

Supplementary Material: R²LIVE: A Robust, Real-time, LiDAR-Inertial-Visual tightly-coupled state Estimator and mapping

A. Perturbation on $SO(3)$

In this appendix, we will use the following approximation of perturbation $\delta \mathbf{r} \rightarrow \mathbf{0}$ on $SO(3)$ [25, 26]:

$$\text{Exp}(\mathbf{r} + \delta \mathbf{r}) \approx \text{Exp}(\mathbf{r})\text{Exp}(\mathbf{J}_r(\mathbf{r})\delta \mathbf{r})$$

$$\text{Exp}(\mathbf{r})\text{Exp}(\delta \mathbf{r}) \approx \text{Exp}(\mathbf{r} + \mathbf{J}_r^{-1}(\mathbf{r})\delta \mathbf{r})$$

$$\mathbf{R} \cdot \text{Exp}(\delta \mathbf{r}) \cdot \mathbf{u} \approx \mathbf{R}(\mathbf{I} + [\delta \mathbf{r}]_{\times}) \mathbf{u} = \mathbf{R}\mathbf{u} - \mathbf{R}[\mathbf{u}]_{\times} \delta \mathbf{r}$$

where $\mathbf{u} \in \mathbb{R}^3$ and we use $[\cdot]_{\times}$ denote the skew-symmetric matrix of vector (\cdot) ; $\mathbf{J}_r(\mathbf{r})$ and $\mathbf{J}_r^{-1}(\mathbf{r})$ are called the *right Jacobian* and the *inverse right Jacobian* of $SO(3)$, respectively.

$$\mathbf{J}_r(\mathbf{r}) = \mathbf{I} - \frac{1 - \cos\|\mathbf{r}\|}{\|\mathbf{r}\|^2} [\mathbf{r}]_{\times} + \frac{\|\mathbf{r}\| - \sin(\|\mathbf{r}\|)}{\|\mathbf{r}\|^3} [\mathbf{r}]_{\times}^2$$

$$\mathbf{J}_r^{-1}(\mathbf{r}) = \mathbf{I} + \frac{1}{2} [\mathbf{r}]_{\times} + \left(\frac{1}{\|\mathbf{r}\|^2} - \frac{1 + \cos(\|\mathbf{r}\|)}{2\|\mathbf{r}\| \sin(\|\mathbf{r}\|)} \right) [\mathbf{r}]_{\times}^2$$

B. Computation of $\mathbf{F}_{\delta \mathbf{x}}$ and $\mathbf{F}_{\mathbf{w}}$

Combing (4) and (6), we have:

$$\begin{aligned} \delta \hat{\mathbf{x}}_{i+1} &= \mathbf{x}_{i+1} \boxminus \hat{\mathbf{x}}_{i+1} \\ &= (\mathbf{x}_i \boxplus (\Delta t \cdot \mathbf{f}(\mathbf{x}_i, \mathbf{u}_i, \mathbf{w}_i))) \boxminus (\hat{\mathbf{x}}_i \boxplus (\Delta t \cdot \mathbf{f}(\hat{\mathbf{x}}_i, \mathbf{u}_i, \mathbf{0}))) \\ &= \left[\begin{array}{c} \text{Log} \left(\left({}^G \hat{\mathbf{R}}_{I_i} \text{Exp}(\hat{\omega}_i \Delta t) \right)^T \cdot \left({}^G \hat{\mathbf{R}}_{I_i} \text{Exp}({}^G \delta \hat{\mathbf{r}}_{I_i}) \text{Exp}(\omega_i \Delta t) \right) \right) \\ \quad {}^G \delta \hat{\mathbf{p}}_{I_i} + {}^G \delta \hat{\mathbf{v}}_i \Delta t \\ \quad {}^I \delta \hat{\mathbf{p}}_{C_i} \\ {}^G \delta \hat{\mathbf{v}}_i + \left({}^G \hat{\mathbf{R}}_{I_i} \text{Exp}({}^G \delta \hat{\mathbf{r}}_{I_i}) \right) \mathbf{a}_i \Delta t - {}^G \hat{\mathbf{R}}_{I_i} \hat{\mathbf{a}}_i \Delta t \\ \quad \delta \hat{\mathbf{b}}_{\mathbf{g}_i} + \mathbf{n}_{\mathbf{b}_{\mathbf{g}_i}} \Delta t \\ \quad \delta \hat{\mathbf{b}}_{\mathbf{a}_i} + \mathbf{n}_{\mathbf{b}_{\mathbf{a}_i}} \Delta t \end{array} \right] \end{aligned}$$

