Chapter 2

Probability

The term **probability** refers to the study of randomness and uncertainty. The discipline of probability provides methods for quantifying the chances, or likelihoods, associated with the various outcomes.

2.1 Sample Spaces and Events

Experiment – an activity or process that whose outcome is subject to uncertainty. That is, experiment lead to random outcomes.

Sample Space – the set of all possible outcomes of an experiment, denoted by S.

Event – any collection (subset) of outcomes of interest denoted by a capital letter. Event is a subset of the sample space. Events can be **simple** (consists of exactly one outcome) or **compound** (consists of more than one outcomes).

Venn diagram – a graphical representation of events that is very useful for illustrating logical relations.

Complement of an event A – denoted by A', is the event that A does not occur.
$A' = \{x \mid x \notin A\}$

Union of two events A and B – denoted by $A \cup B$, is the event that either A or B occurs, or they both occur. That is, at least one of the events occur.
$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

Intersection of two events A and B – denoted by $A \cap B$, is the event that both A and B occur simultaneously.
$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

Null event – denoted by $\emptyset$, consists of no outcomes.

Two events A and B are said to be **mutually exclusive** or **disjoint** if $A \cap B = \emptyset$ so that they have no outcomes in common.

$A \cap A =$	$A \cup A =$	$\emptyset \cup \emptyset =$
$A \cap S =$	$A \cup S =$	$\emptyset \cap \emptyset =$
$A \cap \emptyset =$	$A \cup \emptyset =$	$S \cup S =$
$A \cap A' =$	$A \cup A' =$	$S \cap S =$

Exercise 2.4 Each of a sample of four home mortgages is classified as fixed rate (F) or variable rate (V).

(a) What are the 16 outcomes in S?

(b) Which outcomes are in the event that all four mortgages are of the same type?

(c) Which outcomes are in the event that at most one of the four is a variable-rate mortgage?

(d) What is the union of the events in parts (b) and (c), and what is the intersection of these two events?

2.2 Axioms, Interpretations, and Properties of Probability

Given an experiment and a sample space S, the objective of probability is to assign to each event A a number $P(A)$, called the **probability** of the event A, which will give a precise measure of the chance that A will occur.

Law of Large Numbers – the relative frequency of the number of times that an outcome occurs when an experiment is replicated over and over again approaches the theoretical probability of the outcome.

The probability should satisfy the following **three axioms (basic properties)**:

Axiom 1 For any event A, $P(A) \geq 0$.

Axiom 2 $P(S) = 1$.

Axiom 3 If $A_1, A_2, \cdots$, is an infinite collection of disjoint events, then

$$P\left(A_1 \cup A_2 \cup A_3 \cup \cdots\right) = \sum_{i=1}^{\infty} P(A_i).$$

Properties of Probability

- $P(\emptyset) = 0$

- $A_1, \cdots, A_k$ are disjoint events, then
$$P(A_1 \cup A_2 \cup A_3 \cdots A_k) = \sum_{i=1}^{k} P(A_i)$$

- $P(A) + P(A') = 1$

- $0 \leq P(A) \leq 1$

- For any two events A and B,
  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

  $\Rightarrow$ If events A and B are disjoint events, then
  $P(A \cup B) = P(A) + P(B)$.

- For any three events A, B, and C,

$$\begin{aligned} P(A \cup B \cup C) = & P(A) + P(B) + P(C) \\ & - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ & + P(A \cap B \cap C) \end{aligned}$$

<u>Exercise 2.12</u> Consider randomly selecting a student at a certain university, and let A denote the event that the selected individual has a Visa credit card and B be the analogous event for a MasterCard. Suppose that $P(A) = .5$, $P(B) = .4$, and $P(A \cap B) = .25$.

(a) Compute the probability that the selected individual has at least one of the two types of cards (i.e., the probability of the event $A \cup B$).

(b) What is the probability that the selected individual has neither type of card?

(c) Describe, in terms of A and B, the event that the selected student has a Visa card but not a MasterCard, and then calculate the probability of this event.

