Math 302: Elements in Analysis

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1 1st Class- Introduction, Logistics, Statements, and Proofs

1.1 Intro

1.1.1 Ice Breaker- About myself

I am Tal Malinovitch, and I will be teaching Math 302 Elements in Analysis. A little bit about myself:

1. I am Israeli- 33 yo.

2. I came here from my Ph.D. at Yale (I am a Dr, not a Prof. Feel free to call me Tal).

3. Before my graduate degree, I worked as a physicist in a nuclear reactor - so I always walked the line between physics and math.

4. My handwriting is atrocious. Please let me know if you can't understand what's written.

5. I am a metalhead who loves tv shows, and books and is generally a nerd (and with particular love for history as you will see).

6. I will try to incorporate many references to other fields, and pop culture. Maybe even talk a bit more about the history of the proofs and how they were conceived. If you have any suggestions- tell me!

Now we will do a quick round- each person will say their preferred name (pronouns) and a topic of interest that is not math.

Next, by show of hand, who had any exposure to logic: basic proof techniques, propositional calculus? On the other hand: who has seen things like derivative, sequence, and so on?

You see, many more people saw the content that we are about to learn- but even if you have or haven't- it's ok. The way I think about this course is to teach you a new way of thinking- we would touch on that in a bit. So it doesn't matter if you saw it or not, and how comfortable you feel with it since we are going to start at the very beginning.

1.1.2 Introduction

What is math? What is this subject that we are here to learn?

Math is:

- A language that we use to tell stories.
- A language has stringent rules - the rules of logic deductions.
- We use it to prove many things. But we can't prove everything with it (despite what some people think) -more on it in a bit.
- A way to analyze a problem, a view with merits and drawbacks.

If that is so, what is analysis?

- It is a dialect (think American English vs. British, or even different dialects in the US or the UK) or a certain genre of stories.
- It deals with functions and how they change (continuous, differentiable). In analysis, the story is always about change, and can we somehow say something about it (say, is the change bounded)?
- This dialect has certain building blocks - fundamental to understanding it (such as limits, sequences, and series). Think about it as basic nouns and verbs - that maybe other dialects borrowed- or even also made it intrinsic part of their structure.

In this course, we will learn "elementary analysis", and it will also be an introduction to this language in this dialect. And this is the way I think about this course- you need to learn the symbols (the letters if you want), the nouns and verbs (the basic terminology) as well as the basic grammatical structures (the arguments and proofs in math). This also says something about how maybe you should approach this course. Like a language, it requires repetition, and no matter how much you practice it is still possible to swap an "a" with an "e"- or make small mistakes. So think about trying to practice these exercises even outside the classroom.

We will start by building the language, and once we feel a bit more secure in our standing there, we will start building the basic notions: the real numbers, sequences, and so on. We will slowly become more and more fluent in this dialect, discussing things like convergences, and continuous functions, and eventually, hopefully, we will get to what I considered one of the most important theorems in math: Taylor's theorem. This theorem, in my eyes, represents an organizational principle that is the basis for solving problems in a "physics-y" way. I think the different approaches to this theorem tell you a lot about the "dialect" you are speaking in.

Like any other course, we will have some textbooks that we will follow:

- Elementary Analysis, Kenneth A. Ross, 2nd Edition (2013) - available online.
- Understanding Analysis, Stephen Abbott, 2nd Edition (2015). available online.
- First chapter of Introduction to Real Analysis, Michael J. Schramm, 1996- You don't have to have this one as it is unavailable online.

This week we will start with the last of these books- as we will start talking about proofs and logic. In fact, in the first two weeks, we will not even talk math, in some sense. We will discuss logic! So we will talk about pink elephants and stuff like that.

But first, before we start diving into talking about the subject (not learning), let us go over some logistics and syllabus:

1.1.3 Logistics

The main points from the syllabus:

- I will post the course notes- I hope to post them after each class, but we will see as the semester goes. As you may have noticed English is not my native language, and I will have typos and weird phrases in my notes. Please- if you spot these let me know! Moreover- I am human (SHOCKING!) so even my proofs could be wrong- either since I forgot to multiply by 2, or minus turned to plus, or since I missed subtle points in the proof. You have to keep me honest- especially in the notes. So people that will invest time to correct these, or found a very subtle problem - I will consider rewarding bonus points (1-2) to the final grade.
- As I am building this course on the fly- the problem wet will not be available a long time in advance. Probably I will post them the week before.
- Office hours: TBD. There are several options, out of which 3 will be selected: right before class MWF 1-2 pm, right after class on MW 3-4 pm, on TTh 2-3 pm. I will have a "quiz" on Canvas to see what is the most convenient for y'all. Please fill it out by the end of the week! This week we we have office hours on MWF after class. If you can't make it to the office hours, feel free to email me and we can schedule a meeting.
- There will be homework every week, except midterm week. Your grade will be the average and I will drop your two lowest scores. Late submission only in emergencies and after a discussion with me.
- Since I am new to this course, and to Rice in general- I would love your feedback on the difficulties and the length of the homework. You can write me an email, come talk to me in OH, or write it in mid-semester feedback.

- There will be mid-semester feedback! Details TBA, but I will find a way to make it anonymous. In previous years, I found this tool very useful.
- I don't like exams. I, personally, tend not to do well in exams, and as a consequence, I tend to think less of them. So we will have a mid-term evaluation which is closer to an extended homework assignment than an exam. You do need to do it alone- but no strict time frame.
- Furthermore, the final will be a take-home exam. The exact logistic is in the syllabus.
- Final Grade = $50\% \cdot \mathrm{HW} + 30\% \cdot \max(\mathrm{Final}, \text{mid-semester}) + 20\% \cdot \min(\mathrm{Final}, \text{mid-semester})$
- I want people to feel comfortable here. If you feel uncomfortable for any reason please let me know. Especially in the first couple of weeks- where we will be discussing a lot of examples that are not from math directly.
- Mental health- Though college tends to be a very stressful time in your life, it is of the utmost importance, regardless of the context of this course, to take care of yourselves during this time. This also includes taking care of your mental health and general well-being. Please if you need help- get in touch with someone that could help.
- Absence- you can miss class but let me know.

1.2 A bit of History

Before we would start talking about the numbers, sequences, series, and whatnot- what they are, and proving their properties - I want to start by teaching the language that we will use, and its syntax.

But even before we dive into it I find it useful to understand where it came from, and how this theory of logic came into play. This language is very nontrivial! Many of the theorems that we will learn were known in the 17th century (Think about Newton and Leibniz when they invented calculus)- even though they didn't have the language to talk about these things! That language was developed at the turn of the century (the previous one)! So, we will start with a bit of perspective:

At the end of the 19th century and early 20th century, Mathematics had a crisis of faith (the foundational crisis of mathematics). All of a sudden some paradoxes started appearing in math, for example:

1. The discovery of non-Euclidean geometry: Euclide in his geometry assumed that given a line and a point not on it, there is only one parallel line that goes through the point. That was assumed to be true, but then it was discovered that one can assume there might be no parallel lines or many parallel lines going through the same point.
2. The "set of sets that do not contain themselves" by Bertrand Russell. You can consider a set of things. We will talk about them more next week. Russell asked if this set contains itself or not.

This led to the realization that the foundation of math is not as stable as thought, thus came the "Hilbert Program": To put all mathematics on a solid foundation of a strictly logical system, in other words- to formalize the language of mathematics.

1.3 What did we see today?

- Logistics
- We talk about what is math.
- History- Hilbert's plan.

2 2nd Class- Formalizing arguments, Propositional calculus, Truth table

2.1 Review

So last time we talked a bit about math and why it is a language, and how analysis is actually a dialect that we will use to learn this language. We started discussing how this language came to be, and this is where we will continue.

2.2 A bit of History

So we ended with the realization that the foundation of math is not as stable as thought, thus came the "Hilbert Program": To put all mathematics on a solid foundation of a strictly logical system, in other words- to formalize the language of mathematics. How do we even begin to conceptualize it? First, we start with the realization that mathematics is all about statements. A theorem, lemma, and propositions, are all statements that we can say if they are "true" or not- whatever that means. Then a proof is just a collection of statements.

Now let's do a Think-pair-Share: What do you think will be the requirements for such a system? If you give me a proof- how can I say if it is true or not? What do we consider "a true mathematical statement"?

- We need the statements to obey certain rules, that allow us to deduce from one statement to the other.
- These rules should be devised in such a way that we can't prove something and its negation- this is a consistent system.
- We want to be able to prove everything- or its' negation- this a complete system.
- Furthermore we want our system to be decidable- for each statement, there is an algorithm to decide if it is true or not.
- We want to rely on a small number of axioms- or assumptions about the world. Why is that important?

or those that are interested in the History of Philosophy - this was the height of modern thought- people thought that everything will be explained (soon) by strict mathematical language- with no ambiguity. What lead to this effort is the belief that there is such a unique system of rules, or in Hilbet's own words:

> We are not speaking here of arbitrariness in any sense. Mathematics is not like a game whose tasks are determined by arbitrarily stipulated rules. Rather, it is a conceptual system possessing internal necessity that can only be so and by no means otherwise.

So they wanted to make sure that this language corresponds to the world in some way.

This program yielded the theory of logic as we know it today, but then came the unthinkable: Gödel incompleteness theorem in 1931. He showed that such a system can't be complete and consistent at the same time. This was a revolution similar to that of quantum mechanics in physics, only in mathematics philosophy.

Think about it- what does that tell us that a system can't be complete and consistent at the same time? Why is that so devastating?

That means that we need to add different things as axioms- and each axiom may lead to very different math. So there is no unique system- and we are playing with arbitrary rules.

Nevertheless- it is not a completely arbitrary set of rules- it has to obey a certain inner logic- described above. Which in turn is very useful in many other problems. For different problems, we might have different sets of assumptions or axioms, and that is ok.

2.3 So now let's start formulazing

We begin with some definitions:

Definition 2.1.
- A statement P is a grammatically meaningful sentence to which one or the other (but not both) of the words "true" or "false" can be attached. Examples? All elephants are blue, the sky is blue. non-examples? this statement is false. This sense no.

  Note that we *assume* there is no other possibility other than true or false. In life, this is not always the case. Other forms of logic are built around the assumptions that there are other options (like trough is a number between 0 and 1, or undetermined)

- An *axiom* is a statement P that we regard as true. Examples?

- An argument is a chain of statements one after the other.

To build a proof we need a way to deduce one statement from another, and this will be next class. But for now, we can say that a proof of a statement P(conclusion) from Q (assumption) is a chain of statements, each deduced from the previous one (according to the rules we haven't defined yet) that starts with P and ends with Q.

So we have two notions of "good" proof can you think about what are they? One speaks about the assumptions, are they true? Is it coming from a true statement about the world? The other notion is whether the proof is consistent- that is indeed every statement follows from the previous one.

TPS: Find an example of an inconsistent "proof" of a true statement, and false consistent proof.

2.4 Truth tables

We will now turn into make more complicated statements, and how to deduce a statement from another one.

First, we need to start manipulating the statement, we do this using connectives: not, and, or, if.. then.

The way to understand this manipulation is by a truth table.

We will see how it works, by an example: say we have a statement A, and we and to understand what A (not A) means, then we write a table:

A	$-A$
T	F
F	T

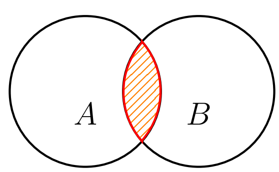
Table 1: Truth table of $-A$

This allows us to modify a statement. Now let's write it for and: say we have two statements A, B, we want to know when $A \wedge B$ (A and B) is true then we write:

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

Table 2: Truth table of $A \wedge B$

Another way to see the same thing is the Van diagram:



This is our definition of this operation, it "just happens" to correspond to what we mean when we say " it is sweet and sour".

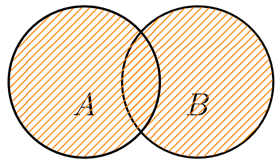
The next connective is the or: How would we define $A \vee B$ (A or B) then?

IMPORTANT: This is not exactly what we mean when we say or in regular speech, since we often mean one or the either but not both together. Mathematical or is always inclusive (there is a notion of xor with is in an exclusive or). Another way to think about it is that you can replace it with or/and.

Another way to see the same thing is the Van diagram:

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

Table 3: Truth table of $A \vee B$



The last connective is the implication, the if.. then statement, we write it as $A \implies B$, we call B conclusion, and A an assumption, or hypothesis. So we define:

A	B	$A \implies B$
T	T	T
T	F	F
F	T	T
F	F	T

Table 4: Truth table of $A \implies B$

This seems rather weird, but think about the following statement:

If it is cool outside, I will walk home

So if it is cool and I walked home, I obeyed the rule. If it was cool and I took my car home- I lied, but what happens if it is hot? can I say that I lied if I still walked? Mathematicians say that the rule is vacuously true- as it is true since the assumption doesn't hold.

This is symbolic logic. An important element is that two equivalent expressions (that is, have the same truth table) are the same!

An example is the expression $A \implies (B \wedge A)$, we check:

A	B	$A \wedge B$	$A \implies (B \wedge A)$
T	T	T	T
T	F	F	F
F	T	F	T
F	F	F	T

Table 5: Truth table of $A \implies B \wedge A$

This is equivalent to just $A \implies B$! so whenever we have this expression we can simplify it to $A \implies B$.

Now let's do the same thing to $(A \wedge B) \implies A$:

A	B	$A \wedge B$	$A \wedge B \implies A$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Table 6: Truth table of $A \wedge B \implies A$

This type of expression is called a tautology, it is always true. Its negation is a contradiction- it is always false! Another example will be $-A \wedge A$:

A	$-A$	$-A \wedge A$
T	F	F
F	T	F

Table 7: Truth table of $-A \wedge A$

Finally, we need a way to deduce from the previous statement. And the only rule we give is modus ponens- the simplest deduction:

$$\begin{array}{l} P \implies Q \\ P \\ \hline Q \end{array}$$

We will not build all the logical implications of it, but this is, in fact, enough to build a logical system.

So we can define

Definition 2.2. A proof of Q from P in an argument that starts with P ends with Q and each step is deduced from the previous.

The only small asterisk is that when we have quantifiers (what we will talk about next time) we have a bit more wiggle room.

2.5 What did we see today?

- History and what are axioms, statements and proofs.
- Truth tables,
- How to show statements are equivalent.
- Deduction, and what are proofs.

3 3rd Class- Quantifiers, order of quantifiers, proof by construction, proof by casework

3.1 Review

So what have we seen so far? we have seen how to start formalizing our language. We discussed what are statements, axioms, arguments, and proofs.

We also saw how can we deduce things, or use equivalences to simplify arguments and check if what we had is proof or not.

3.2 Examples

We will start by checking our first equivalent. We want to find an equivalent to $A \implies B$, using the other relation. We start by looking at the table of $A \implies B$ We note that we

A	B	$A \implies B$
T	T	T
T	F	F
F	T	T
F	F	T

Table 8: Truth table of $A \implies B$

only have one F, which operation also has only 1 false? the $\vee$ (or)! but it is not exactly in the right place right? how do we move it? using negation so we write: There are other

A	B	$A \implies B$	$-A \vee B$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Table 9: Truth table of $A \implies B$ and $-A \vee B$

possibilities, of course. There are many expressions that one can write that are equivalent. TP: find another example.

An important example is the contrapositive! We will talk about it more carefully next week, but in the meanwhile, we look at the truth table of $-B \implies -A$ against $A \implies B$:

A	B	$A \implies B$	$-B \implies -A$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Table 10: Truth table of $A \implies B$ and $-B \implies -A$

They are the same! So another way of saying $A \implies B$ is saying $-B \implies -A$. In other words, "If it is cool I will walk home" is the same as " If I will take the car then it is hot".

This leads to the way for a technique of proof called proof by contradiction. There are different ways to prove it. This equivalency is closely related to the assumption that we did- that statement can either be true or not. This is also called the law of excluded middle. There are different ways to go around it - either by fuzzy logic or by allowing statements that can't be proven as true, instead of false statements.

As an example of argument, we will use the classic Catch-22 argument (taken from Wiki)

> "You mean there's a catch?" "Sure there's a catch," Doc Daneeka replied. "Catch-22. Anyone who wants to get out of combat duty isn't really crazy." There was only one catch and that was Catch-22, which specified that a concern for one's own safety in the face of dangers that were real and immediate was the process of a rational mind. Orr was crazy and could be grounded. All he had to do was ask; and as soon as he did, he would no longer be crazy and would have to fly more missions. Orr would be crazy to fly more missions and sane if he didn't, but if he was sane, he had to fly them. If he flew them, he was crazy and didn't have to; but if he didn't want to, he was sane and had to. Yossarian was moved very deeply by the absolute simplicity of this clause of Catch-22 and let out a respectful whistle.

So we formulate F- flying to missions, S- being sane, and R requesting to stop flying.

Our assumptions are:

1. If you want **not** to fly to missions, you have to request to stop flying and not be sane.
2. If you are insane you won't ask to stop flying.

So we can write them as follows

1. $-F \implies (-S \wedge R)$.
2. $-S \implies -R$

So we get the following logic chain

1. $-F \implies (-S \wedge R)$, Assumption 1.

2. $-S \implies -R$, Assumption 2.
3. $S \vee -R$, equivalent to 2.
4. $-(-S \wedge R)$, equivalent to 3 (HW).
5. $-(-S \wedge R) \implies F$, equivalent to 1.
6. F, deduction 4 and 5.

So we conclude that regardless of your sanity, you will always fly! Let's do an example: Think of an example with 3 statements, and let's analyze it!

Next, Let's construct a proof. We will assume that $A \implies B, B \implies C, -C$, and we will prove A. This is the idea of proof by contradiction!

1. $A \implies B$, Assumption 1.
2. $B \implies C$, Assumption 2.
3. $-C$, Assumption 3.
4. $-B \vee C$ equivalent to 2.
5. $C \vee -B$ equivalent to 4.
6. $-C \implies -B$ equivalent to 5.
7. $-B$ deduction 6 and 3.
8. $-B \implies -A$, equivalent to 1.
9. $-A$ deduction 8 and 7.

We will talk about this type of proof more next time.

3.3 Quantifiers

Now the question arises, is this everything? can we now deal with everything math throws in our way? Well, in math, you have formulas $x^2 + 2x$, or we say "x is nice", or the homework is due on XXX. These all are formulas, in some sense, they are not really statements, as only if you specify the value of x it will give a statement. Maybe " Tal is nice" is false, but "Doc is nice" is true.

These symbols are placeholders, and we will call them variables (surprisingly enough). These types of expressions with variables are called open statements. An open statement is not true or false, since it depends on the value of the variables.

The way we specify the values of the variable is by quantifiers. There are two of them: $\forall$-read "for all", "for every", "whenever" or the universal quantifier, and $\exists$ read "exists", "there is" or existential quantifiers.

When we use these quantifiers we always, but always, have to specify the range of them.

For example, we can write:

$$\forall x \in \text{Elephants} \, , x \in \text{Pink}$$

And we will read "All x that is an elephant, x is pink", or in human language "All elephants are pink". Similarly, we can write

$$\exists x \in \text{Pink} \;\text{ s.t. }\; x \in \text{Elephants}$$

And read "There is x in pink such that x is an elephant" or in human language "There is a pink elephant".

A quantified statement is a statement that **all** its variables are quantified, and when written, it is called the standard form.

A few more examples:

$$\forall x \in \text{Clans} \; \forall y \in \text{Bagpipes} \, , x\text{-marching} \implies y\text{-playing.}$$
$$\exists x \in \text{Kings} \; \exists y \in \text{Snakes} \, , x \text{ is left-handed} \wedge y \text{ is yellow}$$

Some nonexamples

$$\forall x \in \text{Baths} \, , y \text{ is comfortable.}$$
$$\forall x \in \text{Tanks} \, , \exists x \in \text{Towers} \, , x \text{ is blue}$$
$$\forall x, \exists y, x > y$$

Many times we don't say precise statements: "People like the blue color" - is it to say all people, or just there are some people? Or even worse, use the wrong quantifier "Everybody loved secret invasion". In math, we have to be very careful about what quantifiers we use- and whenever we don't quantify, it is understood as universally quantified. For example, "show that a differentiable function is continuous", we always mean that it is true for **all** function of this type.

We saw that we can have several variables, and then the order of quantification matters a lot.

TPS: Consider the following expressions

$$\forall x \in \text{Humans} \, , \exists y \in \text{Heads} \, , y \text{ belongs to } x$$
$$\exists y \in \text{Heads} \, , \forall x \in \text{Humans} \, , y \text{ belongs to } x$$
$$\forall x \in \text{Binders} \, , \forall y \in \text{Stickers} \, , y \text{ can stick to } x$$
$$\forall y \in \text{Stickers} \, , \forall x \in \text{Binders} \, , y \text{ can stick to } x$$

Which ones are the same? which one is different? Now, let's say the first two:

- For every human there exists a head, such that this head belongs to this human.
- There is a head so that for every human, this head belongs to this human

So these two have very different meanings. Whereas when the other two have the same meaning/

There are several rules here, first, we assume that variables that appear later, depend on the ones that came before them. Second, we see that the order is between the different kinds of quantifiers, not among themselves. We will get back to this idea many times in the semester- finding the differences between two statements with different quantifications.

3.4 What did we see today?

- Quantifiers, and standard forms.

4 4th Class- Proof by contradiction and counterexamples, and pitfalls

4.1 Review

So, what have we seen so far? We started formalizing our language. We discussed what are statements, axioms, arguments, and proofs. We also saw how we can deduce things, or use equivalences to simplify arguments and check if what we had is proof or not, and we learned to add quantifiers. We saw how to write quantified statements and talked about their meaning.

Some announcements:

- Email- I am happy to help with HW but don't send me three pages of the exact details and expect me to comb for mistakes there. Try to explain simply what is going on and ask questions about it. If it is more time-consuming- schedule a meeting or come to OH.

- OH will be MW after class The TA will announce their OH later this week.

4.2 What can we prove then?

Now, we have a new type of statement, we need some way to deduce that they are true (or false). We will start with the existence quantifier, how can you show that an existence statement is true?

Well, one way is to show that it exists! That is build an object in the range, that has all the properties. For example, in epistemology (a branch of philosophy) there is a question of whether the external world exists. Here is George E. Moore's argument (as taken from Wikipedia):

- Here is one hand,

- And here is another.

- There are at least two external objects in the world.

- Therefore, an external world exists.

A proof by constructing an example.

A note of caution: there are proofs of existence that are not constructive (do you know any example from calculus?)! Some mathematicians reject nonconstructive proofs- as they usually rely on some axioms that many don't accept for various reasons (the choice axiom that we will not discuss here). And so they are trying to build all the math on constructive proofs.

Next, we have the universal quantifier. How can we prove anything about this one? Well one option is by casework, for example, consider the following statement:

- "Everyone that is named Tal in this room has a beard."

This statement we can just verify: everyone that is named Tal in this room raise your hand. Ok, do they have bread? Yes, great! What about the following statement:

- "Everyone that registered to Elements of Analysis is currently a student at Rice"

Well now we need to start verifying one by one: Are you a student at Rice? And you? and you? Who knows? maybe we have an infiltrator! So proving like that could be very taxing. It is hard to go over all the cases. Sometimes we can shorten the process, say with the following statement:

- All the students are younger than the instructor.

We can break it into groups: Who is the oldest freshman? sophomore? junior? senior? from graduate school?

But still, this takes a lot of time, so how can you prove something? One way, maybe the most common in mathematics, is by choosing an element. For example, we can prove the statement

- "If you are registered in Elements of Analysis you are a student at Rice"

in the following way:

- Let x be registered to this course.
- x had to use the registration system to register.
- This system is only open to students
- therefor x is a student.

We chose an instant of the group and showed something about it.

Now let's think about the following statement: All the students in this course wear black T-shirts! Proof:

- You are a student in this course.
- You are wearing a black T-shirt.
- All students wear black T-shirts.

What is the problem (fallacy) here?

We used a specific property of the instance we used, not a property that is intrinsic to the group (that they are registered to this course).

A Word of caution: We have to be very careful about what type of statement we are proving! if it is existence we just need to find an example, but an example is never proof for a universal statement.

4.3 How do we negate quantifiers?

How can we show that something doesn't exist? For example:

- There is no cat registered to this course

How can we know that? formally written it looks like this:

$$-\exists x \in \text{ Registered to this course }, x \in \text{Cats}$$

The way to show it is to show that everyone registered for this course is not a cat! Similarly, if we want to show the negation of the following statement:

- Everyone in this course is at Lovett College

We just need to bring a counter-example- in other words, to prove the existence of the contrary. A way to remember that:

$$- \forall = \exists -$$
$$- \exists = \forall -$$

4.4 Proof by contradication

Now we are going to talk about proof by contradiction - one of the more useful tools, but one that is hardest usually to grasp. Many people find it a difficult concept- -as it is somewhat obscure. As a result, many people either prove anything by contradiction - even when it is not necessary or use it in a wrong way.

Proof by contradiction is based on the simple proof, that we saw:

1. $A \implies B$, Assumption.
2. $-B$, Assumption.
3. $-B \implies -A$, equivalent to 1.
4. $-A$, deduction, 2 and 3

Let's do an example:

> A burglary happened in Texas A&M somebody stole their mascot's outfit! Sherlock Holmes has two suspects: Prof. Jane Moriarty - a renowned mathematician from Rice University, and Sir Charles Baskerville - A respectable, though sometimes dubious nobleman from Austin. After interrogating both suspects, Sherlock declares that it had to be Sir Baskerville! Dr Watson asks: "But, how did you know it?", they reply: "Say that Moriarty had done it, She then would have been at College Station at 1 pm, and still she made it to her class, at Rice University, at 1:30. This is almost a hundred miles to cover in half an hour! This is impossible, so we have arrived at a contradiction!"

Let's formulate the argument: B -Baskerville is guilty, M - Moriarty is guilty, and P- it took Moriarty 30 minutes to get from College Station to Houston. Then we have:

1. $-B \implies M$, assumption (One of them is guilty)
2. $M \implies P$, assumption.
3. $-P$ - assumption.
4. $-M$, by contradiction.
5. $-B$ by contradiction

That is why Sherlock Holmes Saying is "When you have eliminated the impossible, whatever remains, however improbable, must be the truth". (Think about when it was written!).