with:

$$\hat{\omega}_i = \omega_{m_i} - \hat{\mathbf{b}}_{\mathbf{g}_i}, \quad \omega_i = \hat{\omega}_i - \delta \hat{\mathbf{b}}_{\mathbf{g}_i} - \mathbf{n}_{\mathbf{g}_i} \quad (\text{S1})$$

$$\hat{\mathbf{a}}_i = \mathbf{a}_{m_i} - \hat{\mathbf{b}}_{\mathbf{a}_i}, \quad \mathbf{a}_i = \hat{\mathbf{a}}_i - \delta \hat{\mathbf{b}}_{\mathbf{a}_i} - \mathbf{n}_{\mathbf{a}_i} \quad (\text{S2})$$

And we have the following simplification and approximation from Section. A.

$$\begin{aligned} &\text{Log} \left(\left({}^G \hat{\mathbf{R}}_{I_i} \text{Exp}(\hat{\omega}_i \Delta t) \right)^T \cdot \left({}^G \hat{\mathbf{R}}_{I_i} \text{Exp}({}^G \delta \hat{\mathbf{r}}_{I_i}) \text{Exp}(\omega_i \Delta t) \right) \right) \\ &= \text{Log} \left(\text{Exp}(\hat{\omega}_i \Delta t)^T \cdot \left(\text{Exp}({}^G \delta \hat{\mathbf{r}}_{I_i}) \cdot \text{Exp}(\omega_i \Delta t) \right) \right) \\ &\approx \text{Log} \left(\text{Exp}(\hat{\omega}_i \Delta t)^T \text{Exp}({}^G \delta \hat{\mathbf{r}}_{I_i}) \text{Exp}(\omega_i \Delta t) \cdot \right. \\ &\quad \left. \text{Exp}(-\mathbf{J}_r(\hat{\omega}_i \Delta t) (\delta \hat{\mathbf{b}}_{\mathbf{g}_i} + \mathbf{n}_{\mathbf{g}_i}) \Delta t) \right) \\ &\approx \text{Exp}(\hat{\omega}_i \Delta t) \cdot {}^G \delta \hat{\mathbf{r}}_{I_i} - \mathbf{J}_r(\hat{\omega}_i \Delta t) \delta \hat{\mathbf{b}}_{\mathbf{g}_i} \Delta t - \mathbf{J}_r(\hat{\omega}_i \Delta t) \mathbf{n}_{\mathbf{g}_i} \Delta t \\ &\quad \left({}^G \hat{\mathbf{R}}_{I_i} \text{Exp}({}^G \delta \hat{\mathbf{r}}_{I_i}) \right) \mathbf{a}_i \Delta t \\ &\approx \left({}^G \hat{\mathbf{R}}_{I_i} (\mathbf{I} + [{}^G \delta \hat{\mathbf{r}}_{I_i}]_{\times}) \right) (\hat{\mathbf{a}}_i - \delta \hat{\mathbf{b}}_{\mathbf{a}_i} - \mathbf{n}_{\mathbf{a}_i}) \Delta t \\ &\approx {}^G \hat{\mathbf{R}}_{I_i} \hat{\mathbf{a}}_i \Delta t - {}^G \hat{\mathbf{R}}_{I_i} \delta \hat{\mathbf{b}}_{\mathbf{a}_i} \Delta t - {}^G \hat{\mathbf{R}}_{I_i} \mathbf{n}_{\mathbf{a}_i} \Delta t - {}^G \hat{\mathbf{R}}_{I_i} [\hat{\mathbf{a}}_i]_{\times} {}^G \delta \hat{\mathbf{r}}_{I_i} \Delta t \end{aligned}$$

