PYCLOAK - MATHEMATICAL BACKGROUND

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1. Background

This python module serves the purpose of solving the following general problem. We want to solve the Helmholtz equation

$$\Delta u - k^2 u = 0$$

in $\mathbb{R}^2$ for $k > 0$, but in such a way where we can cancel some incoming solution $u_0(\mathbf{x})$ on a particular region $D_c \subset \mathbb{R}^2$. We present the details as follows.

Let $B_R \subset \mathbb{R}^d$ be the ball of radius $R > 0$. We assume $\mathbf{0} \in D_a \subset B_R$ is the region inside a single antenna with C^2 boundary, ∂D_a. We also let $D_c \subset B_R$ be the control region, which is assumed to satisfy $\overline{D_c} \cap \overline{D_a} = \emptyset$ (see Figure 1). The numerical simulations in the current work are performed for the two dimensional case but implementation of the three dimensional case is in progress.

Consider the function space

$$\Xi = L^2(\partial D_c) \times L^2(\partial B_R),$$

endowed with the scalar product

$$(\phi, \psi)_\Xi = \int_{\partial D_c} \phi_1(\mathbf{y})\overline{\psi}_1(\mathbf{y})\, dS_{\mathbf{y}} + \int_{\partial B_R} \phi_2(\mathbf{y})\overline{\psi}_2(\mathbf{y})\, dS_{\mathbf{y}}, \tag{1}$$

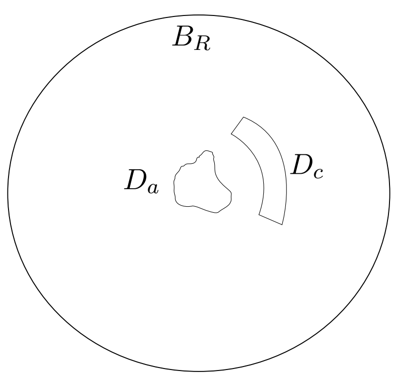

Figure 1. An antenna defined by ∂D_a with a control region D_c and far field region $B_R(\mathbf{0})$

which is a Hilbert space. Consider $K : L^2(\partial D_a) \to \Xi$, the double layer potential operator restricted to ∂D_c and ∂B_R, respectively, defined by

$$K\phi(\mathbf{x},\mathbf{z}) = (K_1\phi(\mathbf{x}), K_2\phi(\mathbf{z})), \quad \phi \in L^2(\partial D_a), \tag{2}$$

where

$$K_1\phi(\mathbf{x}) = \int_{\partial D_a} \phi(\mathbf{y}) \frac{\partial \Phi(\mathbf{x},\mathbf{y})}{\partial \nu_{\mathbf{y}}} ds_{\mathbf{y}}, \text{ for } \mathbf{x} \in \partial D_c,$$
$$K_2\phi(\mathbf{z}) = \int_{\partial D_a} \phi(\mathbf{y}) \frac{\partial \Phi(\mathbf{z},\mathbf{y})}{\partial \nu_{\mathbf{y}}} ds_{\mathbf{y}}, \text{ for } \mathbf{z} \in \partial B_R(\mathbf{0}).$$

Here $\Phi(\mathbf{x},\mathbf{y})$ represents the fundamental solution of the relevant Helmholtz operator, i.e.,

$$\Phi(\mathbf{x},\mathbf{y}) = \begin{cases} \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}, & \text{for } d = 3 \\ \frac{i}{4}H_0^{(1)}(k|\mathbf{x}-\mathbf{y}|), & \text{for } d = 2 \end{cases} \tag{3}$$

with $H_0^{(1)} = J_0 + iY_0$ representing the Hankel function of first type.