When phrased like this- can you think about a problem with this proof method?

You have to make sure you have exhausted all possibilities! This only works if **ONLY** Moriarty or Baskerville could have done this!

How does it work with a quantified statement? For example, we can consider the following statement:

$$\forall n \in \mathbb{N}, \exists p \text{ is prime}, p > n$$

In other words, there is an infinite number of primes. We will build all these things precisely, but for now, let us proceed formally how do we negate this statement?

$$\exists n \in \mathbb{N}, \forall p \text{ is prime}, p \leq n$$

Assuming that- we can write all the primes as $p_1, \ldots, p_m$. Then we consider $Q = p_1 \cdot \ldots \cdot p_m + 1$ this number is not prime (as it is not on the list), so there is some p_i prime that divides it. But then p_i divides Q and $p_1 \cdot \ldots \cdot p_i \cdot \ldots \cdot p_n$, so it also divide $1 = Q - p_1 \cdot \ldots \cdot p_i \cdot .. \cdot p_n$, contradiction. So we eliminated the impossible and we are left with the improbable: there are an infinite number of primes. Note that in this case, it is not a constructive example, as we didn't show that Q is prime on its own merit.

4.5 What did we see today?

- Proof by way of contradiction
- Problems in proof by way of contradiction- and a bit how to deal with them.
- The difference between proof by contradiction and counterexamples.

5 5th Class- the problems with proof by contradiction and sets

5.1 Review

So what have we seen so far? We have talked about statements and proofs. We explained quantifiers and how to treat statements with "for all" or "there exists". And we saw some different methods that we will use to prove such statements: construction, contradiction, and casework. In this class, we are going to start the second part of the preliminaries and that is to discuss set theory.

5.2 The problems with and in proof by contradiction

As I started with people have a lot of difficulty with this way of proof. So we will review some of the common problems with it. Recall the story:

> A burglary happened in Texas A&M somebody stole their mascot's outfit! Sherlock Holmes has two suspects: Prof. Jane Moriarty - a renowned mathematician from Rice University, and Sir Charles Baskerville - A respectable, though sometimes dubious nobleman from Austin. After interrogating both suspects, Sherlock declares that it had to be Sir Baskerville! Dr Watson asks: "But, how did you know it?", they reply: "Say that Moriarty had done it, She then would have been at College Station at 1 pm, and still she made it to her class, at Rice University, at 1:30. This is almost a hundred miles to cover in half an hour! This is impossible, so we have arrived at a contradiction!"

We have the following arguments, for each one try to describe the problem (it is not necessarily a logical problem), and discuss it with your neighbor:

1. Dr. Watson claimed the following:

   > Say that Moriarty had done it, then she would have the outfit in her car, but the costume was found in Baskerville's car, and we have arrived at a contradiction!

2. Inspectress Lestrade claimed the following:

   > Say that Baskerville has done it. He had a game of Cricket at 3 pm in Austin. He would have plenty of time to change clothing and show up for his game. We didn't get a contradiction so he must have done it.

3. One of the constables tries their luck as well:

   > Say that Moriarty has done it. Then to get from College Station to Rice, she would have needed to cross the border to Canada. She didn't cross the border, so she couldn't have done it.

4. Another constable says:

> Say that Baskerville hasn't done it. He said that he lived in Austin for all of his life, yet he didn't know there was a UT Austin in the city. We arrived at a contradiction so he must have done it.

What are the problems here-TPS

1. Here what happened was Dr. Watson has direct proof hiding as proof by way of contradiction. In fact, they have direct evidence that Baskerville has done it, but performed a logical loop, and presents it as though it is proof by contradiction. Though this is technically correct- this is an example of a bad contrapositive proof. Since it hides the true nature of the proof. A proof should be as direct as possible. This is not erroneous, but it is ill-advised.

2. In this case, Inspector Lestrade has no proof by contradiction. They assumed something and didn't arrive at a contradiction. So they have no justification to claim that our assumption is true! Maybe the contradiction hides somewhere else? It just means that it is possible, not that it had to happen!

3. Here I rely on the fact that you know that both cities are in the US, and you don't need to cross the border to get there. Here what happened is that we assumed something in the way of contradiction- and we did get a contradiction. But our contradiction in fact stemmed from an additional assumption that we sneaked in. The contradiction came from this added assumption and nothing in the argument.

4. Here we have arrived at a contradiction, but it has nothing to do with the assumption. The only valid conclusion is that he may be lying- but that does not imply that he has done anything (beyond, maybe lying).

These, of course, are only some of the ways things can go wrong with such proofs. Even though these examples look kind of funny, I can almost guarantee that you will make some of these mistakes throughout this course. Simply because it always happens. For example, the last type of error, in math proofs usually shows up when you make some small error, and the contradiction is just that you forgot a minus somewhere. The second case also happens quite a lot- this is the case when you lose sight of your goal. You forget that you are looking for a contradiction. The first case is unfortunately all too frequent even in papers...

So how do we deal with it? what strategies can we use to check ourselves? There is one very important test, that helps find most of the errors:

- Always check that you used the assumption you want to contradict in a strong way! If you haven't really used it- maybe you have a direct proof for the result- or you just have some error that leads to a contradiction.

A final remark: Is there a difference between proof by contradiction and a counterexample? or are they the same- TPS.

These are very different things. While proof by way of contradiction is a way to prove any kind of statement, a counterexample is a way to show that a universal statement is wrong. In a counterexample, we don't use the contrapositive- we just construct the negation directly.

5.3 Sets

In some way, we already used sets without saying it. A *set* is a collection of objects such as "elephants" "Bakers", and "numbers", the members of the sets are called elements. We will use $x \in A$, where A is a set, to denote a n element in that set. , we can also write $x \notin A$, which means x is not an element of A. When we want to define a set we write expressions like:

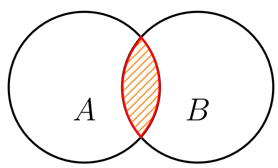
$$A = \{x \in \text{Elephants} \ \mid x \in \text{ Pink objects}\}$$

We read all the elephants that are also pink objects. The rule is $x \in$ some set, and then a line and a condition that will hold true for all these x. Some sets have agreed names such as $\mathbb{N}$ the natural numbers. Some sets we will define in words- though we have to make sure that our definition makes sense.

Now we will consider the following set,

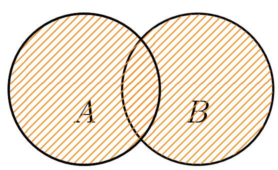
$$C = \{x \mid x \in A \wedge x \in B\}$$

What does that set look like? What is the connection between that and the diagram presented in the first class? This set is the intersection between A and B, and not surprisingly



it is denoted with $A \cap B$.

What the corresponding set will be for the condition $\vee$? This will be the union, denoted with $A \cup B$, and will have the corresponding van diagram:

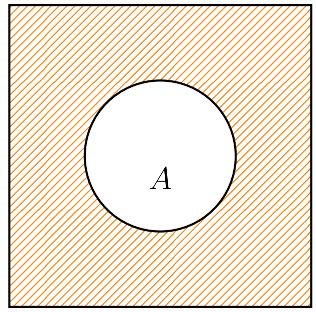


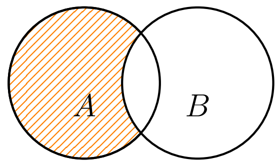
How about $-P$- the negation? So for that, we have the complement set! There are a lot of notions for the complement $-A, A^c, \overline{A}$, and so on, and the way we write it as a van diagram is as follows

The last operation- is just a composition of two operations and that is $A \setminus B = A \cap B^c$, In Van diagram:

5.4 What did we see today?

- we talked about the problems with proof by contradiction
- We defined sets, and saw their basic operation.
- we talked a bit about the connection to logic.





6 6th Class - containment and De Morgan laws

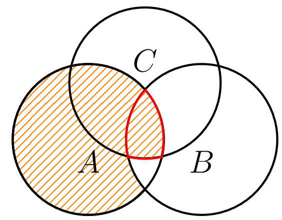
Some announcements, about the HW:

- I will release an answer key- it won't be necessary written in full detail- but you could check that you got the right answers. For example- I will just write the truth table of the full expression- whereas I expect you to show more work.
- Questions about the HW: If it is technical- will you accept this or that- should be emailed to our TAs. Extensions should be directed to me. More involved questions of understanding should be dealt with in OH (if you can't come, then email) Our TAs: Miri Son, and Hyein Choi - emails as usual- first name. last name @rice.edu.
- I gave you 2 assignments that you can drop. Because I want to treat you as adults. So please behave like this. I told you that an extension for HW will be given only in emergencies. Please don't try to use my niceness.
- It is your responsibility to check your submission when you upload! It is on you. I will not accept things late- unless something like your laptop was destroyed by a meteor or something.
- About OH- they are open to whoever wants them- auditing, registered, or just spectators.
- About Pset 2- When I ask for formal proof in 3 I mean like the proof I will show today, or using equalities already established.

6.1 Review

So what have we seen so far? We have talked about statements and proofs. We explained quantifiers and how to treat statements with "for all" or "there exists". And we saw some different methods that we will use to prove such statements: construction, contradiction, and casework. We have defined what are sets and defined the basic operations on these.

So now let's do an example as TPS consider $(A \cap B \cap C) \cup (A \cap B^c)$, and draw its van diagram- then share with your neighbor. It will look like this



We have discussed this a bit- but you have to recall the Vann diagram is not proof! It is an example.

In fact, all the logical operations (that is everything that we learned except for quantifiers) can be expressed using a Van diagram! For those of you who like visualizing more- it may be easier to think about these operations like that!

Why are these things connected this way? Why are we able to think about logical operations similarly to the way we think about set operations?

The reason is simply that logical conditions talk about events in the world! when we ask $P \wedge Q$ we consider the ask for the event that both happen, so if we consider the world of all possible outcomes, and we mark with A the outcomes for which P is true, and by B the outcomes for which Q is true- then $A \cap B$ will exactly these outcomes that P and Q are true! This realization can lead to the probability theory! but we will not deal with that now. Let's do an example, we would like to prove the following:

Claim 6.1. Let A, B, C be sets, then

$$A \cap (B \cap C) = (A \cap C) \cap B$$

Proof. We actually need to show two things: one that $A \cap B = B \cap A$, known as commutative law, and that $A \cap (B \cap C) = (A \cap B) \cap C$- known as associative law.

We start with the first. $x \in A \cap B$, iff $(x \in A \wedge x \in B)$ iff $(x \in B \wedge x \in A)$ iff $x \in B \cap A$. We note that we used that we can commute the logical and- but that can be verified via a truth table.

Next, we need to show that associative law, which has a similar logic: we have that $x \in A \cap (B \cap C)$ iff $x \in A \wedge x \in B \cap C$ iff $x \in A \wedge x \in B \wedge x \in C$ iff $x \in A \cap B \wedge x \in C$ iff $x \in (A \cap B) \cap C$. Finally, to finish off we can write:

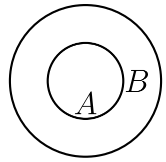
$$A \cap (B \cap C) = A \cap (C \cap B) = (A \cap C) \cap B$$

□

6.2 Containment

Next, we will sometimes want to say that a set is completely contained inside another set. We would write it like this: $A \subset B$. What does that mean? how can we express it precisely? It is the same as all the elements in A are also in B or other words:

$$A \subset B \iff \forall x, x \in A \implies x \in B$$



Or a Van diagram like this:

Not that here I didn't write $\forall x \in$ something. Throughout this section, we will assume that there is some ambient space that the elements belong to. You can think of it as $x \in$ World or something.

Next, we want to talk about equality. How can we say that two sets are equal? In terms of logic, this is the same as the statement $x \in A \iff x \in B$ or in other words $A \subset B$ and $B \subset A$.

Now we would want to prove something quite simple, if $A \subset B$, then $A \cap B = A$. How do we do it?

Proof. We start in the easy direction:

$\underline{\subseteq}$ Let $x \in A \cap B$, then by definition $x \in A$ and $x \in B$. In particular, $x \in A$.

$\underline{\supseteq}$ Let $x \in A$, and $A \subset B$, so we have $x \in A$ and $x \in B$, by definition we have $x \in A \cap B$ $\square$

By the way, proving an " $\iff$ " works similarly. We just prove both directions. We can also talk about infinite intersections or unions. For example, we can define

$$A_n = \{x \in \{\text{People}\} \mid \text{ The age of } x \text{ is greater then } n\}$$

then what will be in $\bigcap\limits_{n\in\mathbb{N}} A_n, \bigcup\limits_{n\in\mathbb{N}} A_n$?

That is right, the first will contain all the people whose age is bigger than **any** natural number - so the empty set (a notion we will use $\emptyset$ to denote), the second is the same as all the people whose age is bigger than **some** natural number, which is equal to A_1!

A remark- we take here that $\mathbb{N}$ starts from 1 which is easier for the following - but the conventions might differ.

Similarly to intersections and unions over the natural numbers, I can also do it over any type of index set. For example:

$$A_i = \{x \in \{\text{students}\} \mid x \text{ is named } i\}$$

Then we can talk about

$$\bigcup_{i\in\{\text{names in math302}\}} A_i, \bigcap_{i\in\{\text{names in math302}\}} A_i$$

The first is the collection of all the students in the class, and the second? is the student that is called by all the names in the class- hopefully, that is no one here, as that will be very confusing.

6.3 De Morgan laws

One of the important rules that we will use in this course is De Morgan's laws, these are tying the knot that connects $\vee$ and $\wedge$. Before we prove them I wanted to introduce the idea of disjoint union: We write $C = A \sqcup B$ to denote two facts: $C = A \cup B$ and $x \in A \iff x \notin B$. This is the same as the xor that we mention in HW. This is useful to determine complements of sets. So we will start

Proposition 6.2 (DeMorgan)**.** $(A \cap B)^c = A^c \cup B^c$

Proof. $\underline{\supset}$: Let $x \in A^c \cup B^c$ that implies $x \in A^c$ or $x \in B^c$, but if $x \in A^c$ then $x \notin A$, and in particular $x \notin A \cap B \subset A$, so we conclude $x \notin A \cap B$ or $x \in (A \cap B)^c$. If $x \in B^c$ then $x \notin B$, and in particular $x \notin A \cap B \subset B$, so we conclude $x \notin A \cap B$ or $x \in (A \cap B)^c$.

$\underline{\subset}$: Let $x \in (A \cap B)^c$, then $x \notin A \cap B$. Now we claim that this implies $x \notin A \vee x \notin B$:

Claim 6.3. If $x \notin A \cap B$ then $x \notin A \vee x \notin B$.

Proof. Let $x \notin A \cap B$, and we will assume by way of contradiction that $x \notin A \vee x \notin B$ is false. $x \notin A \vee x \notin B$ is false when $x \in A$ and $x \in B$ (do a truth table) then so $x \in A \cap B$ in contradiction. $\square$

So we got that $x \notin A \vee x \notin B$ but that is the same as $x \in A^c \vee x \in B^c$, which is the same as $x \in A^c \cup B^c$. $\square$

There is another way to see this:

Proof. Let us denote the world we work in as W. Then we can write:

$$
\begin{aligned}
&W = A \sqcup A^c, \text{ Tautology} \\
&A = (A \cap B) \sqcup (A \setminus B) \text{ In the HW} \\
&W = ((A \cap B) \sqcup (A \cap B^c)) \sqcup A^c \text{ Inserting 1 into 2} \\
&W = (A \cap B) \sqcup ((A \cap B^c) \sqcup A^c) \text{ Associative law} \\
&W = (A \cap B) \sqcup ((A \cup A^c) \cap (B^c \cup A^c)) \text{ Distributive law} \\
&W = (A \cap B) \sqcup (W \cap (B^c \cup A^c)) \text{ This is 1} \\
&W = (A \cap B) \sqcup (B^c \cup A^c) \text{ From earlier}
\end{aligned}
$$

So the complement of $A \cap B$ is $B^c \cup A^c$. $\square$

This is very important when we want to negate statements or sets. We will give only one proof of the other DeMorgan law:

Proposition 6.4 (DeMorgan2)**.** $(A \cup B)^c = A^c \cap B^c$

Proof. Let us denote the world we work in as W. Then we can write:

$$
\begin{aligned}
&W = A \sqcup A^c, \text{ Tautology}\\
&A^c = (A^c \cap B^c) \sqcup (A^c \setminus B^c) \text{ In the HW}\\
&W = A \sqcup ((A^c \cap B^c) \sqcup (A^c \setminus B^c)) \text{ Inserting 1 into 2}\\
&W = (A \sqcup (A^c \cap B)) \sqcup (A^c \cap B^c) \text{ Associative law}\\
&W = ((A \cup B) \cap (A \cup A^c)) \sqcup (A^c \cap B^c) \text{ Distributive law}\\
&W = ((A \cup B) \cap W) \sqcup (A^c \cap B^c) \text{ This is 1}\\
&W = (A \cup B) \sqcup (A^c \cap B^c) \text{ From earlier}
\end{aligned}
$$

So the complement of $A \cap B$ is $B^c \cup A^c$. □

6.4 What did we see today?

- We practiced proof for sets.
- We talked about containment and infinite unions and intersections.
- We saw proofs of De Morgan laws.

7 7th Class - The natural numbers and induction

7.1 Review

So what have we seen so far? We did a lot of preliminaries: Basic logic, quantifiers, sets, operations, and the De Morgan laws. We would want to see an example of a use of De Morgan law, we would want to prove that

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$$

How do we do it?

Proof. Well, we know that :

$$A \setminus (B \cup C) = A \cap (B \cup C)^c$$

Now we can use De Morgan to get:

$$
\begin{aligned}
A \setminus (B \cup C) &= A \cap (B \cup C)^c = A \cap (B^c \cap C^c) = A \cap A \cap (B^c \cap C^c)\\
&= (A \cap B^c) \cap (A \cap C^c) = (A \setminus B) \cap (A \setminus C)
\end{aligned}
$$

As needed. □

7.2 Natural number and induction sets

Now, we are going actually to start talking about numbers! Now, we are starting where most people start this course.

Finally, we introduce the natural numbers. We will denote by the letter $\mathbb{N}$ the natural numbers $\{1, 2, \ldots\}$. We will not consider 0 as a natural number. But what properties do the natural numbers have that make them interesting? Well, they have several properties, and we will denote the successor $S(n) = n + 1$.

1. $1 \in \mathbb{N}$.
2. If $n \in \mathbb{N}$ then it's successor $S(n) \in \mathbb{N}$.
3. 1 is not a successor of any natural number.
4. If $S(n) = S(m)$ then $n = m$,
5. Any inductive subset of $\mathbb{N}$ is equal $\mathbb{N}$

But what is an inductive set? We define

Definition 7.1. A set A is called inductive if:

1. $1 \in A$.
2. If $n \in A$ then $n + 1 \in A$

These 5 axioms are known as Peano axioms. They are the things that allow us to define this set (If you remember when we talked about logic I said in order to get the incompleteness theorem, you should be able to express the natural numbers in the logic system - these are the axioms that you should be able to express). Most of these seem trivial, except for the last one, right? The last axiom is called the induction axiom. Intuitively we have that if a set contains 1 and contains the successors then it should contain and natural number n, which can be proven.But we want to make sure that the natural numbers are minimal with respect to this property. Somehow they a the smallest set for which it works. This axiom is the basis for mathematical induction!

7.3 Induction

Mathematical induction utilizes these axioms to show things for all of $\mathbb{N}$ at once! Let $P_1, P_2, \ldots$ be a collection of statements, such that

1. P_1 is true.
2. If P_n is true then P_{n+1} is true.

Then we know that the $\{i \mid P_i \text{ is true}\}$ contains 1 and if i is in it, so is $i + 1$, so we conclude that it is the whole natural numbers. Therefore we have that P_i are all true.

One important thing is to always check that you are using the induction assumption! If you are not, then you are proving things directly. Sometimes it is possible- but sometimes it means that you have a mistake!

Example 7.2. We want to prove: $1+2+\cdots+n=\frac{1}{2}n(n+1)$

Proof. Our statement is $P_n=$ "$1+2+\cdots+n=\frac{1}{2}n(n+1)$". So we verify it for P_1:

$$P_1=\text{"}1=\frac{1}{2}1\cdot 2\text{"}$$

Which is true. Next, we assume that P_n is true and we build a logical chain that shows that P_{n+1} is true. That is we need to show that $P_n \implies P_{n+1}$.

So we need to assume P_n, and then we look at RHS of P_{n+1}:$1+2+\cdots+n+n+1$, we can use our assumption P_n to get:

$$\begin{aligned}1+2+\cdots+n+n+1&=\frac{1}{2}n(n+1)+n+1=[\frac{1}{2}n+1](n+1)\\&=\frac{1}{2}(n+2)(n+1)=\frac{1}{2}((n+1)+1)(n+1)\end{aligned}$$

As needed. So we conclude that P_n is true for all n, and we conclude by induction principle. □

Not that I can't write things like:

$$1+2+\cdots+n+n+1=^{?}\frac{1}{2}((n+1)+1)(n+1)$$

as this is what we are trying to prove. You might be tempted to manipulate in a way that works only with equality (for example, taking the square of both sides). This is a very strong tool! We can prove all at once an infinite number of statements with a finite computation! I am sure you have done induction (even without noticing)! Let's do another example

Example 7.3. All the numbers of the form 5^n-4n-1 are divisible by 16.

Proof. Our statement is $P_n=$ "5^n-4n-1 is divisible by 16". But how can we translate it into formal language? what does it mean that it is divisible by 16? We can write it as $P_n=$ "$\exists r_n\in\mathbb{N}, 5^n-4n-1=16\cdot r_n$. So we start by checking P_1:

$$\text{LHS}(P_1):5^1-4-1=0$$

Now, is 0 divisible by 16? Usually, we say yes, but it doesn't work with our formulation of the statement (as we declared $0\notin\mathbb{N}$). There are two fixes for that: One option is we can change P_n to be $P_n=$ "$\exists r_n\in\mathbb{N}\cup\{0\}, 5^n-4n-1=16\cdot r_n$", the other is to change P_n to be $P_n:$ "$\exists r_n\in\mathbb{N}, 5^{n+1}-4(n+1)-1=16\cdot r_n$"- then P_1 will correspond to P_2 in the previous formulation.

Let's choose the second option. In this formulation we need to check P_1 :

$$\text{LHS}(P_1):5^2-4\cdot 2-1=25-8-1=16=16\cdot 1$$

as needed. So P_1 is true. Now we assume that $P_n=$ "$\exists r_n\in\mathbb{N}, 5^{n+1}-4(n+1)-1=16\cdot r_n$" is true and we want to prove P_{n+1}:

$$\begin{aligned}\text{LHS}(P_{n+1}):5^{n+2}-4(n+2)-1&=5\cdot 5^{n+1}-4(n+1)-4-1\\&=5\cdot(5^{n+1}-1)-4(n+1)\\&=5\cdot(16\cdot r_n+4(n+1))-4(n+1)\end{aligned}$$

Where the last equality we used the induction assumption. we can now further simplify:

$$\begin{aligned}\text{LHS}(P_{n+1}) : 5^{n+2} - 4(n+2) - 1 &= 16 \cdot 5 \cdot r_n + 5 \cdot 4 \cdot (n+1) - 4 \cdot (n+1) \\ &= 16 \cdot (5 \cdot r_n + n + 1)\end{aligned}$$

So by choosing $r_{n+1} = 5 \cdot r_n + n + 1$, we showed the existence of r_{n+1} such that

$$5^{n+2} - 4(n+2) - 1 = 16 \cdot r_{n+1}$$

As needed. □

What did we see from this example? It has shown us that even though we write P_1, it doesn't mean the base case has to be plugged in $n = 1$, we can also shift to any n that we want. In fact, we can also look at functions of n!

7.4 What did we see today?

- We introduced the natural numbers.
- We learned what is an inductive set.
- We learned how to prove things by induction, and saw some tricks:
  1. We don't have to start with $n = 1$!
  2. Sometimes our induction step requires n to be large enough, sometimes, however, it is the base case that requires us to go to larger n.
  3. We always check that we used the assumption! If we didn't it is not induction!

8 8th Class -Strong induction, and the integers.

Some Announcements:

- OH - we have a classroom now! and they have moved slightly because of it. It will be M 3-4 in HBH 453, and W 3:30-4:30 HBH 447.
- If you notice I haven't released the Pset a week in advance- please let me know! Probably a technical problem.

8.1 Review

Last time, We introduced the natural numbers $\mathbb{N}$ and learned about inductive sets, which led us to start proving statements using induction. We show that for that you need a basis, and show that $\forall n, P_n \implies P_{n+1}$. We start with an example, to warm up:

Example 8.1. We want to prove that $n! > n^2$ for all n is large enough. Recall $n! = 1 \cdot 2 \cdots \cdot n$

Proof. This seems like a weird phrasing- but it is actually very common in math. We need to find some threshold from which this claim is true. This restriction could come from two sources: either our proof works only from some number or we need a higher base case for our induction.

So we will start with our step- to see if there is any requirement on the size for it to work. So we assume P_n : "$n! > n^2$". And we write

$$\text{LHS}(P_{n+1}) = (n+1)! = 1 \cdot \dots \cdot n \cdot (n+1) = n! \cdot (n+1) > n^2 \cdot (n+1)$$

We see that for that to work, we need $n^2 > n+1$. So we need another induction (we can also prove it by claiming something about the quadratic $n^2 - n - 1$- but today's topic is induction- not quadratics).

Example 8.2. We let Q_n : "$n^2 > n+1$".

Proof. This is obviously false for $n = 1$, but it is true for $n = 2$. So we have Q_2 (we can do the same trick and say $\tilde{Q}_n = Q_{n+1}$, and then we have $\tilde{Q}_1$).