To conclude, we have the computation of $\mathbf{F}_{\delta \hat{\mathbf{x}}}$ and $\mathbf{F}_{\mathbf{w}}$ as follows:

$$\mathbf{F}_{\delta \hat{\mathbf{x}}} = \frac{\partial (\delta \hat{\mathbf{x}}_{i+1})}{\partial \delta \hat{\mathbf{x}}_i} \Big|_{\delta \hat{\mathbf{x}}_i = \mathbf{0}, \mathbf{w}_i = \mathbf{0}} = \begin{bmatrix} \text{Exp}(-\hat{\omega}_i \Delta t) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{J}_r(\hat{\omega}_i \Delta t) \Delta t & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{I} \Delta t & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -{}^G \hat{\mathbf{R}}_{I_i} [\hat{\mathbf{a}}_i]_{\times} \Delta t & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & -{}^G \hat{\mathbf{R}}_{I_i} \Delta t \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}$$

$$\mathbf{F}_{\mathbf{w}} = \frac{\partial (\delta \hat{\mathbf{x}}_{i+1})}{\partial \mathbf{w}_i} \Big|_{\delta \hat{\mathbf{x}}_i = \mathbf{0}, \mathbf{w}_i = \mathbf{0}} = \begin{bmatrix} -\mathbf{J}_r(\hat{\omega}_i \Delta t) \Delta t & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -{}^G \hat{\mathbf{R}}_{I_i} \Delta t & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \Delta t & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \Delta t \end{bmatrix}$$

C. The computation of \mathcal{H}

Recalling (15), we have:

$$\mathcal{H} = \frac{(\check{\mathbf{x}}_{k+1} \boxplus \delta \check{\mathbf{x}}_{k+1}) \boxminus \hat{\mathbf{x}}_{k+1}}{\partial \delta \check{\mathbf{x}}_{k+1}} \Big|_{\delta \check{\mathbf{x}}_{k+1} = \mathbf{0}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}_{3 \times 9} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0}_{3 \times 9} \\ \mathbf{0} & \mathbf{0} & \mathbf{B} & \mathbf{0} & \mathbf{0}_{3 \times 9} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0}_{3 \times 9} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{9 \times 9} \end{bmatrix}$$

with the 3×3 matrix $\mathbf{A} = \mathbf{J}_r^{-1}(\text{Log}({}^G \hat{\mathbf{R}}_{I_{k+1}} {}^T {}^G \check{\mathbf{R}}_{I_{k+1}}))$ and $\mathbf{B} = \mathbf{J}_r^{-1}(\text{Log}({}^I \check{\mathbf{R}}_{C_{k+1}} {}^T {}^I \check{\mathbf{R}}_{C_{k+1}}))$.

D. The computation of \mathbf{H}_j^l

Recalling (12) and (15), we have:

$$\begin{aligned} \mathbf{r}_l(\check{\mathbf{x}}_{k+1} \boxplus \delta \check{\mathbf{x}}_{k+1}, {}^L \mathbf{p}_j) &= \mathbf{u}_j^T ({}^G \check{\mathbf{p}}_{I_{k+1}} + {}^G \delta \check{\mathbf{p}}_{I_{k+1}} - \\ \mathbf{q}_j + {}^G \check{\mathbf{R}}_{I_{k+1}} \text{Exp}({}^G \delta \check{\mathbf{r}}_{I_{k+1}}) ({}^I \mathbf{R}_L {}^L \mathbf{p}_j + {}^I \mathbf{p}_L)) \end{aligned} \quad (\text{S3})$$

And with the small perturbation approximation, we get:

$$\begin{aligned} &{}^G \check{\mathbf{R}}_{I_{k+1}} \text{Exp}({}^G \delta \check{\mathbf{r}}_{I_{k+1}}) \mathbf{P}_{\mathbf{a}} \\ &\approx {}^G \check{\mathbf{R}}_{I_{k+1}} \left(\mathbf{I} + \left[{}^G \delta \check{\mathbf{r}}_{I_{k+1}} \right]_{\times} \right) \mathbf{P}_{\mathbf{a}} \\ &= {}^G \check{\mathbf{R}}_{I_{k+1}} \mathbf{P}_{\mathbf{a}} - {}^G \check{\mathbf{R}}_{I_{k+1}} [\mathbf{P}_{\mathbf{a}}]_{\times} {}^G \delta \check{\mathbf{r}}_{I_{k+1}} \end{aligned} \quad (\text{S4})$$

where $\mathbf{P}_{\mathbf{a}} = {}^I \mathbf{R}_L {}^L \mathbf{p}_j + {}^I \mathbf{p}_L$. Combining (S3) and (S4) together we can obtain:

$$\mathbf{H}_j^l = \mathbf{u}_j^T \left[-{}^G \check{\mathbf{R}}_{I_{k+1}} [\mathbf{P}_{\mathbf{a}}]_{\times} \quad \mathbf{I}_{3 \times 3} \quad \mathbf{0}_{3 \times 15} \right]$$