We also introduce the adjoint operator $K^* : \Xi \to L^2(\partial D_a)$, which can be shown to satisfy

$$K^*\psi(\mathbf{x}) = \int_{\partial D_c} \psi_1(\mathbf{y}) \overline{\frac{\partial \Phi(\mathbf{y},\mathbf{x})}{\partial \nu_{\mathbf{x}}}} dS_{\mathbf{y}} + \int_{\partial B_R} \psi_2(\mathbf{y}) \overline{\frac{\partial \Phi(\mathbf{y},\mathbf{x})}{\partial \nu_{\mathbf{x}}}} dS_{\mathbf{y}}, \quad \mathbf{x} \in \partial D_a. \tag{4}$$

Now consider the following problem: for a fixed wave number $k > 0$ and fixed $0 < \mu \ll 1$, find a function $h \in C(\partial D_a)$ such that there exists a $u \in C^2(\mathbb{R}^n \setminus \overline{D_a}) \cap C^1(\mathbb{R}^n \setminus D_a)$ solving

$$\begin{cases} (\Delta + k^2)u(\mathbf{x}) & = 0 \quad \mathbf{x} \in \mathbb{R}^n \setminus \overline{D_a} \\ u & = h \quad \text{on } \partial D_a \\ \|u - f_1\|_{C(\overline{D}_c)} & \leq \mu \\ \|u\|_{C(\mathbb{R}^n \setminus B_R(\mathbf{0}))} & \leq \mu, \end{cases} \tag{5}$$

where f_1 is a solution to the Helmholtz equation in a neighborhood of the control region D_c. This problem is equivalent to: for a fixed wave number $k > 0$ and given a function $f = (f_1, 0) \in \Xi$ and $\mu > 0$, find a density function $\phi \in C(\partial D_a)$ such that the residual $\|K\phi - f\|_\Xi$ is small, i.e., such that

$$\|K\phi - f\|_\Xi \leq \mu \tag{6}$$

Problem (6) is in fact a Fredholm integral equation of the first kind, and it has been proved that the bounded and compact operator K is also one-to-one and has a dense (but not closed) range, thus proving the existence of a class of solutions for (6). Given the fact that K is compact and that its range is not closed, problem (6) is ill-posed. By using regularization, one can approximate a solution to problem (6) with an arbitrary level of accuracy $\mu \ll 1$. There are several methods known in the literature, but we will use the Tikhonov regularization method. This method, when applied to the operator $K : L^2(\partial D_a) \to \Xi$ proposes a solution $\phi_\alpha \in C(\partial D_a)$ of the form

$$\phi_\alpha = (\alpha I + K^* K)^{-1} K^* f, \quad 0 < \alpha \ll 1, \tag{7}$$

where α (the Tikhonov regularization parameter) is a suitably chosen regularization parameter. It is known that $\|K\phi_\alpha - f\|_\Xi \to 0$ as $\alpha \to 0$, but the optimal choice of α is an essential step in designing a feasible method (e.g., finding a minimal norm solution), and there are various modalities to do this. We will use the Morozov discrepancy principle associated to the following weighted residual space:

$$\Xi' = L^2(\partial D_c, \|f_1\|^{-2} dS) \times L^2(\partial B_R, (2\pi R)^{-1} dS). \tag{8}$$

The reasoning behind using the weighted residual space Ξ' for the discrepancy functional defined below (as opposed to Ξ) is as follows. Due to the asymptotic behavior of $\frac{\partial \Phi(\mathbf{x},\mathbf{y})}{\partial \nu_{\mathbf{y}}} = \mathcal{O}(|\mathbf{x}-\mathbf{y}|^{-1/2})$ as $|\mathbf{x}-\mathbf{y}| \to \infty$, we have that given a fixed density ϕ, $\|K\phi\|_{L^2(\partial B_R)} = \mathcal{O}(1)$ as $R \to \infty$. In other words, using the space $L^2(\partial B_R)$ with the standard surface measure is not really suited to the decay properties of double layer potential solutions, because the decay of the normal derivative $\partial_\nu \Phi$ is too weak. Similarly, we use the relative norm

$$\frac{\|[K\phi]_1 - f_1\|_{L^2(\partial D_c)}}{\|f_1\|_{L^2(\partial D_c)}} \tag{9}$$

on ∂D_c because this is a useful quantity for determining how good the control is, regardless of the norm of f_1. Thus, for $f = (f_1, 0) \in \Xi \subset \Xi'$, we will consider the weighted residual $\|K\phi - f\|^2_{\Xi'}$ defined as