Assume Q_n holds. Then we can write:

$$\text{LHS}(Q_{n+1}) = (n+1)^2 = n^2 + 2n + 1 > n + 1 + 2n + 1 = 3n + 2 = 2n + n + 2 > n + 2$$

Note that we have used the induction assumption, and we needed to assume $n > 0$. We got that $\{Q_n\}_{n=2}^{\infty}$ holds. $\square$

With this Q_n claim we get that for $n \geq 2$ we can write

$$\text{LHS}(P_{n+1}) = (n+1)! > n^2 \cdot (n+1) > (n+1)^2$$

As needed.

So our induction step works for any $n \geq 2$. So we need to find a base case under this constraint. We note that for $n = 2$ we have $n! = 2! = 2 < 4 = 2^2$, so continue with $n = 3$:

$$3! = 1 \cdot 2 \cdot 3 = 6 < 9 = 3^2$$

So $n = 3$ doesn't work, how about $n = 4$:

$$\begin{aligned} &4! = 3! \cdot 4 = 6 \cdot 4 = 24 \\ &4^2 = 16 \end{aligned}$$

It works! So we get a base case for the induction. So we get that $\{P_n\}_{n=4}^{\infty}$ is true- as claimed. $\square$

Now, we are going to prove to you that all the horses in the world have the same color!

Example 8.3. Let P_n be the statement "any set of horses of size n has the same color".

Proof. We start with the basis $n = 1$. Of course, any set of a single horse has one color- the color of that horse.

Next, we assume that a set n horses has only one color. Say I have a set of $n+1$ horses, denote it by H. Let $x \in H$, then $H \setminus \{x\}$ is a set of size n so all the horses there have the same color. So we get that any horse in $H \setminus \{x\}$ has the same color. Now let $y \in H \setminus \{x\}$ (we have such horse as $n + 1 > 1$) so now we can consider $H \setminus \{y\}$, again this is a group of n horses, so all of them has the same color. In particular, x has the same color as the rest of them. By induction- all horses have the same color □

This is obviously false! So how did we manage to prove that? TPS

The problem here is that this proof, though completely valid for $n \geq 3$ doesn't work for $n = 2$! Why? Since for $n = 2$ we have $H = \{x, y\}$. It is true that $H \setminus \{x\} = \{y\}$ and $H \setminus \{y\} = \{x\}$ has only one color- but it doesn't have to be the same color. In other words, we assumed that $H \setminus \{x\} \cap H \setminus \{y\} \neq \emptyset$! This is not true in this case. In order for the proof to work, we need to show the basis for $n = 2$, but this is obviously not true. There are sets of two horses in one color- but not any set of two horses has the same color!

This is exactly like the example we saw before: the proof needs to "start working" from the basis up otherwise we can't get the ball rolling.

8.2 Strong Induction

We Will now use Strong induction: This is the case where instead of just assuming P_n and getting P_{n+1} we have to assume that P_k holds for all $k \leq n$. Formally this looks like this $\forall n, (\bigwedge_{k=1}^{n} P_k) \implies P_{n+1}$. Let's give an example- the game Nim!

Example 8.4. In this game, there are two piles of matches, and there are two players. Each player, in their turn, chooses a pile and then chooses how many matches to remove from the pile (at least 1). The winner is the person that takes the last match in the game. We are going to prove that if the game starts with two piles of equal size the second player can always win.

Proof. We start in the case we have two piles of 1 match (this is $P(1)$) - the first player has to take the 1 match from one of the piles, and the second player wins by taking the last match.

Now we assume that if we start with piles of size k for $1 \leq k \leq n$, the second player can win. We will prove this for $n + 1$.

Say we have a game of two piles of size $n+1$. The first player takes some number ℓ from one pile. So we have one pile with $n+1$ and the other with $n+1-\ell$. The second player can then take ℓ from the second pile. Now if $\ell = n + 1$ the second player won. If not, we have a new game with piles of $n + 1 - \ell$, and by our induction assumption, the second player can win! □

This tells us that sometimes we are going to prove things using all the previous assumptions. Sometimes we need only some of them- but still, we will call this a strong form of induction.

8.3 The integers

So of course this is not enough to just talk about the natural numbers. Similar to how we are learning to count as kids - we start building our mathematics by counting $1, 2, \ldots,$ and now we can introduce the integers! these are the natural numbers, with 0, and the negatives. In other words $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup -\mathbb{N}$. But why do we need it? Well, the natural numbers have some nice properties: we can add things and remain in the natural numbers- but we can't subtract, which seem quite natural things to do! The integers have the following properties:

1. $\mathbb{Z}$ is close under addition. That is to say- this operation keeps us on the same playing ground!

2. We have $0 \in \mathbb{Z}$, which is a unique element so that any number $n \in \mathbb{Z}$ we have $n+0 = n$!

3. Furthermore, for every number $n \in \mathbb{Z}$ we have a number $m \in \mathbb{Z}$ so that $n + m = 0$

Let's try and write it formally:

1. $\forall n, m \in \mathbb{Z}, \exists x \in \mathbb{Z}, m + n = x$

2. $\exists!0 \in \mathbb{Z}, \forall n \in \mathbb{Z}, n + 0 = n$ or in other words
   $\exists 0 \in \mathbb{Z}, \forall n \in \mathbb{Z}, n + 0 = n \wedge -(\exists y \in \mathbb{Z} \setminus \{0\} \forall n \in \mathbb{Z}, n + y = n)$

3. $\forall n \in \mathbb{Z} \exists m \in \mathbb{Z}, n + m = 0$

In fact, this is so nice that we give such a structure a name

Definition 8.5. A(n Abelian) group is a set A, with an action on it $+$ and element $0 \in A$ so that the following holds:

1. The action is closed- $\forall n, m \in A \exists x \in A, n + m = x$.

2. Associativity $\forall n, m, k \in A, n + (m + k) = (n + m) + k$.

3. Commutativity $\forall n, m \in A, n + m = m + n$.

4. Identity element for addition $\forall n \in A, n + 0 = n$.

5. The existence of inverse $\forall n \in A \exists m \in A, n + m = 0$

So the integers are a(n Abelian) group! We will write soon that $(A, +, 0)$ is a group, which means that the set A with the operation $+$ and 0 is a group.

We will not prove that rules work- nor we will go in-depth into what it means- beyond the fact that there is a lot of structure here.

8.4 What did we see today?

- We saw some of the problems in Induction
- We saw what is strong induction.
- We introduce the integers.

9 9th Class -Fields and order

Remark 9.1. In this section of the class, we will be very strict. when you write an equality you should write where it came from.

9.1 Review

Last time, we finished talking about induction and started discussing the integers, we saw that they have some very nice algebraic structures. We defined:

Definition 9.2. A(n Abelian) group is a set A, with an action on it $+$ and element $0 \in A$ so that the following holds:

1. The action is closed- $\forall n, m \in A \exists x \in A, n + m = x$.

2. Associativity $\forall n, m, k \in A, n + (m + k) = (n + m) + k$.

3. Commutativity $\forall n, m \in A, n + m = m + n$.

4. Identity element for addition $\forall n \in A, n + 0 = n$.

5. The existence of inverse $\forall n \in A \exists m \in A, n + m = 0$

But they are not enough. We want to be able to multiply and divide. As before- multiplication is okay, but division could lend us outside of the integers! So we want something that is also a group with multiplication!

We want to consider the rational number as all the fractions of numbers in the integers. But we have many fraction that describe the same thing $\frac{4}{2} = \frac{2}{1}, \frac{10}{100} = \frac{1}{10}$ and so on. So we restrict ourselves to fractions of two numbers with no common factor. So we define rational numbers by

$$\mathbb{Q} = \{\frac{m}{n} \mid m \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\}, n, m \text{ has no common factor}\}$$

That is all the things that are written as fractions, where they have no common factor. When $n = \pm 1$, we sometimes write just $\pm m$ instead of $\frac{m}{\pm 1}$.

This has much nicer properties! It closed under addition and multiplication, each of the operations has a unique element that keeps all the numbers in place $(0, 1)$, and you have an inverse for each number in it (except for multiplication inverse for 0- which we can live with).

In fact the rational are very nice set to work with, this type of structure is called a field

Definition 9.3. A field is a set $\mathbb{F}$, with two actions on it $+, \cdot$, called "addition" and "multiplication" respectively, and element $0, 1 \in \mathbb{F}$ so that the following holds:

1. $(\mathbb{F}, +, 0)$ is a group.

2. $(\mathbb{F} \setminus \{0\}, \cdot, 1)$ is a group .

3. $\forall a, b, c \in \mathbb{F}, a \cdot (b + c) = a \cdot b + a \cdot c$

or put differently:

Definition 9.4. A field is a set $\mathbb{F}$, with two actions on it $+, \cdot$, called "addition" and "multiplication" respectively, and element $0, 1 \in A$ so that the following holds:

FA1 Associativity $\forall n, m, k \in \mathbb{F}, n + (m + k) = (n + m) + k$.

FB2 Commutativity $\forall n, m \in \mathbb{F}, n + m = m + n$.

FC3 Identity element for addition $\forall n \in \mathbb{F}, n + 0 = n$.

FD4 The existence of inverse $\forall n \in \mathbb{F} \exists m \in \mathbb{F}, n + m = 0$.

FE5 Associativity $\forall n, m, k \in \mathbb{F} \setminus \{0\}, n \cdot (m \cdot k) = (n \cdot m) \cdot k$.

FF6 Commutativity $\forall n, m \in \mathbb{F} \setminus \{0\}, n \cdot m = m \cdot n$.

FG7 Identity element for multiplication $\forall n \in \mathbb{F} \setminus \{0\}, n \cdot 1 = n$.

FH8 The existence of inverse $\forall n \in \mathbb{F} \setminus \{0\} \exists m \in \mathbb{F} \setminus \{0\}, n \cdot m = 1$.

FI9 $\forall a, b, c \in \mathbb{F}, a \cdot (b + c) = a \cdot b + a \cdot c$

We won't dive deep into the theory of fields, but we will taste a bit about proofs of basic concepts there. For now, we will prove some properties of fields:

Proposition 9.5. *Let $\mathbb{F}$ be an ordered field then we have the following, for any $a, b, c \in \mathbb{F}$*

P1 If $a + c = b + c$ then $a = b$.

P2 $a \cdot 0 = 0$.

P3 $(-a) \cdot b = -ab$.

P4 $(-a)(-b) = ab$.

P5 $ac = bc$ and $c \neq 0$ implies $a = b$.

P6 $ab = 0$ implies either $a = 0$ or $b = 0$.

Proof. Throughout this proof, we let $a, b, c \in \mathbb{F}$.

1. We assume that $a + c = b + c$, then we can write

$$\begin{aligned} a &=^{FC3} a + 0 =^{FD4} a + (c + (-c)) =^{FA1} (a + c) + (-c) \\ &= (b + c) + (-c) =^{FA1} b + (c + (-c)) =^{FD4} b + 0 =^{FC3} b \end{aligned}$$

   where we used our assumption. Note that I write above every equality where it came from. In this part of the class, this is very important (as we want to make sure we use the properties of the field alone).

2. We first write the following

$$0 + a \cdot 0 =^{FC3} a \cdot 0 =^{FC3} a \cdot (0 + 0) =^{FI9} a \cdot 0 + a \cdot 0$$

So we can write

$$0 + a \cdot 0 = a \cdot 0 + a \cdot 0$$

and by P1, we get that $0 = a \cdot 0$.

3. What this statement says is that the additive inverse of a when multiplied by b is the inverse of ab. So we check the condition of "being additive inverse":

$$ab + (-a)b =^{FI9} [(a + (-a)]b =^{FD4} 0 \cdot b = 0 =^{FD4} ab + (-ab)$$

where we used our assumption. So we got:

$$ab + (-a)b = ab + (-ab)$$

And we use P1 to conclude.

4. Left for HW.

5. Left For HW.

6. Assume that $a \cdot b = 0$, and assume that $b \neq 0$ (otherwise we are done), then we can write:

$$0 =^{P2} 0 \cdot b^{-1} = (ab) \cdot b^{-1} =^{FE5} a(b \cdot b^{-1}) =^{FH8} a \cdot 1 =^{FG7} a$$

where we used our assumption.

□

9.2 Ordered fields

Another important property of the rational number is that they are *ordered*! That is we have a relation "$\leq$" satisfying the following properties, for any $a, b, c \in \mathbb{Q}$

- O1 For any $a, b \in \mathbb{Q}$, we have either $a \leq b$ or $b \leq a$!
- O2 If $a \leq b$ and $b \leq a$ then $a = b$.
- O3 If $a \leq b$ and $b \leq c$ then $a \leq c$ - The transitive law!
- O4 If $a \leq b$ then $a + c \leq b + c$ - it "plays nicely" with addition.
- O5 if $a \leq b$ and $0 \leq c$ then $ac \leq bc$- it "plays nicely" with multiplication, note the condition on c.

A field with these properties is called a *ordered field*.

Many of the properties we know from $\mathbb{Q}$ are transferred to any ordered field. Let us prove some of these properties. We will list all the properties- but some will be left as HW:

Proposition 9.6. *Let $\mathbb{F}$ be an ordered field then we have the following, for any $a, b, c \in \mathbb{F}$*

PO1 *If $a \leq b$ then $-b \leq -a$.*

PO2 *If $a \leq b$ and $c \leq 0$ then $bc \leq ac$.*

PO3 *If $0 \leq a$ and $0 \leq b$ then $0 \leq ab$.*

PO4 $0 \leq a^2 = a \cdot a$.

PO5 $0 \leq 1$.

PO6 *If $1 = 0$ then $\mathbb{F} = \{0\}$*

PO7 *If $0 \leq a$ and $a \neq 0$ (denoted $0 < a$) then $a^{-1} > 0$.*

PO8 *If $0 < a < b$ then $0 < b^{-1} < a^{-1}$.*

Proof. Next time! □

9.3 What did we see today?

- We defined fields, and proved some properties of them.
- We introduce the idea of ordered fields.

10 10th Class - Order, functions and triangle inequality

10.1 Review

Last time we discussed fields and introduced ordered fields. We proved some properties of fields. Today we will go order fields, and talk a bit about functions

10.2 Ordered fields

We recall the properties of the order relation $\leq$ for any $a, b, c \in \mathbb{Q}$

O1 For any $a, b \in \mathbb{Q}$, we have either $a \leq b$ or $b \leq a$!

O2 If $a \leq b$ and $b \leq a$ then $a = b$.

O3 If $a \leq b$ and $b \leq c$ then $a \leq c$ - The transitive law!

O4 If $a \leq b$ then $a + c \leq b + c$ - it "plays nicely" with addition.

O5 if $a \leq b$ and $0 \leq c$ then $ac \leq bc$- it "plays nicely" with multiplication, note the condition on c.

A field with these properties is called a *ordered field*.

Many of the properties we know from $\mathbb{Q}$ are transferred to any ordered field. Let us prove some of these properties. We will list all the properties- but some will be left as HW:

Proposition 10.1. *Let $\mathbb{F}$ be an ordered field then we have the following, for any $a, b, c \in \mathbb{F}$*

PO1 If $a \leq b$ then $-b \leq -a$.

PO2 If $a \leq b$ and $c \leq 0$ then $bc \leq ac$.

PO3 If $0 \leq a$ and $0 \leq b$ then $0 \leq ab$.

PO4 $0 \leq a^2 = a \cdot a$.

PO5 $0 \leq 1$.

PO6 If $1 = 0$ then $\mathbb{F} = \{0\}$

PO7 If $0 \leq a$ and $a \neq 0$ (denoted $0 < a$) then $a^{-1} > 0$.

PO8 If $0 < a < b$ then $0 < b^{-1} < a^{-1}$.

Proof. We let $a, b, c \in \mathbb{F}$ throughout this proof.

1. Assume that $a \leq b$, then we can use O4 of order, and add $(-a) + (-b)$ to both sides:

$$a + ((-a) + (-b)) \leq b + ((-a) + (-b))$$

Next, we note that

$$a + ((-a) + (-b)) =^{FA1} (a + (-a)) + (-b) =^{FD4} 0 + (-b) =^{FC3} -b$$

In the same way we have $b + ((-a) + (-b)) = -a$. So we got

$$-b \leq -a$$

2. Let $a \leq b$ and $c \leq 0$, then by PO1 we have that

$$0 = -0 \leq -c$$

Then we can apply O5 to get that

$$-ac =^{P3} a(-c) \leq b(-c) =^{P3} -bc$$

Then we can apply PO1 again to conclude that

$$bc \leq ac$$

3. We assume that $0 \leq a, 0 \leq b$,then we can write, by O4
$$0 \cdot b \leq a \cdot b$$
Then we use P2 to see that $0 = 0 \cdot b$, and we conclude
$$0 \leq ab$$
as needed.

4. TPS. For any a we have either that $0 \leq a$ or $a \leq 0$ (by O1). If $a \geq 0$ we can write, using P2, that
$$0 = 0 \cdot a \leq a \cdot a = a^2$$
If $a \leq 0$, then by PO1 we have that $0 = -0 \leq -a$, and by the previous part we get that $(-a)(-a) \geq 0$, then by P4, we have
$$a^2 = (-a)(-a) \geq 0$$

5. Left as HW.

6. Assume that $1 = 0$, and let $a \in \mathbb{F}$, then we can write
$$a =^{FG7} a \cdot 1 = a \cdot 0 =^{P2} 0$$
where we used the assumption. In fact, one can check that $\mathbb{F} = \{0\}$ with $0 + 0 = 0, 0 \cdot 0 = 0$ is an ordered field- though not a very interesting one.

7. Assume $0 < a$, and assume by way of contradiction that $a^{-1} \leq 0$, so we have that $0 \leq -a^{-1}$ by PO1. Then we can useO5 to get
$$0 = 0 \cdot a \leq -a^{-1} \cdot a = -1$$
But that implies $1 \leq 0$, by PO1, in contradiction to the previous property!

8. Left as HW.

$\square$

Our next topic is a bit of a side track- we will introduce functions, and talk about distance function.

10.3 Functions

So how do we even define function? Our definition comes from Dirichlet in the 1830's. The main advantage of this definition is that it does not depend on our ability to write formulas

Definition 10.2. Given two sets A, B a function $f : A \to B$ (read: f from A to B) is a rule, or mapping, that for each $x \in A$ assigns a *single* element in B. We denote that element $f(x)$. A is called the domain of f, and we also define the *range* of f by

$$\mathrm{Ran}(f) = \{y \in B \mid \exists x \in A, f(x) = y\}$$

We note that $\mathrm{Ran}(f) \subset B$ but not necessarily equal.

Give me a couple of examples.
Let's write formally a few important examples:

Example 10.3. We consider $f : \mathbb{Q} \to \mathbb{Q}$, given by $f(x) = x$. Here the range and then domain are equal to $\mathbb{Q}$.

Example 10.4. We consider $f : \mathbb{Q} \to \mathbb{Q}$, given by $f(x) = 1$. Here we have $\mathrm{Ran}(f) = \{1\}$ and domain is equal to $\mathbb{Q}$.

TPS: Consider the following and write their range and domain:

1. $f(x) = \sin(x)$, Domain $\mathbb{R}$, Range $[-1, 1]$.
2. $f(x) = e^x$, Domain $\mathbb{R}$, Range $(0, \infty)$.
3. $f(x) = \frac{1}{x}$, Domain $\mathbb{R} \setminus \{0\}$, Range $\mathbb{R}$.

Example 10.5. We consider $f : \mathbb{Q} \to \mathbb{Q}$, given by the rule

$$f(x) = \begin{cases} x, & x > 0 \\ -x, & x \leq 0 \end{cases}$$

This function is known as the absolute value function and is usually denoted by $|x| = f(x)$.
Here we have $\mathrm{Ran}(f) = \{x \in \mathbb{Q} \mid x \geq 0\}$ and domain is equal to $\mathbb{Q}$.

Example 10.6. (Dirichlet) We consider $f : \mathbb{R} \to \mathbb{R}$ (we haven't defined the reals yet- but since Dirichlet came up). Given by the rule:

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

We also call this function the indicator of $\mathbb{Q}$ (since it indicates where $\mathbb{Q}$ is).
Here we have $\mathrm{Ran}(f) = \{1, 0\}$ and domain is equal to $\mathbb{R}$.

10.4 What did we see today?

- We finished talking about ordered fields,
- We defined functions.

11 11th Class - Distance function

11.1 Review

Last time we finished with ordered fields, and introduced functions. Today we will focus on the distance function $|x|$.

11.2 The distance function

Let us go back to the second example of $|x|$. This is a very important function which is the model for "distance" between objects. But how so? What properties do we expect from some function that measures distance? TPS.

We will prove that $|x|$ has all of the following properties that show it is a metric- or a distance function.

Proposition 11.1. *For any $x, y \in \mathbb{Q}$ we have*

1. $|x| = 0 \iff x = 0$.
2. $|x| \geq 0$.
3. $|x + y| \leq |x| + |y|$

Wait- where did we get that last demand? This is called the "triangle inequality"-this represents the fact that the distance function "cares" about direction (think about the case $x = 3, y = -2$!). We will see why in a second. So now we need to prove this proposition:

Proof. 1. Do it as TPS! $\underline{\Longleftarrow:}$ If $x = 0$ by definition we have $|x| = 0$, as needed.

$\underline{\Longrightarrow:}$ If $|x| = 0$, then either $x = 0$ or $-x = 0$, so we get $x = 0 = -0$- as needed.

2. We check case by case: If $x > 0$ then $|x| = x > 0$. If $x \leq 0$ then $|x| = -x \geq 0$, by property $PO2$ that we proved last time.

3. First we note that $x \leq |x|$, since if $x \geq 0$ we have $|x| = x \geq x$, and if $x \leq 0$ then $x \leq 0 \leq -x = |x|$.

   Similarly, we can write $x \geq -|x|$, by checking all the cases.

   So we can write:

$$
\begin{aligned}
-|x| &\leq x \\
-|x| + y &\leq x + y
\end{aligned}
$$

   by $O4$, then and similarly we have

$$-|x| - |y| \leq -|x| + y \leq x + y$$

   On the other side, we get that, by a similar argument that

$$x + y \leq |x| + |y|$$

So we get that

$$-(|x| + |y|) \leq x + y \leq |x| + |y|$$

In the HW you will show that if $-b \leq a \leq b$ for $b \geq 0$ then $|a| \leq b$. With this in hand, we can conclude that $|x + y| \leq |x| + |y|$, as needed.

□

Sometimes we will write $d(x, y) = |x - y|$, to emphasize that we are talking about a distance function. Note that we have the following:

Corollary 11.2. *Let $x, y, z \in \mathbb{Q}$ then*

$$d(x, z) \leq d(x, y) + d(y, z)$$

Proof. We write:

$$\begin{aligned} d(x, z) &= |x - z| = |x + 0 - z| = |x + ((-y) + y) - z| \\ &= |(x - y) + (y - z)| \leq |x - y| + |y - z| = d(x, y) + d(y, z) \end{aligned}$$

as needed. □

Now we can see why it is called a triangle inequality! This tells us (if we were working on points on the plane) that we have the following picture: Where we can see that $|B - C| \leq$

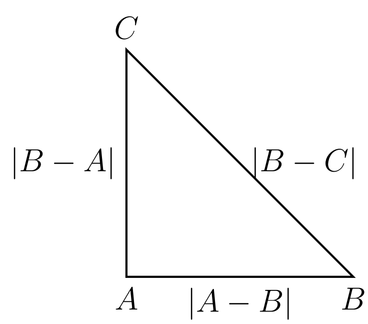


Figure 1: The triangle inequality

$|B - A| + |A - C|$- according to the "triangle inequality" we saw.

11.3 More properties of the distance function

Let's do a TPS: Try to prove the following claim

Claim 11.3. For any $a, b \in \mathbb{Q}$ we have $|ab| = |a||b|$.

Proof. We start by cases: if $a, b \geq 0$ then $ab \geq 0$ by $PO3$. We have:

$$|ab| = ab = |a||b|$$

as needed.

If $a, b \leq 0$ then $-a, -b \geq 0$ then $ab = (-a)(-b) \geq 0$. So we get that

$$|ab| = ab = (-a)(-b) = |a||b|$$

as needed.

If $a \leq 0 \leq b$ then we have that $-a \geq 0$, so we get the $-ab \geq 0$, and in particular $ab \leq 0$. So we have

$$|ab| = -(ab) = (-a)b = |a||b|$$

as needed.

The last case is identical to the previous one. $\square$

Finally, let's recall that we proved many properties of $\mathbb{Q}$, and now we even saw that there is a notion of distance, so this looks like a very good algebraic system! What else could we want? Well, then we can start asking: $\exists x \in \mathbb{Q}, x \cdot x = x^2 = 2$? We all know that the answer is no, but how do you prove it?

Claim 11.4. There is no $x \in \mathbb{Q}$ such that $q^2 = 2$.

Note how can we prove that something is $\mathbb{Q}$? Well, we need to show that it can be written as $\frac{m}{n}$, another way to write this is $x \in \mathbb{Q} \iff \exists n \in \mathbb{Z}, nx \in \mathbb{Z}$! You will need this definition in the HW.

Proof. Assume, by way of contradiction, that such a number exists. Then there are two numbers $n, m \in \mathbb{Z}$ such that $\frac{n^2}{m^2} = 2$, and n, m are relatively prime (they have no common factor). So we write:

$$\begin{aligned}\frac{n^2}{m^2} &= 2 \\ n^2 &= 2m^2\end{aligned}$$

So 2 is a divisor of n^2, and so we get that 2 is a divisor of n (as the square of an odd number is an odd number), so we can write $n = 2k$, for $k \in \mathbb{Z}$. Inserting it we will get

$$\begin{aligned}n^2 &= 2m^2 \\ 4k^2 &= 2m^2 \\ 2k^2 &= m^2\end{aligned}$$

By the same logic, we get that 2 is a divisor of m. But n, m were chosen to have no common factor! Contradiction!

So we conclude there is no $x \in \mathbb{Q}$ such that $q^2 = 2$ $\square$

So not all numbers are rational! so we need to do something more! We have some gaps! The main property of the real numbers is that every bounded subset has a maximum! But how can we define it?

11.4 What did we see today?

- We talked about the distance function and defined some properties of it
- We saw that $\mathbb{Q}$ has some gaps!

12 12th Class - Supremum and infimum and the completeness axiom

12.1 Review

Last time we introduced the distance function and saw some properties of it. Then, at the very end, we saw that there are "gaps" in $\mathbb{Q}$- which we want to start filling up!