E. The computation of \mathbf{H}_s^c and $\mathbf{F}_{\mathbf{P}_s}$

Recalling (16), we have:

$${}^C \mathbf{P}_s = \mathbf{P}_{\mathbf{C}}(\check{\mathbf{x}}_{k+1}, {}^G \mathbf{P}_s) = [{}^C P_{sx} \quad {}^C P_{sy} \quad {}^C P_{sz}]^T$$

where the function $\mathbf{P}_{\mathbf{C}}(\check{\mathbf{x}}_{k+1}, {}^G \mathbf{P}_s)$ is:

$${}^C \mathbf{P}_s = ({}^G \check{\mathbf{R}}_{I_{k+1}} {}^I \check{\mathbf{R}}_{C_{k+1}})^T {}^G \mathbf{P}_s - {}^I \check{\mathbf{R}}_{C_{k+1}}^T {}^I \check{\mathbf{p}}_{C_{k+1}} \quad (\text{S5})$$

$$- ({}^G \check{\mathbf{R}}_{I_{k+1}} {}^I \check{\mathbf{R}}_{C_{k+1}})^T {}^G \check{\mathbf{p}}_{I_{k+1}} \quad (\text{S6})$$

From (20), we have:

$$\mathbf{r}_c(\check{\mathbf{x}}_{k+1}, {}^C \mathbf{p}_s, {}^G \mathbf{P}_s) = {}^C \mathbf{p}_s - \pi({}^C \mathbf{P}_s)$$

$$\pi({}^C \mathbf{P}_s) = \left[f_x \frac{{}^C P_{sx}}{{}^C P_{sz}} + c_x \quad f_y \frac{{}^C P_{sy}}{{}^C P_{sz}} + c_y \right]^T \quad (\text{S7})$$

where f_x and f_y are the focal length, c_x and c_y are the principal point offsets in image plane.

For conveniently, we omit the $(\cdot)|_{\delta\check{\mathbf{x}}_{k+1}=\mathbf{0}}$ in the following derivation, and we have:

$$\mathbf{H}_s^c = -\frac{\partial\pi({}^C\mathbf{P}_s)}{\partial{}^C\mathbf{P}_s} \cdot \frac{\partial\mathbf{P}_C(\check{\mathbf{x}}_{k+1} \boxplus \delta\check{\mathbf{x}}_{k+1}, {}^G\mathbf{P}_s)}{\partial\delta\check{\mathbf{x}}_{k+1}} \quad (\text{S8})$$

$$\mathbf{F}_{\mathbf{P}_s} = -\frac{\partial\pi({}^C\mathbf{P}_s)}{\partial{}^C\mathbf{P}_s} \cdot \frac{\partial\mathbf{P}_C(\check{\mathbf{x}}_{k+1}, {}^G\mathbf{P}_s)}{\partial{}^G\mathbf{P}_s} \quad (\text{S9})$$

where:

$$\frac{\partial\pi({}^C\mathbf{P}_s)}{\partial{}^C\mathbf{P}_s} = \frac{1}{{}^C P_{sz}} \begin{bmatrix} f_x & 0 & -f_x \frac{{}^C P_{sx}}{{}^C P_{sz}} \\ 0 & f_y & -f_y \frac{{}^C P_{sy}}{{}^C P_{sz}} \end{bmatrix} \quad (\text{S10})$$

$$\frac{\partial\mathbf{P}_C(\check{\mathbf{x}}_{k+1}, {}^G\mathbf{P}_s)}{\partial{}^G\mathbf{P}_s} = \left({}^G\check{\mathbf{R}}_{I_{k+1}} {}^I\check{\mathbf{R}}_C \right)^T \quad (\text{S11})$$

