2)^2}{x_1^2 x_2^2}$

$$D: x^\mu \to \lambda x^\mu$$

$$\Rightarrow \langle O_1(x_1) O_2(x_2) O_3(x_3)\rangle \longrightarrow \lambda^{-(\Delta_1+\Delta_2+\Delta_3)}\langle O_1(x_1) O_2(x_2) O_3(x_3)\rangle$$

C_{123} → 3-point function coefficient

$$\Rightarrow \langle O_1(x_1) O_2(x_2) O_3(x_3)\rangle = \frac{C_{123}}{(x_1-x_2)^{a_{12}}(x_1-x_3)^{a_{13}}(x_2-x_3)^{a_{23}}}.$$

$$a_{12} = \Delta_1+\Delta_2-\Delta_3, \quad a_{13} = \Delta_1-\Delta_2+\Delta_3, \quad a_{23} = \Delta_2+\Delta_3-\Delta_1$$

4-point case

$x_i = 1,2,3,4, \qquad x_i \to \lambda x_i$

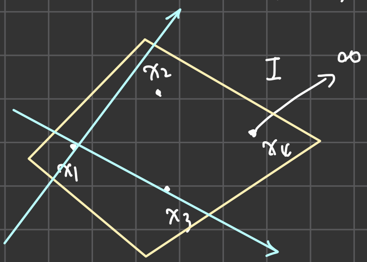
$U = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \qquad U, V \Rightarrow U, V \qquad$ U, V are called conformal cross relations

$V = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \qquad I: U \Rightarrow \dfrac{\frac{x_{12}^2}{x_1^2 x_2^2}\;\frac{x_{34}^2}{x_3^2 x_4^2}}{\frac{x_{13}^2}{x_1^2 x_3^2}\;\frac{x_{24}^2}{x_2^2 x_4^2}} = U$

$x_{ij} = x_i - x_j \qquad$ invariant under conformal transformation

$$V \longrightarrow V$$

$\langle\phi\phi\phi\phi\rangle \Rightarrow \frac{1}{(x_{12}^2 x_{34}^2)^{\Delta\phi}} f(U,V)$



1. Inversion W.R.T. x_4

2. Translate x_1 to 0

3. Use rotation $\quad x_3 = (1, \underline{0}, 0, \cdots, 0)$

$x_2 = (x, \underline{y}, 0, \cdots, 0)$

$x, y \Rightarrow U, V$

$\{x,y\} \Rightarrow \{z, \bar{z}\}$, $U = z\bar{z}$, $V = (1-z)(1-\bar{z})$

permutation symmetry, e.g. $\langle 1234\rangle = \langle 1324\rangle$

$(1 \Leftrightarrow 2$ or $3 \Leftrightarrow 4)$ $\quad f(U,V) = f(U/V, 1/V)$

Crossing symmetry

$(1 \Leftrightarrow 3$ or $2 \Leftrightarrow 4)$ $\quad f(U,V) = \left(\frac{U}{V}\right)^{\Delta\phi} f(U,V)$

We will see $\{\Delta_i, C_{ijk}\}$ defines the CFT

4. Embedding space formalism

In $\mathbb{R}^d$, action of conformal group is nonlinear.

The rotation group $SO(1,d+1)$ in $\mathbb{R}^{1,d+1}$ is linear.

embedding space

$$P^A = (P^{-1}, P^0, \overbrace{P^1, \cdots, P^d}^{\mu=1,\cdots d})$$

$$P\cdot P = -(P^{-1})^2 + (P^0)^2 + \cdots + (P^d)^2$$

$SO(1,d+1)$

$$P \rightarrow gP \qquad g \in SO(1,d+1)$$

How to connect P with x ?

Need to eliminate two component

Impose: $P\cdot P = 0 \qquad d+2 \Rightarrow d+1$

$P \sim \lambda P \qquad \lambda \neq 0$

"gauge redundancy"

Claim: $x^\mu \in \mathbb{R}^d \Leftrightarrow$ null ray in $\mathbb{R}^{1,d+1}$.

To explicitly relate P to x^μ, we choose a "gauge".

$$P^+ \equiv P^{-1} + P^0 = 1$$

$$\Rightarrow P^A = \left(\frac{1+x^2}{2}, \frac{1-x^2}{2}, x^\mu\right)$$

Now let's see how $g\in SO(1,d+1)$ leads to conformal transformation.

$$P \xrightarrow{g} gP$$

g may not preserve the gauge $P^+=1$

$$gP \xrightarrow{\text{rescale}} \frac{gP}{(gP)^+}$$

$$\left(\frac{1+x^2}{2}, \frac{1-x^2}{2}, x^\mu\right) = P \longrightarrow gP \longrightarrow \frac{gP}{(gP)^+} = \left(\frac{1+x'^2}{2}, \frac{1-x'^2}{2}, x'^\mu\right)$$

x to x' is a conformal transformation.

Examples:

1. g only act P^μ, $(P^{-1}, P^0, \overbrace{P^1 \cdots P^d}^{e})$ $\quad g=\begin{pmatrix}1&&\\&1&\\&&e\end{pmatrix}$

$x'^\mu = e^\mu{}_\nu x^\nu \Rightarrow$ Rotations in $\mathbb{R}^d$

Minkowski

2. $$g^A{}_B = \begin{pmatrix} \boxed{1+\frac{a^2}{2}} & \boxed{\frac{a^2}{2}} & a^\nu \\ \boxed{-\frac{a^2}{2}} & \boxed{1-\frac{a^2}{2}} & -a^\nu \\ a^\mu & a^\mu & g^{\mu\nu} \end{pmatrix}$$ $a^\nu \to d$, $d \downarrow a^\mu$, $g^{\mu\nu} \to d\times d$

$a^\mu \in \mathbb{R}^d$, infinitesimal

Check $g^A{}_C\, \eta^{CD} (g^T)_D{}^B = \eta^{AB}$

$$P \Rightarrow gP = \left(\frac{1+(x+a)^2}{2}, \frac{1-(x+a)^2}{2}, x^\mu + a^\mu\right)$$

$(gP)^+=1$, $x^\mu \Rightarrow x^\mu + a^\mu$

translation

3. $$g=\begin{pmatrix} \frac{1+\lambda^2}{2\lambda} & \frac{1-\lambda^2}{2\lambda} & 0 \\ \frac{1-\lambda^2}{2\lambda} & \frac{1+\lambda^2}{2\lambda} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ λ is a number

$$P \Rightarrow gP = \left(\frac{\lambda^{-1}+\lambda x^2}{2}, \frac{\lambda^{-1}-\lambda x^2}{2}, x^\mu\right), \quad (gP)^+ = \lambda^{-1}$$ gauge do not perserve

rescale by λ, $gP \Rightarrow \lambda gP = \left(\frac{1+\lambda^2 x^2}{2}, \frac{1-\lambda^2x^2}{2}, \lambda x^\mu\right)$

$$x^\mu \Rightarrow \lambda x^\mu \quad \longrightarrow \text{Dilatation}$$

4. $$g=\begin{pmatrix} 1+\frac{b^2}{2} & -\frac{b^2}{2} & -b^\nu \\ \frac{b^2}{2} & 1-\frac{b^2}{2} & -b^\nu \\ -b^\mu & b^\mu & g^{\mu\nu} \end{pmatrix}$$

$$P \Rightarrow gP = \left(\frac{1-2b\cdot x + b^2x^2 + x^2}{2}, \frac{1-2b\cdot x + b^2 x^2 - x^2}{2}, x^\mu - b^\mu x^2\right)$$

$$(gP)^+ = 1-2b\cdot x + b^2 x^2.$$

$$gP \longrightarrow (1-2b\cdot x + b^2x^2)^{-1} gP = \left(\frac{1-2b\cdot x+b^2x^2+x^2}{2(1-2b\cdot x+b^2x^2)}, \frac{1-2b\cdot x+b^2x^2-x^2}{2(1-2b\cdot x)+b^2x^2)}, \frac{x^\mu - b^\mu x^2}{1-2b\cdot x+b^2x^2}\right)$$

$x^\mu \longrightarrow x'^\mu = \dfrac{x^\mu - b^\mu x^2}{1-2b\cdot x + b^2 x^2}$ special conformal transformation

$$x^\mu \xrightarrow{I} \frac{x^\mu}{x^2} \xrightarrow{\text{transformation by } -b^\mu} \frac{x^\mu}{x^2} - b^\mu \xrightarrow{I} \frac{x^\mu - b^\mu x^2}{1-2b\cdot x + b^2 x^2}$$

4.2 operators in embedding space & correlation functions

- Focus on scalar operators
- $x^\mu \in \mathbb{R}^d \longrightarrow$ null ray in $\mathbb{R}^{1,d+1}$ $\quad P \sim \lambda P$
- Claim: scalar operators are defined as $O(\lambda P) = \lambda^{-\Delta} O(P)$

$O(P)\big|_{P^+=1} = O(x)$

correct definition and gives the expected transformation of scalar operators

How does the operator transform?

- Embedding space

$$P \Rightarrow gP \longrightarrow \frac{gP}{(gP)^+}$$

$$O(P) \longrightarrow O(gP) \longrightarrow \frac{1}{[(gP)^+]^\Delta} O\left(\frac{gP}{(gP)^+}\right)$$

- In $\mathbb{R}^d$

$$U_g O(x) U_g^{-1} = \Omega^\Delta(x') O(x')$$

Same if $x'^\mu = \dfrac{(gP)^\mu}{(gP)^+}$, $\Omega(x') = \dfrac{1}{(gP)^+}$

$$ds^2 = dx\cdot dx = dP\cdot dP = (d(g\cdot P))^2 = (d(\lambda(P')P'))^2 = (\lambda dP' + P'(D\lambda \cdot DP'))^2$$

$P' = \dfrac{g\cdot P}{(gP)^+}$, $\lambda(P') = (gP)^+$

$$= \lambda^2 (dP')^2 + \cancel{2\lambda(D\lambda\cdot dP')\, dP'\cdot P'} + \cancel{(P')^2(D\lambda\cdot dP')^2}$$

$(P')^2 = 0$

$dP'\cdot P' = 0$

$$= \lambda^2 (dP')^2 = \lambda^2 ds'^2$$

$ds'^2 = \left(\frac{1}{\lambda^2}\right) ds^2 \Rightarrow \Omega = \dfrac{1}{(gP)^+}$ done!