12.2 Maximum and minimum

We start with some basic notions of min and max:

Definition 12.1. Let A be an ordered set. Then

- We say that $x \in A$ is the *maximum* of A if
$$\forall s \in A, s \leq x$$
and then we write $x = \max A$.
- We say that $x \in A$ is the *minimum* of A if
$$\forall s \in A, x \leq s$$
and then we write $x = \min A$

Some examples

Example 12.2. If $A = \{1, 5, 200, 3000\}$, then $\min A = 1, \max A = 3000$.

In fact, every finite set has a maximum and a minimum!

Example 12.3. If $A = \{n \in \mathbb{Z} \mid -4 < n \leq 100\}$, then $\min A = -3, \max A = 100$.

Example 12.4. What happens if $A = \mathbb{N}$, or $A = \mathbb{Z}$? In the first case, there is a minimum but no maximum, and in the latter, there are no maximum or minimum.

Example 12.5. What happens if $A = \{x \in \mathbb{Q} \mid x^2 < 2\}$? There is no minimum obviously. Since we don't have $\sqrt{2} \in \mathbb{Q}$- we don't have a maximum! The maximum is somehow outside of this set!

Thinking about the last example, we still would like to say that this set is *bounded from above* somehow. From now on we will define properties of a subset of $\mathbb{R}$. Similarly to the other construction, we will give a list of properties. Later in the course we will circle back to this point and show how to formally define $\mathbb{R}$ from $\mathbb{Q}$.

Definition 12.6. A set $\emptyset \neq A \subset \mathbb{R}$,

- If there is $M \in \mathbb{R}$ such that $\forall x \in A, x \leq M$, then we say that A is *bounded from above*, and we call M an *upper bound*.
- If there is $m \in \mathbb{R}$ such that $\forall x \in A, x \geq m$, then we say that A is *bounded from below*, and we call M an *lower bound*.
- We call A a *bounded* set if it is bounded from above and below.

So now we need examples. TPS: For each of these sets, determine if it is bounded from above, below, or both.

1. For $a, b \in \mathbb{R}$ we define $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$.
2. For $a, b \in \mathbb{R}$ we define $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$.
3. For $a, b \in \mathbb{R}$ we define $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$.
4. For $a, b \in \mathbb{R}$ we define $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$.
5. $A = \{x \mid x \in \mathbb{Q} \wedge x^2 < 4\}$
6. $\mathbb{N}$
7. $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$.

So of course we have that all first sets are bounded. Note that there are many upper bounds, for example in the first set b is an upper bound, but so is $b + 1000$ or $50b$ (if $b > 0$). $x^2 < 4$ gives us an interval (given by $|x| < 2$). The natural numbers are bounded from below, and the last example is bounded from both directions.

Finally we can define sup and inf

Definition 12.7. Let $A \subset \mathbb{R}$ non-empty, then we define

- If A is bounded from above, we call $S \in \mathbb{R}$ the supremum of A, or the least upper bound of A, if S is an upper bound with the property that if S' is another upper bound then $S \leq S'$ (it is the smallest upper bound). We denote then $\sup A = S$.
- If A is bounded from below, we call $s \in \mathbb{R}$ the infimum of A, or the greatest lower bound of A, if s is a lower bound with the property that if s' is another upper bound then $s' \leq s$ (it is the biggest upper bound). We denote then $\inf A = s$.

Remark 12.8. Some people use *glb* A instead of $\inf A$ (for greater lower bound), or *lup* A instead of $\sup A$ (least upper bound). We will not use this notation as it is very confusing.

Question: What is the difference between maximum and supremum, and infimum and minimum? Let's go back to the example above, and let's discuss if they have max/min/sup/inf.

1. In this case, there is no maximum nor minimum but a is the infimum and b is the supremum.

2. In this case, there is no maximum but a is the infimum and the minimum and b is the supremum.

3. $a = \inf(a, b], b = \sup(a, b] = \max(a, b]$, and no minimum.

4. $a = \min[a, b] = \inf[a, b], b = \sup[a, b] = \max[a, b]$.

5. $a = \min A = \inf A, b = \sup A = \max(a, b]$.

6. $1 = \inf \mathbb{N} = \min \mathbb{N}$, and no max or sup.

7. $\max A = \sup A = 1, \inf A = 0$ no minimum.

So we see that, unlike min/max, sup/inf does not have to be part of the set. They live outside of it.

12.3 The completeness axiom

Now we can give some sort of defining property for the real numbers! So now we define a complete field:

Definition 12.9. We say that $\mathbb{F}$, an ordered field, is complete if, for every $A \subset \mathbb{F}$ nonempty and bounded from above, there is $s \in \mathbb{F}$ such that $s = \sup A$.

So now we can give the defining property of $\mathbb{R}$(we will actually build $\mathbb{R}$ later in the course), also known as the completeness axiom:

Completeness Axiom. $\mathbb{R}$ *is a complete ordered field.*

For example, we can consider the upper bounds of the $A = \{r \in \mathbb{Q} \mid r^2 < 2\}$, and we can try and find the least upper bound- but it will not be in $\mathbb{Q}$. So if our world is just $\mathbb{Q}$ -then we have no $\sup A$.This number will actually be irrational, and we call $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ the irrational numbers.

Naturally, one may ask, what about bounded from below? Well for that we have the following corollary

Corollary 12.10. *Every $A \subset \mathbb{R}$ nonempty, that is bounded from below has an infimum $s \in \mathbb{R}$.*

Proof. How do you think we can do it? Well, we need to invert the set somehow! So we will look at the following set:

$$-A = \{x \in \mathbb{R} \mid -s \in A\}$$

Now the set A has a lower bound, as it is bounded from below. Let $n \in \mathbb{R}$ be such a lower bound. Then we have

$$\forall a \in A, n \leq a$$

So we can conclude

$$\forall a \in A - n \geq -a$$

or in other words, $-n$ is an upper bound for $-A$! So $-A$ is a subset of $\mathbb{R}$ that is bounded from above. By the completeness axiom, $-A$ has a supremum, so we write:

$$\sup -A = M$$

So now we want to prove $\inf A = -\sup -A$. Which I will leave as to the PSet. $\square$

Let us start with a useful face about suprema:

Claim 12.11. Let $A \subset \mathbb{R}$ a bounded subset, then

$$M = \sup A \iff \forall \varepsilon > 0, \exists a \in A, M - \varepsilon < a \wedge M \text{ is an upper bound.}$$

This is a useful property- any number smaller than $\sup A$ can't be an upper bound!

Proof. $\underline{\implies:}$ Let $M = \sup A$, and let $\varepsilon > 0$ be arbitrary. Since $M - \varepsilon < M$, it can't be an upper bound, let's prove that (otherwise we had an upper bound smaller than the sup). In particular, there is some $a \in A$ such that $a > M - \varepsilon$, as needed.

$\underline{\impliedby:}$ Assume that M is an upper bound with this property. Let $b \in \mathbb{R}$ be an upper bound of A. Assume by way of contradiction that $b < M$, then we can choose $\varepsilon = M - b$, and we get that there is some $a \in A$ such that $a > M - (M - b) = b$, in contradiction to b being an upper bound. So we conclude that $M \leq b$, and thus $M = \sup A$, as needed. $\square$

Next time we will see some consequences of completeness.

12.4 What did we see today?

- We defined a lot of things: max/min/sup/inf
- We gave the defining property of the real numbers!
- we started seeing what does this property mean!

13 13th Class -Consequences of the completeness axiom

13.1 Review

Last time we defined the sup and inf of sets and formulated the defining property of $\mathbb{R}$- that it is complete. Today we will dive deep into the consequences of this property.

Now we are going to prove some consequences of completeness:

13.2 Nested interval property

We are going to prove the following:

Theorem 13.1. *For each $n \in \mathbb{Z}$, we define*

$$I_n = [a_n, b_n] = \{x \in \mathbb{R}, a_n \leq x \leq b_n\}$$

Assume that $I_n \subset I_{n+1}$. Then, the resulting nested sequence of closed intervals:

$$I_1 \supset I_2 \supset \dots$$

has a nonempty intersection, that is $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$

How can even prove that? So we want to find $x \in \mathbb{R}$, such that $x \in I_n$ for all n. We want to use the axiom of completeness, so we would like to take some sup or inf of a set. Which sets make sense? We can look at $\{a_n\}$ and $\{b_n\}$, and show there is something between them! So let's formulate it!

Proof. We denote $A = \{a_n\}_{n=1}^{\infty}$. Then we want to show that A is bounded from above. We claim that b_1 is an upper bound. Let $a_n \in A$, then we have that $a_n \leq b_n$, and since the intervals are nested (this is a corollary of something you will see in the Pset) we have $b_n \leq b_1$. So we have by transitivity $a_n \leq b_1$. So we write

$$x = \sup A$$

Now we claim that $x \in I_n$ for any n. Since $x = \sup A$ we have that $a_n \leq x$, so we just need to show that $x \leq b_n$. For this, we claim that b_n is also an upper bound for A (if b_n is an upper bound that $x = \sup A \leq b_n$ and we conclude). Let $a_m \in A$. Then if $m \geq n$ we have that

$$a_m \leq b_m \leq b_n$$

since the intervals are nested. If $m \leq n$ then we have

$$a_m \leq a_n \leq b_b$$

From being nested. So we conclude that b_n is an upper bound, and $a_n \leq x \leq b_n$, and so $x \in I_n$. ☐

13.3 Archimedean Property

We would like to be able to claim the following properties:

$$\forall a > 0, \exists n \in \mathbb{Z}, a > \frac{1}{n} \tag{1}$$

$$\forall b > 0 \exists n \in \mathbb{Z}, b < n \tag{2}$$

This may seem trivial- but actually, it is not that obvious. In fact, this does not come from just the axioms of an ordered field! There are counter-examples! This is actually the Archimedean property of the real numbers

Theorem 13.2. *Let $a, b > 0$, there for some $n \in \mathbb{N}$ we have $na > b$.*

In other words (From Beals book), given enough time, one can empty a tub with a spoon! The above claims are just a special case of this. So now we get to the proof.

Proof. How do we approach it? It isn't obvious, but since we want to use completeness, maybe it will make sense to start by contradiction. Assume that the Archimedean property fails. What is the negation of it?

$$P : \forall a, b, \exists n \in \mathbb{N}, an > b$$
$$-P : \exists a, b, \forall n \in \mathbb{N}, an \leq b$$

So we assume $-P$ then can define $A = \{an \mid n \in \mathbb{N}\}$, and by $-P$ we have b is an upper bound of A. So we can take $M = \sup A$. Since $a > 0$ we have that $M - a \leq M$, and by the property we started the day with there is some $n_0 \in \mathbb{N}$ so that $M - a \leq an_0$ (as these are the elements of A), so we get that

$$M \leq a(n_0 + 1)$$

But $a(n_0 + 1) \in A$, as $n_0 + 1 \in \mathbb{N}$, and $M = \sup A$, so we got a contradiction. □

13.4 Density

Finally, we will prove a very interesting property: the rational numbers are "dense" in some sense in $\mathbb{R}$. More precisely we claim

Theorem 13.3. *Let $a, b \in \mathbb{R}$ and $a < b$ then there is a rational $r \in \mathbb{Q}$ such that $a < r < b$*

We want to find some $a < \frac{m}{n} < b$ or in other words

$$an < m < bn$$

So we have some Archimedean property hiding! So this is how we do it

Proof. Since $b - a > 0$, by the Archimedean property we can find some $n \in \mathbb{N}$, such that

$$(b - a)n > 1 \implies bn > an + 1$$

Now that seems to prove that there is some natural number between an and bn, but how do we prove it? we will want to capture all the relevant integers in the game.

We look at $|an| + |bn|$, by the Archimedean there is some integer $k > |an| + |bn|$. So we consider $K = \{j \in \mathbb{Z} \mid -k \leq j \leq k\}$, and $L = \{j \in K \mid an < j\}$. Both at not empty since $k \in K$ and $k \in L$. Since L is finite, we have that it has a minimum $m = \min L$.

Then we have that $m > an$, and since $an > -k$, we have that $m - 1 \in K$ also. Since m is minimal, $m - 1 \notin L$, and in particular:

$$m - 1 \leq an \implies m \leq an + 1$$

And we say that $an + 1 < bn$, so we got $m \leq an$, so we have:

$$an < m < bn$$

which implies what we wanted. □

Finally, we want to show that there is $\sqrt{2} \in \mathbb{R}$:

Theorem 13.4. *There is $x \in \mathbb{R}$ such that $x^2 = 2$*

Proof. We consider the set

$$A = \{y \in \mathbb{R} \mid y^2 < 2\}$$

It is bounded from above, (by, say, 2). Then we write $x = \sup A$. We will eliminate the possibilities of $x^2 > 2$ and $x^2 < 2$.

Assume that $x^2 < 2$, and we consider the following

$$(x + \frac{1}{n})^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2} \leq x^2 + \frac{2x}{n} + \frac{1}{n} = x^2 + \frac{2x+1}{n}$$

Now by the Archimedean property, we have some N such that

$$\frac{1}{N} < \frac{2 - x^2}{2x+1}$$

note that $2 - x^2 > 0$, so we get that

$$(x + \frac{1}{N})^2 \leq x^2 + \frac{2x+1}{N} < x^2 + 2 - x^2 = 2$$

but we have that

$$x < x + \frac{1}{N} \in A$$

In contradiction to x being an upper bound.

Similarly, we have that

$$(x - \frac{1}{n})^2 = x^2 - \frac{2x}{n} + \frac{1}{n^2} \geq x^2 - \frac{2x}{n}$$

Since $x^2 - 2$ is positive there is some number N such that

$$\frac{1}{N} < \frac{x^2 - 2}{2x}$$
$$-\frac{1}{N} > \frac{2 - x^2}{2x}$$

So we get that

$$(x - \frac{1}{N})^2 \geq x^2 - \frac{2x}{N} > x^2 + 2 - x^2 = 2$$

Now we claim that $x - \frac{1}{N}$ is also an upper bound on A. So let $y \in A$, and a assume that $x - \frac{1}{N} < y$. Then we have that

$$(x - \frac{1}{N})^2 < (x - \frac{1}{N})y < y^2 < 2$$

In contradiction to the way that N was chosen. So we get that $x - \frac{1}{N}$ is an upper bound, but x is the least upper bound- contradiction. So we conclude that $x^2 = 2$.

$\square$

13.5 What did we see today?

- We saw the nested interval property.
- We saw the Archimedean property.
- We proved the density of the rationals in the real numbers!
- We saw that we have $\sqrt{2} \in \mathbb{R}$.

14 14th Class -Cardinality

14.1 Review

Last time we saw some consequences of completeness, and we finished by showing that we had filled the gaps in the rational numbers, we also proved that the rational numbers are dense in $\mathbb{R}$.

In the HW (Question 5) you will also prove that the irrational $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ are also dense in $\mathbb{R}$. Then it is tempting to think about them in the same way- they are sort of two sets that are on equal footing, and a homogenous mixture of them creates $\mathbb{R}$.

This is actually not true! This astonishing result is due to Georg Cantor who developed, in the late 19th century, the theory of cardinality, or sizes, for infinite sets. The techniques used here, and specifically Cantor's diagonal method are a new type of proof that is very useful in many cases. I will also remark that this work is on the basis of the incompleteness theorems we mentioned in the first class.

14.2 Cardinality

We want, somehow, to compare the sizes of sets. For a finite set, this makes sense and is quite easy you can just attach some natural number to it.

For example, the set of seasons of "Stranger Things" (4 seasons) is smaller "Twilight" movies { Twilight, New Moon, Ecplise, Breaking Dawn – Part 1, Breaking Dawn – Part 2}. Since 5 movies are more than 4 seasons. But how do we make it so it can work for infinite sets?

Cantor's idea is to say that two sets have the same size if there is a $1-1$ correspondence between the elements of the set. If we put all the Twilight movies in a row (say on DVD- or USB drives with one movie) and put in a similar way the Stranger Things seasons, we will have a longer row of movies! Or we can say that the cardinality of people that the number of stars in The Last of Us is the same as the number of movies in Barbenheimer phenomenon. So how do we define it precisely? We will now use some notations that you have seen in the last Pset.

Definition 14.1. • A function $f : A \to B$ is called *injective* (or 1-1) if

$$\forall a, b \in A, f(a) = f(b) \implies a = b$$

- A function $f : A \to B$ is called *surjective* (or onto) if

$$\forall b \in B, \exists a \in A f(a) = b$$

- A function that is both is called a bijection, or bijective function.

Why is this what we want? Think about what each of these means:

- An injective function tells us that it doesn't "mix" elements. Or you can think about it as though it doesn't lose information moving from A.

- A surjective function tells us that it doesn't miss information in B. We can always trace it back to A.

- A function that is both tells us that we didn't lose information, and we have all the information from A and we covered all B- so they are the same in some sense.

So now we can define

Definition 14.2. We say two sets A, B have the same cardinality if there exists $f : A \to B$ such that it is bijective. Then we write $A \sim B$. The cardinality of a set A is denoted by $|A|$.

We say that $|A| \leq |B|$ there is $f : A \to B$ such that f is injective.

Though we will not prove it, this relation between sets is an equivalence relation (which I am not going to define). But it means that we can treat it as a sort of equality- two sets that have the same cardinality, can't be distinguished in terms of "size".

Let's give some examples:

Examples 14.3.

- Let $A = \{1, 2, \ldots 10\}$ and $B = \{20, 30, 40, \ldots, 100, 110\}$, do they have the same cardinality?

  Of course, we can define a function that is $1 - 1$ and onto. Choose $f(x) = 10x + 10$ and we get what we wanted. It is easy to check that if $f(x) = f(y)$ we have the $x = y$, and that we can get any $y \in B$ from any $x \in A$.

- Let $E = \{2, 4, 6, \ldots\}$ the set of even numbers, we claim $\mathbb{N} \sim E$. That is weird! how can a subset of a set have the same cardinality? Well, this is a bias from finite sets. Infinite sets can be much more interesting. The function is of course $f(n) = 2n$. These properties are proven easily.

- Let's make it even more interesting- we will show that $\mathbb{Z} \sim \mathbb{N}$! We define, $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = \begin{cases} \frac{n-1}{2}, & n \text{ is odd} \\ -\frac{n}{2}, & n \text{ is even} \end{cases}$$

  Now we will show that this function is injective and surjective. Let $m, n \in \mathbb{N}$ and assume that $f(m) = f(n)$. If m is odd, then $f(m) \geq 0$, and so is $f(n) > 0$ so we get that we have

$$\frac{n-1}{2} = \frac{m-1}{2}$$

Which implies $m = n$.

If m is even then $f(m) < 0$ and so is $f(n) < 0$ so we get that we have

$$-\frac{n}{2} = -\frac{m}{2}$$

which again implies $m = n$.

Let $x \in \mathbb{Z}$. If $x > 0$ then $2x+1 \in \mathbb{N}$ is an odd number, and then we have $f(2x+1) = x$. If $x = 0$ then $1 \in \mathbb{N}$ and we have $f(1) = 0$. If $x < 0$ then $-2x \in \mathbb{N}$ and even, so $f(-2x) = x$.

So it is a bijection!

- Another way to phrase the last example is Hilbert's Hotel paradox: Consider a hotel with an infinite number of rooms, numbered by the natural numbers. Say the hotel is currently full, that is every room has a guest. Then an infinite bus arrives with all the guests of the nearby hotel (also infinite, in a similar way). Even though your hotel is full- you can host all of them (send them to the odd rooms and tell the guests in n to move to $2n$).

So now we can further define:

Definition 14.4. A set A is called countable if $A \sim \mathbb{N}$. An infinite set that is not countable is uncountable.

So a natural question is whether all infinite sets are countable or not. So now we get this shocking result:

Theorem 14.5. $\mathbb{Q}$ *is countable.*

There are uncountable sets. Moreover, an easy exercise (in the HW) will show that $A \subset B$ then the cardinality of B is bigger than A! so we may conclude $\mathbb{R}$ is much bigger than $\mathbb{Q}$!

Proof. We will show that $\mathbb{Q} \sim \mathbb{Z}$ which will be enough, as we already know that $\mathbb{N} \sim \mathbb{Z}$. We will use the fact that any number $x \in \mathbb{N}$ can be written a unique multiplication of prime factors. So any rational number can be written as:

$$0 < x \in \mathbb{Q} \implies x = p_1^{a_1} p_2^{a_2} \dots p_n^{a_n} q_1^{-b_1} \dots q_m^{-b_m}$$

where $p_1, \dots p_n, q_1, \dots q_n$ are prime and non repeating, i.e. $i \neq j \implies p_i \neq p_j, q_i \neq q_j$ and $\forall i, j, p_i \neq q_j$. So we define

$$f(x) = \begin{cases} p_1^{2a_1} p_2^{2a_2} \dots p_n^{2a_n} q_1^{2b_1+1} \dots q_m^{2b_m+1}, & x = p_1^{a_1} p_2^{a_2} \dots p_n^{a_n} q_1^{-b_1} \dots q_m^{-b_m} > 0 \\ 0 & x = 0 \\ 1 & x = 1 \\ -p_1^{2a_1} p_2^{2a_2} \dots p_n^{2a_n} q_1^{2b_1+1} \dots q_m^{2b_m+1}, & x = p_1^{a_1} p_2^{a_2} \dots p_n^{a_n} q_1^{-b_1} \dots q_m^{-b_m} < 0 \end{cases}$$

This is injective, as the composition of primes is unique, and it covers all $\mathbb{Z}$ since every number has prime decomposition, and you can always group them to even and odd powers. So we proved the first part.

□

14.3 What did we see today?

- We talked about cardinality.
- We saw that rationals are countable and even just $(0, 1)$ is uncountable.

15 15th Class- Sequences

15.1 Review

Last time we talked about cardinality and saw that there are different types of infinity. Today we will focus on the countable sets. We will introduce the idea of sequences and the limit of a sequence. But first, we need to prove Cantor's theorem:

Theorem 15.1. $(0, 1)$ *is an uncountable set*

Proof. The more interesting part is the second part. This is where Cantor's diagonal argument comes into play. Assume by way of contradiction that there is an injective and surjective function $f : \mathbb{N} \to [0, 1]$. So we can enumerate all the elements in $[0, 1]$ -the nth element is just the image of $f(n)$, so we write $x^i = f(i)$.

Moreover, we can write each x^i with an infinite sequence of decimal expansion! For example 0.500000, or 0.12345678912345.... and so on so we write:

x^i	Expansion
x^1	$0.x^1_1 x^1_2 x^1_3 \dots x^1_n \dots$
x^2	$0.x^2_1 x^2_2 x^2_3 \dots x^2_n \dots$
$\vdots$	$\vdots$
x^m	$0.x^m_1 x^m_2 x^m_3 \dots x^m_n \dots$
$\vdots$	$\vdots$

where $x^i_j \in \{0, 1, 2, 3, \dots, 9\}$. Now we define:

$$y^j_i = \begin{cases} x^j_j + 1, & x^j_j < 9 \\ 0 & x^j_i = 9 \end{cases}$$

And we look at

$$y = 0.y^1_1 y^2_2 \dots y^n_n$$

This is a real number $0 < y < 1$, and assume there is some m such that $x_m = y$ then we have that $x^m_m = y^m_m$, in contradiction to the construction!

We conclude that $(0, 1)$ is not countable! $\square$

15.2 Sequences

A sequence is a function from the natural numbers to some set A: $f : \mathbb{N} \to A$. We usually denote the elements $f_n = f(n)$. We sometimes call $f : \{n \in \mathbb{N} \mid n \geq m\} \to A$ a sequence as well. We also denote $(f_n)_{n\in\mathbb{N}}, (f_n)_{n=m}^{\infty}$ to denote sequences. We will mostly consider sequences of real numbers, that is $A = \mathbb{R}$. Some examples:

- $f_n = n^2$, this is the sequence $(1, 4, 9, \dots)$ the round parenthesis meant to distinguish this as a sequence. The set of values is of course $\{1, 4, 9, \dots\}$.
- Consider $((-1)^n)_{n=1}^{\infty}$ this is the sequence $(-1, 1, -1, 1, \dots)$. The set of values is $\{-1, 1\}$.
- (a_n) where $a_n = \frac{1}{n}$, for $n \in \mathbb{N}$.
- (x_n) where $x_1 = x_2 = 1$ and $x_{n+1} = x_n + x_{n-1}$.

Now we would like to define a limit of sequence. But how do we define it formally? We want to say that somehow the sequence is getting closer and closer to this number, as n increases.

For example, we would want to say the sequence a_n converges to 0 somehow. We note that it never quite equals 0, but just getting closer. The sequence $(-1)^n$ is weird since there are infinitely many n for which it actually equals 1, and infinitely many that it equals 1. So do we think it has a limit or not? It is unclear. We will say it does not have a limit. So we will define:

Definition 15.2. We say that a sequence of real numbers $(a_n)_{n=1}^{\infty}$ converges to L if the following holds:

$$\forall \epsilon > 0, \exists N > 0 \text{ s.t. } n > N \implies |a_n - L| < \epsilon$$

And then we can write $\lim\limits_{n\to} a_n = L, a_n \xrightarrow{n\to\infty} L, a_n \to a$. If there is no such L we say that (a_n) diverges.

Remark 15.3. We use ϵ - the Greek letter epsilon, as it is the usual notation. We usually think about ϵ as a positive small number.

This definition is a mouthful. So we need to slowly get used to it. One way to think about it is using the language of ϵ -neighborhood.

Definition 15.4. Let $a \in \mathbb{R}$ then we call $V_\epsilon(a) = \{x \in \mathbb{R} \mid |x - a| < \epsilon\}$ the ϵ -neighborhood of a.