$$\|K\phi - f\|^2_{\Xi'} = \frac{1}{\|f_1\|^2_{L^2(\partial D_c)}} \|K_1\phi - f_1\|^2_{L^2(\partial D_c)} + \frac{1}{2\pi R}\|K_2\phi\|^2_{L^2(\partial B_R)}, \tag{10}$$

and then make use of the Tikhonov regularization with Morozov's discrepancy principle for the unique choice of α, i.e. such that

$$\|K\phi_\alpha - f\|^2_{\Xi'} = \delta^2, \tag{11}$$

with $\delta^2 \le \mu^2 \min\left\{\frac{1}{2\|f_1\|^2_{L^2(\partial D_c)}}, \frac{1}{4\pi R}\right\}$. Please note that ϕ_α in (11) is given by the Tikhonov regularization for the operator $K : L^2(\partial D_a) \to \Xi$ as described in (7).

We will account for noise and measurement errors and will consider (11) with $f = (f_1, 0) \in \Xi$ replaced by $f_\epsilon = (f_{\epsilon,1}, f_{\epsilon,2}) \in \Xi$, given by

$$f_\epsilon = (f_1 + \epsilon \widehat{\nu} \|f_1\|_{L^2(\partial D_c)}, 0) \in \Xi, \tag{12}$$

where $\widehat{\nu} \in L^2(\partial D_c)$ is a random perturbation with $\|\widehat{\nu}\| = 1$ and $f_1 \in L^2(\partial D_c)$ the far field of a far field observer. The goal of this program is to numerically compute the minimal norm solution uniquely determined by (11), analyze its stability for given noisy data in Ξ and, in the case of data corresponding to a point source, analyze its sensitivity with respect to parameters such as: mutual distances between D_a, D_c and $B_R(\mathbf{0})$; wave number k; and the location of the point source with respect to $B_R(\mathbf{0})$.

2. Implementation Details

We discretize the integral operator K via the method of moment collocation. First we choose an approximate basis of functions for $L^2(\partial D_a)$. To do this we suppose the domain D_a can be parametrized in polar coordinates by points

$$(s(\tau)\cos\tau, s(\tau)\sin\tau)), \quad \tau \in [0, 2\pi]$$

where $s : \mathbb{R} \to \mathbb{R}_+$ is a 2π-periodic smooth function. Using these coordinates, any function ϕ defined on ∂D_a can be realized via the pullback as a function of τ:

$$\phi(s(\tau)\cos\tau, s(\tau)\sin\tau).$$

For convenience, let us use the notation $\widehat{\tau} = (\cos\tau, \sin\tau)$ and $\widehat{\tau}^\perp = (-\sin\tau, \cos\tau)$.

Now let $n_a \in \mathbb{N}$ and let $\tau_j = \frac{2\pi j}{n}$, $0 \le j \le n_a - 1$ be n_a equally spaced points on the interval $[0, 2\pi)$. We then use the exponential basis functions $\{e^{il\tau}\}_{l=0}^{n_a-1}$ for $L^2([0, 2\pi])$ and approximate a given $\phi \in L^2(\partial D_a)$ via interpolation at the points $\{\tau_j\}_{j=0}^{n_a-1} \subset [0, 2\pi]$. Note that

$$\begin{aligned}\int_{\partial D_a} \phi(\mathbf{y}) \frac{\partial \Phi}{\partial \nu_{\mathbf{y}}}(\mathbf{x}, \mathbf{y})\, dS_{\mathbf{y}} = \int_0^{2\pi} &\phi(s(\tau)\cos\tau, s(\tau)\sin\tau) \frac{\partial \Phi}{\partial \nu_y}(\mathbf{x}, (s(\tau)\cos\tau, s(\tau)\sin\tau)) \\ &\cdot \sqrt{s(\tau)^2 + s'(\tau)^2}\, d\tau. \end{aligned} \tag{13}$$

