Ken Miller's Question

Qn 1.

$$\frac{dH}{d\Delta} = \frac{d - (P_j \log_2 P_j + P_k \log_2 P_k)}{d\Delta}$$

$$= \frac{d - \left(\frac{P_j \ln P_j}{\ln(2)} + \frac{P_k \ln P_k}{\ln(2)}\right)}{d\Delta}$$

$$= -\left(\frac{d \, \frac{(P_{mean}+\Delta)\ln(P_{mean}+\Delta)}{\ln(2)}}{d\Delta} + \frac{d \, \frac{(P_{mean}-\Delta)\ln(P_{mean}-\Delta)}{\ln(2)}}{d\Delta} \right)$$

$$= -\left(\frac{\ln(P_{mean}+\Delta)+1}{\ln(2)} - \frac{\ln(P_{mean}-\Delta)+1}{\ln(2)} \right)$$

$$= -\left(\frac{\ln(P_{mean}+\Delta) - \ln(P_{mean}-\Delta)}{\ln(2)} \right)$$

$$= -\left(\frac{\ln P_j - \ln P_k}{\ln(2)} \right)$$

As $P_j > P_k$, $\ln P_j - \ln P_k > 0$.

Thus, $\frac{dH}{d\Delta}$ is always negative. When Δ decreases, H increases.

Qn 2.

$$\sum_i H(q) - \sum_i H(q \mid s(t_i))$$

$$= \sum_i (-P_{sp}\log_2 P_{sp} - P_{Np}\log_2 P_{Np}) - \sum_i (-P_{sp|s}\log_2 P_{sp|s} - P_{Np|s}\log_2 P_{Np|s})$$

$$= \sum_i \left(-r_m\Delta t \log_2 r_m\Delta t - (1-r_m\Delta t)\log_2(1-r_m\Delta t)\right) + \sum_i \big(r(t_i)\Delta t \log_2 r(t_i)\Delta t + (1-r(t_i)\Delta t)\log_2(1-r(t_i)\Delta t) \big)$$

$$= \sum_i \left(-\frac{r_m\Delta t \log r_m\Delta t}{\log 2} + \frac{(1-r_m\Delta t) r_m\Delta t}{\log 2} + \frac{r(t_i)\Delta t \log r(t_i)\Delta t}{\log 2} - \frac{(1-r(t_i)\Delta t) r(t_i)\Delta t}{\log 2} \right)$$

Since $\sum_i r(t_i) = \sum_i r_m$, $\sum_i \frac{r_m \Delta t \log r_m \Delta t}{\log_2} = \sum_i r_m \cdot \sum_i \frac{\Delta t \log r_m \Delta t}{\log_2}$

$$= \sum_i r(t_i) \cdot \sum_i \frac{\Delta t \log r_m \Delta t}{\log_2}$$

$$= \sum_i \frac{r(t_i)\Delta t \log r_m \Delta t}{\log_2}$$

By similar reasoning, $\frac{(1-r_m\Delta t) r_m \Delta t}{\log_2} = \frac{(1-r(t_i)\Delta t) r(t_i)\Delta t}{\log_2}$

Thus, the equation $= \sum_i \left(-\frac{r(t_i)\Delta t \log r_m\Delta t}{\log_2} + \frac{r(t_i)\Delta t \log r(t_i)\Delta t}{\log 2} \right)$

$$= \sum_i \left(\frac{\log\left(\frac{r(t_i)\Delta t}{r_m \Delta t}\right) r(t_i)\Delta t}{\ldots} \right) = \sum r(t_i)\Delta t \log_2 \frac{r(t_i)}{r_m} = r_m \int^T \frac{r(t)}{r_m} \log_2 \frac{r(t)}{r_m}$$

Qn3 Fisher information:

$$I_F(s) = -\int dr\, P[r|s] \frac{d^2 \log P[r|s]}{ds^2} \quad (1)$$

$$P(r|s) = \frac{1}{\sqrt{(2\pi)^N \mathrm{Det}\, Q}} \mathrm{Exp}\left(-\frac{1}{2}(r - f(s))^T Q^{-1} (r - f(s))\right) \quad (2)$$

Substitute (2) in (1):

$$I_F(s) = f'(s)^T Q^{-1}(s) f'(s) + \frac{1}{2} \mathrm{Tr}\left[Q'(s) Q^{-1}(s) Q'(s) Q^{-1}(s)\right]$$

Tr: trace operation.

