Review

Relations

Definition

- An *equivalence relation* on a set S is a set *R* of ordered pairs of elements of S such that

$(a,a) \in R$ for all a in S. *Reflexive*

$(a,b) \in R$ implies $(\mathrm{b}, \mathrm{a}) \in \mathrm{R}$ *Symmetric*

$(a,b) \in R$ and $(b,c) \in R$ imply $(a,c) \in R$ *Transitive*

Antisymmetry

- A relation R on a set A is *antisymmetric* if for all $a, b \in A$,

$$(a, b) \in R \text{ and } (b, a) \in R \rightarrow a = b.$$

- This is equivalent to

$$(a, b) \in R \text{ and } a \neq b \rightarrow (b, a) \notin R.$$

- The following relations are antisymmetric.
  - $b \mid a$, on $\mathbb{Z}^+$.
  - $x \leq y$, on $\mathbb{R}$.

Partial Order Relations

- A relation R on a set A is a *partial order* relation if
  - R is reflexive.
  - R is antisymmetric.
  - R is transitive.
- We use $\leq$ as the generic symbol for a partial order relation.

Problem 2. (10 points). You are given three relations $P, Q, R \subseteq \{a, b, c, d\} \times \{a, b, c, d\}$:

P	a	b	c	d
a	Y	N	Y	N
b	N	Y	N	Y
c	Y	N	Y	N
d	N	Y	N	Y

Q	a	b	c	d
a	Y	Y	N	Y
b	N	Y	N	Y
c	N	N	Y	Y
d	N	N	N	Y

R	a	b	c	d
a	Y	N	N	N
b	N	N	N	Y
c	N	N	N	Y
d	N	N	Y	N

For each relation tell (write Y or N) whether it is:

	Reflexive	Transitive	Symmetric	Partial order	Equivalence
P					
Q					
R					

Logic

Problem 3. (10 points). For each sentence (a)-(e) below, tell which of the sentences (i)-(v) is its negation.

(a) **"If X is green, then X is a vegetable."**

(i) "X is not green and X is not a vegetable."
(ii) "X is not green or X is a vegetable."
(iii) "X is green and X is not a vegetable."
(iv) "If X is green then X is not a vegetable."
(v) None of the above.

(b) "$\forall x\, \exists y : y < x + 10$"

(i) "$\exists x\, \exists y : y > x + 10$."
(ii) "$\forall x\, \exists y : y \geq x + 10$."
(iii) "$\forall y\, \exists x : x + 10 < y$."
(iv) "$\exists x\, \forall y : y > x + 10$."
(v) None of the above.

For each of the statements below, tell whether it is true or false.
Justify your answer.

statement	T/F
$\exists\, x \in \mathbb{R} \,:\, x^2 + x = 2$	
$\exists\, x \in \mathbb{R} \,:\, x^2 + x = -2$	
$\forall\, x \in \mathbb{R} \,:\, (x^2 > 4) \implies (x > 2)$	
$\forall\, x \in \mathbb{R}\; \exists\, y \in \mathbb{R} \,:\, xy^2 + x = 1$	
$\exists\, x \in \mathbb{R}\; \forall\, y \in \mathbb{R} \,:\, xy^2 + 2^x = 1$	

For each of the statements below, tell whether it is true or false.
Justify your answer.

statement	T/F
$\exists\, x \in \mathbb{R} \,:\, x^2 + x = 2$	T
$\exists\, x \in \mathbb{R} \,:\, x^2 + x = -2$	F
$\forall\, x \in \mathbb{R} \,:\, (x^2 > 4) \implies (x > 2)$	F
$\forall\, x \in \mathbb{R}\; \exists\, y \in \mathbb{R} \,:\, xy^2 + x = 1$	F
$\exists\, x \in \mathbb{R}\; \forall\, y \in \mathbb{R} \,:\, xy^2 + 2^x = 1$	T

Mathematical Induction

Principle of Mathematical Induction

Let P(n) be a predicate defined for int. $n \in N_0$.

1. Base case:

   P(n) is true for n = a ($a \in N_0$)

2. Induction hypothesis:

   P(n) is true for n = k ($k \geq a$)

   implies that

3. Inductive step:

   P(n) is true for n = k + 1.

Then for all integers $n \geq a$, P(n) is true.

Example: Sum of Odd Integers

$1 + 3 + \ldots + (2n-1) = n^2$ (P(n))

for all integers $n \geq 1$.