Furthermore, since $(s'(\tau)\cos\tau - s(\tau)\sin\tau, s(\tau)\cos\tau + s'(\tau)\sin\tau)$ is a tangent vector to ∂D_a, we have that

$$\begin{aligned}\nu(\mathbf{y}) = \nu(\tau) &= \frac{(s(\tau)\cos\tau + s'(\tau)\sin\tau,\ s(\tau)\sin\tau - s'(\tau)\cos\tau)}{\sqrt{s(\tau)^2 + s'(\tau)^2}} \\ &= \frac{s(\tau)\widehat{\tau} - s'(\tau)\widehat{\tau}^\perp}{\sqrt{s(\tau)^2 + s'(\tau)^2}}.\end{aligned}$$

is the unit outward normal vector to ∂D_a. It is then straightforward to compute in the case of Helmholtz equation in 2-D that

$$\begin{aligned}&\frac{\partial \Phi}{\partial \nu_{\mathbf{y}}}(\mathbf{x}, (s(\tau)\cos\tau, s(\tau)\sin\tau)) \\ &= \nabla_y \Phi(\mathbf{x}, (s(\tau)\cos\tau, s(\tau)\sin\tau)) \cdot \nu(\tau) \\ &= \frac{ik}{4} H_0^{(1)'}(k|\mathbf{x} - s(\tau)\widehat{\tau}|) \frac{s(\tau)\widehat{\tau} - \mathbf{x}}{\sqrt{s(\tau)^2 + |\mathbf{x}|^2 - 2s(\tau)\mathbf{x}\cdot\widehat{\tau}}} \cdot \frac{s(\tau)\widehat{\tau} - s'(\tau)\widehat{\tau}^\perp}{\sqrt{s(\tau)^2 + s'(\tau)^2}}\end{aligned}$$

Let $n_c \in \mathbb{N}$ be the total number of sample points on ∂D_c. Also let n_R be the total number of sample points on ∂B_R. We write the $2 \times (n_c + n_R)$ matrix of points

$$\mathbf{X} := [x_1, \dots, x_{n_c+n_R}],$$

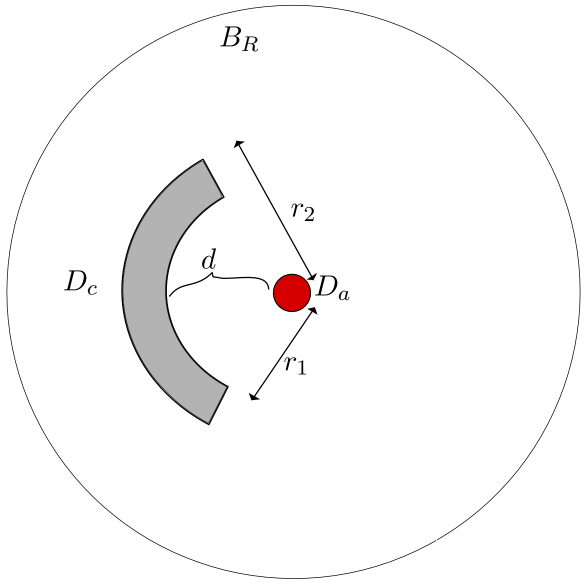

FIGURE 2. Antenna and control region geometry used for numerical experiments.

where each x_j is a 2-vector, $\{x_j\}_{j=1}^{n_c} \subset \partial D_c$ and $\{x_j\}_{j=n_c+1}^{n_c+n_R} \subset \partial B_R$. In the case when D_c is the sector of an annulus, we have

$$x_j = \left\{ \begin{array}{rl} \begin{bmatrix} r_i \cos\left((j-1/2)\Delta\tau_1\right) \\ r_i \sin\left((j-1/2)\Delta\tau_1\right) \end{bmatrix} & 1 \leq j \leq n_{arc_1} \\ (j - n_{arc_1} - 1/2)\Delta t[\cos(\tau_f),\, \sin(\tau_f)]^T & n_{arc_1}+1 \leq j \leq n_{arc_1}+n_s \\ \begin{bmatrix} r_f \cos\left((n_{arc_1}+n_{arc_2}+n_s-j+1/2)\Delta\tau_2\right) \\ r_f \sin\left((n_{arc_1}+n_{arc_2}+n_s-j+1/2)\Delta\tau_2\right) \end{bmatrix} & n_{arc_1}+n_s+1 \leq j \leq n_c - n_s \\ (n_{arc_1}+n_{arc_2}+2n_s-j+1/2)\Delta t[\cos(\tau_i),\, \sin(\tau_i)]^T & n_c - n_s + 1 \leq j \leq n_c \end{array} \right\}.$$