Ω^2

2-point function

$O_i(\lambda_i P_i) = \lambda_i^{-\Delta_i} O_i(P_i)$ $\quad \lambda_i$ can be different for different i

$\langle O_1(P_1) O_2(P_2)\rangle \qquad \langle O_1(P_1) O_2(P_2)\rangle = \lambda_1^{-\Delta_1}\lambda_2^{-\Delta_2}\langle O_1(P_1) O_2(P_2)\rangle$

conformal invariance $\longleftrightarrow$ rotation invariance

$$-2P_1\cdot P_2 = x_{12}^2 \quad \Rightarrow \quad \frac{a}{(-2P_1\cdot P_2)^c}$$

scaling respect to x_1, $c = \Delta_1$

scaling respect to x_2, $c = \Delta_2$

$$\Rightarrow \langle O_1(P_1) O_2(P_2)\rangle = \frac{\delta_{\Delta_1\Delta_2}}{(-2P_1\cdot P_2)^{\Delta_1}}$$

3-point function

$$\langle O_1(P_1) O_2(P_2) O_3(P_3)\rangle = \frac{C_{123}}{(-2P_1\cdot P_2)^{\alpha_{12}}(-2P_1\cdot P_3)^{\alpha_{13}}(-2P_2\cdot P_3)^{\alpha_{23}}}$$

$P_i \to \lambda_i P_i$

$$\begin{aligned} \alpha_{12}+\alpha_{13} &= \Delta_1 \\ \alpha_{12}+\alpha_{23} &= \Delta_2 \\ \alpha_{13}+\alpha_{23} &= \Delta_3 \end{aligned} \quad \longrightarrow \quad \begin{aligned} \alpha_{12} &= \frac{\Delta_1+\Delta_2-\Delta_3}{2} \\ \alpha_{13} &= \frac{\Delta_1-\Delta_2+\Delta_3}{2} \\ \alpha_{23} &= \frac{\Delta_2+\Delta_3-\Delta_1}{2} \end{aligned}$$

$-2P_i\cdot P_j = x_{ij}^2$

rotation invariance + rescaling

4-point function

$$U = \frac{(-2P_1\cdot P_2)(-2P_3\cdot P_4)}{(-2P_1\cdot P_3)(-2P_2\cdot P_4)}, \quad V = \frac{(-2P_1\cdot P_4)(-2P_2\cdot P_3)}{(-2P_1\cdot P_3)(-2P_2\cdot P_4)}$$

For n-point, let us consider $\prod_{i<j}(-2P_iP_j)^{-\beta_{ij}}$

$\cdot\, P_i \to \lambda_i P_i \cdots$

$\beta_{ij}=\beta_{ji},\ \beta_{ii}=0.$ $\quad \sum_i \beta_{ij}=0$ $\quad$ unknows $\frac{n(n-1)}{2}$ $\Rightarrow$ solutions $\frac{n(n-1)}{2}-n = \frac{n(n-3)}{2}$

$M_{ij} = -2P_i\cdot P_j$ $\quad n\times n$

$n > d+2$

$\det M = 0 \Rightarrow \det(-2P_i\cdot P_j)=0.$

\# of cross ratios

$$\begin{cases} \text{If } n>d+2, & \# = nd - \frac{(d+2)(d+1)}{2} \\ \text{If } n\le d+2, & \# = \frac{n(n-3)}{2} \end{cases}$$

$x^\mu \in \mathbb{R}^d$, n-point

$$n\cdot d - \underbrace{\frac{(d+2)(d+1)}{2}}_{\#\text{ of generators of } SO(1,d+1)}$$

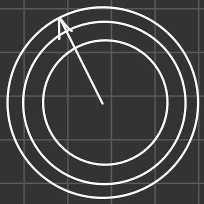
Not correct if $n\le d+2$, because there is a stability group.

1. Use conformal symmetry, move 2 points to 0 & ∞ still have $n-2$
2. Can Find $(n-2)$ dimension plane that goes through 0.
3. Transverse rotation $SO(d-n+2)$, add back its generators

$$nd - \frac{(d+2)(d+1)}{2} + \frac{(d-n+2)(d-n+1)}{2} = \frac{n(n-3)}{2}$$

5. Radial quantization & state-operator corresponding

In CFT, we have scale invariance, it's advantageous to choose quantization that respect symmetry.



Foliate spacetime into spheres around origin
Radius $\longrightarrow$ Time
S^{d-1}: defines $\mathcal{H}$

} Radial quantization

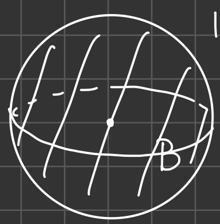
Time ordering $\to$ radial ordering

$$\langle O_1(x_1)\cdots O_n(x_n)\rangle = \langle 0|\mathcal{R}\{O_1(x_1)\cdots O_n(x_n)\}|0\rangle$$

$$=\theta(|x_n|-|x_{n-1}|)\cdots\theta(|x_2|-|x_1|)\langle 0|O_n(x_n)\cdots O_1(x_1)|0\rangle + \text{permutation correlations}$$

5.2 State-operator

To prepare a state



$|\psi\rangle$

Nothing in B $\to |0\rangle$

Operator in B $\to$ nontrivial state

Operator $\longrightarrow$ state

State $\to$ operator

$|\psi\rangle$ can be demcomposed into eigenstates of D. $D|O_i\rangle = \Delta_i|O_i\rangle$

$r \leftrightarrow$ time

Dilatation $\longleftrightarrow$ Time translation

$e^{-rD} \to e^{-r\Delta_i}$ $\quad |O_i\rangle$ not changed but multiplied by a factor

Use e^{-rD} to shrink B to a point

$|O_i\rangle \to O_i$ inserted at origin

How conformal generators act on states in radial quantization.

$[K_\mu, O(0)] = 0$, $\qquad |O\rangle = O(0)|0\rangle$

$K_\mu|0\rangle = 0 \qquad K_\mu|O\rangle = K_\mu O(0)|0\rangle = [K_\mu, O(0)]|0\rangle = 0$

$[D, O(0)] = \Delta\, O(0) \longleftrightarrow D|O\rangle = \Delta|O\rangle$

$[M_{\mu\nu}, O(0)] = S_{\mu\nu}O(0) \longleftrightarrow M_{\mu\nu}|O\rangle = S_{\mu\nu}|O\rangle$

$\begin{cases} D|0\rangle = 0 \\ M_{\mu\nu}|0\rangle = 0 \end{cases}$

Primary state $|O\rangle$,

Desendant states

$|0\rangle,\ P_\mu|0\rangle,\ P_\mu P_\nu|0\rangle, \cdots$

$\underbrace{\qquad\qquad\qquad\qquad}_{\text{conformal multiplet}}$

$$\partial_\mu \mathcal{O}(x=0)|0\rangle = [P_\mu, \mathcal{O}(x=0)]|0\rangle = P_\mu \mathcal{O}(0)|0\rangle = P_\mu|\mathcal{O}\rangle$$

5.3 Cylinder quantization

Dilatation $\longrightarrow$ Weyl factor

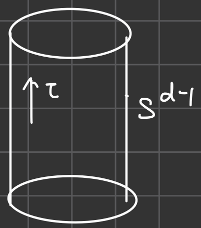
Can get rid of it by Weyl rescaling $g \to \Omega^2(x)\, g$

Conformal killing $\longrightarrow$ Killing

rescaling $\longrightarrow$ Isometry

$$ds^2_{\mathbb{R}^d} = dr^2 + r^2 ds^2_{S^{d-1}} = r^2\left(\frac{dr^2}{r^2} + ds^2_{S^{d-1}}\right) \qquad r = e^\tau$$

$$= e^{2\tau}\underbrace{(d\tau^2 + ds^2_{S^{d-1}})}_{ds^2_{\mathbb{R}\times S^{d-1}}} \longrightarrow ds^2_{\mathbb{R}\times S^{d-1}}$$



Dilatation:

$r \to \lambda r \quad \Rightarrow$ Translation $\tau \to \tau + \log\lambda$

under Weyl scaling $g \to \Omega^\tau g$

$$\frac{\langle \mathcal{O}_1(x_1)\cdots \mathcal{O}_n(x_n)\rangle_g}{\langle 1\rangle_g} = \prod_i \Omega(x_i)^{\Delta_i}\,\frac{\langle \mathcal{O}_1(x_1)\cdots \mathcal{O}_n(x_n)\rangle_{\Omega^2 g}}{\langle 1\rangle_{\Omega^2 g}}$$

$\hookrightarrow$ natural to define cylinder operators

$$\mathcal{O}_{cyl}(\tau,\vec{n}) = e^{\Delta\tau}\,\mathcal{O}_{flat}(\vec{x} = e^\tau \vec{n})$$

5.4 Unitary bounds

In Loretzian signature and unitary theories, conserved charges are Hermitian.

In Euclidean, unitarty is replaced by reflection positively.

Consider Lorentzian

$$\mathcal{O}_L(t,\vec{x}) = e^{iHt - i\vec{x}\cdot\vec{P}_L}\,\mathcal{O}_L(0,0)\,e^{-iHt + i\vec{x}\cdot\vec{P}_L} \qquad H, \vec{P}_L \text{ are Hermitian.}$$

Wick rotation to Euclidean,

$$\mathcal{O}_E(t_E,\vec{x}) = \mathcal{O}_L(-it_E,\vec{x}) = e^{Ht_E - i\vec{x}\cdot\vec{P}_L}\,\mathcal{O}_L(0,0)\,e^{-Ht_E + i\vec{x}\cdot\vec{P}_L}$$

Take $\dagger$

$$\Big\downarrow \dagger$$

$$e^{-Ht_E - i\vec{x}\cdot\vec{P}_L}\,\mathcal{O}_L^\dagger(0,0)\,e^{Ht_E + i\vec{x}\cdot\vec{P}_L} \qquad \mathcal{O}_L^\dagger(0,0) = \mathcal{O}_L(0,0)$$

$$\Rightarrow \mathcal{O}_E^\dagger(t_E,\vec{x}) = \mathcal{O}_E(-t_E,\vec{x}) \text{ for scalar } \mathcal{O}$$

For spinning operators, e.g. vector

$$O_E^0(t_E,\vec{x}) = -i\,O_L^0(-it_E,\vec{x}), \qquad O_E^i(t_E,\vec{x}) = O_L^i(-it_E,\vec{x})$$

$$O_E^{\mu_1\cdots\mu_\ell}(t_E,\vec{x})^\dagger = \Theta^{\mu_1}{}_{\nu_1}\cdots\Theta^{\mu_\ell}{}_{\nu_\ell}\,O_E^{\nu_1\cdots\nu_\ell}(-t_E,\vec{x})$$

$$\Theta^\mu{}_\nu = \delta^\mu_\nu - 2\delta^\mu_0\delta^0_\nu$$

E.g. stress tensor

$$T_E^{i0}(0,\vec{x})^\dagger = -T_E^{i0}(0,\vec{x}) \qquad T_E^{00}(0,\vec{x})^\dagger = T_E^{00}(0,\vec{x})$$

$$P^\mu = -\int d^{d-1}\vec{x}\; T_E^{\mu 0}(0,\vec{x}) \qquad P^0 = H,\; P^i = -iP_L^i$$

$\Rightarrow P^0$ is Hermitian, P^i is anti-Hermitian.

