Final Project Report

„Application of the Fourth-Order Runge-Kutta Method to Solve Second-Order Differential Equations”

Created by: Hubert Durnaś
Date: December 10, 2022

Introduction

The goal of this project was to apply the Fourth-Order Runge-Kutta (RK4) method to solve second-order differential equations describing the behavior of an RLC circuit (resistor, inductor, capacitor) under the influence of a square wave input signal. The project involved implementing a numerical algorithm, simulating the circuit, and comparing the results with those obtained using MATLAB's ode45 function.

To further analyze the behavior of the RLC circuit, I extended the project to examine three damping cases: overdamping, critical damping, and underdamping. Damping significantly affects the oscillatory characteristics and stability of the system, which is crucial for practical applications.

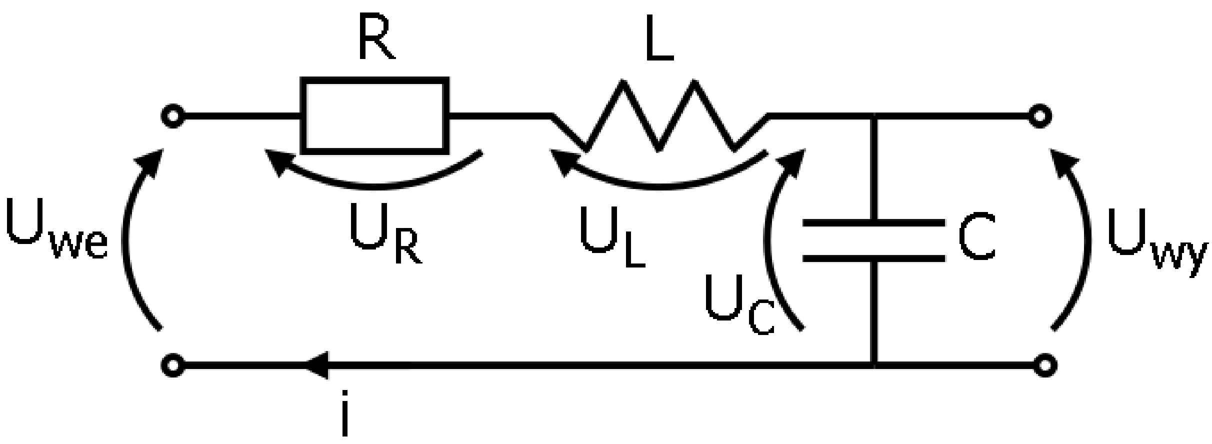
Runge-Kutta Method

The Runge-Kutta method is a family of numerical algorithms used to solve differential equations. In this project, the four-step version of the method was used. RK4 integrates differential equations using discrete time steps, where each step computes a weighted average of different derivative approximations, providing higher accuracy compared to simpler methods.

For each time step i, the input signal $U(i)$ is generated as a square wave, and the RK4 steps $k1$, $k2$, $k3$, $k4$ are computed using the function $f(t, Y, R, L, C, U)$, where Y represents the state variable vector. Each subsequent step depends on the previous one, following the RK4 algorithm.

Mathematical Description of the System

The quadrant under study presents itself as follows:



The premise of our function was to calculate the output voltage U_{wy}, given data : R, L, C and the input voltage U_{we} forced by a rectangular wave with amplitude U.

The analyzed RLC circuit can be described by the following second-order differential equations:

$u(t)$ - represents the voltage across the capacitor, which is also the output voltage U_{wy}.
$v(t)$ - represents the current through the inductor.
R is the resistance.
L is the inductance.
C is the capacitance.
U is the amplitude of the input square wave.

1. $u'(t) = v(t)$
2. $v'(t) = -\frac{R}{L}v(t) - \frac{1}{LC}u(t) + \frac{U}{LC}$

Damping in RLC Circuits

In this project, I focused on the damping characteristics of the RLC system. Damping can be categorized into three types:

Overdamping ($\zeta > 1$): $\boldsymbol{R > 2\sqrt{L/C}}$

- Rapid decay of oscillations.
- The system returns to equilibrium without overshooting.

Critical damping ($\zeta = 1$): $\boldsymbol{R = 2\sqrt{L/C}}$

- The fastest possible decay without oscillatory behavior.
- Ideal for systems where overshoot must be avoided.

Underdamping ($\zeta < 1$): $\boldsymbol{R < 2\sqrt{L/C}}$

- Slower decay with oscillatory behavior.
- The system overshoots before stabilizing.