Here $\Delta\tau_1 = \frac{\tau_2-\tau_1}{n_{arc_1}}$, $\Delta\tau_2 = \frac{\tau_2-\tau_1}{n_{arc_2}}$, and $\Delta t = \frac{r_2-r_1}{n_s}$, where $n_s = \left\lceil \frac{r_2-r_1}{r_1\Delta\tau_1} \right\rceil$. Moreover, n_{arc_1} is a chosen positive integer denoting the number of sample points to use on the inner arc of the control region, and from this we determine $ds_1 = r_i(\tau_2-\tau_1)/n_{arc_1}$ and $n_{arc_2} = \lceil r_2(\tau_2-\tau_1)/ds_1 \rceil$. In this case, $n_c = n_{arc_1}+n_{arc_2}+2n_s$. For z_j we simply have $z_j = [R\cos\left(\frac{2\pi}{R}(j-1)\right), R\sin\left(\frac{2\pi}{R}(j-1)\right)]^T$, $1 \leq j \leq n_R$. See Figure 2 for details.

For each $1 \leq j \leq n_c+n_R$ and each $0 \leq l \leq n_a-1$, we compute $K[e^{il\tau}](x_j)$ via the approximation

$$\frac{2\pi}{n_a}\sum_{m=0}^{n_a-1} \frac{\partial\Phi(x_j, [s(\tau_m)\cos(\tau_m), s(\tau_m)\sin(\tau_m)]^T)}{\partial\nu_{\mathbf{y}}} e^{il\tau_m}\sqrt{s(\tau_m)^2 + s'(\tau_m)^2}.$$

If we fix j and vary l, we see that the above sum is equivalent to computing the discrete fourier transform of the n_a-vector

(14)
$$\mathbf{v}_j := \left[\frac{\partial\Phi(x_j, [s(\tau_m)\cos(\tau_m), s(\tau_m)\sin(\tau_m)]^T)}{\partial\nu_{\mathbf{y}}}\sqrt{s(\tau_m)^2 + s'(\tau_m)^2}\right]_{m=0}^{n_a-1},$$

which can be computed efficiently using the Fast Fourier Transform algorithm. In particular

$$[K\{e^{il\tau}\}(x_j)]_{l=0}^{n_a-1} \approx 2\pi\mathsf{FFT}(\mathbf{v}_j),$$

where FFT is defined on n_a-vectors $\mathbf{v} = [v_m]_{m=1}^{n_a}$ by

$$\mathsf{FFT}(\mathbf{v})_j = \frac{1}{n_a}\sum_{m=1}^{n_a} v_m e^{\frac{2\pi i(m-1)(j-1)}{n_a}}. \tag{15}$$

So the matrix representation of K is then the $n_a \times (n_c + n_R)$ matrix

$$A := [2\pi\mathsf{FFT}(\mathbf{v}_1), \cdots, 2\pi\mathsf{FFT}(\mathbf{v}_{n_c+n_R})]. \tag{16}$$

Now, in order to approximately solve the ill-posed problem $K\phi = f$, we attempt to solve the linear system

$$\begin{aligned} K_1\phi(x_j) &= f_1(x_j), \quad 1 \le j \le n_c \\ K_2\phi(x_j) &= f_2(x_j), \quad 1 \le j \le n_R. \end{aligned}$$

Since A is computed with respect to the functions $\{e^{il\tau}\}$, we approximate the coefficients of ϕ with respect to the given basis:

$$\begin{aligned} c_l &:= \frac{2\pi}{n_a}\sum_{m=0}^{n_a-1} e^{-il\tau_m}\phi(s(\tau_m)\cos(\tau_m), s(\tau_m)\sin(\tau_m)) \\ &\approx \frac{1}{2\pi}\int_0^{2\pi} e^{-il\tau}\phi(s(\tau)\cos(\tau), s(\tau)\sin(\tau))\,d\tau. \end{aligned}$$

Let

$$h = [c_l]_{l=0}^{n_a-1}.$$

We now numerically compute the Tikhonov regularized solution

$$h_\alpha := (A^*A + \alpha I)^{-1}A^*f,$$

with $\alpha > 0$. The solution h_α gives an approximation of the coefficients c_l of the desired density ϕ with respect to the functions $\{e^{il\tau}\}_{l=0}^{n_a-1}$. We obtain the density ϕ_α corresponding to h_α sampled at the angles τ_m on ∂D_a by the formula

$$\phi_\alpha(\tau_m) := \sum_{l=0}^{n_a-1}[h_\alpha]_l e^{il\tau_m}.$$

Note that we have yet to specify how the regularization parameter α is chosen.