Unitary Lorentian $\xrightarrow[\text{Wick rotation}]{}$ Euclidean

An important property: norms of states are positive

$$|\psi\rangle = O(-t_{E,1})\cdots O(-t_{E,n})|0\rangle, \quad \langle\psi| = \langle 0|O(t_{E,n})\cdots O(t_{E,1})$$

$\langle\psi|\psi\rangle \geq 0$. $\quad | \; t\to -t \quad$ reflection positivity

$\hookrightarrow \langle 0|O(t_{E,n})\cdots O(t_{E,1})\,O(-t_{E,1})\cdots O(-t_{E,n})|0\rangle \geq 0$

In an Euclidean theory, Wick rotated from a unitary Lorentzian theory, reflection positivity is satisfied.

On the other hand,

reflection positivity Euclidean theory $\xrightarrow[\text{Reconstruction}]{\text{Osterwalder–Schrader}}$ Unitary Lorentzian theory

(Analytic continuation)

Hermitian conjugation in cylinder quantization

Scalar:

$$O_{cyl}(\tau,\vec{n})^\dagger = O_{cyl}(-\tau,\vec{n})$$

In radial quantization: $O_{flat}^\dagger(x) = x^{-2\Delta}\,O_{flat}\left(\frac{x^\mu}{x^2}\right)$

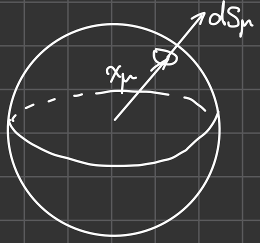
$\dagger$ in radial quantization is implemented by $I: x^\mu \to \frac{x^\mu}{x^2}$

For spinning case, $O^{\mu_1\cdots\mu_\ell}(x)^\dagger = I^{\mu_1}{}_{\nu_1}(x)\cdots I^{\mu_\ell}{}_{\nu_\ell}(x)\,x^{-2\Delta}\,O^{\nu_1\cdots\nu_\ell}\left(\frac{x}{x^2}\right)$

$$I^\mu{}_\nu = \delta^\mu_\nu - \frac{2x^\mu x_\nu}{x^2}$$

$T^{\mu\nu}$: $\quad T^{\mu\nu\dagger}(x) = \frac{1}{x^{2d}}T^{\mu\nu}(x') - \frac{2x^\mu x_\rho}{x^{2+2d}}T^{\rho\nu}(x') - \frac{2x^\nu x_\rho}{x^{2+2d}}T^{\rho\mu}(x') + \frac{4x^\mu x^\nu x_\rho x_\sigma}{x^{4+2d}}T^{\rho\sigma}(x')$

$x \in$ unit sphere , $x^2=1$. I leaves unit sphere invariant



$Q_\epsilon = -\int dS_\mu \epsilon_\nu T^{\mu\nu}$. dS_μ is proportional to x^μ

Under inversion, $dS_\mu \Rightarrow -dS_\mu$,

Extra minus sign from the definion of charge.

$$\Rightarrow x_\mu \epsilon_\nu T^{\mu\nu} \xrightarrow{\dagger} ?$$

$$x_\mu \epsilon_\nu (T^{\mu\nu})^\dagger = x_\mu \epsilon_\nu \left(\frac{1}{x^{2d}} T^{\mu\nu}(x') - \frac{2x^\mu x_\rho}{x^{2+2d}} T^{\rho\nu}(x') - \frac{2x^\nu x_\rho}{x^{2+2d}} T^{\rho\mu}(x') + \frac{4x^\mu x^\nu x_\rho x_\sigma}{x^{4+2d}} T^{\rho\sigma}(x') \right)$$

$$= x_\mu \epsilon_\nu T^{\mu\nu} - 2x_\rho \epsilon_\nu T^{\rho\nu} - 2(x\cdot\epsilon)\, x_\mu x_\rho T^{\rho\mu} + 4(x\cdot\epsilon)\, x_\rho x_\sigma T^{\rho\sigma}$$

$$= 2(x\cdot\epsilon) x_\mu x_\nu T^{\mu\nu} - x_\mu \epsilon_\nu T^{\mu\nu}$$

$$\epsilon_\mu \xrightarrow{\dagger} 2(x\cdot\epsilon) x_\mu - \epsilon_\mu$$

Hermitian conjugation of charges

Translation, $\epsilon_\mu = a_\mu$, $a_\mu \longrightarrow -a_\mu + 2(x\cdot a) x_\mu$

$$P_\mu = \partial_\mu , \quad k_\mu = 2x_\mu (x\cdot\partial) - x^2 \partial_\mu \quad \Rightarrow P_\mu^\dagger = k_\mu , \quad k_\mu^\dagger = P_\mu$$

Dilatation, $d = x^\mu \partial_\mu$

$$\lambda x^\mu \longrightarrow 2(\lambda x^2) x^\mu - \lambda x^\mu = \lambda x^\mu \quad \Rightarrow D^\dagger = D$$

Rotation, $m_{\mu\nu} = x_\nu \partial_\mu - x_\mu \partial_\nu$

$$x_\nu \omega^{[\mu,\nu]} \longrightarrow 2(\underbrace{\cancel{x_\rho x_\nu \omega^{[\rho,\nu]}}}_{0}) x_\mu - x_\nu \omega^{[\mu,\nu]} \quad \Rightarrow M_{\mu\nu}^\dagger = -M_{\mu\nu}$$

$$\begin{cases} P_\mu^\dagger = k_\mu , \quad k_\mu^\dagger = P_\mu \\ D^\dagger = D \\ M_{\mu\nu}^\dagger = -M_{\mu\nu} \end{cases}$$

$P_\mu |\mathcal{O}\rangle$

$k_\mu |\mathcal{O}\rangle = \langle \mathcal{O}| P = 0$

because $\mathcal{O}$ is primary

$$|P_\mu|\mathcal{O}\rangle|^2 = (P_\mu|\mathcal{O}\rangle)^\dagger (P_\mu|\mathcal{O}\rangle) = \langle \mathcal{O}| k_\mu P_\mu |\mathcal{O}\rangle$$

$$= \langle \mathcal{O}| [k_\mu, P_\mu] |\mathcal{O}\rangle$$

μ is not summed over

$$= 2\langle \mathcal{O}| D |\mathcal{O}\rangle = 2\Delta \langle \mathcal{O}|\mathcal{O}\rangle$$

$\longrightarrow \Delta \geq 0$

$P_\mu P^\mu |\mathcal{O}\rangle$ μ is summed over

Exercise

$$|P_\mu P^\mu |\mathcal{O}\rangle|^2 = \langle \mathcal{O}| k_\mu k^\mu P^\nu P_\nu |\mathcal{O}\rangle = 8 d \Delta \left(\Delta - \frac{d-2}{2}\right) \langle \mathcal{O}|\mathcal{O}\rangle \geq 0$$

$\Delta = 0, \quad \mathcal{O} = \mathrm{id}$

$\Delta \geq \frac{d-2}{2}$ unitarity bounds for scalars

Spinning operator in irreducible representation ρ of $SO(d)$ $|\mathcal{O}^a\rangle$

$$(P_\mu |\mathcal{O}^a\rangle)^\dagger P^\nu |\mathcal{O}^b\rangle = \langle \mathcal{O}_a | K_\mu P^\nu |\mathcal{O}^b\rangle = 2\Delta\, \delta_\mu^\nu\, \delta_a^b - 2(M_\mu^\nu)_a{}^b$$

tensor product $\square \otimes \rho \rightarrow$ irreducible reprensentation

$\hookrightarrow$ vector representation

Unitary $\Delta \geq$ max-eigenvalue of $(M_\mu^\nu)_a{}^b$

Review M as

$$(M_\mu{}^\nu)_a{}^b = \frac{1}{2}\sum_{A,B} (L^{AB})_\mu{}^\nu (S_{AB})_a{}^b = \sum_{\substack{I=AB \\ A<B}} \underbrace{(L^I)_\mu{}^\nu}_{\square} \underbrace{(S_I)_a{}^b}_{\rho}$$

$$\sum_I L^I \otimes S_I = \frac{1}{2}\Big((L\otimes 1 + 1\otimes S)^2 - (L\otimes 1)^2 - (1\otimes S)^2 \Big)$$

$$= \frac{1}{2}\Big(-\underbrace{C_2(\square\otimes\rho)}_{\text{min}} + C_2(\square)\otimes 1 + 1\otimes C_2(\rho) \Big)$$

C_2: quadratic Casimir $\qquad V_{l-1} \subseteq \square \otimes V_l$

ρ: spin-l symmetric traceless

$\rho = V_l \quad , \quad C_2(V_l) = l(l+d-2)$

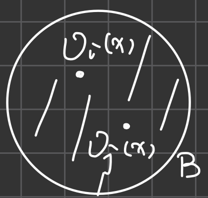
$$\Delta \geq \frac{1}{2}\Big(-C_2(V_{l-1}) + C_2(\square) + C_2(V_l)\Big) = l+d-2$$

Unitary bounds:

$$\begin{cases} \Delta = 0 \ \text{ or } \ \Delta \geq \frac{d-2}{2}, & l = 0 \\ \Delta \geq l+d-2, & l > 0 \end{cases}$$

6. Operator product Expansion (OPE)

Consider radial quantization



Path integral inside B defines a state on ∂B state-operator correspondence

$$\mathcal{O}_i(x)\mathcal{O}_j(0)|0\rangle = \sum_k C_{ijk}(x, P)\, \mathcal{O}_k(0)|0\rangle$$

OPE

$\hookrightarrow$ If no other operators in B

$$\mathcal{O}_i(x)\, \mathcal{O}_j(0) = \sum_k C_{ijk}(x, \partial)\, \mathcal{O}_k(0)$$

$$\mathcal{O}_i(x_1)\,\mathcal{O}_j(x_2) = \sum_k C'_{ijk}(x_{12}, x_{23}, \partial)\,\mathcal{O}_k(x_3)$$

C_{ijk} is restrict by conformal symmetry

Consider $\mathcal{O}_{i,j,k}$ are scalars, show

$$C_{ijk}(x,\partial) = |x|^{\Delta_k-\Delta_i-\Delta_j}\left(a_0 + a_1 x^\mu\partial_\mu + a_2 x^\mu x^\nu\partial_\mu\partial_\nu + a_3 x^2\partial^2 + \cdots\right)$$