Damping plays a significant role in oscillatory systems such as RLC circuits. Observing different ζ values allows for a comprehensive analysis of how damping affects the system's stability and response.

Final Results Analysis

By modifying resistance values R, I analyzed different damping behaviors in the output signal.

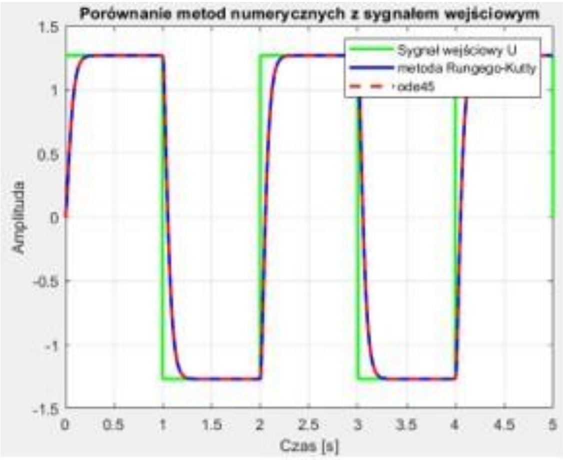


Figure 1. Critical attenuation

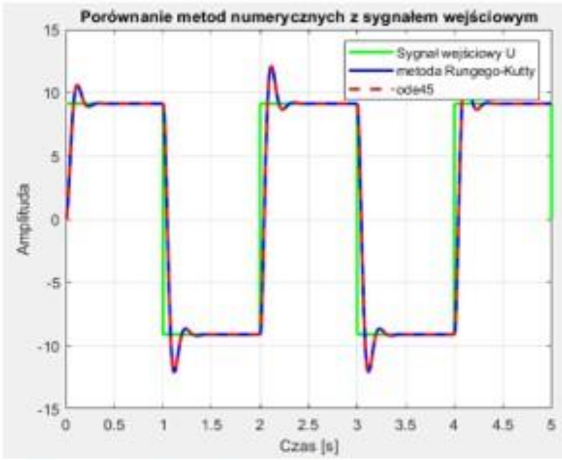


Figure 2. Supercritical attenuation

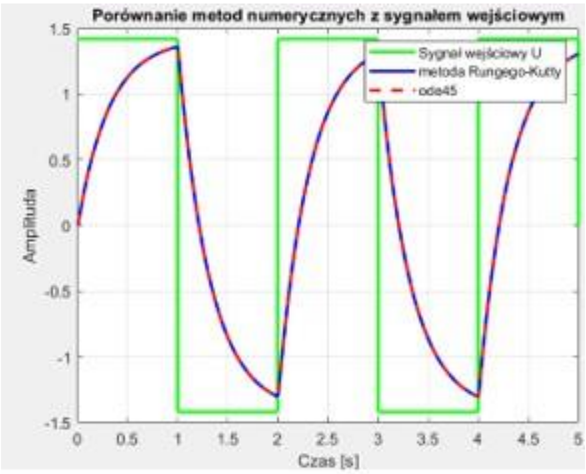
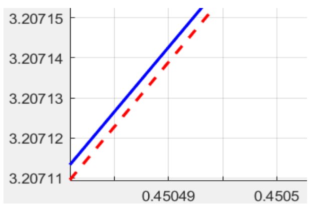


Figure 3. Subcritical attenuation

Simulations for various ζ values revealed the impact of damping on oscillatory characteristics. As expected, damping varies based on resistance R and significantly alters the signal shape. The damping coefficient strongly influenced the rate of oscillation decay and system stability.

To verify the accuracy of our results, I compared them with solutions obtained using MATLAB's ode45 function. The results showed high consistency, confirming the correct implementation of the Fourth-Order Runge-Kutta method. The error margin was approximately 10e-5, proving the method's effectiveness.



Conclusions

This project enabled the application and understanding of the Runge-Kutta method in solving differential equations, which is essential for modeling dynamic physical processes. Additionally, analyzing damping provided insights into how resistance impacts oscillatory systems, which has practical applications in electronics and engineering.

Exploring different damping coefficients allowed me to evaluate various system behaviors, which is valuable for electronic circuit design in terms of stability and performance optimization. The final results, along with the comparison to MATLAB's ode45 function, confirm that RK4 is a useful tool for analyzing dynamic electrical system.