This truly describes all the numbers that are ϵ close (recall the distance function) to a! Then we can redefine limits as:

Definition 15.5. We will say that $a_n \to a$, if for all $\epsilon > 0$ we have some $N > 0$ such that $a_n \in V_\epsilon(a)$ for all $n > N$. In other words $V_\epsilon(a)$ contains all but finitely many (a_n)

Let's do some examples

Example 15.6. We consider $a_n = \frac{1}{n^2}$, we would think it goes to 0. . If we chose $\epsilon = 1$ then all of a_n is in $V_1(0) = [-1, 1]$! If we chose $\epsilon = \frac{1}{100}$, then we have $\forall n > 10, a_n \in V_{\frac{1}{100}}(0) = [-\frac{1}{100}, \frac{1}{100}]$. So choosing $N = 11$ will make the limit statement hold! What happens if we chose $\epsilon = \frac{1}{400}$? Then we want some $\frac{1}{n^2} < \frac{1}{400}$, so we can find $n^2 > 400$, which tells us that we want $n > 20$

So now we can prove a claim

Claim 15.7. We have that

$$\lim_{n\to\infty} \frac{1}{n^2} = 0$$

Proof. How do we approach this? We let $\epsilon > 0$ arbitrary (since we want to show a statement with $\forall$!), then we have that $\sqrt{\epsilon} > 0$ is some positive real number. By the Archimedean property, there is some natural number N such that

$$N > \frac{1}{\sqrt{\epsilon}} > 0$$

.Then we have that for any $n > N$

$$n > \frac{1}{\sqrt{\epsilon}} > 0 \qquad\qquad 0 < \frac{1}{n} < \sqrt{\epsilon}$$

$$|a_n| = \frac{1}{n^2} < \epsilon$$

So for any $\epsilon > 0$ we found an $N > 0$ such that for all $n > N$ we have $|a_n - 0| < \epsilon$. So we conclude

$$a_n \to 0$$

$\square$

Let's do another example, this time for a non-zero limit

Claim 15.8. We have $\lim\limits_{n\to\infty} \frac{1+n}{n} = 1$

Now we can discuss how to approach these proofs in general. My way is to do "back and forth". You start "informally"- on some scrap paper, you write what you want to get:

$$|\frac{1+n}{n} - 1| < \epsilon$$

$$|\frac{1+n-n}{n}| < \epsilon$$

$$\frac{1}{n} < \epsilon$$

$$n > \frac{1}{\epsilon}$$

And now we do the actual proof (this is what is expected on the Psets)

Proof. Let $\epsilon > 0$, By the Archimedean property we have some $N > \frac{1}{\epsilon}$, then we have that if $n > N$ then

$$\frac{1}{n} < \frac{1}{N} < \epsilon$$

So we can write:

$$|\frac{1+n}{n} - 1| = |\frac{1}{n}| = \frac{1}{n} < \epsilon$$

as needed. □

Remark 15.9. Final remark, now we see why we have spent so much time on quantifiers and the order of quantifiers! Here it is crucial that we have $\forall\epsilon\exists N$ and not $\exists N\forall\epsilon$!

15.3 What did we see today?

- We saw Cantor's theorem about the cardinality of $(0, 1)$.
- We defined sequences.
- we gave a formal definition of a limit.

16 16th Class- Limit algebra

16.1 Review

Last time we introduced the notion of limits. Today we will start expanding our world, introducing the algebra of limits.

But first, we will be making sure that we can talk about "the" limit of a sequence we have to show that this is unique!

Proposition 16.1. *Let (a_n) such that $a_n \to L$, then L is unique.*

Remark 16.2. We will prove this using a very useful method called the 2 epsilon method.

Proof. Assume that $a_n \to L$ and $a_n \to S$. We will show that $S = L$. Let $\epsilon > 0$ then we have by the definition of a limit that there is some $N_1 > 0$ such that for all $n > N_1$

$$|a_n - L| < \frac{\epsilon}{2}$$

(why $\frac{\epsilon}{2}$? since we have two epsilons!)

And we have some $N_2 > 0$ such that for all $n > N_2$

$$|a_n - S| < \frac{\epsilon}{2}$$

Then, let $n > \max(N_1, N_2)$, we can write- using the triangle inequality:

$$|S - L| < |S - a_n| + |a_n - L| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

So we have shown that for any $\epsilon > 0$ we have $|S - L| < \epsilon$. But we still need to show that this implies $S = L$.

Assume by way of contradiction that $S \neq L$, then $S - L \neq 0$ and then $|S - L| > 0$, in particular, there is some rational number $r \in \mathbb{Q}$ such that $0 < r < |S - L|$. Pick $\epsilon = r$, we got

$$r < |S - L| < \epsilon = r$$

Contradiction!

So we got S=L. □

We will now show that the sequence $a_n = (-1)^n$ doesn't converge!

Claim 16.3. Define $a_n = (-1)^n$, then a_n diverges.

Proof. We can prove this by contradiction, but in this case, we will prove it directly from the negation! Recall we want for some L

$$-(\forall \epsilon > 0, \exists N > 0, \forall n > N, |a_n - L| < \epsilon)$$

So the negation will be:

$$\exists \epsilon > 0 \forall N > 0 \exists n > N |a_n - L| \geq \epsilon$$

The second part just means that for that epsilon we can find infinitely many n's such that $|a_n - L| \geq \epsilon$. In particular, we will choose $\epsilon = 1$. Assume that $|(-1) - L| \leq 1$, then we have that

$$|(-1)^{2n} - L| = |1 - L| \geq ||1 - (-1)| - |(-1) - L|| = |2 - |(-1) - L||$$

We note that $|(-1) - L| \leq 1$, so $|2 - |(-1) - L|| \geq 1$. so we got

$$|a_{2n} - L| > 1$$

And of course, if $|(-1) - L| > 1$ we have that that $|a_{2n+1} - 1| > 1$! So we found an infinite sequence that is "far away " (that is at a distance of at least 1) from any L- and so we don't have a limit. □

16.2 Limit algebra

We start with a definition:

Definition 16.4. We say that a sequence is bounded if it is bounded as a set.

Now we will show the following:

Theorem 16.5. *Every convergent (to a finite limit) sequence is bounded.*

Proof. Let (a_n) a convergent sequence. Denote $\lim_{n\to\infty} a_n = a$. Then we can take $\epsilon = 1$, and get that there is some N such that for all $n > N$ we have:

$$|a_n - a| < 1$$

Which implies that

$$|a_n| = |a_n - a + a| \leq |a_n - a| + |a| \leq 1 + |a|$$

Now we define $M = \max\{1 + |a|, a_1, \ldots, a_N\}$ (the maximum exists as it is a finite set).Now we note that if $n > N$ we have $|a_n| \leq 1 + |a| \leq M$. And if $0 \leq n \leq N$, then $|a_n| \leq M$. So in particular, we have

$$\forall n \in \mathbb{N}, |a_n| < M$$

And we saw that this means

$$\forall n \in \mathbb{N}, -M < a_n < M$$

In particular, the sequence is bounded from above and below. □

Next, we want to show that multiplying by a constant commutes with limits:

Theorem 16.6. *Let (a_n) be a sequence that convergence $\lim a_n = a$. let $k \in \mathbb{R}$, denote $b_n = ka_n$, then we have $\lim b_n = \lim ka_n = ka$*

Proof. We start with the easy case: if $k = 0$, then $b_n = ka_n = 0$ and $\lim 0 = 0$.

Let $k \neq 0$, and $\epsilon > 0$. We will want $|ka_n - ka| < \epsilon$, so we will look at $\frac{\epsilon}{|k|}$ (we now that $k \neq 0$!). So by the limit definition, we have some $N > 0$ so for any $n > N$ we have

$$|a_n - a| < \frac{\epsilon}{|k|}$$

Which implies

$$|ka_n - ka| = |k||a_n - a| \leq \epsilon$$

as needed. □

Next, we will treat the addition of limits:

Theorem 16.7. *Let $\lim a_n = a, \lim b_n = b$ then $\lim(a_n + b_n) = a + b$. In other words*

$$\lim(a_n + b_n) = \lim a_n + \lim b_n$$

Proof. Let $\epsilon > 0$, we need to show

$$|a_n + b_n - a - b| < \epsilon$$

So, again we will use the 2ϵ method!

There is some $N_1 > 0$ such that

$$\forall n > N_1 |a_n - a| < \frac{\epsilon}{2}$$

and some $N_2 > 0$ such that

$$\forall n > N_2 |b_n - b| < \frac{\epsilon}{2}$$

So we have for $N_3 = \max\{N_1, N_2\}$ if $n > N_3$:

$$|a_n + b_n - a - b| \leq |a_n - a| + |b_n - b| < \epsilon$$

where we used that $n > \max\{N_1, N_2\}$. □

But first, we show multiplication:

Theorem 16.8. *Let* $\lim a_n = a, \lim b_n = b$ *then* $\lim(a_n b_n) = ab$. *In other words*

$$\lim(a_n b_n) = \lim a_n \lim b_n$$

Proof. Let $\epsilon > 0$, we need to show

$$|a_n b_n - ab| < \epsilon$$

how can we do it? We will insert now the term ab_n! So, again we will use the 2ϵ method!

Then we have some $N_2 > 0$ such that

$$\forall n > N_2 |b_n - b| < \frac{\epsilon}{2(|a| + 1)}$$

By the previous claim, we know that b_n is bounded, as it is convergent. Denote this bound by $M > 0$. Now we have some $N_1 > 0$ such that

$$\forall n > N_1 |a_n - a| < \frac{\epsilon}{2|M|}$$

So we have for $N_3 = \max\{N_1, N_2\}$ if $n > N_3$:

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \leq |a_n - a||b_n| + |a||b_n - b| \\ &< \frac{\epsilon}{2|M|}|M| + |a|\frac{\epsilon}{2(|a| + 1)} < \epsilon \end{aligned}$$

where we used that $n > \max\{N_1, N_2\}$. □

16.3 What did we see today?

- We saw that limits are unique.
- Showed that $(-1)^n$ has no limit.
- We've proved the rules of limit algebra!

17 17th Class- Convergence to infinity

Reminder: This is your responsibility to make sure that you have enough time to upload your Pset, and to check that everything works properly. From now on I will not accept any excuse for late submission.

17.1 Review

Last time we continued our discussion about limits. We proved some sequences don't have a limit, and show the algebra of limits. Today we will talk about sequences that converge, only $\pm\infty$.

But what about division? That will be covered by the HW.

Now let's give some examples:

1. $\lim a^n = 0$ for $|a| < 1$,
2. $\lim n^{\frac{1}{n}} = 1$

Proof.

1. If $a = 0$ it is obvious. Otherwise, since $|a| < 1$, we can write $|a| = \frac{1}{1+b}$. Then we note that by the binomial theorem (that you proved in the HW!)

$$(1+b)^n = 1 + nb + \cdots > nb$$

So we get

$$|a^n - 0| = |a|^n = \frac{1}{(1+b)^n} < \frac{1}{bn}$$

Let $\epsilon > 0$, then we can choose $N = \frac{1}{\epsilon b}$, and we have that for all $n > N$:

$$|a^n - 0| < \frac{1}{bn} < \epsilon$$

as needed.

2. We will look at $a_n = n^{\frac{1}{n}} - 1$, we note that $a_n > 0$. If we show $\lim a_n = 0$ we are done by limit arithmetic (the addition rule in this case). Now we use, again, the binomial theorem

$$n = (1 + a_n)^n = 1 + na_n + \frac{1}{2}n(n-1)a_n^2 + \cdots > \frac{1}{2}n(n-1)a_n^2$$

So we get, for $n \geq 2$

$$a_n^2 < \frac{2}{n-1} \implies |a_n| \leq \sqrt{\frac{2}{n-1}}$$

Then let $\epsilon > 0$, we can choose $N > \frac{2}{\epsilon^2} + 1$ which will imply $|a_n| < \epsilon$.

□

17.2 Convergence to infinity

We begin by defining convergence to ∞

Definition 17.1. We say that a sequence (a_n) converges to ∞, or write $\lim_{n\to\infty} a_n = \infty$, if we have:

$$\forall M > 0, \exists N > 0, \forall n > N, a_n > M$$

This means that if you give me a number, I can find a place in the sequence, such that the sequence is bigger than the given number after that place! the terminology is a bit problematic- as some people don't consider ∞ a limit, so they will say the sequence diverges to ∞.

A few examples:

Examples 17.2.

- We have that $\lim_{n\to\infty} n^2 = \infty$:

  Proof. Let $M > 0$, then choose $N = \sqrt{M}$, we have then if $n > N$ then

  $$a_n = n^2 > N^2 = M$$

  as needed. □

- the sequence $(-1)^n$ divergence but not to infinity! It does not have a limit- it is not that its limit is $\pm\infty$.

- Let $a_n = \frac{n^2+5}{n+1}$, then $a_n \to \infty$. We start with a bit of a discussion: Here we will want to compare it to n so we would want to have

  $$\frac{n^2+5}{n+1} > \frac{n}{2}$$

  say (we can choose any number other than 2, as long as it works). In this case, we would write

  $$\frac{n^2+5}{n+1} > \frac{n^2}{n+1}$$

  So we will need $n + 1 < 2n$, which of course will work for $n > 1$. So it will be enough to have $\frac{n}{2} > M$. Now we can start writing an actual proof:

  Proof. Let $M > 0$, choose $N > 2M$, then we have that for any $n > M$ we have $\frac{n}{2} > M$, so we get

  $$\frac{n^2+5}{n+1} > \frac{n^2}{n+1} > \frac{n^2}{2n} = \frac{n}{2} > M$$

  as needed. □

Now we will prove some algebra of limits:

Theorem 17.3. *Let a_n, b_n be sequences such that $\lim a_n = \infty, \lim b_n > 0$ (b_n can be infinity or not) then we have*

$$\lim a_n b_n = \infty$$

How do we prove it? We will need to make sure that b_n is somehow not too small, and then a_n will "win" and will take us to infinity.

Proof. Let $M > 0$, and let $m \in \mathbb{R}$ be such that $0 < m < \lim b_n$- if $\lim b_n < \infty$ then we have it from density, if $\lim b_n = \infty$, we chose m to be any number.

In your HW you will prove that this implies that there is some $N_1 > 0$ such that $\forall n > N_1, b_n > m$.

Since $a_n \to \infty$ there is some $N_2 > 0$ such that for all $n > N_2$ we have

$$a_n > \frac{M}{n}$$

So we get that for any $n > \max\{N_1, N_2\}$ we have

$$a_n b_n > \frac{M}{m} m = M.$$

as needed. ☐

Now we can use this to get a faster proof that $\frac{n^2+5}{n+1}$ converges to ∞.

Proof. We write $\frac{n^2+5}{n+1} = \frac{n+\frac{5}{n}}{1+\frac{1}{n}}$. Then we can write:

$$a_n = n + \frac{5}{n}, b_n = \frac{1}{1+\frac{1}{n}}$$

It is easy to see that $a_n \to \infty$ and $b_n \to 1$. We conclude by the theorem. ☐

17.3 Order of limits

Now we would want to show that limits "play nicely" with order

Theorem 17.4. *Let $a_n \to a, b_n \to b$ then we have*

- *If $a_n \geq 0$ for all $n \in \mathbb{N}$ then $a \geq 0$*
- *If $a_n \leq b_n$ for all $n \in \mathbb{N}$ then $a \leq b$.*
- *If there is $c \in \mathbb{R}$ such that $b_n > c$ for all $n \in \mathbb{N}$ then $b \geq c$, and similarly if $c < a_n$ then $c \leq a$.*

Proof. We begin with the first part: Assume by way of contradiction that $a < 0$, then we choose $\epsilon = -a > 0$. By the limit definition we have some $N > 0$ such that for all $n > N$ we have

$$|a_n - a| < |a|$$

In particular, it implies that $a_{N+1} < 0$. in contradiction with the assumption that $a_{N+1} \geq 0$. So we conclude that $a \geq 0$.

For the second part, we consider $c_n = b_n - a_n \geq 0$ and by limit algebra, it converges to $c_n \to b - a$. By the first part, we have that $b - a \geq 0$.

The last part is given by taking the constant sequence $a_n = c$ (or $b_n = c$) and applying the second part to it. □

Remark 17.5. We note that in the above we didn't really need to have these conditions on all $n \in \mathbb{N}$, we need them only to be true from someplace in the sequence. This leads us to the idea of a "tail " of sequence. the convergence is a property of the tail of the sequence- not of any finite number of elements of it.

Now we would like to put these things together and show that that is enough to bound a sequence from below for convergence to infinity:

Theorem 17.6. *Let $(a_n), (b_n)$ be sequences, such that $a_n \leq b_n$ from some $N > 0$. Assume further that $a_n \to \infty$ then $b_n \to \infty$.*

Proof. Let $M > 0$ then there is some $N_1 > 0$ such that for all $n > N_1$ we have that

$$a_n > M$$

Choose $N_2 = \max\{N, N_1\}$, then we have for all $n > N_2$

$$b_n \geq a_n > M$$

as needed. □

This gives us a way to show that $n + \frac{5}{n}$ diverges! Which we have used in the previous proof.

17.4 What did we see today?

- We finished the algebra of limits.
- We have talked about convergence to infinity.
- We started talking about the order of limits

18 18th Class- Squeeze and monotone sequences

Some announcements:

- The graders asked me to remind you: when you upload your question to Gradescope, make sure that you mark what question is on which page.
- The mid-semester assignment is coming up. I remind you: it will be published on Saturday the 7th, and you will have to submit it by Saturday the 14th! the deadline will be 8 pm as usual. There will be no Pset due this week.
- Please check the file you upload several times! and make sure you have enough time to upload it!
- This is an individual assignment- you can use the course notes, the Psets, and their solutions, but don't talk to your peers about it.
- There will be no time limit- though I recommend that you clear two and a half hours from your schedule (just to be sure), and treat it as an exam. Of course, if you have time extension, clear more accordingly.
- Next week we will only meet on Wednesday, for one class. There will be no OH - as you are not supposed to discuss with your peers. Questions should be emailed to me.
- There will also be a mid-semester feedback, that will be released tonight. It will be an assignment on Canvas- with a link to the survey. It shouldn't take long- 4 topics, with ranking questions and then room for comments. You don't have to fill it out, but I ask that you do. It will be very helpful to me.

18.1 Review

Last time we continued our discussion about limits. We finished the limit algebra discussion and discussed convergence to ∞. and start talking about order and limits. Today we will continue talking about the order of limits and monotone sequences.

18.2 Squeeze theorem

Finally, we will want to prove the squeeze theorem:

Theorem 18.1. *Let $(a_n)(b_n),(c_n)$ be sequences such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$, and we have :*

$$\lim a_n = \lim c_n$$

Then $\lim b_n$ exists and is equal to the limits above.

Proof. First, we note that we can use the order of limits directly as we don't know that b_n converges. First, we denote the limit $\lim a_n = L$. Next, we note that $a_n - c_n \to 0$, by the algebra of limits. Let $\epsilon > 0$, so there is some $N > 0$ such that for all $n > N$ we have that

$$|a_n - c_n| < \frac{\epsilon}{2}$$

We also have some $N_1 > 0$ such that for all $n > N_1$ we have

$$|a_n - L| < \frac{\epsilon}{2}$$

We note that $c_n - a_n, b_n - a_n > 0$ so we have that

$$|b_n - a_n| = b_n - a_n \leq c_n - a_n = |c_n - a_n|$$

Then we write :

$$|b_n - L| \leq |b_n - a_n| + |a_n - L| \leq |c_n - a_n| + \frac{\epsilon}{2} \leq \epsilon$$

as needed. □

18.3 Monotone Sequences

So far we have seen that how to generate more limits from existing limits. But we want to have a way to know that a sequence converges without knowing what its limit is! So we start with a definition

Definition 18.2. A sequence (a_n) or real numbers is called (weakly) increasing if

$$\forall n, a_n \leq a_{n+1}$$

and (weakly) decreasing if

$$\forall n, a_n \geq a_{n+1}$$

We say that a sequence is monotone if it is either (weakly) increasing or (weakly) decreasing.

Remark 18.3. Sometimes we use the terminology of non-decreasing, for weakly increasing. We will say that a sequence is strongly increasing if $a_n < a_{n+1}$ and similarly for decreasing.

Then we have the following strong theorem:

Theorem 18.4 (Monotone Convergence)**.** *If a_n is monotone then it converges.*

Proof. We will focus on the case of an increasing sequence, but the decreasing case is done similarly.

Let (a_n) be an increasing sequence. If (a_n) is bounded from above, then we can write

$$L = \sup\{a_n\}$$

We will show that $a_n \to L$. Let $\epsilon > 0$, by the properties of sup we saw that

$$\exists N > 0, L - \epsilon < a_N \leq L$$

(a_n) is increasing so we get that

$$\forall n > N, a_n \geq a_N > L - \epsilon$$

So we have

$$L - \epsilon < a_n \leq L < L - \epsilon$$

So we conclude, for all $n > N$

$$|a_n - L| < \epsilon$$

as needed.

If (a_n) is not bounded, then for every number $M > 0$ we have some $N > 0$ such that $a_N > M$. But (a_n) is increasing and so

$$\forall n > N, a_n \geq a_N > M$$

And in particular

$$\forall M > 0 \exists N > 0 \forall n > N, a_n > M$$

as needed. □

Let's give an example

Example 18.5. Consider the sequence, defined as follows:

$$a_1 = 1$$
$$a_n = a_{n-1} + \frac{1}{n^2}$$

Then a_n converges for a finite limit.

Proof. Well, a_n is obviously increasing, as $\frac{1}{n^2} > 0$ To show that it is bounded, we need to work a bit. we write

$$a_n = a_{n-1} + \frac{1}{n^2} < a_{n-1} + \frac{1}{n(n-1)} = a_{n-1} + \frac{1}{n-1} - \frac{1}{n}$$

So we can write it a telescopic sum! That means:

$$a_n < 1 + 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{n-1} - \frac{1}{n}$$

From the third term, every term cancels out, so we get

$$a_n < 1 + 1 - \frac{1}{n} < 2$$

So it is bounded, and therefore converges! □

This example is the first glance into the world of series! we can define $\sum_{n=1}^{\infty} \frac{1}{n^2}$. what we proved above is exactly that sequence of partial sums converges! So what about this sequence

Example 18.6. Consider $a_k = \sum_{n=1}^{k} \frac{1}{n}$, what does it converges to?

Proof. Well by the theorem we know it converges to something. The previous guess of 2 doesn't work since

$$a_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 2$$

we can actually extend this computation, to show that $a_{2^k} > 1 + \frac{k}{2}$.

How do we do it? By induction! Assume that it holds for k and then

$$\begin{aligned} a_{2^{k+1}} = a_{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}-1} + \frac{1}{2^{k+1}} &> 1 + \frac{k}{2} + \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} + \cdots + \frac{1}{2^{k+1}} \\ &= 1 + \frac{k}{2} + \frac{2^k}{2^{k+1}} = 1 + \frac{k+1}{2} \end{aligned}$$

as needed. So we have that

$$a_{2^k} > 1 + \frac{k}{2}$$

In particular, it means that it is an unbounded sequence. We note that this doesn't mean that $a_k > 1 + \frac{k}{2}$, so we can't use order. But if it is unbounded from above, it means it can't converge to a finite number, but it has to converge- so it converges to ∞! $\square$

18.4 What did we see today?

- We proved the squeeze theorem.
- We talked about monotone sequences.
- We saw that they always converge

19 19th Class- Limsup and Liminf and Cauchy sequence

19.1 Review

Last time we saw the squeeze theorem, and we also discussed monotone sequences and saw that all monotone sequences converge. Today we will give another criterion for convergence. But before that, we need to talk about two important concepts: $\limsup$ and $\liminf$. We will also return to them later.

19.2 Limsup and Liminf

One of the things that we can see from the definition of limit is that it somehow only cares about the tail of the sequence- not what happens at the beginning of it.

So Let (a_n) be a sequence. Then we build from it the following sequences

$$M_N = \sup\{a_n \mid n > N\}, m_n = \inf\{a_n \mid n > N\},$$

Where we allow the sup to infinity. So these sequences are defined (might by infinity - but we allow it for now). One can show that if $\lim a_n$ exists then it must be in the interval $[m_N, M_N]$. What can we say about these intervals?

They are nested! We can write:

$$A_N = \{a_n \mid n > N\}$$

Then it is clear that $A_N \supset A_{N+1}$, and by the HW you have shown that

$$\inf A_N \leq \inf A_{N+1} \leq \sup A_{N+1} \leq \sup A_N$$

So by the nested interval property, we know that there is something in all the intervals!

Moreover, we get that M_N, and m_N are both monotone! So we can talk about their limit! We define

$$\limsup a_n = \lim_{N\to\infty} \sup\{a_n \mid n > N\}, \liminf a_n = \lim_{N\to\infty} \inf\{a_n \mid n > N\}$$

We note that these are always defined (though not always finite), unlike the limit. But what happens if we have a limit?

Theorem 19.1. *Let a_n be a sequence or real numbers. Then we have*

$$\lim a_n = a \iff \limsup a_n = \liminf a_n = a$$

where $a \in \mathbb{R} \cup \{\pm\infty\}$

Remark 19.2. Note that the left-hand side also claims that the limit exists!

Proof. $\underline{\implies:}$ We assume that $\lim a_n = +\infty$, then for every $M > 0$ there is some $N > 0$ such that for all $n > N$ we have

$$a_n > M$$

in particular,

$$M \leq \inf\{a_n \mid n > N\} \leq \sup\{a_n \mid n > N\}$$

since m_n is monotone we have

$$\forall M > 0.\exists N > 0, \forall n > N, m_n > M$$

Or $\liminf a_n = \lim m_n = \infty$ And since $M_n \geq m_n$ by order of limits proposition, we have that $\limsup a_n = \infty$, as needed.

For $\lim a_n = -\infty$ this is very similar.

Next we consider $a_n \to a$ for $a \in \mathbb{R}$. Let $\epsilon > 0$, then we have some $N > 0$ such that for all $n > N$ we have

$$|a_n - a| < \epsilon$$
$$a - \epsilon < a_n < a + \epsilon$$

So we have that

$$M_N = \sup\{a_n \mid n > N\} \leq a + \epsilon$$
$$m_N = \inf\{a_n \mid n > N\} \geq a - \epsilon$$

And from monotonicity, we have that

$$\forall n > N, M_n \leq a + \epsilon, m_n \geq a - \epsilon$$

So we have that

$$\forall \epsilon, \limsup a_n = \lim M_n \leq a + \epsilon, \liminf a_n = \lim m_n \geq a + \epsilon,$$

Since this is true for all epsilon, we get that

$$a \leq \lim m_n = \liminf a_n \leq \limsup a_n = \lim M_n \leq a$$

So all numbers have to be equal and we get $\limsup a_n = \liminf a_n = a$, as needed.

