Intro theory class: Homework

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1. The discrete-time stochastic integrate and fire model is given by the equation

$$V_{i+1} = V_i\left(1 - \frac{\Delta}{\tau}\right) + \frac{\Delta}{\tau}\left(E + \vec{k}\cdot\vec{x}_i\right) + \sigma\sqrt{\Delta}\varepsilon_i, \quad \varepsilon_i \sim \mathcal{N}(0,1)$$

with input $\vec{x}_i$, preferred stimulus $\vec{k}$, rest potential E, membrane time constant τ, time-step duration Δ, and noise scale σ. We can simplify this to

$$V_{i+1} = \vec{z}_i \cdot \phi + \sigma\sqrt{\Delta}\varepsilon_i$$

if we make the following substitutions:

$$\vec{z}_i \equiv \begin{bmatrix} V_i \\ 1 \\ \vec{x}_i \end{bmatrix}; \quad \phi \equiv \begin{bmatrix} 1 - \frac{\Delta}{\tau} \\ \frac{\Delta}{\tau}E \\ \frac{\Delta}{\tau}\vec{k} \end{bmatrix}.$$

The value of the parameter σ is the amount we would expect the voltage to drift in 1 second based on noise alone. But what is noise? While there are true noise sources (e.g., quantum fluctuations in the openings and closings of ion channels), we also use noise to subsume structure in the data that our model fails to account for (e.g., nonlinearities, et cetera). So estimating this parameter gives important information both about the data that we are modeling and about the model.

Show that the maximum likelihood solution for σ is

$$\sigma^* = \sqrt{\frac{1}{\Delta(N-1)}(\vec{V} - \mathbf{Z}\phi)^{\mathrm{T}}(\vec{V} - \mathbf{Z}\phi)}$$

where

$$\vec{V} = \begin{bmatrix} V_2 \\ \vdots \\ V_N \end{bmatrix}; \quad \mathbf{Z} = \begin{bmatrix} z_1^{\mathrm{T}} \\ \vdots \\ z_{N-1}^{\mathrm{T}} \end{bmatrix}.$$

2. Typically the transformation between the instantaneous firing rate function f and a spike train is modeled with Poisson noise. As discussed in class, fitting the parameters of such a model is a concave optimization if f is convex and log-concave. In the limit of vanishing Δ, a Poisson model will always result in at most 1 spike per time step, but with finite Δ the possibility of multiple simultaneous spikes is given non-zero probability. One way to enforce a refractory period and to speed computation is to use a longer Δ

but to explicitly enforce at most 1 spike per time step. This results in the "Bernoulli" spiking model, an alternative to the Poisson:

$$p(y = \text{no spike}) = e^{-f(u)\Delta}; \quad p(y = \text{spike}) = 1 - e^{-f(u)\Delta}.$$

The likelihood function is then

$$\begin{aligned} L(\theta) &= \sum_i \ln p(y_i) \\ &= \sum_{i \in \text{spikes}} \ln p(y_i = \text{spike}) + \sum_{i \notin \text{spikes}} \ln p(y_i = \text{no spike}) \\ &= \sum_{i \in \text{spikes}} \ln \left[1 - e^{-f(u_i)\Delta}\right] - \sum_{i \notin \text{spikes}} f(u_i)\Delta. \end{aligned}$$

Prove that if we apply the same conditions for f as in the Poisson model (convexity and log-concavity), this is sufficient to guarantee that the Bernoulli model is also concave. Note that if f is log-concave, that implies the following:

$$0 \geq (\ln f)'' = \left(\frac{f'}{f}\right)' = \frac{ff'' - (f')^2}{f^2} \quad \text{or} \quad -ff'' \geq -(f')^2.$$

3. Fit the parameters of the discrete time stochastic integrate and fire model (see problem 1). In the file `Problem3.mat`, you will find `V` (a row vector with the membrane potential in volts at every time step), `Spikes` (a row vector of with 1's at the spike times and 0's otherwise, and `X` (a stimulus matrix) . `X` has dimensions $50 \times 500,000$ where 50 is the number of stimulus dimensions and 500,000 is the number of time steps (500 s with Δ of 0.001 s). I have also provided `fitLIF_template.m` for you to use if you wish with commented sections indicating where you need to edit the file. This saves you from having to deal with the refractory periods (0.002 s in this case). All you need to do is construct a data matrix `S` (called $\mathbf{Z}$ in problem 1), then perform least-squares regression ($\phi = (\mathbf{Z}^{\mathrm{T}}\mathbf{Z})^{-1}\mathbf{Z}^{\mathrm{T}}\vec{V}$) and extract τ, E, and $\vec{k}$ from ϕ. Finally, use the formula in problem 1 to get the maximum likelihood solution for σ.

   The submittables for this problem are your code and the maximum likelihood values for τ, E, $\vec{k}$, and σ. Submit $\vec{k}$ as a plot (`plot(k)`).

4. Fit a generalized linear model to data

$$p(y_i|\theta) = \frac{(\lambda_i\Delta)^{y_i} e^{-\lambda_i\Delta}}{y_i!}; \quad \lambda_i = f(\vec{k} \cdot \vec{x}_i + b)$$

   where y_i and $\vec{x}_i$ are the number of spikes and the stimulus at time step i, Δ is the time step size, $\vec{k}$ is the stimulus filter, and $f(b)$ is the background firing rate in the absence of a stimulus.

   In the files `Problem4_train.mat` and `Problem4_test.mat`, you will find 4 variables. `Spikes_train` and `Spikes_test` are row vectors where the entries in the vector are the number of spikes in each time step. `X_train` and `X_test` are matrices of dimension $49 \times 200,000$ where 49 is the number of stimulus dimensions and 200,000 is the number of time steps (200 s with Δ of 0.001 s). Fit the model with 3 nonlinearities: the exponential ($f(u) = \exp u$), the smooth rectified linear ($f(u) = \ln(1 + \exp u)$), and the rectified quadratic ($f(u) = [u]_+^2$). Note that for the rectified quadratic, add a small value ε so that $\ln f(u)$ is never $-\infty$ (e.g., $f(u) = [u]_+^2 + \varepsilon$ with $\varepsilon = 10^{-5}$). Then calculate the log-likelihood of each of these three fits on the test data. When you train with some data, and test the fit with other data, it is called cross validation, and this allows us to identify the best model among the fits (i.e., the model with the highest cross validated log-likelihood). Once you have identified the the best model, plot $\vec{k}$ for this model using the following command `imagesc(reshape(k,7,7))`.

You may wish to use the following template files for this problem: `fitGLM_template.m` and `GLM_LL_template.m`. The former is a template for performing Newton–Raphson optimization to get the maximum likelihood solution for $\vec{k}$. The latter is a template for calculating the log-likelihood given the data and parameters, and can thus be used to get the cross validated log-likelihoods.

The submittables for this problem are your code, the three cross-validated log-likelihoods for the test data, and the plot of $\vec{k}$ and value of the background firing rate $f(b)$ for the best model.