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1. (i) Prove $C_i P_i = I$, assuming $C_{i-1} P_{i-1} = I$

$$C_i P_i = (r(t)r(t)^T + C_{i-1}) P_{i-1} - (r(t)r(t)^T + C_{i-1}) \frac{P_{i-1} r(t) r(t)^T P_{i-1}}{1 + r(t)^T P_{i-1} r(t)}$$

$$= r(t) r(t)^T P_{i-1} + \underbrace{C_{i-1} P_{i-1}}_{= I} - \frac{(r(t) r(t)^T P_{i-1})^2 + \underbrace{C_{i-1} P_{i-1}}_{=I} r(t) r(t)^T P_{i-1}}{1 + r(t)^T P_{i-1} r(t)}$$

$$= r(t) r(t)^T P_{i-1} + I - \frac{r(t) r(t)^T P_{i-1} \, (r(t) r(t)^T P_{i-1} + 1)}{1 + r(t)^T P_{i-1} r(t)}$$

$\therefore$ For $C_i P_i \equiv I$, need to prove:

$$r(t) r(t)^T P_{i-1} = \frac{r(t) r(t)^T P_{i-1} (r(t) r(t)^T P_{i-1} + 1)}{1 + r(t)^T P_{i-1} r(t)}$$

$$(r(t) r(t)^T P_{i-1})(1 + r(t)^T P_{i-1} r(t)) = r(t) r(t)^T P_{i-1} (r(t) r(t)^T P_{i-1} + 1)$$

As $r(t)^T P_{i-1} r(t)$ has a shape of a scalar,

$$LHS = r(t) r(t)^T P_{i-1} + r(t) r(t)^T P_{i-1} r(t) r(t)^T P_{i-1}$$

$$= r(t) r(t)^T P_{i-1} (r(t) r(t)^T P_{i-1} + 1) = RHS$$

done

(ii). Prove $W_i = A_i C_i^{-1} = W_{i-1} - (W_{i-1} r(t) - y(t))) r(t)^T P_i$

As $C_i P_i = I$, the question is equivalent to prove $W_i = A_i P_i$

similar to (i), let's assume $A_{i-1} P_{i-1} = W_{i-1}$

$$A_i P_i = (y(t) r(t)^T + A_{i-1}) P_i = y(t) r(t)^T P_i + A_{i-1} P_i \quad (1)$$

$$W_i = W_{i-1} - W_{i-1} r(t) r(t)^T P_i + y(t) r(t)^T P_i \quad (2)$$

For $(1) \equiv (2)$, $A_{i-1} P_i = W_{i-1} (1 - r(t) r(t)^T P_i)$

$$= A_{i-1} P_{i-1} (1 - r(t) r(t)^T P_i)$$

$$P_i = P_{i-1} - P_{i-1} r(t) r(t)^T P_i$$

$$P_i = \frac{P_{i-1}}{1 + P_{i-1} r(t) r(t)^T}$$

The question has turned to proving:

$$\frac{P_{i-1}}{1 + P_{i-1} r(t) r(t)^T} = P_{i-1} - \frac{P_{i-1} r(t) r(t)^T P_{i-1}}{1 + r(t)^T P_{i-1} r(t)}$$

$$P_{i-1} = P_{i-1} + P_{i-1} r(t) r(t)^T P_{i-1} - \frac{P_{i-1} r(t) r(t)^T P_{i-1}}{1 + r(t)^T P_{i-1} r(t)}$$

$$- \frac{P_{i-1} r(t) r(t)^T P_{i-1} r(t) r(t)^T P_{i-1}}{1 + r(t)^T P_{i-1} r(t)}$$

$$= P_{i-1} + \frac{P_{i-1} r(t) r(t)^T P_{i-1} + P_{i-1} r(t) r(t)^T P_{i-1} r(t) r(t)^T P_{i-1} - P_{i-1} r(t) r(t)^T P_{i-1} - P_{i-1} r(t) r(t)^T P_{i-1} r(t) r(t)^T P_{i-1}}{1 + r(t)^T P_{i-1} r(t)}$$

$$= P_{i-1}$$

done