Because of the asymptotic behavior of $\frac{\partial\Phi(\mathbf{x},\mathbf{y})}{\partial\nu_{\mathbf{y}}} = \mathcal{O}(|\mathbf{x}-\mathbf{y}|^{-1/2})$ as $|\mathbf{x}-\mathbf{y}| \to \infty$, we have that given a fixed density ϕ, $\|K\phi\|_{L^2(\partial B_R)} = \mathcal{O}(1)$ as

$R \to \infty$. In other words, using the space $L^2(\partial B_R)$ with the standard surface measure is not really suited to the decay properties of double layer potential solutions, because the decay of the normal derivative $\partial_\nu \Phi$ is too weak. To this end, we instead use the averaged L^2 norm on ∂B_R given by

$$\begin{aligned} \|f\|_{L^2(\partial B_R, \frac{1}{\sqrt{2\pi R}} dS)} &= \frac{1}{\sqrt{2\pi R}} \left(\int_{\partial B_R} |f(x)|^2 \, dS_x \right)^{1/2} \\ &\approx \frac{1}{\sqrt{N}} \left(\sum_{j=0}^{N-1} |f(\widehat{\tau}_j)|^2 \right)^{1/2}, \quad N \to \infty, \end{aligned}$$

where

$$\widehat{\tau}_j = \left(R \cos\left(\frac{2\pi j}{N}\right), R \sin\left(\frac{2\pi j}{N}\right) \right).$$

For fixed f, we also use the relative norm

$$\frac{\|[K\phi]_1 - f_1\|_{L^2(\partial D_c)}}{\|f_1\|_{L^2(\partial D_c)}} \tag{17}$$

on ∂D_c. Altogether, we will consider the double layer potential operator K as a map from $L^2(\partial D_a)$ to the space

$$\Xi' = L^2(\partial D_c, \|f_1\|^{-2} dS) \times L^2(\partial B_R, (2\pi R)^{-1} dS). \tag{18}$$

After computing the residual $K\phi - f$, we will then need to compute

$$\|K\phi - f\|_{\Xi'}^2 = \frac{1}{\|f_1\|_{L^2(\partial D_c)}^2} \|[K\phi]_1 - f_1\|_{L^2(\partial D_c)}^2 + \frac{1}{2\pi R} \|[K\phi]_2 - f_2\|_{L^2(\partial B_R)}^2.$$

We define the discrepancy function $F(\alpha)$ by

$$F(\alpha) = \|K\phi_\alpha - f\|_{\Xi'}^2 - \delta^2 \tag{19}$$

where $\delta > 0$ is a fixed error parameter. This function is not globally increasing for every $\alpha > 0$, but it can be experimentally shown to be monotonically increasing for a range of α that includes the optimal regularization parameter with typical domain setups.

Note that if we split the matrix A into two blocks A_{near} (n_c by n_a) and A_{far} (n_R by n_a) so that

$$A = \begin{bmatrix} A_{near} \\ A_{far} \end{bmatrix},$$

then $[A\phi]_1 = A_{near}\phi$, $[A\phi]_2 = A_{far}\phi$, and $A^*A = A_{near}^* A_{near} + A_{far}^* A_{far}$. In the discretized setting, instead of (17) we take

$$F(\alpha) = \frac{1}{\|f_1\|^2} \|A_{near} h_\alpha - f_1\|_{L^2(\partial D_c)}^2 + \frac{1}{2\pi R} \|A_{far} h_\alpha - f_2\|_{L^2(\partial B_R)}^2 - \delta^2 \tag{20}$$

with

$$h_\alpha = (A^*A + \alpha I)^{-1} A^* f = (A_{near}^* A_{near} + A_{far}^* A_{far} + \alpha I)^{-1} \left(A_{near}^* f_1 + A_{far}^* f_2 \right). \tag{21}$$