$\underline{\Longleftarrow:}$ If $\limsup a_n = \liminf a_n = \pm\infty$, then it almost directly form definition. We will do $+\infty$- if $\liminf a_n = +\infty$, that means that for all $M > 0$ we have some $N \in \mathbb{N}$ such that

$$\inf\{a_\ell \mid \ell > n\} = m_n > M$$

In particular, for all $\ell > N$ we have

$$a_\ell > \inf\{a_\ell \mid \ell > N\} > M$$

And we conclude that $\lim a_n = \infty$.

If $\limsup a_n = \liminf a_n = a$ for $a < \infty$. Let $\epsilon > 0$, then we have some $N_1 \in \mathbb{N}$ such that for all $N > N_1$

$$|M_N - s| < \epsilon \implies \sup\{a_n \mid n > N\} < s + \epsilon$$

which implies, in turn, that for $n > N_1$

$$a_n \leq \sup\{a_n \mid n > N_1\} < s + \epsilon$$

Similarly there is some N_2 so that for all $N > N_2$ we have

$$|m_N - s| < \epsilon \implies \inf\{a_n \mid n > N\} > s - \epsilon$$

So for $n > N_2$ we have

$$a_n \geq \inf\{a_n \mid n > N_2\} < s - \epsilon$$

So we get that for $n > \max\{N_1, N_2\}$

$$s - \epsilon < a_n < s + \epsilon$$
$$|a_n - s| < \epsilon$$

as needed. □

So it tells us that a sequence converges if starting from somewhere all the numbers are huddled together. Maybe this means that we can turn that into a criterion.

19.3 Cauchy Sequence

We will now introduce Cauchy sequences. I will say that this concept, in many ways is the basis of my Thesis. This is a very useful way to show convergence. In many cases- this is the only way to get that some sequence converges.

Definition 19.3. A sequence (a_n) is called a Cauchy sequence if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall m, n > N, |a_n - a_m| < \epsilon$$

Let's try to think about it for a second: this is telling us that the values of the sequence are very (in fact ϵ) close together. In other words, they are all in an ϵ neighborhood of each other. We will want to show that a sequence is convergent iff it is Cauchy. We will do it next time. But for now, we start setting the stage.

Lemma 19.4. *Cauchy sequences are bounded.*

Proof. This proof is very similar to the proof that convergent sequences are bounded! Let (a_n) be a Cauchy sequence, and let $\epsilon = 1$, then we get that there is some $N \in \mathbb{N}$ such that

$$\forall n, m > N, |a_n - a_m| < 1$$

In particular, we have that

$$|a_n - a_{N+1}| < 1$$
$$|a_n| < |a_{N+1}| + 1$$

And so we can define $M = \max\{a_1, \ldots, a_N, |a_{N+1}| + 1\}$. That will be a bound for all $n \in \mathbb{N}$ □

19.4 What did we see today?

- We defined lim sup and lim inf
- We started a discussion about Cauchy sequence

20 20th Class- Cauchy sequence and sub-sequences

Regarding Pset 5 Question 7- I understand it was quite hard, so let's go over it: In this question we will prove that if $A \subset B$ and B is countable then A is either finite or countable

We have a function $f : \mathbb{N} \to B$ which is a bijection, let A be an infinite set. Let $n_1 = \min\{n \in \mathbb{N} \mid f(n) \in A\}$. We will define $g : \mathbb{N} \to A$. We start with $g(1) = f(n_1)$, proceed inductively, and show that g is a bijection. (Hint: to show that g is surjective use induction).

Proof. Start the construction as above, and assume we defined $g(1), \ldots, g(k)$. We define $n_{k+1} = \min\{n \in \mathbb{N} \mid n > n_k\}$, and we define $g(k+1) = f(n_{k+1})$.

Assume that we have $g(m) = g(\ell)$, then we have

$$f(n_m) = f(n_\ell)$$

Since f is injective we have that $n_m = n_\ell$. If $m > \ell$ we have $n_m > n_\ell$ - contradiction, similarly if $m < \ell$ we have $n_m < n_\ell$ - contradiction. So we conclude that $m = \ell$.

Let $x \in A$, then since f is a bijection we have some m such that $f(m) = x$. We claim that $x \in g(\{1, \ldots, m\})$. We prove it by induction.

For $m = 1$ we have $f(1) \in A$, so $n_1 = 1$ and $g(1) = f(1)$.

Assume that if $x \in A$ is such that $f(k) = x$ then $x \in g(\{1, \ldots, k\})$, for all $k \leq m$. And assume that $f(m+1) = x$. If $f(\{1, \ldots, m\}) \cap A = \emptyset$ then $n_1 = m+1$ and $g(1) = f(m+1) = x$, and in particular $x \in g(\{1, \ldots, k\})$. If $f(\{1, \ldots, m\}) \cap A \neq \emptyset$, then there is some maximal $1 \leq \ell < m$ such that $f(\ell) \in A$. By the induction assumption, $f(\ell) \in g(\{1, \ldots, \ell\})$. Then we have that $n_{\ell+1} = \min\{n \in \mathbb{N} \mid n > n_\ell\}$, and so $n_{\ell+1} = m$, so we get that $g(\ell + 1) = f(n_{\ell+1}) = f(m) = x$, as needed.

So we showed for all $x \in A$ that there is some $m \in \mathbb{N}$ such that $g(m) = x$, and we conclude that g is a bijection. □

20.1 Review

We are still expanding our "limit" universe and finding different ways to find sequences that converge.

Last time we started the discussion about Cauchy sequences. We saw that any Cauchy sequence is bounded. Today we will prove the juicy part! A sequence converges if and only if it is Cauchy! That will allow us to actually construct $\mathbb{R}$ from $\mathbb{Q}$- that we will do next time. Recall

Definition 20.1. A sequence (a_n) is called a Cauchy sequence if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall m, n > N, |a_n - a_m| < \epsilon$$

Then we have

Theorem 20.2. *Let (a_n) be a sequence then a_n converge to a finite number L if and only if it is Cauchy.*

Proof. $\underline{\Longrightarrow:}$ (TPS) Assume $a_n \to L$, then we can use the 2ϵ trick! Let $\epsilon > 0$ then there is some $N \in \mathbb{N}$ such that

$$\forall n > N, |a_n - a| < \frac{\epsilon}{2}$$

So for any $n, m > N$ we have that

$$|a_n - a_m| \leq |a_n - a| + |a - a_m| < \epsilon$$

And we got that (a_n) is Cauchy.

$\underline{\Longleftarrow:}$ Let (a_n) be Cauchy. We will show that $\limsup a_n = \liminf a_n$, which by proof in the previous class is enough.

Let $\epsilon > 0$, by definition we have some $N \in \mathbb{N}$ such that for all $n, m > N$ we have

$$|a_n - a_m| < \epsilon$$

In particular, we have $a_n < a_m + \epsilon$, for all $n, m > N$. And so we have that $a_m + \epsilon$ is an upper bound for the set $\{a_n \mid n > N\}$. So $\sup\{a_n \mid n > N\} = M_N < a_m + \epsilon$. Now we have that

$$\forall m > N, a_m > M_N - \epsilon$$

in particular, that $\inf\{a_n | n > N\} = m_N > M_N - \epsilon$. So we got that

$$\limsup a_n \leq M_N < m_N + \epsilon \leq \liminf a_n + \epsilon$$

This holds for any $\epsilon > 0$, so we get

$$\limsup a_n \leq \liminf a_n$$

and since we always have $\liminf a_n \leq \limsup a_n$, we conclude that two are equal, and the sequence converges. $\square$

This idea is very strong, it tells us that we don't need the limit to show that a sequence converges. Now we turn to another important concept: sub-sequences.

20.2 Sub-sequences

We start with the definition

Definition 20.3. Let (a_n) be a sequence. We call (b_k) a subsequence of (a_n) if there is some strictly increasing sequence (n_k), where $\forall k, n_k \in \mathbb{N}$

$$n_1 < n_2 < \cdots < n_k < n_{k+1} < \ldots$$

such that

$$b_k = a_{n_k}$$

So it just means that we just choose some indices as we move along the sequence. Or just choosing some infinite subset of $\mathbb{N}$, and treating this as our sequence.

Let's give some examples:

Examples 20.4. We consider $a_n = \frac{1}{n}$, give examples of subsequence!

- Then $(1, \frac{1}{4}, \frac{1}{9}, \dots)$ is a subsequence.
- Another subsequence $(\frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \dots)$.
- But $(1, 1, \frac{1}{3}, \frac{1}{3}, \dots)$ is not!
- Even $(\frac{1}{100}, 1, \frac{1}{1000}, \frac{1}{100}, \dots)$ is not a subsequence.

20.3 What did we see today?

- We continued our discussion on Cauchy sequences.
- We started talking about subsequences.

21 21st Class- Cauchy sequence and the construction of the real numbers

21.1 Review

Today we are going to go back a bit and use the language of Cauchy sequences to show how can one build the real line. Which we have really done. We will build $\mathbb{R}$, and show that our new construction, $\tilde{\mathbb{R}}$, has the following properties

- It is a field.
- Contains $\mathbb{Q}$, in the following sense: there is some $f : \mathbb{Q} \to \tilde{\mathbb{R}}$ such that f is injective, and preserves the actions.
- $\tilde{\mathbb{R}}$ is complete. i.e. every bounded set and a sup.

This will be enough to show this is a construction of $\mathbb{R}$. Next time, on Monday, we will continue in our regular program.

21.2 Construction of the real numbers

The idea is simple- we will consider the set of all Cauchy sequences made from $\mathbb{Q}$! The problem is we need to define what it means to be "equal", the action, and the order!

Definition 21.1. We define

$$\tilde{\mathbb{R}} = \{(a_n) \mid a_n \in \mathbb{Q}, a_n \text{ is Cauchy}\}$$

We that $a_n, b_n \in \tilde{\mathbb{R}}$ are equal, if the sequence $c_n = a_n - b_n$, converges to 0, as a sequence in $\mathbb{Q}$.

Recall that our definition of convergence makes sense even in $\mathbb{Q}$! So we can define it. The problem is that not every Cauchy sequence has a limit in $\mathbb{Q}$, so we will treat a number as sequence.

First, is this even a field? We need to define the actions!

Definition 21.2. Let $(a_n), (b_n) \in \tilde{\mathbb{R}}$, then we define

$$(a_n) + (b_n) = (a_n + b_n)$$
$$(a_n) \cdot (b_n) = (a_n \cdot b_n)$$

In the HW, you will show that since $(a_n), (b_n)$ is Cauchy we will have that $(a_n + b_n), (a_n b_n)$ are also Cauchy. The properties of the field will then follow from these properties about rational numbers. We will not prove it- but you can take this as an exercise. This Next, we want to embed $\mathbb{Q}$ in $\tilde{\mathbb{R}}$ this can be done by defining $f : \mathbb{Q} \to \tilde{\mathbb{R}}$

$$f(x) = (x, x, x, x, \dots)$$

that is mapping to the constant sequence. We note that the constant sequence is Cauchy (since $|a_n - a_m| = |x - x| = 0$), so it is well-defined, injective, and preserves the actions!

Next, in order to talk about completeness, we need to order elements, and so we define

Definition 21.3. Let $(a_n), (b_n) \in \tilde{\mathbb{R}}$, we will say that $(a_n) \leq (b_n)$ if

$$\exists N \in \mathbb{N}, \forall n > N, a_n \leq b_n$$

So, the only thing we need to verify then is that $\tilde{\mathbb{R}}$ is complete. For this, we first must show that the rational sequences are dense! We will use a slightly different definition of dense, which will be more convenient.

Theorem 21.4. *Let $x \in \tilde{\mathbb{R}}$ and let $0 < \epsilon \in \mathbb{Q}$ then we have some rational $r \in \mathbb{Q}$ such that*

$$|x - f(r)| < f(\epsilon)$$

In other words

$$\exists N \in \mathbb{N}, \forall n < N, |x_n - r| < \epsilon$$

Proof. Let $\epsilon \in \mathbb{Q}$, since $x \in \tilde{\mathbb{R}}$ we have some $N > 0$ such that for all $n, m > N$ we have

$$|x_m - x_n| < \epsilon$$

Then we choose $r = x_\ell$ for some $\ell > N$. So $r \in \mathbb{Q}$ and we have that

$$|x_m - r| < \epsilon$$

as needed. $\square$

Remark 21.5.

- if $x \in \tilde{\mathbb{R}}$, then can find $r \in \mathbb{Q}$ such that

$$|x + 1 - f(r)| < f(\frac{1}{4})$$

So in particular we found $r \in \mathbb{Q}$ such that $f(r) > x$, similarly we can find $p \in \mathbb{Q}$ such that $f(p) < x$.

- If $x > 0$. That means that starting at some $N \in \mathbb{N}$ we have $x_n > 0$, for all $n > N$ .Since $x \neq 0$, we have that there is some $\epsilon > 0$ and some subsequence such that $x_{n_k} > \epsilon$. Since it is Cauchy we have that from someplace $x_n > \frac{\epsilon}{2}$. And so we have some $\frac{\epsilon}{2} > r > 0$, where $r \in \mathbb{Q}$, and then we have $x > f(r) > 0$.

Now we show that this implies the regular density

Theorem 21.6. *Let $a < b$ both in $\tilde{\mathbb{R}}$, then we have some $r \in \mathbb{Q}$ such that*

$$a < f(r) < b$$

Proof. We note that $b - a > 0$ so there is some $0 < r \in \mathbb{Q}$ such that $b - a > f(r) > 0$, and so $b - a - f(r) > 0$. In particular, there is some N_1 such that for all $n > N_1$ we have

$$b_n > a_n + r$$

Since a is Cauchy we have some N_2 such that for all $n, m > N_2$ we have

$$|a_n - a_m| < \frac{r}{4}$$

So choose some $\ell > \max\{N_2, N_1\}$. We let $q = a_\ell + \frac{r}{2}$. Then we have that for $n > \max N_2, N_1$

$$q - a_n = \frac{r}{2} + a_\ell - a_n > \frac{r}{2} - |a_\ell - a_n| > \frac{r}{4}$$

And in particular $q > a$.

On the other hand, for $n > \max N_2, N_1$

$$q = a_\ell + \frac{r}{2} = a_\ell - a_n + a_n + \frac{r}{2} < |a_\ell - a_n| + a_n + \frac{r}{2} < a_n + \frac{3r}{4} < a_n + r < b_n$$

So in particular, we have

$$a < f(q) < b$$

as needed. □

So now we have our main theorem for today

Theorem 21.7. *$\tilde{\mathbb{R}}$ is complete. That is if $A \subset \tilde{\mathbb{R}}$ non empty, such that $U \in \tilde{\mathbb{R}}$ is an upper bound, then $\sup A \in \tilde{\mathbb{R}}$ exists.*

Proof. Let $A \subset \tilde{\mathbb{R}}$ be such that $U \in \tilde{\mathbb{R}}$ is an upper bound. By the above, we can find $u \in \mathbb{Q}$ such that $f(u)$ is an upper bound of A. Since A is not empty we have some $L \in A$, so we can find $\ell \in \mathbb{Q}$ so that $f(\ell) < L$. Now we define two sequences in the following inductive scheme

- $u_1 = u, \ell_1 = \ell$.
- Say that u_n, ℓ_n are defined. Then we denote $m_n = \frac{u_n + \ell_n}{2}$.
- If $f(m_n)$ is an upper bound of A then define $u_{n+1} = m_n$ and $\ell_{n+1} = \ell_n$.
- Otherwise, $u_{n+1} = u_n$, $\ell_{n+1} = m_n$.

We note that since $\ell < u$, we can show (by induction) that u_n is decreasing and ℓ_n is increasing. Also, we always have that u_n is an upper bound, and ℓ_n is not an upper bound.

Now we will show that u_n is Cauchy. We note that for every n we have

$$u_n - \ell_n < \frac{1}{2}(u_{n-1} - \ell_{n-1}) < \cdots < 2^{-n+1}(u_1 - \ell_1) < 2^{-n+1}(u - \ell)$$

Then let $0 < \epsilon \in \mathbb{Q}$ then there is some $N \in \mathbb{N}$ such that

$$2^{-n+1}(u - \ell) < \epsilon$$

We also note that $\forall n, u_n \geq \ell_n$.

Then we have that for any $m > n > N$ that

$$|u_n - u_m| = u_n - u_m < u_n - \ell_m < u_n - \ell_n < 2^{-n+1}(u - \ell) < \epsilon$$

A similar argument will show that ℓ_n is also Cauchy.

Now we want to show that u is an upper bound. Assume that it is not, so there is some $x \in A$, such that $x > u$. By density, we have some $q \in \mathbb{Q}$ such that $u < f(q) < x$. So there is some $N_1, N_2 \in \mathbb{N}$ such that

$$\forall n > N_1, x_n > q$$
$$\forall n > N_2, q > u_n$$

In particular, choose for some $k > N_2$, then we have $q > u_k$, and then for all $n > N_1$, we have $x_n > q > u_k$. So we have that $f(u_k) < x$ in contradiction to the fact that it was an upper bound.

Next, we want to show it is the least upper bound. Assume that we have some $s < u$, that is also an upper bound. Then by density, we can find $r \in \mathbb{Q}$ such that $s < f(r) < u = \ell$. So there is some $N_1, N_2 \in \mathbb{N}$ such that

$$\forall n > N_1, \ell_n > r$$
$$\forall n > N_2, r > s_n$$

Then pick some $k > N_1$ then $\ell_k > r$, and so for all $n > N_2$ we have

$$s_n < r < \ell_k$$

So $s < f(\ell_k)$, and since s is an upper bound on A we have that $f(\ell_k)$ is an upper bound on A. In contradiction to the fact that it is not an upper bound. so $u = \ell = \sup A$, as needed. $\square$

21.3 What did we see today?

- We build the real number from $\mathbb{Q}$!

22 22nd Class- Finishing completeness and subsequences

22.1 Review

Last time we constructed the reals from the rational, but we haven't proved the main theorem- that $\tilde{\mathbb{R}}$ is complete. So, we will finish the proof from last time:

Theorem 22.1. *$\tilde{\mathbb{R}}$ is complete. That is if $A \subset \tilde{\mathbb{R}}$ non empty, such that $U \in \tilde{\mathbb{R}}$ is an upper bound, then $\sup A \in \tilde{\mathbb{R}}$ exists.*

Proof. Let $A \subset \tilde{\mathbb{R}}$ be such that $U \in \tilde{\mathbb{R}}$ is an upper bound. By the above, we can find $u \in \mathbb{Q}$ such that $f(u)$ is an upper bound of A. Since A is not empty we have some $L \in A$, so we can find $\ell \in \mathbb{Q}$ so that $f(\ell) < L$. Now we define two sequences in the following inductive scheme

- $u_1 = u, \ell_1 = \ell$.
- Say that u_n, ℓ_n are defined. Then we denote $m_n = \frac{u_n + \ell_n}{2}$.
- If $f(m_n)$ is an upper bound of A then define $u_{n+1} = m_n$ and $\ell_{n+1} = \ell_n$.
- Otherwise, $u_{n+1} = u_n$, $\ell_{n+1} = m_n$.

We note that since $\ell < u$, we can show (by induction) that u_n is decreasing and ℓ_n is increasing. Also, we always have that u_n is an upper bound, and ℓ_n is not an upper bound.

Now we will show that u_n is Cauchy. We note that for every n we have

$$u_n - \ell_n < \frac{1}{2}(u_{n-1} - \ell_{n-1}) < \cdots < 2^{-n+1}(u_1 - \ell_1) < 2^{-n+1}(u - \ell)$$

Then let $0 < \epsilon \in \mathbb{Q}$ then there is some $N \in \mathbb{N}$ such that

$$2^{-n+1}(u - \ell) < \epsilon$$

We also note that $\forall n, u_n \geq \ell_n$.

Then we have that for any $m > n > N$ that

$$|u_n - u_m| = u_n - u_m < u_n - \ell_m < u_n - \ell_n < 2^{-n+1}(u - \ell) < \epsilon$$

A similar argument will show that ℓ_n is also Cauchy. Which, together with the previous estimate gives that $(u_n) = (\ell_n)$.

Now we want to show that u is an upper bound. Assume that it is not, so there is some $x \in A$, such that $x > u$. By density, we have some $q \in \mathbb{Q}$ such that $u < f(q) < x$. So there is some $N_1, N_2 \in \mathbb{N}$ such that

$$\forall n > N_1, x_n > q$$
$$\forall n > N_2, q > u_n$$

In particular, choose for some $k > N_2$, then we have $q > u_k$, and then for all $n > N_1$, we have $x_n > q > u_k$. So we have that $f(u_k) < x$ in contradiction to the fact that it was an upper bound.

Next, we want to show it is the least upper bound. Assume that we have some $s < u$, that is also an upper bound. Then by density, we can find $r \in \mathbb{Q}$ such that $s < f(r) < u = \ell$. So there is some $N_1, N_2 \in \mathbb{N}$ such that

$$\forall n > N_1, \ell_n > r$$
$$\forall n > N_2, r > s_n$$

Then pick some $k > N_1$ then $\ell_k > r$, and so for all $n > N_2$ we have

$$s_n < r < \ell_k$$

So $s < f(\ell_k)$, and since s is an upper bound on A we have that $f(\ell_k)$ is an upper bound on A. In contradiction to the fact that it is not an upper bound. so $u = \ell = \sup A$, as needed. $\square$

22.2 Subsequences

We started talking about subsequences, so we recall

Definition 22.2. Let (a_n) be a sequence. We call (b_k) a subsequence of (a_n) if there is some strictly increasing sequence (n_k), where $\forall k, n_k \in \mathbb{N}$

$$n_1 < n_2 < \cdots < n_k < n_{k+1} < \dots$$

such that

$$b_k = a_{n_k}$$

So we have the following very natural theorem:

Theorem 22.3. *Let a_n be a sequence.*

1. *Let $L \in \mathbb{R}$ then there is a subsequence (a_{n_k}) of (a_n) that converges to L if and only if the set $A_\epsilon = \{n \in \mathbb{N} \mid |a_n - L| < \epsilon\}$ is infinite for all $\epsilon > 0$.*
2. *(a_n) is not bounded from above if and only if it has a subsequence that converges to ∞.*
3. *(a_n) is not bounded from below if and only if it has a subsequence that converges to $-\infty$.*

In fact that subsequence is monotone!

Proof. We let a_n be a sequence.

- $\underline{\implies:}$ Assume that there is some subsequence such that $a_{n_k} \xrightarrow{k\to\infty} L$. Let $\epsilon > 0$ then we now that there is some $K \in \mathbb{N}$ such that

  $$\forall k > K, |a_{n_k} - L| < \epsilon$$

  So for all $k > K$ we have $n_k \in A_\epsilon$. In particular, this is an infinite set.

  $\underline{\impliedby:}$ We assume now that for every $\epsilon > 0$ we have that A_ϵ is infinite.

  We note that that if $n > m$

  $$A_{\frac{1}{n}} \subset A_{\frac{1}{m}}$$

  Now we define n_1 by choosing $n_1 \in A_{\frac{1}{1}}$. Next, we proceed with an inductive definition: say that we have defined $n_1, \dots n_k$, such that $n_1 < n_2 < \cdots < n_k$, and $n_j \in A_{\frac{1}{j}}$. we will now define n_{k+1} by the following: We consider $B_{k+1} = A_{\frac{1}{k}} \setminus \{1, 2, \dots n_k\}$. Since $A_{\frac{1}{k}}$ is infinite, by assumption, removing a finite number of elements will still be an infinite set. So we can choose $n_{k+1} = \min B_{k+1}$. We are guaranteed that $n_{k+1} > n_k$, and that $n_{k+1} \in A_{\frac{1}{k+1}}$.

  In particular, if we consider the sequence $b_k = |a_{n_k} - L|$ we get that $0 \leq b_k \leq \frac{1}{k}$. Since $\frac{1}{k} \to 0$, with $k \to \infty$, we get by the squeeze theorem that $b_k \to 0$. By definition, (think why) that implies that $a_{n_k} \xrightarrow{k\to\infty} L$. as needed.

- We will only deal with the case of $+\infty$.

  $\underline{\implies}$ Assume that (a_n) is not bounded from above. So for every $M > 0$ there is some $n \in \mathbb{N}$ such that $a_n > M$. we do a similar construction- only slightly easier. Let $n_1 = 1$, and assume we found $n_1 < n_2 < \dots n_k$ such that $a_{n_i} > i$. Then we let $M_{k+1} = \max\{k, n_k\}$. Since (a_n) is unbounded there is some n_{k+1} such that

  $$a_{n_{k+1}} > M_{k+1}$$

  In particular we have that $a_{n_k} > k$, and so we get that $a_{n_k} \to \infty$.

  $\underline{\impliedby}$ Assume some $a_{n_k} \xrightarrow{k\to\infty} \infty$, then let $M > 0$ then there is some $K \in \mathbb{N}$ such that for all $k > K$

  $$a_{n_k} > M$$

  And in particular we have some $n = n_k$ such that $a_{n_k} > M$

$\square$

22.3 What did we see today?

- We continued our discussion on Cauchy sequences.
- We started talking about subsequences.

23 23rd Class-Bolzano-Weierstrass and Subsequential limits

23.1 Review

Last time, we went back to subsequences. Today, we will discuss the Bolzano-Weierstrass Theorem, which will tell us that every bounded sequence has a convergent subsequence! Then, we will start a more extensive discussion about the limits of subsequences.

23.2 Bolzano-Weierstrass

So now we will continue building our tools: We will see that every sequence has a monotone sequence.