Act with D

$$D\,\mathcal{O}_i(x)\,\mathcal{O}_j(0)|0\rangle = (x^\mu\partial_\mu + \Delta_i + \Delta_j)\,\mathcal{O}_i(x)\,\mathcal{O}_j(0)|0\rangle$$

$$= \sum_k (x^\mu\partial_\mu + \Delta_i + \Delta_j)\,C_{ijk}(x,P)\,\mathcal{O}_k(0)|0\rangle$$

Focus on $\mathcal{O}_k$

$$RHS = D\,C_{ijk}(x,P)\,\mathcal{O}_k|0\rangle = [D, C_{ijk}(x,P)]\,\mathcal{O}_k(0)|0\rangle + \Delta_k C_{ijk}(x,P)\,\mathcal{O}_k(0)|0\rangle$$

$$\Rightarrow (x^\mu\partial_\mu + \Delta_i + \Delta_j - \Delta_k)\,C_{ijk}(x,P) = [D, C_{ijk}(x,P)]$$

After extracting $|x|^{\Delta_k-\Delta_i-\Delta_j}$

$$x^\mu\partial_\mu \longleftrightarrow [D, \cdots]$$

$x^{\mu_1}\cdots x^{\mu_n}$ $\;n x\;$ counting # of P

counting of x

Working out the coefficients

$$\langle\mathcal{O}_i(x_1)\,\mathcal{O}_j(x_2)\,\mathcal{O}_k(x_3)\rangle = C_{ijk}(x_{12},\partial_2)\,\langle\mathcal{O}_k(x_2)\,\mathcal{O}_k(x_3)\rangle$$

$\Delta_i = \Delta_j = \Delta_\phi$, $\Delta_k = \Delta$

$$\Rightarrow \frac{C_{ijk}}{x_{12}^{2\Delta_\phi-\Delta}\,x_{23}^{\Delta}\,x_{13}^{\Delta}} = C_{ijk}(x_{12},\partial_2)\,x_{23}^{-\Delta}$$ expand in small $|x_{12}|/|x_{13}|$

$$\Rightarrow a_0 = C_{ijk},\quad \frac{a_1}{a_0} = \frac{1}{2},\quad \frac{a_2}{a_0} = \frac{\Delta+2}{8(\Delta+1)},\quad \frac{a_3}{a_0} = -\frac{\Delta}{16(\Delta+1)(\Delta-\frac{d-2}{2})}$$

Special case 1D

$$C_{ijk}(x,\partial) = C_{ijk}\,|x|^{\Delta-2\Delta_\phi}\left(\beta_0 + \beta_1 x\partial + \beta_2 x^2\partial^2 + \beta_3 x^3\partial^3 + \cdots\right)$$

$$\beta_m = \frac{(\Delta)_m}{(1)_m(2\Delta)_m},\qquad (a)_b = \frac{\Gamma(a+b)}{\Gamma(a)}$$ Pochhammer symbol

$$C_{ijk}(x,\partial) = C_{ijk}\,x^{\Delta-2\Delta_\phi}\,{}_1F_1(\Delta, 2\Delta, x\partial)$$

$$= C_{ijk}\,x^{\Delta-2\Delta_\phi}\,\frac{\Gamma(2\Delta)}{\Gamma(\Delta)}\int_0^1 dt\, t^{\Delta-1}(1-t)^{\Delta-1}e^{tx\partial_x}$$

$$C_{ijk}(x_{12},\partial_2) = \langle\mathcal{O}(x_2),\,\mathcal{O}(x_3)\rangle$$

$$= C_{ijk}\, x_{12}^{\Delta-2\Delta_\phi} \int_0^1 dt\, t^{\Delta-1}(1-t)^{\Delta-1} e^{t x_{12}\partial_2} \frac{1}{x_{23}^{2\Delta}}$$

$$= C_{ijk}\, x_{12}^{\Delta-2\Delta_\phi} \int_0^1 dt\, t^{\Delta-1}(1-t)^{\Delta-1} \frac{1}{(x_{23}+t x_{12})^{2\Delta}} \qquad e^{a\partial} f(x) = f(x+a)$$

$$= C_{ijk}\, x_{12}^{\Delta-2\Delta_\phi} x_{23}^{-2\Delta} \left(1-\frac{x_{21}}{x_{23}}\right)^{-\Delta}$$

$$= \langle O_1(x_1) O_2(x_2) O_2(x_3)\rangle = \sum_k C_{12k}(x_{12},\partial_2)\langle O_k(x_2)\cdots O_n(x_n)\rangle$$

$\langle \underbrace{O_1(x_1)\, O_2(x_2)}\cdots O_n(x_n)\rangle$

Repeat until $n=3$

$\{\Delta_i, C_{ijk}\}$ = CFT data

Determines all n-point correlators

7. Conformal blocks

7.1 From OPE

Consider 4-point function, focus on identical scalars

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{G(U,V)}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}}$$

Consider OPE of $\phi(x_1)$ & $\phi(x_2)$

$O(x_2)$ transforms in spin-l symmetric traceless representation of $SO(d)$

$$\phi(x_1)\phi(x_2) = \sum_O C_{\phi\phi O}\, C(x_{12},\partial_2) O(x_2)$$

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \sum_O C_{\phi\phi O}^2\, C(x_{12},\partial_2)\, C(x_{34},\partial_4)\langle O(x_2) O(x_4)\rangle$$

$$= \frac{1}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} \sum_O C_{\phi\phi O}^2\, g_{\Delta_O, l_O}(U,V)$$

where $g_{\Delta_O, l_O} \equiv x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}\, C(x_{12},\partial_2)\, C(x_{34},\partial_4)\langle O(x_2), O(x_4)\rangle$

conformal block

Captures the contribution of exchanging O, $\quad G(U,V) = \sum_O C_{\phi\phi O}^2\, g_{\Delta_O, l_O}(U,V)$

Computing conformal blocks

1. Use OPE (In general, very difficult!)

C_{ijk}? Resum

CFT_1 Only one cross ratio z, $z = \frac{x_{12}x_{34}}{x_{13}x_{24}}$, $U=z^2$, $V=(1-z)^2$

$$C(x_{12},\partial_2) = \frac{\Gamma(2\Delta)}{\Gamma(\Delta)} x_{12}^{\Delta-2\Delta_\phi} \int_0^1 dt\, t^{\Delta-1}(1-t)^{\Delta-1} e^{t x_{12}\partial_2}$$

$$g_\Delta^{1d}(z) = \frac{\Gamma(2\Delta)}{\Gamma(\Delta)^2}\int_0^1 dt\, t^{\Delta-1}(1-t)^{\Delta-1} \frac{x_{12}^\Delta x_{34}^\Delta}{(x_{23}+t x_{12})^\Delta (x_{24}+x_{12})^\Delta}$$

$\to$ A function of z

z is invariant under conformal transformation

$\{x_1, x_3, x_4\} \to \{0, 1, \infty\} \quad x_2 \leftrightarrow z$

$$= \frac{\Gamma(2\Delta)}{\Gamma(\Delta)^2}\int_0^1 dt\, t^{\Delta-1}(1-t)^{\Delta-1}\left(1-\frac{z}{z-1}t\right)^{-\Delta}\left(\frac{z}{1-z}\right)^{\Delta}$$

$$= \left(\frac{z}{1-z}\right)^{\Delta} {}_2F_1\left(\Delta,\Delta;2\Delta;\frac{z}{z-1}\right)$$

$$= z^\Delta {}_2F_1(\Delta,\Delta;2\Delta;z)$$

Integral representation

$${}_2F_1(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 dt\, t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}$$

$${}_2F_1(a,b;c;z)=(1-z)^{-a}\,{}_2F_1\left(a,b;c;\frac{z}{1-z}\right)$$

7.2. Use conformal casimir

Choose an origin such that $|x_{3,4}|>|x_{1,2}|$

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle=\langle 0|R\{\phi(x_3)\phi(x_4)\}R\{\phi(x_1)\phi(x_2)\}|0\rangle$$

For each primary operator O, define an projector

$$|O|=\sum_{\substack{\alpha,\beta=O,PO,\\ PPO\cdots}}|\alpha\rangle N_{\alpha\beta}^{-1}\langle\beta|\ ,\quad N_{\alpha\beta}=\langle\alpha|\beta\rangle\ ,\quad \mathbb{1}=\sum_O|O|$$

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle=\sum_O\underbrace{\langle 0|R\{\phi(x_3)\phi(x_4)\}|O|R\{\phi(x_1)\phi(x_2)\}|0\rangle}_{\frac{C_{\phi\phi O}^2}{x_{12}^{2\Delta\phi}x_{34}^{2\Delta\phi}}g_{\Delta_O,l_O}(U,V)}$$

Use conformal Casimir

$SO(1,d+1)$ generators L_{AB}, $A,B=-1,0,1,\cdots d$, $L_{AB}=-L_{BA}$

$$Cas=-\frac{1}{2}L^{AB}L_{AB}=-\frac{1}{2}M^{\mu\nu}M_{\mu\nu}-\frac{1}{2}P\cdot K-\frac{1}{2}K\cdot P+D^2$$

Same eigenvalue for every state in the conformal multiplet.

$$Cas|O\rangle=\lambda_{\Delta,l}|O\rangle\ ,\quad \lambda_{\Delta,l}=\Delta(\Delta-d)+l(l+d-2)$$

$$Cas|O|=|O|Cas=\lambda_{\Delta,l}|O|$$

Denote the action of L_{AB} $\phi(x_i)$ by $\mathcal{L}_{AB}^{(i)}$

$$L_{AB}\phi(x_1)\phi(x_2)|0\rangle=\Big([L_{AB},\phi(x_1)]\phi(x_2)+\phi(x_1)[L_{AB},\phi(x_2)]\Big)|0\rangle=(\mathcal{L}_{AB}^{(1)}+\mathcal{L}_{AB}^{(2)})\phi(x_1)\phi(x_2)|0\rangle$$

$$Cas\,\phi(x_1)\phi(x_2)|0\rangle=-\frac{1}{2}(\mathcal{L}_{AB}^{12})^2\phi(x_1)\phi(x_2)|0\rangle\ ,\quad \mathcal{L}_{AB}^{12}=\mathcal{L}_{AB}^{(1)}+\mathcal{L}_{AB}^{(2)}.$$

$$-\frac{1}{2}(\mathcal{L}_{AB}^{12})^2\langle 0|R\{\phi(x_3)\phi(x_4)\}|O|R\{\phi(x_1)\phi(x_2)\}|0\rangle$$ Differential operator of x_1

$$=\langle 0|R\{\phi(x_3)\phi(x_4)\}|O|\,Cas\,R\{\phi(x_1)\phi(x_2)\}|0\rangle$$

$$=\lambda_{\Delta,l}\langle 0|R\{\phi(x_3)\phi(x_4)\}|O|R\{\phi(x_1)\phi(x_2)\}|0\rangle$$

$$\mathcal{D}g_{\Delta,l}(U,V)=\lambda_{\Delta,l}\,g_{\Delta,l}(U,V)$$

$\downarrow$

2nd

Compute conformal blocks by solving PDEs!