We compute

$$F'(\alpha) = \frac{-2\alpha}{\|f_1\|^2_{L^2(\partial D_c)}} \mathrm{Re}\left(\frac{\partial h_\alpha}{\partial \alpha}, h_\alpha\right)$$

$$+ \left(\frac{1}{\pi R} - \frac{2}{\|f_1\|^2_{L^2(\partial D_c)}}\right) \mathrm{Re}\left(\frac{\partial h_\alpha}{\partial \alpha}, A^*_{far} A_{far} h_\alpha - A^*_{far} f_2\right) \tag{22}$$

$$\frac{\partial h_\alpha}{\partial \alpha} = -(A^*A + \alpha I)^{-1} h_\alpha, \tag{23}$$

where $(\cdot, \cdot)$ denotes the L^2 inner product on ∂D_a.

The function f_1 defined on ∂D_c will typically be the trace of a plane wave or of the fundamental solution to the Helmholtz equation. Also, in all numerical examples presented herein, we assume $f_2 \equiv 0$ on ∂B_R. A plane wave with frequency k and direction ξ ($\|\xi\| = 1$) is given by

$$e^{ik\xi\cdot\mathbf{x}}, \tag{24}$$

and a spherical source is represented as

$$\frac{i}{4} H_0^{(1)}(k|\mathbf{x} - \mathbf{x}_0|), \tag{25}$$

where $\mathbf{x}_0$ is the source point (typically outside of B_R). Both are solutions to $(\Delta + k^2)u = 0$.

For such an f_1, there are some quantities in which we will be interested so as to determine the effectiveness of a given density ϕ in solving the problem $K\phi = f$. These are: the relative error of $K\phi$ on ∂D_c; the L^2 average of the norm of $K\phi$ on ∂B_R; the relative and absolute stability of ϕ when applying a small perturbation to f_1; the norm of ϕ on ∂D_a, which is directly related to the power of the antenna. In other words, we will measure

$$\frac{\|K_1\phi - f_1\|_{L^2(\partial D_c)}}{\|f_1\|_{L^2(\partial D_c)}}, \quad \frac{1}{\sqrt{2\pi R}} \|K_2\phi\|_{L^2(\partial B_R)}, \tag{26}$$

$$\frac{\|\phi^\epsilon - \phi^0\|_{L^2(\partial D_a)}}{\|\phi^0\|_{L^2(\partial D_a)}}, \quad \|\phi^\epsilon - \phi^0\|_{L^2(\partial D_a)}, \tag{27}$$

and

$$\|\phi\|_{L^2(\partial D_a)}, \tag{28}$$

where ϕ^ϵ is the Tikhonov regularized solution to $K\phi = (f_1^\epsilon, 0)$ with $\|f_1 - f_1^\epsilon\|_{L^2(\partial D_c)} = \epsilon \|f_1\|_{L^2(\partial D_c)}$, and ϕ^0 is the solution with unperturbed f_1. Furthermore, the regularization parameters α used to determine ϕ^0 and ϕ^ϵ are chosen via Newton's Method using (22), (23) such that

$$\begin{aligned} \|K\phi^0 - f\|_{\Xi'} &= \delta \\ \|K\phi^\epsilon - f^\epsilon\|_{\Xi'} &= \delta. \end{aligned} \tag{29}$$

2.1. **Noise.** When adding noise to the desired data $(f_1, 0)$, since we always want the trace on ∂B_R to be 0, we only add noise to f_1. We choose a random perturbation $\eta \in L^2(\partial D_c)$ and set

$$f_1^\epsilon = f_1 + \epsilon\widehat{\eta}\|f_1\|_{L^2(\partial D_c)}, \tag{30}$$

where $\epsilon > 0$ represents the relative percentage of noise added. In the discrete case, η is chosen to be a vector of n_c pseudorandom numbers on the interval $(-1, 1)$. Furthermore, for reproducibility, whenever generating η using a pseudorandom number generator, we always reset the seed to the same value.