When the sample space S is either finite or "countably infinite", the probability of any event A is computed by adding together the individual probabilities for each outcome in A.

If $A = \{E_1, E_2, \cdots, E_k\}$, then $P(A) = \sum_{i=1}^{k} P(E_i)$.

When all the outcomes in the sample space S are equally likely to happen,

$$P(A) = \frac{\text{number of outcomes within A}}{\text{total number of outcomes}} = \frac{N(A)}{N}.$$

Example There are 2 black balls and 2 white balls in a box. Suppose you close your eyes and randomly draw 2 balls from the box. What is the probability that you get exactly 1 black ball and 1 white ball?

2.3 Counting Techniques

Product Rule for Ordered Pairs

If the first element of an ordered pair can be selected in n_1 ways, and for each of these n_1 ways the second element of the pair can be selected in n_2 ways, then the number of pairs is $n_1 n_2$.

Example Suppose that a box contains 7 red balls and 5 blue balls. If two balls are drawn out at random, how many possible ordered pairs are there? What is the probability that the first one is red and the second one is blue?

General Product Rule for k-Tuples

Suppose a set consists of ordered collections of k elements (k-tuples) and that there are n_1 possible choices for the first element; for each choice of the first element, there are n_2 possible choices of the second element; $\ldots$; for each possible choice of the first $k-1$ elements, there are n_k possible choices of the kth element. Then there are a total of $n_1 n_2 \cdots n_k$ possible k-tuples.

Example Suppose a package of 30 Milk Chocolate M&M candies contain 9 brown, 10 green and 11 yellow M&M candies. If you randomly choose 3 candies, what's the probability of getting all three colors?

Permutations and Combinations

Permutation – order does matter
An ordered subset is called a **permutation**. The number of permutations of size k that can be formed from the n individuals or objects in a group will be denoted by $P_{k,n}$.

$$P_{k,n} = \frac{n!}{(n-k)!}$$

Combination – order does NOT matter
An unordered subset is called a **combination**. The number of permutations of size k that can be formed from the n individuals or objects in a group will be denoted by $C_{k,n}$, or $\binom{n}{k}$.

$$C_{k,n} = \binom{n}{k} = \frac{P_{k,n}}{k!} = \frac{n!}{k!(n-k)!}$$

Example: Suppose we have 18 people competing in a game. How many ways can we award a 1st, 2nd and 3rd place prize among the 18 contestants?

Example: Suppose we need to select a committee of size 3 from a group of 18 people. How many ways can the committee of size 3 be selected?

Exercise 2.38 A box in a certain supply room contains four 40-W light bulbs, five 60-W bulbs, and six 75-W bulbs. Suppose that three bulbs are randomly selected.

(a) What is the probability that exactly two of the selected bulbs are rated 75 W?
(b) What is the probability that all three of the selected bulbs have the same rating?
(c) What is the probability that one bulb of each type is selected?
(d) Suppose now that bulbs are to be selected one by one until a 75-W bulb is found. What is the probability that it is necessary to examine at least six bulbs?

2.4 Conditional Probability

Example The probability of observing an even number (event A) on a toss of a fair die is 0.5, where $S = \{1, 2, 3, 4, 5, 6\}$ and $A = \{2, 4, 6\}$. Suppose we're given the information that on a particular throw of the die the result was a number less than or equal to 3 (event B), where $B = \{1, 2, 3\}$. Would the probability of observing an even number on that throw of the die still be equal to 0.5?

For any two events A and B with $P(B) > 0$, the **conditional probability of A given B has occurred** is defined by

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Multiplication Rule for $P(A \cap B)$

$$P(A \cap B) = P(A \mid B) \cdot P(B) = P(B \mid A) \cdot P(A)$$

Exercise 2.47 Return to the credit card scenario of Exercise 2.12, where $A = \{\text{Visa}\}$, $B = \{\text{MasterCard}\}$, $P(A) = .5$, $P(B) = .4$, and $P(A \cap B) = .25$. Calculate and interpret each of the following probabilities .