How to compute $\mathcal{D}$?

Use embedding space, $\mathcal{L}^{(i)}_{AB} = P_{iA}\frac{\partial}{\partial P_{iB}} - P_{iB}\frac{\partial}{\partial P_{iA}}$

$$\langle 0| R\{\phi(x_3)\phi(x_4)\}|O|R\{\phi(x_1)\phi(x_2)\}|0\rangle = \frac{C^2_{\phi\phi O}}{(-2P_1\cdot P_2)^{\Delta\phi}(-2P_3\cdot P_4)^{\Delta\phi}} g_{\Delta,l}(U,V)$$

$$U = \frac{P_{12}P_{34}}{P_{13}P_{24}}, \quad V = \frac{P_{14}P_{23}}{P_{13}P_{24}}$$

$$\mathcal{D} = 2(z^2(1-z)\partial_z^2 - z^2\partial_z) + 2(\bar{z}^2(1-\bar{z})\partial_{\bar{z}}^2 - \bar{z}^2\partial_{\bar{z}}) + 2(d-2)\frac{z\bar{z}}{z-\bar{z}}((1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}})$$

Boundary condition: Given by OPE $U\to 0$ $g_{\Delta,0}(U,V) = U^{\Delta/2}(1+\cdots)$
$l=0$

Solutions in 2d

$$\mathcal{D} = \mathcal{D}_z + \mathcal{D}_{\bar{z}}, \quad \mathcal{D}_z = 2(z^2(1-z)\partial_z^2 - z^2\partial_z)$$

$$\lambda_{\Delta,l}\Big|_{d=2} = \Delta(\Delta-2) + l^2 = 2h(h-1) + 2\bar{h}(\bar{h}-1), \quad h = \frac{\Delta+l}{2}, \quad \bar{h} = \frac{\Delta-l}{2}$$

We can now solve the casimir equation by separation of variables. A general solution will be sum of $f(z)\bar{f}(\bar{z})$

$$\mathcal{D}_z f(z) = h(h-1)f(z), \quad \mathcal{D}_{\bar{z}}\bar{f}(\bar{z}) = \bar{h}(\bar{h}-1)\bar{f}(\bar{z})$$ ODE

$$f(z) = A\, k_{2h}(z) + B\, k_{2(1-h)}(z)$$

arbitrary constants

$k_{2h}(z) = z^h {}_2F_1(h,h;2h;z) \to$ 1D conformal block

$$\underbrace{SO(2,2)}_{\text{2D conformal}} = \underbrace{SO(1,2)}_{\text{1D conformal}} \times SO(1,2)$$

Boundary condition $\to g^{2d}_{\Delta,l} \propto k_{2h}(z)k_{2\bar{h}}(\bar{z}) + k_{2\bar{h}}(z)k_{2h}(\bar{z})$

$d=4$

$$g^{(4d)}_{\Delta,l} \propto \frac{z\bar{z}}{z-\bar{z}}(k_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z}) - k_{\Delta-l-2}(z)k_{\Delta+l}(\bar{z})).$$

In even dimensions, solution is easy

In odd dimensions, solution is difficult. No closed form solutions.
$(d\neq 1)$

l is even

$g_{\Delta,l}$ are invariant under $z\to\frac{z}{z-1}$, $\bar{z}\to\frac{\bar{z}}{\bar{z}-1}$ or $U\to U/V$, $V\to 1/V$

$$g_{\Delta,l}(U,V) = g_{\Delta,l}\left(\frac{U}{V},\frac{1}{V}\right) \quad l \text{ even}$$

$x_1 \leftrightarrow x_2$ or
$x_3 \leftrightarrow x_4$

8. Conformal bootstrap

$\{\Delta_i, C_{ijk}\}$ cannot be arbitrary. Must satisfy consistency conditions.

Conformal bootstrap's idea: Use these conditions to solve CFT.

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \qquad \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \qquad \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$$

Equivalence of different ways to compute 4-point function: OPE is associative

$\{U,V\}\to\{V,U\}$ $\quad 1\leftrightarrow 3$ or $2\leftrightarrow 4$, $\quad G(U,V)=\left(\frac{U}{V}\right)^{\Delta_\phi}G(V,U)$

$\{U,V\}\to\{U/V,\frac{1}{V}\}$ $\quad 1\leftrightarrow 2$ or $3\leftrightarrow 4$, $\quad G(U,V)=G(U/V,1/V)$

crossing equation

12 OPE

$$\langle\phi\cdots\phi\rangle=\frac{1}{x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}}G(U,V),\quad G(U,V)=\underbrace{\sum_{O\in\phi\times\phi}C_{\phi\phi O}^2\, g_{\Delta_O,l_O}(U,V)}_{\text{s-channel}}=\left(\frac{U}{V}\right)^{\Delta_\phi}\underbrace{\sum_{O\in\phi\phi}C_{\phi\phi O}^2\, g_{\Delta_O,l_O}(V,U)}_{\text{t-channel}}$$

$$\sum_{O\in\phi\times\phi}C_{\phi\phi O}^2\, g_{\Delta_O,l_O}(U,V)=\sum_{O\in\phi\times\phi}C_{\phi\phi O}^2\, g_{\Delta_O,l_O}\left(\frac{U}{V},\frac{1}{V}\right)$$

trivial does not lead to constraints

$\phi\times\phi$ OPE: only even l can appear

$g_{\Delta,l}(U,V)=g_{\Delta,l}(U/V,1/V)$

cannot be satisfied by ~~a single~~ any finite number conformal block $\longrightarrow$ CFT must have an infinite number of conformal primaries!

Consider $d=2$

$$k_{2h}(z)=z^h\,{}_2F_1(h,h;2h;z)\xrightarrow{z\to 0}z^h \qquad h=\frac{\Delta+l}{2},\ \bar h=\frac{\Delta-l}{2}$$

$$g_{\Delta,l}=k_{2h}(z)k_{2\bar h}(\bar z)+k_{2h}(\bar z)k_{2\bar h}(z)$$

set $z=\bar z$, expand in small z

$$\phi\times\phi=1+\cdots=\text{leading contribution}$$

LHS $=1+\cdots$

$U=z\bar z,\ V=(1-z)(1-\bar z)$ $\quad U\leftrightarrow V\ \Leftrightarrow\ z\leftrightarrow 1-z$

$$g_{\Delta,l}(V,U)\Big|_{z=\bar z}=2k_{\Delta+l}(1-z)k_{\Delta-l}(1-z)\sim\log^2 z+\cdots$$

The crossing equation tells us

$$1\sim z^{2\Delta_\phi}\sum_O(\log^2 z+\cdots)$$ Contradiction?

$$\lim_{z\to 0}z^{2\Delta_\phi}\log^2 z=0$$

Need infinitely many $\mathcal{O}$, cannot exchange the order of sum and limits.

Consider the following integral

Bessel K

$$K_0(2lz) = -(\log z + \log l + \gamma_E) - l^2(\log z + \log l + \gamma_E - 1)\, z^2 + \mathcal{O}(z^4)$$

l: spin $\qquad K_0(2lz) \sim \log z$

$$\sum_{\mathcal{O}} \leftrightarrow \sum_{l} \xrightarrow[l]{\text{large}} \int_0^{+\infty} dl$$

$$\int_0^{+\infty} dl\, \underbrace{l^{2\Delta_\phi - 1}}_{\text{OPE coefficients}} K_0(2lz) = \frac{\Gamma^2(\Delta_\phi)}{4} z^{-2\Delta_\phi}$$

logarithmic sigularity can be enhanced into a power singularity with infinitely many terms.

Rewrite the crossing equation as

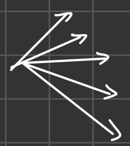
$$\sum_{\mathcal{O}\in\phi\times\phi} C^2_{\phi\phi\mathcal{O}} \Big(\underbrace{V^{\Delta_\phi} g_{\Delta_\mathcal{O}, l_\mathcal{O}}(U,V) - U^{\Delta_\phi} g_{\Delta_\mathcal{O}, l_\mathcal{O}}(V,U)}_{F^{\Delta_\phi}_{\Delta_\mathcal{O}, l_\mathcal{O}}(U,V)} \Big) = 0$$

$$\sum_{\mathcal{O}\in\phi\times\phi} C^2_{\phi\phi\mathcal{O}} F^{\Delta_\phi}_{\Delta_\mathcal{O}, l_\mathcal{O}}(U,V) = 0$$

We can abstractly think of $F^{\Delta_\phi}_{\Delta_\mathcal{O}, l_\mathcal{O}}$ as an vector in infinitely dimension vector space of U and V.

Unitarity requires:

$$C_{\phi\phi\mathcal{O}} \in \mathbb{R} \Rightarrow C^2_{\phi\phi\mathcal{O}} \geq 0$$

 $=0$

$\sum_{\mathcal{O}} c^2 F = 0$ may not hold $\Rightarrow$ Constraints on the CFT data.

Algoritm for bounding operator dimensions

1. We make a guess for dimensions and spins Δ, l of operators appearing in the $\phi\times\phi$ OPE.
2. We assume for a linear functional α which is non-negative on all $\vec{F}^{\Delta_\phi}_{\Delta_\mathcal{O}, l_\mathcal{O}}$.

$$\alpha(\vec{F}^{\Delta_\phi}_{\Delta_\mathcal{O}, l_\mathcal{O}}) \geq 0.$$

And stricitly positive on the operator (e.g. identity operator)

$$\alpha(\vec{F}^{\Delta_\phi}_{\Delta_\mathcal{O}, l_\mathcal{O}}) > 0$$

3. If we can find such an α, it means our assumption is wrong.
   We can therefore bound the spectrum by exclusion.

Difficulties:

1. Space of functionals is infinitely dimensional.
2. Possible spectrum is also infinite. $\begin{cases} \text{spin is infinite} \\ \Delta \text{ is continuous} \end{cases}$

In numerical conformal bootstrap:

For ① Expand crossing equation around $z = \bar{z} = \frac{1}{2}$

$$\alpha(F)=\sum_{m+n\le\Lambda} a_{mn}\,\partial_z^m\partial_{\bar z}^n F(z,\bar z)\Big|_{z=\bar z=\frac{1}{2}}$$

$\Lambda \rightsquigarrow$ cut-off

For ②. For spins, assume l_{max}, α positive for $l\le l_{max}$ (Check also $l>l_{max}$)

For Δ, cutoff Δ_{max}, discretize with a small spacing.