2.2. **Algorithm.** In summary, algorithm 1 gives the rough approach we use to solving the control problem.

2.3. **Typical Parameters Used for Numerical Experiments.** Here we describe some of the parameters used for the various numerical experiments presented. In the research paper associated with this work, we usually assume that ∂D_a is a circle with radius given by either $a = 0.01$ or $a = 0.1$, and that ∂D_c is a sector of an annulus with $\theta_1 = 3\pi/4$ and $\theta_2 = 5\pi/4$. For the collocation method, we used $n_a = 256$ sample points on ∂D_a, $n_{\text{inner arc}} = 256$ (number of points on inner arc of nearfield control region ∂D_c), and $n_R = 256$ (number of sample points on ∂B_R). We note that increasing $n_{\text{inner arc}}$ or n_R will put more emphasis on matching f on ∂D_c or ∂B_R, respectively. The discrepancy parameter δ used for Tikhonov regularization we chose at either 0.01 or 0.02. The number of points used on the other boundary segments of ∂D_c are chosen so that the arc length differential is approximately constant. The key variables under consideration are $d = r_1 - a$ (distance from ∂D_c to ∂D_a), k, ϵ (perturbation parameter for adding noise to f_1), and ξ (direction of plane wave solution).

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Algorithm 1 Basic Collocation Method with Tikhonov Regularization Approach

Require: $k > 0$, tol > 0, $\delta > 0$, $\alpha_i > \alpha_{min} > 0$, $f = (f_1, f_2) \in \Xi'$, $s(\tau)$.
Choose collocation points $\{y(\tau_j)\}_{j=1}^{n_a} \subset \partial D_a$, $\{x_j\}_{j=1}^{n_c+n_R} \subset \partial D_c \times \partial B_R$.
Choose line search step parameter $\beta > 1$.

▷ Compute matrix representation A of $K : L^2(\partial D_a) \to \Xi'$

for $q = 1$ to $n_c + n_R$ **do**

$$\begin{aligned} A[j,:] &= \left[\frac{i}{4} \sum_{l=0}^{n-1} e^{im\tau_l} \frac{\partial \Phi_k(x_j, y(\tau_l))}{\partial \nu_{\mathbf{y}}} \sqrt{s(\tau_l)^2 + s'(\tau_l)^2}\, \frac{2\pi}{n} \right]_{m=0}^{n_a-1} \\ &= 2\pi \mathsf{FFT}(\mathbf{v}_j). \end{aligned}$$

end for
$\mathbf{f}_1 = [f_1(x_j)]_{j=1}^{n_c}$, $\mathbf{f}_2 = [f_2(x_j)]_{j=n_c+1}^{n_c+n_R}$, $\mathbf{f} = [\mathbf{f}_1; \mathbf{f}_2]$
Set $\alpha = \alpha_i$
Set $h_\alpha \leftarrow (A^*A + \alpha I)^{-1} A^* \mathbf{f}$
while $F(\alpha) := \|Ah_\alpha - \mathbf{f}\|_{\Xi'}^2 - \delta^2 > 0$ **and** $\alpha \geq \alpha_{min}$ **do**
  $\alpha \leftarrow \alpha / \beta$
  Recompute h_α
end while
if $F(\alpha) \leq 0$ **then**

▷ Use current value of α to start Newton's method

  while $|F(\alpha)| > $ tol **do**
    $\partial_\alpha h_\alpha \leftarrow -(A^*A + \alpha I_{n_c+n_R})^{-1} h_\alpha$

$$F'(\alpha) \leftarrow \frac{-2\alpha}{\|f_1\|^2} \mathrm{Re}\,(h_\alpha,\, \partial_\alpha h_\alpha) + \left(\frac{1}{\pi R} - \frac{2}{\|f_1\|^2} \right) \mathrm{Re}\left(\partial_\alpha h_\alpha,\, A_{far}^* A_{far} h_\alpha - A_{far}^* f_2 \right)$$

    $\alpha \leftarrow \alpha - \frac{F'(\alpha)}{F(\alpha)}$
  end while
end if
Compute ϕ_α from h_α.