Theorem 23.1. *Let (a_n) be a sequence, then it has a monotonic subsequence.*

Proof. We will say that the element a_n is dominant if

$$\forall m > n, a_m < a_n$$

now we divide into two cases:

- Assume that there are infinitely many dominant terms. Then we define a_{n_k} as just these terms. Since a_{n_k} is dominant, in particular, we have that $a_{n_{k+1}} < a_{n_k}$ and this is a decreasing sequence.
- Assume that there are only finitely many dominant terms. Then there is some N such that a_n has no dominant terms for $n > N$. Choose $n_1 > N$. Assume that we found $a_{n_1} \leq a_{n_2} \leq \cdots \leq a_{n_k}$, and $n_1 < n_2 < \ldots n_k$. Since a_{n_k} is not dominant- there is some $n_{k+1} > n_k$ such that

  $$a_{n_{k+1}} \geq a_{n_k}$$

  and that will be our next term in the sequence. So we have that $a_{n_k} \leq a_{n_{k+1}}$, and this is an increasing subsequence of a_n.

□

As an immediate conclusion, we have the Bolzano-Weierstrass Theorem

Theorem 23.2. *Every bounded sequence has a convergent subsequence to a finite number.*

Proof. Let a_n be a bounded sequence; by the previous theorem, it has a monotone subsequence. The subsequence is also bounded, so by monotone convergence, we have that it converges. $\square$

This is very powerful! We now know that every sequence has a convergent subsequence! If it is bounded, it has a convergent sub-sequence to a finite number- if it is unbounded, then there is a subsequence that converges to either $+\infty$ or $-\infty$.

There are more general ways to prove the Bolzano-Weierstrass theorem. One can prove it in a way that will allow one to prove it even in higher dimensions (for example, on the plane). But this is more direct.

23.3 Subsequential limits

We want to start connecting subsequences with the ideas of $\limsup$ and $\liminf$. For this, we need some preparations. So, we start with a definition:

Definition 23.3. Let (a_n) be a sequence. Then a subsequential limit is $x \in \mathbb{R} \cup \{\pm\infty\}$, such that there is some subsequence of (a_n) such that $a_{n_k} \xrightarrow{k\to\infty} x$.

We expect that the collection of subsequential limits will capture the behavior of the sequence. Try to see what all the possible subsequences of the following sequence (no proof) - TPS

- $a_n = (-1)^n n^2$
- $b_n = \sin(\frac{n\pi}{3})$

Let's discuss:

Examples 23.4.
- For $a_n = (-1)^n n^2$, we can consider $a_{2n} = (2n)^2$ which converges to $+\infty$, and the subsequence of $a_{2n+1} = -(2n+1)^2$ which converges to $-\infty$. One can show that all the subsequences with a limit converge to $+\infty$ or $-\infty$. So we get that $\{\pm\infty\}$ is the set of all subsequential limits.
- Consider $b_n = \sin(\frac{n\pi}{3})$. The we have the subsequence $b_{3n} = \sin(n\pi) = 0$, the subsequence $b_{6n+1} = \sin(2n\pi + \frac{\pi}{3}) = \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$, and the subsequence $b_{6n+4} = \sin(\frac{4\pi}{3}) = -\frac{\sqrt{3}}{2}$. And again, we can show that this is all the possible limits.

So now we will prove the connection between subsequential limits and $\limsup$ and $\liminf$

Theorem 23.5. *Let (a_n) be a sequence, then there is a monotone subsequence that converges to $\limsup a_n$, and also a monotone subsequence that converges to $\liminf a_n$*

Proof. Recall that it is enough to show that a subsequence converges to $\limsup / \liminf$ as this subsequence always has a monotone subsequence. So we will get what we want.

Recall that a_n is not bounded from above; it has a subsequence converging to ∞, and that construction was monotone! A similar argument works for $-\infty$, so we will focus on cases where the sequence is bounded from above or below.

We will deal only with the case where (a_n) is bounded from above. So we have that $t = \limsup a_n < \infty$. Let $\epsilon > 0$ there is some $N_0 \in \mathbb{N}$ such that for all $N > N_0$

$$\sup\{a_k \mid k > N\} < t + \epsilon$$

So we get that since $\forall n > N_0, a_n \in \{a_k \mid k > N_0\}$ we have $\forall n > N_0, a_n < t + \epsilon$. Now we denote

$$A_\epsilon = \{n \in \mathbb{N} \mid t - \epsilon < a_n < t + \epsilon\}$$

We claim that A_ϵ is infinite. Assume that it is finite. Then there is some $N_1 > N_0$ such that $a_n \leq t - \epsilon$ (since we know that $a_n < t + \epsilon$!). So we get that

$$\forall N > N_1, \sup\{a_n | n > N\} < t - \epsilon$$

But then $\limsup a_n \leq t - \epsilon < t$ a contradiction.

So, by the theorem from Monday, we know that the set $A_\epsilon = \{n \in \mathbb{N} \mid |a_n - t| < \epsilon\}$ is infinite, for every $\epsilon > 0$ so there is a subsequence that converges to t. $\square$

23.4 What did we see today?

- We proved the Bolzano-Weierstrass theorem.
- We started talking about subsequential limits.

24 24th Class-Limsup and Liminf

24.1 Review

Last time, we started discussing subsequential limits and saw their connection to lim sup and lim inf. Today, we will continue exploring it. We start with this natural theorem

Theorem 24.1. *Let (a_n) be a sequence, denote by S the set of subsequential limits of (a_n). Then we have the following:*

1. $S \neq \emptyset$
2. $\sup S = \limsup a_n, \inf S = \liminf a_n$.
3. $\lim a_n$ *exists, and is equal* L *if and only if* $S = \{L\}$.

Proof. For the first- this is exactly the content of the theorem from last time! We have that $\limsup a_n, \liminf a_n \in S$.

Next, we want to show the second item. Let a_{n_k} be a subsequence that converges to $L \in S$. We saw that that means that

$$L = \liminf a_{n_k} = \limsup a_{n_k}$$

by construction we have that $n_k \geq k$, and so we have that

$$\{a_{n_k} | k > N\} \subset \{a_n \mid n > N\}$$

So we have that

$$\liminf a_n \leq \liminf a_{n_k} = L = \limsup a_{n_k} \leq \limsup a_n$$

This holds for any $L \in S$. So we have that

$$\liminf a_n \leq \inf S \leq \sup S \leq \limsup a_n$$

But $\liminf a_n \in S$! A number that is less than the inf and in S has to be the minimum. So $\inf S = \liminf a_n$. Similarly, for sup.

Finally, if $\lim a_n$ exists, we have $\sup S = \limsup a_n = \liminf a_n = \inf S$, so S contains one element, which is that limit. And if S contains one element, then since $\limsup a_n, \liminf a_n \in S$, they have to be equal, and the sequence converges to it. $\square$

This tells us that the subsequential limits somehow encode all the sequence's limit information! Moreover - if we know S, we can compute lim sup and lim inf of the sequence! Moreover, if we can somehow compute the sup of all the subsequences- we have a way to compute the lim sup. Let's return to the examples from last time.

Examples 24.2.

- For $a_n = (-1)^n n^2$. We saw that $S = \{\pm\infty\}$, so $\limsup a_n = +\infty$, and $\liminf a_n = -\infty$.
- For $a_n = \sin(\frac{n\pi}{3})$, we saw $S = \{0, \pm\frac{\sqrt{3}}{2}\}$, and so $\limsup a_n = \sup S = \frac{\sqrt{3}}{2}$, and $\liminf a_n = \inf S = -\frac{\sqrt{3}}{2}$.

24.2 Liminf and Limsup-reprised

So recall the way we defined the lim sup and lim inf of a sequence:

Definition 24.3. Let (a_n) be a sequence, then we define

$$\limsup a_n = \lim_{N\to\infty} \sup\{a_n \mid n > N\}, \liminf a_n = \lim_{N\to\infty} \inf\{a_n \mid n > N\}$$

So, if we focus on lim sup, it was introduced as the tail's limit of sup. Then we saw that it is sup of the limits! If we denote by S the set of subsequential limits, then $\limsup a_n = \sup S$. We also saw that there is a subsequence that converges to lim sup! So it is truly the limit of the supremum subsequence! So, we will want to see more properties that will come into play in the following.

Theorem 24.4. *Let a_n be a sequence such that $\lim a_n = a$, where $0 < a \in \mathbb{R}$ and let b_n be any sequence. Then*

$$\limsup a_n b_n = a \limsup b_n$$

Where we allow $a(+\infty) = +\infty$, and similarly for $-\infty$, as $a > 0$.

Proof. We will begin by showing that

$$\limsup a_n b_n \geq a \limsup b_n$$

If $\limsup b_n \in \mathbb{R}$, then there is a subsequence $b_{n_k} \to \limsup b_n$. In addition $\lim a_{n_k} = \lim a_n = a$, so we get that $\lim a_{n_k} b_{n_k} = a \limsup b_n$. So we get

$$a \limsup b_n \leq \limsup a_n b_n$$

as we found a subsequence that converges to the right-hand side.

If $\limsup b_n = +\infty$, then we have some subsequence $\lim b_{n_k} = +\infty$, and similarly we have $\lim a_{n_k} = \lim a_n$, and so $\lim a_{n_k} b_{n_k} = +\infty$.

If $\limsup b_n = -\infty$, then $a \limsup b_n = -\infty$, and then whatever is on the left-hand side, it will be true.

Next, we want to show the other inequality. Since $a_n \to a$, and $a > 0$ we saw that we have some $N \in \mathbb{N}$ such that for all $n > N$ we have $a_n > 0$. So we can assume that $a_n \neq 0$ for all n (as the first terms don't matter for limit). By algebra of limits, we have $\lim \frac{1}{a_n} = \frac{1}{a}$. Now we get that

$$\limsup b_n = \limsup[(\frac{1}{a_n}(a_n b_n)] \geq \frac{1}{a} \limsup a_n b_n$$

where we used the previous inequality that we proved. Reorganizing gives us

$$a \limsup b_n \geq \limsup a_n b_n$$

as needed. $\square$

We saw that we used quite strongly that $a > 0$. What happens if $a = 0$? Well- we can think about $a_n = -\frac{1}{n}$, and $b_n = n$. Then, we will have

$$\limsup a_n b_n = \limsup -1 = -1$$

And on the other hand, we will have

$$\limsup b_n = \infty$$

So we get

$$-1 = 0 \times \infty$$

Regardless of how you define $0 \times \infty$, it will not be -1.

24.3 What did we see today?

- We saw more connections between the subsequential limits and lim sup and lim inf
- We recalled the different definitions and characteristics of lim sup and lim inf.
- We showed some lim sup algebra.

25 25th Class-Subsequential limits and open sets

25.1 Review

Last time, we continued our discussion of lim sup and lim inf. Today, we will continue with this discussion and start talking about the topology of the real line. First some comments about the mid-semester assignment:

- The grades are a bit on the low side- but still reasonable. We have a mean of 80.7 and a median of 84.
- Many of you made the mistake of saying that unbounded sequence is the same as $\lim a_n = \infty$- This is not the definition; there is a difference between unbounded and limit $= \infty$.
- I will also want to go over 4(a), where you needed to show that $\alpha < \beta$.

Proof. By the nested interval property, there is some $x \in \bigcap_{n=1}^{\infty} I_n$. In particular, for all $n \in \mathbb{N}$ we have

$$a_n \leq x \leq b_n$$

So x is an upper bound of (a_n) and lower bound of (b_n) then by the properties of sup and inf we have that

$$\alpha \leq x \leq \beta$$

Then we can conclude that $\alpha \leq \beta$. $\square$

25.2 Series preparation

Next, we will prove the following useful property we will use later when we learn about series:

Theorem 25.1. *Let a_n be a sequence of nonzero real numbers. Then*

$$\liminf |\frac{a_{n+1}}{a_n}| \leq \liminf |a_n|^{\frac{1}{n}} \leq \limsup |a_n|^{\frac{1}{n}} \leq \limsup |\frac{a_{n+1}}{a_n}|$$

Proof. We will only prove the right most inequality- and you will do the other one in the Pset.

Let $a = \limsup |a_n|^{\frac{1}{n}}$, and $L = \limsup |\frac{a_{n+1}}{a_n}|$, we need to show $a \leq L$. If $L = \infty$, it will always hold, so we assume $L < \infty$. Let $\epsilon > 0$. Then we have that

$$\lim_{N \to \infty} \sup\{|\frac{a_{n+1}}{a_n}| \mid n > N\} = L < L + \epsilon$$

And so there is some $N \in \mathbb{N}$ such that

$$\sup\{|\frac{a_{n+1}}{a_n}| \mid n > N\} < L + \epsilon$$

which allows us to get that for $n > N$

$$|\frac{a_{n+1}}{a_n}| \leq \sup |\frac{a_{n+1}}{a_n}| < L + \epsilon$$

So for $n > N$ we write

$$|a_n| = |\frac{a_n}{a_{n-1}}||\frac{a_{n-1}}{a_{n-2}}||\frac{a_{n-2}}{a_{n-3}}| \dots |\frac{a_{N+1}}{a_N}||a_N| < (L+\epsilon)^{n-N}|a_N| = (L+\epsilon)^n \frac{|a_N|}{(L+\epsilon)^N}$$

Now the right-hand side is fixed! It doesn't depend on n; it is some positive constant, call it C. So we get

$$|a_n|^{\frac{1}{n}} < C^{\frac{1}{n}}(L+\epsilon)$$

you can show that $\lim C^{\frac{1}{n}} = 1$, for any $C > 0$ (and you will do it in the HW). So, we conclude that

$$a = \limsup |a_n|^{\frac{1}{n}} < L + \epsilon$$

for any $\epsilon > 0$. So we conclude that $a \leq L$, as needed. □

Interestingly, we have the following corollary.

Corollary 25.2. *If* $\lim |\frac{a_{n+1}}{a_n}| = L$ *, then* $\lim |a_n|^{\frac{1}{n}} = L$

Since the far sides of the inequality are equal, so are the ones in the middle- which proves that claim! This will be very useful when we start a series of real numbers soon.

25.3 Closed sets-prelude

But before that, we want to show a last property of S - the set of all subsequential limits. This set has an interesting property- it contains all limits of sequences from S:

Theorem 25.3. *Let (a_n) be a sequence and S be the set of all subsequential limits. Let b_n be a sequence in $S \cap \mathbb{R}$, such that* $\lim b_n = L$. *Then $L \in S$.*

Proof. We start with the case of $L \in \mathbb{R}$. Let $\epsilon > 0$, then there is some $n \in \mathbb{N}$ such that

$$\begin{aligned} &|L - b_n| < \epsilon \\ &L - \epsilon < b_n < L + \epsilon \end{aligned}$$

We write $\delta = \min\{L + \epsilon - b_n, b_n - L + \epsilon\}$. So now we have

$$\begin{aligned} b_n + \delta &< b_n + L + \epsilon - b_n = L + \epsilon \\ b_n - \delta &> b_n - b_n + L - \epsilon = L - \epsilon \end{aligned}$$

We have that $b_n \in S$, so the set

$$\{m \in \mathbb{N} \mid |a_m - b_n| < \delta\} = \{m \in \mathbb{N} \mid b_n - \delta < a_m < b_n + \delta\}$$

is infinite, so in particular, the set

$$\{m \in \mathbb{N} \mid L - \epsilon < a_n < L + \epsilon\}$$

is infinite, and by a theorem, we saw in class, that means that $L \in S$.

If $t = +\infty$ $(-\infty)$, then a_n is not bounded from above (below), and so (a_n) has a subsequence that has limit $+\infty$ $(-\infty)$. Thus $+\infty \in S$ $(-\infty \in S)$. $\square$

But why should we care about this property? This property is connected to the "shape" of $\mathbb{R}$- what is known as the topology of $\mathbb{R}$, which we will start studying in the next couple of classes.

25.4 What did we see today?

- We prepared the ground for series.
- We saw some prelude for closed sets.

26 26th Class-Topology of the real line.

26.1 Review

Last time, we saw that S has some weird property. It contains all the limits I can make from the sequences inside of it. Today, we will start exploring similar ideas- open and closed sets. Before, I want to address question 3b in Pset 7 - we have the sequence $y_{n+1} = 3 - y_n$- and the question was, why can't we compute the limit by taking the limit of this condition? The answer is simply since the limit doesn't exist!

26.2 Open sets

We start by defining a seemingly unrelated property:

Definition 26.1. A set $A \subset \mathbb{R}$ is *open* if for any $a \in A$, there is $\epsilon > 0$ such that the ϵ-neighborhood is contained in A. That is :

$$(a - \epsilon, a + \epsilon) \subset A$$

What sets are open? TPS

Examples 26.2.

- $\mathbb{R}$ is open! since if we pick any $a \in \mathbb{R}$ any ϵ neighborhood is contained on $\mathbb{R}$.
- $\emptyset$ is also open, since we don't have any $a \in \emptyset$, therefor- the condition always hold!
- More useful is to consider an open interval (a, b): if we have $x \in (a, b)$ then there is some $\epsilon < \min\{x - a, b - x\}$ and then $V_\epsilon(x) \subset (a, b)$.

This explains the terminology of an open set. Open sets are a way to generalize the idea of open intervals to more general settings.

Next, we will see how this notion "plays" with union and intersections

Theorem 26.3. *1. Any union of open sets is open.*

2. A finite intersection of open sets is open.

Proof. So let $\{O_\lambda | \lambda \in \Lambda\}$ a collection of opens sets and set $O = \bigcup_{\lambda \in \Lambda} O_\lambda$. Then let $a \in O$, then$a \in O_\lambda$ for some $\lambda \in \Lambda$, since O_λ is open we have some $\epsilon > 0$ such that $V_\epsilon(a) \subset O_\lambda \subset O$. So we have that $V_\epsilon(a) \subset O$ as needed.

Next, we let $\{O_1, \ldots, O_N\}$ be a finite collection of open sets. Then we let $a \in \bigcap_{i=1}^N O_i$, then for all $1 \leq i \leq N$ we have $a \in O_i$. Since O_i is open we have $\epsilon_i > 0$ such that $V_{\epsilon_i}(a) \subset O_i$. But we need to find a single ϵ, so we choose $\epsilon = \min\{\epsilon_i\}_{i=1}^N$. Then we have that for all $1 \leq i \leq N$ we have $V_\epsilon \subset V_{\epsilon_i} \subset O_i$. So we have that $V_\epsilon \subset \bigcap_{i=1}^N O_i$, as needed. □

Why is it essential to have an intersection of only finitely many open intervals? TPS! Can you think of an example of an infinite intersection that results in something that is not open? Consider $I_n = (-\frac{1}{n}, \frac{1}{n})$, then $\bigcap_{n=1}^\infty I_n = \{0\}$.

26.3 Closed sets

We continue with the idea of closed set

Definition 26.4. We say that a set $A \subset \mathbb{R}$ is closed if A^c is open.

So, what are some closed sets?

Examples 26.5.

- Weirdly enough both $\mathbb{R}$ and $\emptyset$ are also closed!
- A closed interval is a closet set (as its complement is a union of open intervals).

We will soon give more examples, but first, we want to note the following natural theorem

Theorem 26.6. *1. Any intersection of closed sets is close.*

2. A finite union of closed sets is close.

Do this as TPS!

Proof. This is a direct application of DeMorgan laws! Since we have

$$(\bigcup_{i=1}^N A_n)^c = \bigcap_{i=1}^N A_n^c$$
$$(\bigcap_{i \in \mathcal{I}} A_n)^c = \bigcup_{i \in \mathcal{I}} A_n^c$$

which gives ass the wanted result. □

and we will also define the following:

Definition 26.7. $x \in \mathbb{R}$ is called a limit point of $A \subset \mathbb{R}$ if there is a sequence (x_n) such that $\forall n, x_n \in A$, and $\lim x_n = x$.

Then we have the following characterization:

Theorem 26.8. *A set A is closed if and only if A contains all of its limit points*

Proof. $\underline{\Longrightarrow}$ Let A be close, and let $x \in \mathbb{R}$ a limit point A. So we have that (x_n) such that $x_n \to x$, and $x_n \in A$. Assume by way of contradiction that $x \in A^c$. Since A^c is open there is some ϵ-neighborhood such that $V_\epsilon(x) \subset A^c$. But for that ϵ there is some $x_n \in A$ such that $|x - x_n| < \epsilon$. In other words $x_n \in V_\epsilon(x)$. So $x_n \in A$ and $x_n \in A^c$ - contradiction. So $x \in A$.

$\underline{\Longleftarrow}$ Let A be a set containing all its limit points. We want to show that A^c is open. Assume by way of contradiction that is not open. So there is some $x \in A^c$ such that for any ϵ, we have $V_\epsilon \not\subset A^c$. In particular, choose $\epsilon = \frac{1}{n}$, so for any n we have some $x_n \in V_{\frac{1}{n}}$ such that $x_n \in A$. In particular, we have that

$$|x - x_n| < \frac{1}{n}$$

So we have that $x_n \to x$, and $x_n \in A$. So x is a limit point of A, and by assumption, then $x \in A$- in contradiction. □

So, what we saw was that S is a closed set!

26.4 What did we see today?

- We talked about open sets.
- We defined closed sets

27 27th Class-Closure and open covers

27.1 Review

Last time, we defined closed sets and saw some properties of them. Open sets are "nice" because every point has a neighborhood, so they play well with notions like limits. Also, they work great under unions. On the other hand, closed sets are "neat" since they contain all their limits, so they end sharply, and that way, they play nicely with intersections. Today, we will define another topological notion- open coves.

Before we start with that, we want to define the closure of a set:

Definition 27.1. Let $A \subset \mathbb{R}$ a set, and let L be the set of all the limit points of A. Then we denote by $\overline{A}$ the closure of A, defined as:

$$\overline{A} = A \cup L$$

That is- we take a set and add to it all of its limit points! Let's give some examples -TPS

Examples 27.2.
- What is $\overline{\mathbb{Q}}$?

- What is $\overline{\mathbb{I}}$?
- Let $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$, what is $\overline{A}$.
- Let $B = (a, b)$, what is $\overline{B}$.

So:

Examples 27.3.
- We saw how to build $\mathbb{R}$ from $\mathbb{Q}$, and this was precisely this process! So, in fact, the closure of $\mathbb{Q}$ is $\mathbb{R}$.
- We can also approximate every rational number with an irrational one! This is the same thing as being dense (think why). So we have $\overline{\mathbb{I}} = \mathbb{R}$.
- In this case we will just have $\overline{A} = A \cup \{0\}$.
- Naturally we will have $\overline{B} = [a, b]$.

Before we start talking about that, we want to show the characteristics of the closure of a set

Theorem 27.4. *Let $A \subset \mathbb{R}$ the closure $\overline{A}$ is a closed set, the smallest closed set containing A.*

Smallest in what sense? Just like in sup- if there is some closed set B such that $A \subset B$, then $\overline{A} \subset B$.

Proof. Let $A \subset \mathbb{R}$ be a subset, L be the set of limit points of A, and $\overline{A} = A \cup L$ the closure. Then, we need to show that $A \cup L$'s set of limit points of $A \cup L$ is contained in L. Maybe L has some limit point that is not a limit point of A.

First, we will show that L is close! Let $a_n \in L$ be such that $a_n \to a$. For every n, we have $a_n \in L$ and so there is a sequence $b_{n,k}$ such that $\forall k \in \mathbb{N}, b_{n,k} \in A$ and $b_{n,k} \to a_n$.

In particular, for each n there is some $k_n \in \mathbb{N}$ such that

$$|b_{n,k_n} - a_n| < \frac{1}{n}$$

Denote $c_n = b_{n,k_n}$. Let $\epsilon > 0$, there there is some N_2 such that for all $n > N_1$ we have

$$|a_n - a| < \frac{\epsilon}{2}$$

And we have some N_2 such that

$$|c_n - a_n| < \frac{\epsilon}{2}$$

and so we get that for $n > \max\{N_1, N_2\}$

$$|c_n - a| < \epsilon$$

Finally, we note $A \subset L$, as we can always choose the constant sequence. So, we have proven that $\overline{A}$ is closed.

Now let $A \subset B$, B closed. Since B is closed, it will contain L (as limit points of A are also limit points of B), so $\overline{A} \subset B$. □

27.2 Open covers

Many times, especially when discussing more advanced analysis, one needs to take a cover of some set, usually by open sets- and many properties relate to that. So, we start with a definition of an open cover of a set.

Definition 27.5. Let $A \subset \mathbb{R}$ be a set. We say that a collection $\{U_i\}_{i\in\mathcal{I}}$ of subsets $U_i \subset \mathbb{R}$ is an open cover if :

- For all $i \in \mathcal{I}$, U_i is open.
- $A \subset \bigcup\limits_{i\in\mathcal{I}} U_i$

Let's do some examples

Examples 27.6.

- For $(0,1)$, consider $U_i = (0, 1-\frac{1}{i})$, for $i \geq 2$. Is $\{U_i\}$ an open cover?
- For $[0,1]$, consider $U_i = (0, 1-\frac{1}{i})$, for $i \geq 2$. Is $\{U_i\}$ an open cover?
- For $(0,1)$, consider $U_x = (\frac{x}{2}, 1)$, for $x \in (0,1)$. Is $\{U_x\}$ an open cover?
- For $[0,1]$, consider $U_x = (\frac{x}{2}, 1)$, for $x \in (0,1)$, and we add $U_0 = (-\frac{1}{10}, \frac{1}{10})$ and $U_1 = (1-\frac{1}{10}, 1+\frac{1}{10})$. Is $\{U_x\}$ an open cover?

Note something interesting about the last example: even though it is an open cover- it seems to have a lot of redundancies! We can choose a much smaller set of U_x to cover all $[0,1]$! For example we can choose $U_0, U_1, U_{\frac{1}{10}}$. So, we can choose a finite cover from that infinite (even uncountable) cover! That is a very striking property- and you might think it is some property of the cover we choose- but it is a property of the set we are covering!

27.3 Compact set

So, we define the following property

Definition 27.7. A set $A \subset \mathbb{R}$ is called compact, if from any open cover $\{U_i\}_{i\in\mathcal{I}}$, there is some $N \in \mathbb{N}$, and some indices $i_1, \dots i_N \in \mathcal{I}$, such that

$$A \subset \bigcup_{j=1}^{N} U_{i_j}$$

This is called a finite subcover of A.

A couple of remarks:

Remark 27.8. This property is beneficial! Many properties don't pass through an infinite union or intersection (like being closed or open), so choosing a finite cover enables us to ensure that these properties will remain.