Proof (by induction on n):

1. Base case:

   The statement (P(n)) is true for $n = 1$: $1=1^2$.

2. Induction hypothesis:

   Assume the statement is true for $n = k \geq 1$:

   $1 + 3 + \ldots + (2k-1) = k^2$

3. Inductive step:

   Show that the assumption implies that P(n) is true for $k + 1$:

   $1 + 3 + \ldots + (2(k+1) -1) = (k + 1)^2$

Example: Sum of Odd Integers

Proof (cont.):

The statement is true for k:

$$1+3+\ldots+(2k-1) = k^2 \qquad \textbf{(1)}$$

We need to prove:

$$1+3+\ldots+(2(k+1)-1) = (k+1)^2 \qquad \textbf{(2)}$$

Proof: $1+3+\ldots+(2(k+1)-1) = 1+3+\ldots+(2k+1) =$

$= (\ 1+3+\ldots+(2k-1)\ )+(2k+1) =$

by (1)

$= k^2 + (2k+1) =$

$= (k+1)^2$.

We proved that:

P(n) is true for n = 1, and $P(k) \rightarrow P(k + 1)$ for all $n \geq 1$,

thus P(n) is true for all integers $n \geq 1$.

Important theorems proved by mathematical induction

➢**Theorem 1 (*Sum of the first n integers*):**

For all integers n≥1,

$$1+2+\ldots+n=\frac{n(n+1)}{2}$$

➢**Theorem 2 (*Sum of a geometric series*):**

For any real number r except 1, and any integer n≥0,

$$\sum_{i=0}^{n} r^{i}=\frac{r^{n+1}-1}{r-1}$$

Proving a divisibility property by math. induction

Prove, that for any integer $n \geq 1$, $7^n - 2^n$ is divisible by 5. (P(n))

- **Proof** (*by induction*):

1. Base case:

   The statement is true for n=1: (P(1))

   $7^1 – 2^1 = 7 - 2 = 5$ is divisible by 5.

2. Induction hypothesis:

   Assume that P(n) is true for $n = k$ ($k \geq 1$):

   $7^k - 2^k$ is divisible by 5. (P(k))

3. Inductive step: show that P(n) is true for $n = k+1$:

   $7^{k+1} - 2^{k+1}$ is divisible by 5. (P(k+1))

Proving a divisibility property by math. induction

Proof (cont.): We are given that $7^k - 2^k$ is divisible by 5. (1)

Then $7^k - 2^k = 5b$ for some $b \in \mathbf{N}$. *(by definition)* (2)

We need to prove:

$7^{k+1} - 2^{k+1}$ is divisible by 5. (3)

$7^{k+1} - 2^{k+1} = 7 \cdot 7^k - 2 \cdot 2^k = (5 + 2) \cdot 7^k - 2 \cdot 2^k = 5 \cdot 7^k + 2 \cdot 7^k - 2 \cdot 2^k =$

$= 5 \cdot 7^k + 2 \cdot (7^k - 2^k) = 5 \cdot 7^k + 2 \cdot 5b$ *(by (2))*

$= 5 \cdot (7^k + 2b)$ which is divisible by 5. *(by def.)*

Thus, P(n) is true for all integers $n \geq 1$. ■

The sum of consecutive squares

Prove, using math. induction, the closed form expression for the sum of consecutive squares

$$\sum_{i=1}^{n} i^2 = \frac{(2n+1)(n+1)n}{6}$$

Mathematical Induction

False Theorem 5.1.3. *In every set of $n \geq 1$ horses, all the horses are the same color.*

Mathematical Induction

Bogus proof. The proof is by induction on n. The induction hypothesis $P(n)$ will be

$$\text{In every set of } n \text{ horses, all are the same color.} \tag{5.3}$$

Base case: ($n = 1$). $P(1)$ is true, because in a size-1 set of horses, there's only one horse, and this horse is definitely the same color as itself.

Mathematical Induction

Inductive step: Assume that $P(n)$ is true for some $n \geq 1$. That is, assume that in every set of n horses, all are the same color. Now suppose we have a set of $n + 1$ horses:

$$h_1,\ h_2,\ \ldots,\ h_n,\ h_{n+1}.$$

By our assumption, the first n horses are the same color:

$$\underbrace{h_1,\ h_2,\ \ldots,\ h_n}_{},\ h_{n+1}$$

Also by our assumption, the last n horses are the same color:

$$h_1,\ \underbrace{h_2,\ \ldots,\ h_n,\ h_{n+1}}_{\text{same color}}$$