$$Q'(s) = \frac{dQ(s)}{ds} \qquad f'(s) = \frac{df(s)}{ds}$$

Assume Q is independent of S, then simplify:

$$I_F(s) = f'(s)^T Q^{-1}(s) f'(s) \quad (3)$$

Solve for $Q^{-1}(s)$ by assume $Q^{-1}_{ij} = a + \delta_{ij} b$, $\sum_j Q_{ij} Q^{-1}_{jk} = \delta_{ik}$

$$Q^{-1}_{ij} = \frac{\delta_{ij}(Nc + 1 - c) - c}{\sigma^2 c(1-c)(Nc + 1 - c)} \quad (4)$$

put (4) into (3), $$I_F(x) = \frac{cN^2[F_1(x) - F_2(x)] + (1-c) N F_1(x)}{\sigma^2 (1-c)(Nc + 1 - c)}$$

$$F_1(x) = \frac{1}{N}\sum_i (f_i'(x))^2, \quad F_2(x) = \left(\frac{1}{N}\sum_i f_i'(x)\right)^2$$

Sean's homework

Qn1. $V_{i+1} = \vec{Z}_i \cdot \phi + \sigma\sqrt{\Delta}\,\varepsilon_i \,, \quad \varepsilon_i \sim \mathcal{N}(0,1)$

$$\vec{Z}_i \equiv \begin{bmatrix} V_i \\ \vec{x}_i \end{bmatrix} ; \qquad \phi \equiv \begin{bmatrix} 1-\frac{\Delta}{\tau} \\ \frac{\Delta}{\tau}E \\ \frac{\Delta}{\tau}\vec{k} \end{bmatrix}$$

Thus, $V_{i+1} \sim \mathcal{N}(\vec{Z}_i \cdot \phi,\ \sigma^2\Delta)$

From Sean's lecture notes: $L(\hat{\theta}) = \sum_i \ln \mathcal{N}(V_{i+1} \mid \vec{z}_i \cdot \phi,\ \sigma^2\Delta)$

$$= \text{const.} - \frac{1}{2\sigma^2\Delta}(\vec{V} - Z\phi)^T(\vec{V} - Z\phi)$$

where $\vec{V} = \begin{bmatrix} V_2 \\ \vdots \\ V_N \end{bmatrix}$; $Z = \begin{bmatrix} Z_1^T \\ \vdots \\ Z_{N-1}^T \end{bmatrix}$

To maximize $L(\hat{\theta})$ is to minimize $\frac{1}{2\sigma^2\Delta}(\vec{V} - Z\phi)^T(\vec{V} - Z\phi)$.

This gives us:

$$\sigma^* = \sqrt{\frac{1}{\Delta(N-1)}(\vec{V} - Z\phi)^T(\vec{V} - Z\phi)}$$

Qn2. To prove concavity is to prove that the second derivative is always negative.

$$L = \sum_{i \in \text{spikes}} \ln\left[1 - e^{-f(u_i)\Delta}\right] - \sum_{i \in \text{spikes}} f(u_i)\Delta$$

To simplify: $f(u_i) = e^{f(u_i)}$, thus $f(u_i) > 0$. $L = \sum_{i \in \text{spikes}} \left(\ln[1 + f\Delta] - f\Delta\right)$

$$L' = \sum_{i \in \text{spikes}} \left(\frac{f'\Delta}{1 + f\Delta} - f'\Delta\right)$$

$$L'' = \sum_{i \in \text{spikes}} \left(\frac{(1 + f\Delta)f''\Delta - (f')^2\Delta^2}{(1 + f\Delta)^2} - f''\Delta\right)$$

$$= \sum_{i \in \text{spikes}} \left(\frac{f''\Delta + ff''\Delta^2 - (f')^2\Delta^2}{(1 + f\Delta)^2} - f''\Delta\right)$$

$$= \sum_{i \in \text{spikes}} \left[f''\Delta\left(\frac{1}{(1 + f\Delta)^2} - 1\right) + \frac{\Delta^2(ff'' - (f')^2)}{(1 + f\Delta)^2}\right]$$

- $f\Delta > 0$, $(1 + f\Delta)^2 > 1$, Thus $\left(\frac{1}{(1 + f\Delta)^2} - 1\right) < 0$
- f is convex, $f'' > 0$.

Thus, $f''\Delta\left(\frac{1}{(1 + f\Delta)^2} - 1\right) < 0$.

- As $-ff'' \geq -(f')^2$, $ff'' - (f')^2 \leq 0$

Thus, $\frac{\Delta^2(ff'' - (f')^2)}{(1 + f\Delta)^2} < 0$.

Thus, L'' = Sum of negative terms < 0.