(a) $P(B \mid A)$

(b) $P(B' \mid A)$

(c) $P(A \mid B)$

(d) $P(A' \mid B)$

(e) Given that the selected individual has at least one card, what is the probability that he or she has a Visa card?

Exercise 2.56 For any events A and B with $P(B) > 0$, show that
$P(A \mid B) + P(A' \mid B) = 1$.

A set of events $A_1, A_2, \cdots, A_k$ is **exhaustive** if at least one of the events must occur.

$$A_1 \cup A_2 \cup \cdots \cup A_k = \bigcup_{i=1}^{k} A_i = S$$

Law of Total Probability

Let $A_1, A_2, \cdots, A_k$ be mutually exclusive and exhaustive events. The unconditional probability of B, $P(B)$, can then be written as a weighted average of the conditional probabilities of B given A_i, $P(B \mid A_i)$, as follows:

$$P(B) = \sum_{i=1}^{k} P(B \mid A_i) P(A_i)$$

Bayes' Theorem

Let $A_1, A_2, \cdots, A_k$ be a collection of k mutually exclusive and exhaustive events with **prior probabilities** $P(A_i)$ ($i = 1, \cdots, k$). Then for any other event B for which $P(B) > 0$, the **posterior probability** of A_j given that B has occurred is

$$P(A_j \mid B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B \mid A_j)P(A_j)}{\sum_{i=1}^{k} P(B \mid A_i)P(A_i)},$$

for $j = 1, \cdots, k$.

$\Rightarrow$ For any two events A and B with $P(B) > 0$,

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid A')P(A')}$$

<u>Exercise 2.59</u> At a certain gas station, 40% of the customers use regular gas (A_1), 35% use plus gas (A_2), and 25% use premium (A_3). Of those customers using regular gas, only 30% fill their tanks (event B). Of those customers using plus, 60% fill their tanks, whereas of those using premium, 50% fill their tanks.

(a) What is the probability that the next customer will request plus gas and fill the tank ($A_2 \cap B$)?

(b) What is the probability that the next customer fills the tank?

(c) If the next customer fills the tank, what is the probability that regular gas is requested? Plus? Premium?

Exercise 2.60 Seventy percent of the light aircraft that disappear while in flight in a certain country are subsequently discovered. Of the aircraft that are discovered, 60% have an emergency locator, whereas 90% of the aircraft not discovered do not have such a locator. Suppose a light aircraft has disappeared.

(a) If it has an emergency locator, what is the probability that it will not be discovered?

(b) If it does not have an emergency locator, what is the probability that it will be discovered?

2.5 Independence

Two events A and B are said to be **independent** (denoted by $A \perp B$) if $P(A \mid B) = P(A)$ and are **dependent** otherwise.

$P(A \mid B) = P(A) \iff P(B \mid A) = P(B)$

Multiplication Rule for $P(A \cap B)$

A and B are **independent** if and only if (iff) $P(A \cap B) = P(A)P(B)$

More generally, events $A_1, \cdots, A_n$ are **mutually independent** if for every k $(k = 2, \cdots, n)$ and every subset of indices $i_1, \cdots, i_k$,

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_k})$$

Example: If A and B are disjoint with $P(A) > 0$ and $P(B) > 0$, are they independent?

Exercise 2.74 Suppose that the proportions of blood phenotypes in a particular population are as follows:

A	B	AB	O
.42	.10	.04	.44

Assuming that the phenotypes of two randomly selected individuals are independent of one another, what is the probability that both phenotypes are O? What is the probability that the phenotypes of two randomly selected individuals match?

<u>Exercise 2.80</u> Consider the system of components connected as in the accompanying picture. Components 1 and 2 are connected in parallel, so that subsystem works iff either 1 or 2 works; since 3 and 4 are connected in series, that subsystem works iff both 3 and 4 work. If components work independently of one another and P(component works) = .9, calculate P(system works).