Find an upper bound for scalar operator with lowest Δ

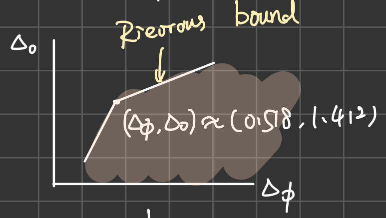
1. Assume scalars in $\phi\times\phi$ OPE have $\Delta\ge\Delta_0$
2. We search for a_{mn}

$$\sum_{m+n\le\Lambda} a_{mn}\,\partial_z^m\partial_{\bar z}^n F(z,\bar z)\Big|_{z=\bar z=1/2}\ge 0$$

for all $l=0,2,\cdots,l_{max}$ and $\Delta\ge\Delta_0$ (If $l=0$) or $\Delta\ge l+d-2$ (If $l>0$)

3. If we can find solution, it means there must exist a scalar with dimension smaller than Δ_0

In 3d, people found an upper bound for $\Delta_0(\Delta_\phi)$



very close to dimensions of $\Delta_\sigma, \Delta_\epsilon$ in 3d Ising

σ: spin operator, ϵ: energy operator

9. Holographic CFTs

9.1 Introduction to AdS/CFT correspondence

CFT in general are strongly coupled

- Conformal bootstrap
- AdS/CFT, gravitational theory in AdS $\longleftrightarrow$ CFT

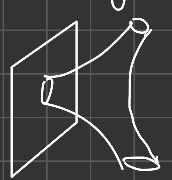
Basic idea: low energy description of D-brane system from equivalent perspectives of closed and open string

N D3 branes in Minkowski spacetime $\mathbb{R}^{1,9}$

Closed strings can interact with D3 in two equivalent ways

(a) Open string perspective: D3 branes are submanifold on which open strings can end.

Closed string can break into open string to end on D3



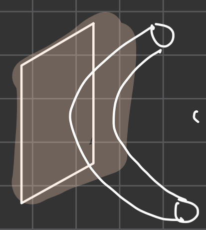
(a)

(b) Closed string perspective: D3 are solitonic solutions in string theory.

They create nontrivial background on which closed strings can propagate.

Open - Closed duality

in low energy limit



(b)

String length $l_s \to 0$, keep string coupling g_s.

Number of D3 N and energy fixed.

(a) The low energy excitations from two decoupled sectors.

Massless closed strings in $\mathbb{R}^{1,9}$ + Massless open strings ending on D3

(IIB SUGRA) (4d $\mathcal{N}=4$ super Yang-Mills)

(b) Background create by D3

$$ds^2 = \frac{1}{\sqrt{H(r)}}\underbrace{\eta_{\mu\nu}dx^\mu dx^\nu}_{\text{Metric of } \mathbb{R}^{1,3}} + \sqrt{H(r)}\,(dr^2 + r^2 d\Omega_5^2)$$

$H(r) = 1 + \frac{R^4}{r^4}$, $R^4 = 4\pi g_s N l_s^4$

$l_s \to 0$, need to be careful about $r=0$, high energy excitations at $r=0$ are red-shift at infinity.

Correct way: $z = R^2/r$ kept fixed as $l_s \to 0$

Near horizon geometry

$$ds^2 = R^2 \frac{dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu}{z^2} + R^2 d\Omega_5^2 \qquad AdS_5 \times S^5 \text{ with radius } R$$

Two decoupled sectors IIB SUGRA in $\mathbb{R}^{1,9}$ + Full type IIB string theory on $AdS_5 \times S^5$

4d $N=4$ Super Yang-Mills $\Longleftrightarrow$ IIB string theory in $AdS_5 \times S^5$

$$g_{YM}^2 = 4\pi g_s, \quad \frac{R^4}{l_s^4} = g_{YM}^2 N = \lambda \quad \text{'t Hooft coupling}$$

4d $N=4$ super Yang-Mills is conformal for any g_{YM} & N

A_μ, 4 Weyl fermions $\Psi^{a=1,\cdots 4}$, 6 scalars $\Phi^{I=1,\cdots 6}$. in adjoint of $SU(N)$

$$\mathcal{L} = \frac{1}{g_{YM}^2}\mathrm{tr}\Big[\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}(D^\mu\Phi^I)^2 + \bar{\Psi}^a\sigma^\mu D_\mu \Psi_a - \tfrac{1}{4}[\Phi^I,\Phi^J]^2 - C_I^{ab}\Psi_a[\Phi^I,\Psi_b]$$
$$- \bar{C}_{Iab}\bar{\Psi}^a[\Phi^I,\bar{\Psi}^b]\Big]$$

D_μ: covariant derivative $\qquad SO(6) = SU(4)$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$

10d $N=1$ SYM

$\hookrightarrow$ Dimensional reduction

Symmetries:

Bosonic

SYM: $\underbrace{SO(2,4)}_{\text{conformal}} \times \underbrace{SO(6)}_{\text{R-symmetry}}$ AdS$_5 \times S^5$: $\underbrace{SO(2,4)}_{\text{Isometry of AdS}_5} \times \underbrace{SO(6)}_{\text{Isometry of } S^5}$

Also the same fermonic symmetries on both sides.

superconformal = SUSY + conformal

AdS/CFT is a strong/weak duality.

Tractable limit: 't Hooft limit $N \to \infty$, λ fixed

|| ⋏ field theory, planner diagrams only

AdS: $g_s \to 0$ We can ignore string loops

In field theory, we want small λ in order to use perturbation theory. This corresponds to

$$R \ll l_s$$

But the world sheet string theory is strongly coupled in AdS, we want $R \gg l_s$.

This corresponds to the SUGRA limit, But the field theory is strongly interacting!

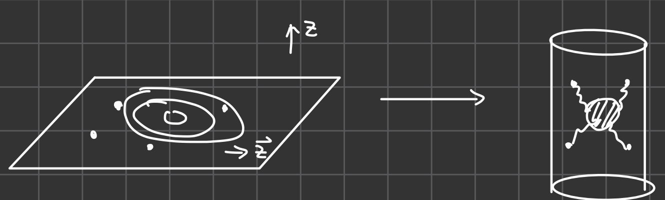
We will focus on the large λ limit.

9.2 Perturbation theory in AdS

tree-level, Witten diagrams

Correlation functions in CFT $\longleftrightarrow$ On-shell scattering amplitutes in AdS

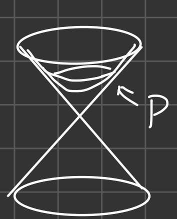
weakly coupled in bulk



Feynman diagrams $\longleftrightarrow$ Witten diagrams

AdS$_{d+1}$ space can be described in $\mathbb{R}^{1,d+1}$ embedding space (Euclidean)

$$-(X^{-1})^2 + (X^0)^2 + \sum_{\mu=1}^{d} (X^\mu)^2 = -R^2$$



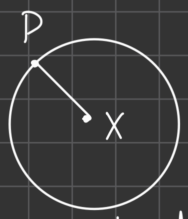
A useful set of coordinates is Poincare coordinates

$$X^{-1} = R\,\frac{1+z_0^2+\vec{z}^{\,2}}{2z_0}, \quad X^0 = R\,\frac{1-z_0^2-\vec{z}^{\,2}}{2z_0}, \quad \vec{X} = R\,\frac{\vec{z}}{z_0} \quad (z_0>0)$$

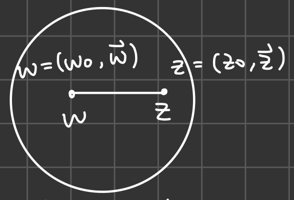
$$ds^2 = R^2\,\frac{dz_0^2 + d\vec{z}^{\,2}}{z_0^2}$$

From now on, we will set $R=1$, $P = \lim_{z_0 \to 0} z_0 X$, $X \to \left(\frac{1+\vec{z}^{\,2}}{2}, \frac{1-\vec{z}^{\,2}}{2}, \vec{z}\right) = P$

Witten diagrams are also constructed from propagators & vertices



bulk to boundary



bulk to bulk

$$u = (w-z)^2 = w^2 + z^2 - 2W\cdot Z = -2-2wz$$

$$= \frac{(z-w)^2}{z_0 w_0} \qquad \text{chordal distance}$$

$$= \frac{(\vec{z}-\vec{w})^2 + (z_0-w_0)^2}{z_0 w_0}$$

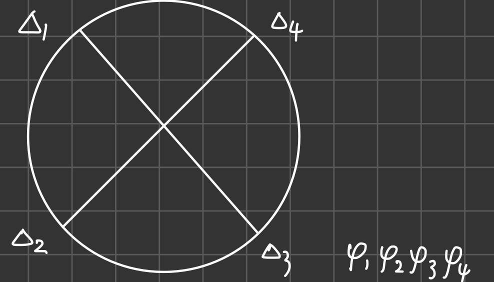
$$G^{\Delta}_{B\partial}(P,X) = \frac{1}{(-2P\cdot X)^{\Delta}} \qquad G^{\Delta}_{BB}(X,Z)$$

$$= C_\Delta u^{-\Delta} {}_2F_1\left(\Delta, \Delta-\frac{d}{2}+\frac{1}{2}; 2\Delta-d+1; -4u^{-1}\right)$$

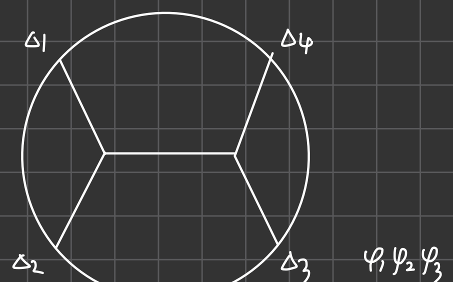
$$C_\Delta = \frac{\Gamma(\Delta)}{2\pi^{d/2}\Gamma(\Delta-\frac{d}{2}+1)}$$

$$(\Box_Z - m^2) G^{\Delta}_{BB}(Z,W) = \delta(Z,W), \qquad m^2 = \Delta(\Delta-d)$$

Take the limit $W \to P$, $\quad G^{\Delta}_{BB}(Z,W) \longrightarrow G^{\Delta}_{B\partial}(Z,P)$



contact Witten diagram



exchange Witten diagram

Contact Witten diagrams:

"D-function"

$$W_{con} = \int dZ \prod_{i=1}^{4} G^{\Delta_i}_{B\partial}(P_i, Z) = D_{\Delta_1\Delta_2\Delta_3\Delta_4}(P_i)$$