According to Section. A, we have the following approximation of $\mathbf{P}_C(\check{\mathbf{x}}_{k+1} \boxplus \delta\check{\mathbf{x}}_{k+1}, {}^G\mathbf{P}_s)$:

$$\begin{aligned} & \mathbf{P}_C(\check{\mathbf{x}}_{k+1} \boxplus \delta\check{\mathbf{x}}_{k+1}, {}^G\mathbf{P}_s) \\ &= \left({}^G\check{\mathbf{R}}_{I_{k+1}} \text{Exp} \left({}^G\delta\check{\mathbf{r}}_{I_{k+1}} \right) {}^I\check{\mathbf{R}}_{C_{k+1}} \text{Exp} \left({}^I\delta\check{\mathbf{r}}_{C_{k+1}} \right) \right)^T {}^G\mathbf{P}_s \\ &- \left({}^I\check{\mathbf{R}}_{C_{k+1}} \text{Exp} \left({}^I\delta\check{\mathbf{r}}_{C_{k+1}} \right) \right)^T \left({}^I\check{\mathbf{p}}_{C_{k+1}} + {}^I\delta\check{\mathbf{p}}_{C_{k+1}} \right) \\ &- \left({}^G\check{\mathbf{R}}_{I_{k+1}} \text{Exp} \left({}^G\delta\check{\mathbf{r}}_{I_{k+1}} \right) {}^I\check{\mathbf{R}}_{C_{k+1}} \text{Exp} \left({}^I\delta\check{\mathbf{r}}_{C_{k+1}} \right) \right)^T \left({}^G\check{\mathbf{p}}_{I_{k+1}} + {}^G\delta\check{\mathbf{p}}_{I_{k+1}} \right) \\ &\approx \mathbf{P}_b(\check{\mathbf{x}}_{k+1}, {}^G\mathbf{P}_s) - \left({}^I\check{\mathbf{R}}_{C_{k+1}} \right)^T \left({}^G\check{\mathbf{R}}_{I_{k+1}} \right)^T {}^G\delta\check{\mathbf{p}}_{I_{k+1}} \\ &+ \left({}^I\check{\mathbf{R}}_{C_{k+1}} \right)^T \left[{}^G\check{\mathbf{R}}_{I_{k+1}}^T ({}^G\mathbf{P}_s - {}^G\check{\mathbf{p}}_{I_{k+1}}) \right]_{\times} {}^G\delta\check{\mathbf{r}}_{I_{k+1}} \\ &+ \left[\mathbf{P}_b(\check{\mathbf{x}}_{k+1}, {}^G\mathbf{P}_s) \right]_{\times} {}^I\delta\check{\mathbf{r}}_{C_{k+1}} - \left({}^I\check{\mathbf{R}}_C \right)^T {}^I\delta\check{\mathbf{p}}_{C_{k+1}} \end{aligned} \quad (\text{S12})$$

With this, we can derive:

$$\begin{aligned} & \frac{\partial\mathbf{P}_C(\check{\mathbf{x}}_{k+1} \boxplus \delta\check{\mathbf{x}}_{k+1}, {}^G\mathbf{P}_s)}{\partial\delta\check{\mathbf{x}}_{k+1}} = [\mathbf{M}_A \quad \mathbf{M}_B \quad \mathbf{M}_C \quad \mathbf{M}_D \quad \mathbf{0}_{3 \times 12}] \quad (\text{S13}) \\ & \mathbf{M}_A = \left({}^I\check{\mathbf{R}}_{C_{k+1}} \right)^T \left[{}^G\check{\mathbf{R}}_{I_{k+1}}^T ({}^G\mathbf{P}_s - {}^G\check{\mathbf{p}}_{I_{k+1}}) \right]_{\times} \\ & \mathbf{M}_B = - \left({}^I\check{\mathbf{R}}_{C_{k+1}} \right)^T \left({}^G\check{\mathbf{R}}_{I_{k+1}} \right)^T \\ & \mathbf{M}_C = \left[{}^C\mathbf{P}_s \right]_{\times} \\ & \mathbf{M}_D = - \left({}^I\check{\mathbf{R}}_C \right)^T \end{aligned} \quad (\text{S14})$$

Substituting (S10), (S11) and (S13) into (S8) and (S9), we finish the computation of \mathbf{H}_s^c and $\mathbf{F}_{\mathbf{P}_s}$.