Schwinger parameterization

$$\frac{1}{A^\Delta} = \frac{1}{\Gamma(\Delta)}\int_0^{+\infty}\frac{dt}{t}\, t^\Delta e^{-tA}.$$

$$D_{\Delta_1\Delta_2\cdots\Delta_n} = \int \frac{dz_0\, d^d\vec{z}}{z_0^{d+1}} \prod_{i=1}^{n}\left(\frac{z_0}{z_0^2 + (\vec{z}-\vec{x}_i)^2}\right)^{\Delta_i}$$

$$= \int \frac{dz_0\, d^d\vec{z}}{z_0^{d+1}} \int_0^{+\infty}\prod_{i=1}^{n}\frac{dt_i\, t_i^{\Delta_i-1}}{\Gamma(\Delta_i)}\, z_0^{\sum_i \Delta_i}\, e^{-\sum_i t_i (z_0^2 + (\vec{z}-\vec{x}_i)^2)}$$

$$= \int_0^{+\infty} dz_0\, z_0^{\sum \Delta_i - d - 1}\int_0^{+\infty}\prod_{i=1}^{n} dt_i\, \frac{t_i^{\Delta_i-1}}{\Gamma(\Delta_i)}\left(\frac{\pi}{\sum_i t_i}\right)^{d/2} e^{-\sum_i t_i z_0^2 - \frac{1}{2\sum_i t_i}\sum_{i,j} t_i t_j \vec{x}_{ij}^{\,2}}$$

perform the z_0 integral

$$D_{\Delta_1\Delta_2\cdots\Delta_n} = \frac{\Gamma\left(\frac{\sum \Delta_i - d}{2}\right)}{2}\int_0^{+\infty}\prod_{i=1}^{n} dt_i\, \frac{t_i^{\Delta_i-1}}{\Gamma(\Delta_i)}\, \pi^{d/2}\Big(\underset{\mu}{\cancel{\sum_i t_i}}\Big)^{-d/2} e^{-\frac{1}{2\underset{\mu}{\cancel{\sum_i t_i}}}\sum_{ij} t_i t_j x_{ij}^2}$$

Insert $1 = \int_0^{+\infty} d\mu\, \delta(\mu - \sum_i t_i)$ allows us to replace $\sum_i t_i$ by μ.

We then rescale $t_i \to \mu t_i$

$$D_{\Delta_1\Delta_2\cdots\Delta_n} = \frac{\Gamma\left(\frac{\sum t_i - d}{2}\right)}{2}\int_0^{+\infty}\prod_{i=1}^n dt_i\,\frac{t_i^{\Delta_i-1}}{\Gamma(\Delta_i)}\,\pi^{d/2}\,\delta\left(\sum_i t_i-1\right)\int_0^{+\infty} d\mu\,\mu^{\sum\Delta_i-d/2-1}\,e^{-\frac{\mu}{2}\sum_{i<j}t_it_jx_{ij}^2}$$

$$=\frac{\pi^{d/2}\,\Gamma\left(\frac12\sum_i\Delta_i-\frac12 d\right)\Gamma\left(\frac12\sum_i\Delta_i\right)}{2\prod_{i=1}^n\Gamma(\Delta_i)}\int_0^{+\infty}\prod_{i=1}^n dt_i\,t_i^{\Delta_i-1}\,\frac{\delta\left(\sum_i t_i-1\right)}{\left(\sum_{i<j}t_it_jx_{ij}^2\right)^{\sum_i\Delta_i/2}}$$

?

$$\frac{\partial}{\partial x_{ij}^2}D_{\Delta_1\cdots\Delta_n} = \frac{\pi^{d/2}\,\Gamma\left(\frac{\Sigma-d}{2}\right)\Gamma\left(\frac{\Sigma}{2}\right)}{2\prod_{i=1}^n\Gamma(\Delta_i)}\int_0^{+\infty}\prod_{i=1}^n dt_i\,t_i^{\Delta_i-1}\,\frac{\delta\left(\sum_i t_i-1\right)(-t_it_j)}{\left(\sum_{i<j}t_it_jx_{ij}^2\right)^{\Sigma/2+1}}\cdot\left(\frac{\Sigma}{2}\right)$$

$$D_{\Delta_1\cdots\Delta_i+1\cdots\Delta_j+1\cdots\Delta_n} = \frac{d-\Sigma}{\theta\Delta_i\,\partial\Delta_j}\,\frac{\partial}{\partial x_{ij}^2}D_{\Delta_1\cdots\Delta_i\cdots\Delta_j\cdots\Delta_n}\qquad \begin{matrix}\Delta_i\to\Delta_i+1\\ \Delta_j\to\Delta_j+1\end{matrix}\qquad \Sigma\to\Sigma+2$$

We can relate D-functions of different weights by taking derivatives.

Comment 1: D-functions are secretly the same as flat space conformal integrals

$$\langle O_{\Delta_1}\cdots O_{\Delta_n}\rangle = I_n = \int d^Dx\prod_{i=1}^n\frac{1}{(x-x_i)^{2\Delta_i}}\qquad \mathbb{R}^D$$

$x_i\to\lambda x_i$

$x\to\lambda x$

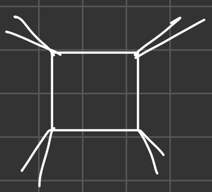
$\lambda^{-\Sigma}$ A special example $\Delta_i=1,\ n=4 \Rightarrow D=4$

$2\Sigma-D=\Sigma$

$\Rightarrow D=\sum_i\Delta_i$ $\quad \frac{1}{(x-x_i)^{2\Delta_i}}\to\frac{1}{\lambda^{2\Delta_i}(x-x_i)^{2\Delta_i}}$

$x_i-x_{i-1}=p_i$

$$I_4=\int d^4x\prod_{i=1}^4\frac{1}{(x-x_i)^2} = \text{one-loop box diagram in 4d}$$



$$D_{1111}\propto\frac{1}{x_{13}^2x_{24}^2(z-\bar z)}\left(2\mathrm{Li}_2(z)-2\mathrm{Li}_2(\bar z)+\log(z\bar z)\log\left(\frac{1-z}{1+z}\right)\right)$$

Comment 2

$$\begin{tikzcd}
D_{\Delta_1\Delta_2\Delta_3\Delta_4} \arrow[r, "\frac{\partial^2}{\partial x_{12}^2}"] \arrow[dr, "\frac{\partial}{\partial x_{13}^2}"'] & D_{\Delta_1+1,\Delta_2+1,\Delta_3\Delta_4} \arrow[r, "\frac{\partial}{\partial x_{34}^2}"] & D_{\Delta_1+1,\Delta_2+1,\Delta_3+1,\Delta_4+1} \\
& D_{\Delta_1+1,\Delta_2,\Delta_3+1,\Delta_4} \arrow[ur, "\frac{\partial}{\partial x_{24}^2}"'] &
\end{tikzcd}$$

D-functions are invariant under conformal Yangian

Impose nontrivial constraints on D-functions as PDES

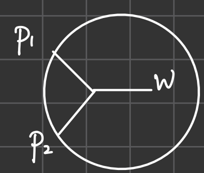
$\Longleftrightarrow$ Yangian invariance

Exchange Witten diagrams

$$(\Box_w-m^2)G_{BB}^{\Delta}(z,w)=\delta(z,w)$$

$$W_{\Delta,0}(P_i)=\int dz\,dw\;G_{B\partial}^{\Delta_1}(P_1,z)\,G_{B\partial}^{\Delta_2}(P_2,z)\,G_{BB}^{\Delta}(z,w)\,G_{B\partial}^{\Delta_3}(P_3,w)\,G_{B\partial}^{\Delta_4}(P_4,w)$$

$$A_{\Delta,0}(P_1,P_2,w)=\int dz\;G_{B\partial}^{\Delta_1}(P_1,z)\,G_{B\partial}^{\Delta_2}(P_2,z)\,G_{BB}^{\Delta}(z,w)$$



cross ratio $\chi=\frac{-2P_1\cdot P_2}{(-2P_1\cdot W)(-2P_2\cdot W)}=\frac{\vec x_{12}^{\,2}\,w_0^2}{[w_0^2+(\vec w-\vec x_1)^2][w_0^2+(\vec w-\vec x_2)^2]}$

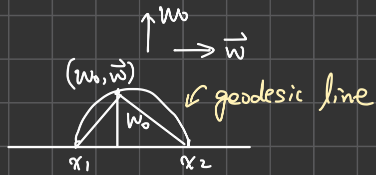
$$\left(\mathcal{L}^{(1)}_{AB}+\mathcal{L}^{(2)}_{AB}+\mathcal{L}^{W}_{AB}\right)A_{\Delta,0}(P_1,P_2,W)=0$$

$$\mathcal{L}^{(1)}_{AB}=P_{iA}\frac{\partial}{\partial P_{iB}}-P_{iB}\frac{\partial}{\partial P_{iA}}$$

$$\mathrm{Cas}\,A_{\Delta,0}=-\frac{1}{2}\left(\mathcal{L}^{(1)}_{AB}+\mathcal{L}^{(2)}_{AB}\right)^2A_{\Delta,0}=-\frac{1}{2}\left(\mathcal{L}^{W}_{AB}\right)^2A_{\Delta,0}=\Box_W A_{\Delta,0}$$

$$(\mathrm{Cas}-m^2)A_{\Delta,0}=(\Box_W-m^2)A_{\Delta,0}=G^{\Delta_1}_{B\partial}(P_1,W)\,G^{\Delta_2}_{B\partial}(P_2,W)$$

$$A_{\Delta,0}=\frac{1}{(-2P_1\cdot W)^{\Delta_1}(-2P_2\cdot W)^{\Delta_2}}f(\gamma)$$



EOM implies ODE for $f(\gamma)$

$$4\gamma^2(\gamma-1)f''(\gamma)+\gamma\left[4(\gamma-1)(\Delta_1+\Delta_2+1)+2d\right]f'(\gamma)+\left[(\Delta-\Delta_1-\Delta_2)(-d+\Delta+\Delta_1+\Delta_2)+4\Delta_1\Delta_2\gamma\right]f(\gamma)=0$$

Boundary conditions

1. $\gamma\to 0$, $f(\gamma)\to\gamma^{\frac{\Delta-\Delta_1-\Delta_2}{2}}$ (from ODE)
2. $\gamma\to 1$, regular

Let us assume $f(\gamma)$ has a power series expansion, $f(\gamma)=\sum_k a_k\gamma^k$

$$a_{k-1}=\frac{\left(k+\frac{\Delta-\Delta_1-\Delta_2}{2}\right)\left(k+\frac{\Delta+\Delta_1+\Delta_2}{2}-\frac{d}{2}\right)}{(k+\Delta_1-1)(k+\Delta_2-1)}a_k$$

$\to k-k_*$,

$$k_{max}=-1,\quad a_{k_{max}}=\frac{1}{4(\Delta_1-1)(\Delta_2-1)}\qquad a_{k^*-1}=0,\quad a_k=0\ (k\le k^*)$$

If $\underbrace{\Delta_1+\Delta_2-\Delta}_{2k_*}\in 2\mathbb{Z}_{>0}$, $f(\gamma)$ truncates into a polynomial

$$A_{\Delta,0}(P_1,P_2,W)=\sum_{k=k_{min}}^{k=k_{max}}\frac{a_k(-2P_1\cdot P_2)^k}{(-2P_1\cdot W)^{k+\Delta_1}(-2P_2\cdot W)^{k+\Delta_2}}$$

$$=\sum_{k=k_{min}}^{k=k_{max}}a_k x_{12}^{2k}\,G^{\Delta_1+k}_{B\partial}(P_1,W)\,G^{\Delta_2+k}_{B\partial}(P_2,W)$$

$$W_{\Delta,0}=\int dW\sum_k a_k x_{12}^{2k}\,G^{\Delta_1+k}_{B\partial}(P_1,W)\,G^{\Delta_2+k}_{B\partial}(P_2,W)\,G^{\Delta_3}_{B\partial}(P_3,W)\,G^{\Delta_4}_{B\partial}(P_4,W)$$

$$=\sum_k a_k x_{12}^{2k}D_{\Delta_1+k,\Delta_2+k,\Delta_3,\Delta_4}$$

General case

${}_3F_2$ + ${}_2F_1$ $\qquad \cancel{{}_2F_1}$

special solution

$f(\gamma)\to\sum_k\gamma^k$

$\Rightarrow$ Infinite sum of D-functions

Melling space "momentum space for AdS" Mellin transform

Consider n-point correlators of scalar operators

$$\langle O_1(x_1)\cdots O_n(x_n)\rangle = \int [d\delta_{ij}] \left(\prod_{i<j} (-2P_i\cdot P_j)^{-\delta_{ij}} \Gamma(\delta_{ij}) \right) \underbrace{M(\delta_{ij})}_{\text{Mellin amplitude}}$$

Set $\delta_{ij}=\delta_{ji}$, $\delta_{ii}=-\Delta_i$

$[\delta_{ij}]$ over independent δ_{ij} along imaginary axis

$\sum_j \delta_{ij}=0$ (from $P\to\lambda P$)

$$\frac{n(n-1)}{2}-n=\frac{n(n-3)}{2}$$

Constraints of δ_{ij} can be automatically solved if we write δ_{ij} as $\delta_{ij}=\vec{k}_i\cdot\vec{k}_j$

And impose $\sum_i \vec{k}_i=0$, $\vec{k}_i^2=-\Delta_i$ $(-m_i^2)$

We can view δ_{ij} as Mandelstam variables.

$\mathbb{R}^d$:

$$\#(\text{Mandelstam}) = \begin{cases} \frac{n(n-3)}{2} & \text{if } n\le D+1 \\ n(D-1)-\frac{(D+1)D}{2} & \text{if } n>D+1 \end{cases}$$

$D=d+1$: $n\le d+2$; $nd-\frac{(d+2)(d+1)}{2}$ $n>d+2$

Witten diagrams in Mellin space

Symanzik formula

$$I_n = \frac{\pi^{\frac{D}{2}}}{\prod_i \Gamma(\Delta_i)} (2\pi i)^{\frac{n(n-3)}{2}} \int [d\delta_{ij}] \prod_{i<j} \Gamma(\delta_{ij})\, x_{ij}^{-2\delta_{ij}}.$$

$D=\sum_i \Delta_i$ $\quad I_n \sim D_{\Delta_1,\cdots\Delta_n}$

Contact Witten diagram are constant in Mellin space.

$$M_{con}=\frac{\pi^{d/2}\Gamma(\frac{\Sigma-d}{2})}{\prod_i \Gamma(\Delta_i)}$$

$$D_\mu G_{B\partial}^{\Delta_1} D^\mu G_{B\partial}^{\Delta_2} = \Delta_1\Delta_2\left(G_{B\partial}^{\Delta_1} G_{B\partial}^{\Delta_2} - 2x_{12}^2\, G_{B\partial}^{\Delta_1+1} G_{B\partial}^{\Delta_2+1}\right)$$

$$W_{con,2d} = \Delta_1\Delta_2\left(D_{\Delta_1\Delta_2\cdots\Delta_n} - 2x_{12}^2 D_{\Delta_1+1,\Delta_2+1,\cdots,\Delta_n}\right)$$

$D_{\Delta_1+1,\Delta_2+1\cdots\Delta_n}$

δ_{ij} statisfy different relations

$$\sum_{j=2}^n \delta_{1j}=\Delta_1+1,\quad \sum_{j\neq 2}^n \delta_{2j}=\Delta_2+1 \quad\cdots\quad \sum_{j\neq i}\delta_{ij}=\Delta_i \ (\text{If } i\neq 1,2)$$

We can shift $\delta_{12}\longrightarrow \delta_{12}+1$, then δ_{ij} satisfy the same relations.

$$(x_{ij}^2)^{-\delta_{ij}} \qquad x_{12}^2(x_{12}^2)^{-\delta_{ij}} \longrightarrow x_{12}^2(x_{12}^2)^{-\delta_{ij}-1}=(x_{12}^2)^{-\delta_{12}}$$

$$\frac{\Gamma(\delta_{12}+1)}{\Gamma(\delta_{12})}=\delta_{12}$$

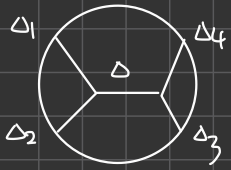
$$\mathcal{M}_{con,2\partial} = \frac{\pi^{d/2}\,\Gamma\left(\frac{\Sigma-d}{2}\right)}{\prod_i \Gamma(\Delta_i)}\left(\Delta_1\Delta_2-(\Sigma-d)\delta_{12}\right)$$

$2L-\partial$ contact $\longrightarrow$ degree-L polynomial

Exchange Witten diagrams

$$\delta_{12} = \frac{\Delta_1+\Delta_2-s}{2}, \quad \delta_{14} = \frac{\Delta_1+\Delta_4-t}{2}, \quad \delta_{13} = \frac{\Delta_1+\Delta_3-u}{2} \qquad s+t+u = \Delta_1+\Delta_2+\Delta_3+\Delta_4$$

$$\delta_{34} = \frac{\Delta_3+\Delta_4-s}{2}, \quad \delta_{23} = \frac{\Delta_2+\Delta_3-t}{2}, \quad \delta_{24} = \frac{\Delta_2+\Delta_4-u}{2}$$



Consider $\Delta = \Delta_1+\Delta_2-2$

$$W_{\Delta,0} \propto \frac{1}{x_{12}^2} D_{\Delta_1-1,\Delta_2-1,\Delta_3\Delta_4}$$

$$\frac{\Gamma(\delta_{12}-1)}{\Gamma(\delta_{12})} = \frac{1}{\delta_{12}-1} = \frac{1}{\frac{\Delta_1+\Delta_2-s}{2}-1} = \frac{-2}{s-(\Delta_1+\Delta_2-2)} = -\frac{2}{s-\Delta}$$

$\Delta = \Delta_1+\Delta_2-2m$

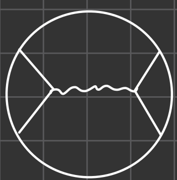
$$(x_{12}^2)^{-a} D_{\Delta_1-a;\Delta_2-a,\Delta_3,\Delta_4} \qquad \frac{\Gamma(\delta_{12}-a)}{\Gamma(\delta_{12})} = \frac{1}{(\delta_{12}-1)\cdots(\delta_{12}-a)}$$

$$\mathcal{M}_{\Delta=\Delta_1+\Delta_2-2m,0} \propto \left(\frac{1}{s-\Delta} + \underbrace{\frac{a_1}{s-\Delta-2} + \cdots + \frac{a_m}{s-\Delta-2m}}_{\text{conformal desendants}}\right)$$

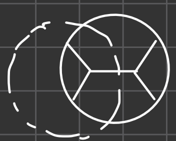
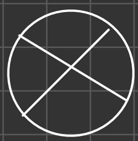
General Δ

$$\mathcal{M}_{\Delta,0} = \sum_{m=0}^{+\infty}\frac{a_m}{s-\Delta-2m}$$

Exchange spinning operators



$$\mathcal{M}_{\Delta,l} = \sum_{m=0}^{+\infty}\frac{Q_{l,m}(t)}{s-(\Delta-l)-2m} + P_{l-1}(s,t)$$

 $= \Sigma$ 

$$(\Box_w - m^2)\, G_{BB}(z,w) = \delta(z,w)$$

$$(Cas - m^2)\, A_{\Delta,0}(P_1,P_2,W) = G_{B\partial}^{\Delta_1}(P_1,W)\, G_{B\partial}^{\Delta_2}(P_2,W)$$

$$(Cas - m^2)\, W_{\Delta,0} = D_{\Delta_1\Delta_2\Delta_3\Delta_4}$$

$$W_{\Delta,0} = \frac{1}{(x_{12}^2x_{34}^2)^{\Delta_\phi}}\,\mathcal{W}_{\Delta,0}(U,V)$$

$$\mathcal{D}\,\mathcal{W}_{\Delta,0}(U,V) = \bar{D}(U,V)$$

$$G_4(x_i) = \frac{1}{(x_{12}^2x_{34}^2)^{\Delta_\phi}}\int_{-\infty}^{+\infty}\frac{ds\,dt}{(4\pi i)^2}\,U^{s/2}V^{\frac{t}{2}-\Delta_\phi}\,\Gamma^2\left(\frac{2\Delta_\phi}{2}\right)\Gamma^2\left(\frac{2\Delta_\phi}{2}\right)\Gamma^2\left(\frac{2\Delta_\phi-u}{2}\right)\mathcal{M}(s,t)$$

$$\widehat{U\partial_U} \longrightarrow \frac{s}{2}x, \quad \widehat{V\partial_V} \longrightarrow \left(\frac{t}{2}-\Delta_\phi\right)x, \quad \widehat{U^mV^n} \longrightarrow \mathcal{M}(s-2m,t-2m) = \frac{I_{stu}(s-2m,t-2m)}{I_{stu}(s,t)}$$

$\mathcal{D} \longrightarrow \hat{\mathcal{D}}$ difference operator