27.4 What did we see today?

- We defined closure of a set
- We saw some characterization of closed sets.
- We defined compact sets.

28 28th Class- Compact sets and Heine Borel

28.1 Review

A comment about the Pset- there were a bunch of typos, and I am sorry about it

- In Question 3 it should be $|a|\ell$ (in the second part). And, of course a_n should be defined w.r.t to C_{n-1} so

$$a_n = \begin{cases} 1, & x \in \text{ the left third of the interval chosen of } C_{n-1} \\ 0, & x \in \text{ the right third of the interval chosen of } C_{n-1} \end{cases}$$

- In question 2, it is supposed to be $A_n \supset A_{n+1}$, not $n-1$, as written. These are nested closed bounded sets (should remind you of nested intervals).
- In question 4a, the hint referred to the previous questions, not necessarily the previous question.

Last time, we defined open covers and compact sets. Today- we will explore these sets more.

28.2 Compact set

So, we define the following property

Definition 28.1. A set $A \subset \mathbb{R}$ is called compact, if from any open cover $\{U_i\}_{i\in\mathcal{I}}$, there is some $N \in \mathbb{N}$, and some indices $i_1, \ldots i_N \in \mathcal{I}$, such that

$$A \subset \bigcup_{j=1}^{N} U_{i_j}$$

This is called a finite subcover of A.

A couple of remarks:

Remark 28.2. This property is beneficial! Many properties don't pass through an infinite union or intersection (like being closed or open), so choosing a finite cover enables us to ensure that these properties will remain.

This property seems very hard, which happens only for very few sets. But we have many sets like that, as we shall later see. First, we will see the following useful property: a closed subset of a compact set is also compact!

Lemma 28.3. *Let $A \subset \mathbb{R}$ be a compact subset. Let $B \subset A$ be a closed subset, then B is compact.*

Proof. Let A, B be as above, and let $\{U_i\}$ be an open cover B. We note that B^c is an open set, and so we get that $\{U_i\} \cup \{B^c\}$ is a cover of A. Since A is compact we have that a finite cover $\{V_j\}_{j=1}^{N}$, Then we have that $\{V_i\} \setminus \{B^c\}$ is a sub-cover of B made from $\{U_i\}$. So B is compact. $\square$

28.3 Heine Borel

Theorem 28.4 (Heine Borel)**.** *A set $A \subset \mathbb{R}$ is compact if and only if it is closed and bounded.*

Proof. $\underline{\implies}$ Let $A \subset \mathbb{R}$ compact. Then we define

$$\forall x \in A, U_x = V_1(x)$$

That is the open 1-neighborhood of $x \in A$. This is an open cover of A, and since A is compact, there is some finite cover U_{x_i} for $1 \leq i \leq N$.So if we define $M = \max_{1 \leq i \leq N}\{x_i + 1\}$, and $m = \min_{1 \leq i \leq N}\{x_i - 1\}$, we will have that

$$\forall x \in A, \exists 1 \leq i \leq N, x \in U_{x_i} \implies x_i - 1 \leq x \leq x_i + 1 \implies m < x < M$$

So A is bounded.

Showing that A is closed is a bit harder. We will assume that A is not close, so by the properties of closed sets, there is a sequence $(y_n) \subset A$ that $y_n \to y$, and $y \notin A$.

So we have that for all $x \in A$ we have $|y - x| > 0$, so we define the following

$$\forall x \in A, U_x = \{z \in \mathbb{R} \mid |x - z| < \frac{|y - x|}{2}\}$$

the set of points that their distance from x is smaller than half their distance to y. This is an open cover of A, and since A is compact, there is a finite subcover. We denote it by U_{x_i} for $1 \leq i \leq N$. Then, we can choose

$$\epsilon = \min_{1 \leq i \leq N}\{\frac{|x_i - y|}{2}\}$$

Since $y_n \to y$ there is some n such that $|y_n - y| < \epsilon$, but that means that

$$|y_n - x_i| > ||y - x_i| - |y_n - y|| > |2\epsilon - \epsilon| = \epsilon$$

And in particular, $y_n \notin U_{x_i}$ for all $1 \leq i \leq N$, and we got a contradiction to the fact that U_{x_i} is a cover!

$\underline{\impliedby}$ Let A be a bounded closed set. Since A is bounded, it is contained in some $I = [m, M]$. By the previous lemma, It is enough to show that I is compact. Assuming that I is not compact, there is some cover $\mathcal{U} = \{U_i\}$ that we can't find a finite subcover that will cover I. We will show that in fact you can build a finite subcover or it.

We define the following sequence of intervals:

- Let $I_0 = [a_0, b_0] = I = [m, M]$.
- If we found $I_n = [a_n, b_n]$, that doesn't have a finite subcover from $\mathcal{U}$. Then consider $J_1 = [a_n, \frac{a_n+b_n}{2}], J_2 = [\frac{a_n+b_n}{2}, b_n]$. We claim that one of them does not have a finite subcover from $\{U_i\}$. If both had a finite subcover, then the union of these subcovers would have been a finite subcover of I_n, in contradiction to the assumption. So, if J_m, for $m \in \{1, 2\}$, is an interval without a finite subcover, we set $I_{n+1} = J_m$.

So we have a sequence of nested intervals $I_n \supset I_{n+1}$. We note that the length of I_n denoted ℓ_n, is

$$\ell_n = 2^{-n}(M - n)$$

So, by a question from the problem set, we have that the intersection of all intervals, $\bigcap I_n$, contains a single point a.

There is some i_0 such that $a \in U_{i_0}$, since U_{i_0} is open there is some $\epsilon > 0$ such that $(a - \epsilon, a + \epsilon) \subset U_{i_0}$.

Since $\ell_n \to 0$, there is some $N \in \mathbb{N}$ such that

$$\forall n > N, I_n \subset (a - \epsilon, a + \epsilon) \subset U_{i_0}$$

In particular, I_{N+1} is covered by a single element U_{i_0}! This contradicts the fact that I_{N+1} has no finite subcover. So we conclude that I is compact, and since $A \subset I$ is closed, we get that A is compact. $\square$

28.4 What did we see today?

- We defined compact sets.
- We proved Heine -Borel!

29 29th Class- Series

29.1 Review

Last time, We talked about Heine Borel- and saw that there are a lot of compact sets in $\mathbb{R}$. Today, we are shifting gears again- and start talking about a particular type of sequence - a series!

29.2 Series

We already introduced the "Sigma notation," but recall that

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

So, we will define, for any sequence, the sequence of partial sums

Definition 29.1. Let (a_n) be a sequence; then the partial sums sequence is defined as:

$$s_n = \sum_{i=1}^{n} a_n$$

We can define the infinite series as follows:

Definition 29.2. Let (a_n) be a sequence, then when we write

$$\sum_{n=1}^{\infty} a_n = L$$

we mean that $\lim s_n = L$, or

$$\lim_{N\to\infty} \sum_{n=1}^{N} a_n = L$$

We can also say that the sequence converges to $\pm\infty$ if the limit is $\pm\infty$. And a series that doesn't converge- diverges.

What can we say if $a_n \geq 0$ about the sequence of the partial sums? TPS

Then we have that s_n is increasing monotonic, so it will either converge to a real number or infinity. So we get that $\sum |a_n|$ always exists! It could be infinite, though.

Remark 29.3. A word about terminology- unlike sequences, in series, when we say that the series diverges, we usually mean that it converges to $+\infty$. Precisely since $\sum |a_n|$ always converges to something.

Definition 29.4. We say that a series $\sum a_n$ converges absolutely if $\sum |a_n|$ converges to a real number. If $\sum a_n$ converges to a real number and $\sum |a_n|$ converges to ∞, the series conditionally converges.

Let's give some examples!

Examples 29.5.

- In Pset 3, you have proven that, for all $n \in \mathbb{N}$

$$\sum_{n=0}^{N} 2^{-n} = 2 - \frac{1}{2^N}$$

So we have that

$$\sum_{n=0}^{\infty} 2^{-n} = \lim_{N\to\infty} 2 - \frac{1}{2^N} = 2$$

This is an example of the geometric series! In general, we have, if $|r| > 1$

$$\sum_{n=0}^{\infty} r^{-n} = \frac{1}{1 - \frac{1}{r}}$$

- Another important is given by $\sum_{n=1}^{\infty} \frac{1}{n^p}$. We already saw that for $p = 1$, the series converges to ∞, and for $p = 2$, it converges to a finite number. The full result is that this series converges if and only if $p > 1$.

Remark 29.6. From now on, we will use both in their full generality. That is, when we have $|r| < 1$ we can conclude that $\sum r^{-n}$ converges, and if $p \geq 1$ then $\sum \frac{1}{n^p}$ diverges, and if $p < 1$ then $\sum \frac{1}{n^p}$ converges.

We will continue with the following natural theorem

Theorem 29.7 (Algebra of series)**.** *Let $\sum_{n=1}^{\infty} a_n = A, \sum_{n=1}^{\infty} b_n = B$, for $A, B \in \mathbb{R}$, i.e. the limits exist, and they are finite. Then we have*

- *For $c \in \mathbb{R}$ we have $\sum_{n=1}^{\infty} ca_n = cA$.*
- $\sum_{n=1}^{\infty} a_n + b_n = A + B$

Proof.

- Denote $s_n = \sum_{k=1}^{n} a_k$, then we note that

$$cs_n = c\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} ca_k$$

note that we can only use distribution for finite sums! It is not trivial to use this property whenever you have a limit! So we have that

$$\sum_{n=1}^{\infty} ca_n = \lim_{N\to\infty} \sum_{k=1}^{N} ca_k = \lim_{N\to\infty} cs_N = cA$$

by the algebra of limits.

- Similarly here we will denote $s_n = \sum_{k=1}^{n} a_k, r_n = \sum_{k=1}^{n} b_k$. Then we have that

$$\sum_{k=1}^{n} a_k + b_k = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = s_n + r_n$$

and so

$$\sum_{n=1}^{\infty} a_n + b_n = \lim_{N\to\infty} \sum_{k=1}^{N} a_k + b_k = \lim_{N\to\infty} s_N + r_n = A + B$$

as needed.

□

What about multiplication? This is a bit more delicate- and we will address it next time.
Finally, we will show that you can use the Cauchy criterion for convergence of series

Theorem 29.8. *A series $\sum_{n=1}^{\infty} a_n$ converges if and only if for all $\epsilon > 0$ there is some $N \in \mathbb{N}$ such that for all $n > m \geq N$ we have*

$$|\sum_{k=m+1}^{n} a_k| < \epsilon$$

Proof. If we denote the partial sums by s_n, we note that

$$\sum_{k=m+1}^{n} a_k = \sum_{k=1}^{n} a_k - \sum_{k=1}^{m} a_k = s_n - s_m$$

So the statement is (s_n) converges iff (s_n) is Cauchy- which we have already proven. $\square$

Corollary 29.9. *If $\sum_{n=1}^{\infty} a_n = L$, for $L \in \mathbb{R}$ then $a_n \to 0$*

Proof. If the series converges- then it is Cauchy, and specifically, we can choose $n = m + 1$: for all $\epsilon > 0$, we have some $N \in \mathbb{N}$ such that if $m > N$ we have

$$|a_{m+1}| = |\sum_{k=m+1}^{m+1} a_k| < \epsilon$$

as needed. $\square$

Does the converse hold? That is if $a_n \to 0$, then we have that $\sum_{n=1}^{\infty} a_n$ converges? No! Think about the Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$!

29.3 Convergence tests

Ee will start showing some converges tests:

Theorem 29.10 (Comparison Test)**.** *Let $(a_n), (b_n)$, where $\forall n, 0 \leq a_n \leq b_n$ then*

1. *If $\sum b_n$ converges then $\sum a_n$ converges.*
2. *If $\sum a_n = \infty$ then $\sum b_n = \infty$*

Proof. The first part follows from the Cauchy criterion since we have that

$$|\sum_{k=m+1}^{n} a_k| < |\sum_{k=m+1}^{n} b_k|$$

So if $\sum b_n$ converges, then $\sum a_n$ has to converge.

For the second part, we have that $\sum_{n=1}^{N} a_n \leq \sum_{n=1}^{N} b_n$, and so since the partial sums of the first converges to $+\infty$, by order of limits, the second also converges to ∞. $\square$

Let's do some examples as TPS- For each series, say if it converges or diverges (converge to $+\infty$)

Examples 29.11.

- $\sum \frac{n}{n^3+2}$
- $\sum \frac{n^2+n+5}{n^4+2}$
- $\sum r^n$, for $r \geq 1$.

29.4 What did we see today?

- We saw some algebra of series.
- We saw the Cauchy criterion for convergence of series.
- We saw the first convergence test!

30 30th Class- Series test

30.1 Review

Last time, we discussed series and saw the Cauchy criterion for convergence and the comparison test. Let's do some examples as TPS- For each series, say if it converges or diverges (converge to $+\infty$)

Examples 30.1.
- $\sum \frac{n}{n^3+2}$
- $\sum \frac{n^2+n+5}{n^4+2}$
- $\sum r^n$, for $r \geq 1$.

So, let's see what is going on:

Proof.
- $\frac{n}{n^3+2} < \frac{n}{n^3} = \frac{1}{n^2}$, since $\sum \frac{1}{n^2}$ converges, we get that series converges as well.
- Here, we do the following

$$\frac{n^2+n+5}{n^4+2} = \frac{n^2}{n^4+2} + \frac{n}{n^4+2} + \frac{5}{n^4+2}$$

and we note

$$\frac{n^2}{n^4+2} \leq \frac{n^2}{n^4} = \frac{1}{n^2}$$
$$\frac{n}{n^4+2} \leq \frac{n}{n^4} = \frac{1}{n^3}$$
$$\frac{5}{n^4+2} \leq \frac{5}{n^4}$$

Since all the series on the LHS converges, we have that each of the summands converges, and so their sum converges.

- For $r \geq 1$ we have

$$r^n \geq 1$$

Since we have that $\sum_{n=1}^{N} 1 = N \to \infty$, we get that this series diverges.

$\square$

An immediate corollary from this is the absolute convergence test:

Theorem 30.2. *Let $\sum a_n$ be a series, then if $\sum |a_n|$ converges then $\sum a_n$ converges.*

Proof. This is immediate as $a_n \leq |a_n|$ for all n. □

Is the converse true? Actually not! We can consider an alternating series $\sum(-1)^{n+1}\frac{1}{n}$:

Example 30.3. The series $\sum(-1)^{n+1}\frac{1}{n}$ converges!

Proof. Define $S_N = \sum_{n=1}^{N}(-1)^{n+1}\frac{1}{n}$, and call $a_n = (-1)^{n+1}\frac{1}{n}$ Then we note that following

$$\begin{aligned} S_{2n+2} - S_{2n} = a_{2n+2} + a_{2n+1} &= (-1)^{2n+2+1}\frac{1}{2n+1} + (-1)^{2n+1+1}\frac{1}{2n+1} \\ &= -\frac{1}{2n+2} + \frac{1}{2n+1} = \frac{2n+2-2n-1}{(2n+1)(2n+2)} > 0 \end{aligned}$$

So S_{2n} is increasing, and we note that

$$S_{2n+1} - S_{2n} = a_{2n+1} = \frac{1}{2n+1} > 0$$

Similarly, we have

$$\begin{aligned} S_{2n+3} - S_{2n+1} = a_{2n+3} + a_{2n+2} &= (-1)^{2n+3+1}\frac{1}{2n+3} + (-1)^{2n+2+1}\frac{1}{2n+2} \\ &= \frac{1}{2n+3} - \frac{1}{2n+2} < 0 \end{aligned}$$

So S_{2n+1} is monotone decreasing. So we have that $S_{2n} < S_{2n+1} < S_1$, so it is an increasing and bounded sequence. So $\lim S_{2n} = L$ exists, so we get that

$$\lim S_{2n+1} = \lim S_{2n} + a_{2n+1} = \lim S_{2n} + \lim \frac{1}{2n+1} = L + 0 = L$$

We have that $\lim S_{2n} = \lim S_{2n+1} = L$; in the Pset, you will show that it means that $\lim S_n = L$- as needed. □

Remark 30.4. In the Pset, you will generalize this to conclude the alternating series test.

30.2 Root test

We will prove the root test, giving us another tool to prove convergence.

Theorem 30.5. *Let $\sum a_n$ a series and denote $\alpha = \limsup |a_n|^{-\frac{1}{n}}$ then*

1. *If $\alpha < 1$, the series converges absolutely*
2. *If $\alpha > 1$, it diverges.*
3. *If $\alpha = 1$, the test is inconclusive, and we have no information.*

Proof. 1. If $\alpha < 1$, let $\epsilon > 0$, such that $\alpha + \epsilon < 1$. Then we write

$$\lim_{N\to\infty} \sup\{|a_n|^{\frac{1}{n}} \mid n > N\} = \alpha$$

and so there is some $N \in \mathbb{N}$ such that

$$\alpha - \epsilon < \sup\{|a_n|^{\frac{1}{n}} \mid n > N\} < \alpha + \epsilon$$

In particular for $n > N$, we have $|a_n|^{\frac{1}{n}} < \alpha + \epsilon$, and so

$$|a_n| < (\alpha + \epsilon)^n$$

we note that $\sum(\alpha + \epsilon)^n$ converges (as $\alpha + \epsilon < 1$), and by the comparison test we have that $\sum |a_n|$ converges, as claimed.

2. If $\alpha > 1$, then we can find a subsequence of $|a_n|^{\frac{1}{n}}$ such that

$$\lim_{k\to\infty} |a_{n_k}|^{\frac{1}{n_k}} > 1$$

So we have infinitely many n for which $|a_n|^{\frac{1}{n}} > 1$, of in other words $|a_n| > 1$ for infinitely many n. In particular, that means that a_n doesn't converge to 0, and so by the contrapositive of the theorem, we saw- $\sum a_n$ diverges.

3. Recall that we saw that $\lim n^{\frac{1}{n}} = 1$, and so we have that

$$\lim(\frac{1}{n})^{\frac{1}{n}} = 1$$
$$\lim(\frac{1}{n})^{\frac{2}{n}} = 1$$

So in particular for $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$, we have that

$$\limsup |\frac{1}{n}|^{\frac{1}{n}} = 1$$
$$\limsup |\frac{1}{n^2}|^{\frac{1}{n}} = 1$$

So we have that in both cases $\alpha = 1$, but one converges and one diverges. So, α can't give us any information about convergence or divergence.

□

30.3 What did we see today?

- We started seeing that conditional convergence is very different from absolute converges.
- We proved the root test!

31 31st Class- The ratio and root test and rearrangements

31.1 Review

Last time, we started proving different tests for convergence: the comparison test, the alternating series test (which you will prove in the pset), and the root test. Today, we will talk about the ratio test!

31.2 Ratio test

We will now use the root test to prove the ratio test, using the inequality that we proved several classes ago:

$$\liminf |\frac{a_{n+1}}{a_n}| \leq \liminf |a_n|^{\frac{1}{n}} \leq \limsup |a_n|^{\frac{1}{n}} \leq \limsup |\frac{a_{n+1}}{a_n}|$$

We recall the root test

Theorem 31.1. *Let $\sum a_n$ a series and denote $\alpha = \limsup |a_n|^{-\frac{1}{n}}$ then*

1. *If $\alpha < 1$, the series converges absolutely*
2. *If $\alpha > 1$, it diverges.*
3. *If $\alpha = 1$, the test is inconclusive, and we have no information.*

Theorem 31.2. *Let $\sum a_n$ be a series of non-zero terms, then*

1. *If $\limsup |\frac{a_{n+1}}{a_n}| < 1$ the series converges.*
2. *If $\liminf |\frac{a_{n+1}}{a_n}| > 1$ the series diverges.*
3. *If $\liminf |\frac{a_{n+1}}{a_n}| \leq 1 \leq \limsup |\frac{a_{n+1}}{a_n}|$ the test is inconclusive.*

Proof. If $\limsup |\frac{a_{n+1}}{a_n}| < 1$, by the inequality that we reminded you at the beginning of the class, we have

$$\limsup |a_n|^{\frac{1}{n}} \leq \limsup |\frac{a_{n+1}}{a_n}| < 1$$

And by the root test, the series converges.

If $\liminf |\frac{a_{n+1}}{a_n}| > 1$, then by the same inequality

$$1 < \liminf |\frac{a_{n+1}}{a_n}| \leq \limsup |a_n|^{\frac{1}{n}}$$

And by the root test, the series diverges.

The inconclusive part again will come from looking at $\sum \frac{1}{n}, \sum \frac{1}{n^2}$:

For $a_n = \frac{1}{n}$, we have that

$$|\frac{a_{n+1}}{a_n}| = \frac{n}{n+1} \to 1$$

And similarly we have that, for $b_n = \frac{1}{n^2}$

$$|\frac{a_{n+1}}{a_n}| = \frac{n^2}{(n+1)^2} \to 1$$

So, both have the same result: one converges, and one diverges. $\square$

Now we have seen some tests, we need to do some examples:

Examples 31.3. For all the following, say if the converge or diverge

1. $\sum \frac{n!}{2^n}$, recall $n! = 1 \cdot 2 \cdot \dots \cdot n$.
2. $\sum (\frac{n^4+n^2+1}{2n^4+3n^3+5})^n$.
3. $\sum \frac{n}{3^n}$.

Proof. 1. It diverges from the ratio test! We have

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{2^{n+1}}}{\frac{n!}{2^n}} = \frac{(n+1)!}{n!} \frac{2^n}{2^{n+1}} = \frac{n+1}{2}$$

So $\limsup |\frac{a_{n+1}}{a_n}| = \lim |\frac{a_{n+1}}{a_n}| = \infty > 1$.

2. It converges from the root test! We have

$$|a_n|^{\frac{1}{n}} = |\frac{n^4+n^2+1}{2n^4+3n^3+5}| = \frac{1+\frac{1}{n^2}+\frac{1}{n^4}}{2+3\frac{3}{n}+\frac{5}{n^4}}$$

by limit algebra we conclude that $\limsup |a_n|^{\frac{1}{n}} = \limsup |a_n|^{\frac{1}{n}} = \frac{1}{2} < 1$.

3. It converges from the ratio test! We have

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{3^{n+1}}}{\frac{n}{3^n}} = \frac{n+1}{n} \cdot \frac{3^n}{3^{n+1}} = (1+\frac{1}{n})\frac{1}{3} \to \frac{1}{3} < 1$$

We can also use the root test as

$$|a_n|^{\frac{1}{n}} = \frac{n^{\frac{1}{n}}}{3}$$

We now the numerator goes to 1, and we also get $\frac{1}{3} < 1$.

$\square$

31.3 Rearrangements

We will now show something that looks pretty straightforward- that if a series converges absolutely, then it doesn't matter in what order you sum the elements. For that, we need to define a rearrangement of a series

Definition 31.4. Let $\sum a_n$ be a series, a series $\sum b_k$ will be called a rearrangement of $\sum a_n$ if there exists a bijection $f : \mathbb{N} \to \mathbb{N}$ such that $\forall n \in \mathbb{N}, b_{f(n)} = a_n$.

So, a rearrangement is simply a permutation of the indices. Think about the case of rearrangement of $\{1, \ldots, n\}$! We have the following natural statement

Theorem 31.5. *If $\sum a_n$ converges absolutely to L, then any rearrangement of a_n, denote b_k, will have $\sum b_k = L$.*

Proof. We assume that $\sum a_n = L$ absolutely and let $\sum b_k$ a rearrangement. We will denote

$$S_N = \sum_{n=1}^{N} a_n \qquad\qquad T_K = \sum_{k=1}^{K} b_k$$

as the partial sums. We know that $S_n \to L$ and want to show that $T_k \to L$.

Let $\epsilon > 0$, so there is some $N_1 > 0$ such that for all $n > N$ we have

$$|S_n - L| < \frac{\epsilon}{2}$$

the series converges absolutely, so by the Cauchy criterion, we have that there is some $N_2 > 0$ such that for all $n > m > N_2$:

$$\sum_{k=m+1}^{n} a_k < \frac{\epsilon}{2}$$

Take $N > \max\{N_1, N_2\}$. We consider

$$M = \max\{f(k) \mid 1 \leq k \leq N\}$$

This exists as this is a finite list. So we know that the set $\{a_1, \ldots, a_N\}$ must be contained in $\{b_1, \ldots, b_M\}$. So for $m > M$ we have that

$$\{b_1, \ldots, b_m\} \setminus \{a_1, \ldots, a_N\} \subset \{a_n \mid n > N\}$$

That means

$$|T_m - S_n| \leq \sum_{k=N+1}^{\infty} |a_k| < \frac{\epsilon}{2}$$

So, combining everything we have for $m > M$

$$|T_m - L| \leq |T_m - S_N| + |S_N - L| < \epsilon$$

as needed. $\square$

31.4 What did we see today?

- We proved the ratio test.
- We defined rearrangements!
- And saw that if the series converges absolutely, the order doesn't matter!

32 32nd Class- Rearrangements

32.1 Review

Before we start today's lecture, I want to go over 2 in Pset 9- which I understand many had a problem with: Recall that A_n is a sequence of closed bounded sets, such that

$$\forall n \in \mathbb{N}, A_n \supset A_{n+1}$$

Denote $A = \bigcap_{n=1}^{\infty} A_n$.

1. Show that A is bounded and closed.

   Proof. This is an infinite intersection of closed sets; therefore, it is closed. Since A_1 is bounded we have some $x < y$ such that $\forall a \in A_1, x < a < y$. Since $A \subset A_1$, we get that $\forall a \in A, x < a < y$, and so A is bounded. □

2. Use Bolzano-Weierstrass to show that A is not empty.

   Proof. We choose a sequence by choosing an element $a_n \in A_n$. We note that since $A_n \subset A_{n-1} \subset \cdots \subset A_1$. We have that $\forall n, a_n \in A_1$, which is bounded in particular. Then, by Bolzano-Weierstrass, there is a subsequence, a_{n_k}, that converges to some limit, denote it by a. We claim that $a \in A$. Let $N \in \mathbb{N}$ we will show that $a \in A_N$. Since n_k is increasing there some K such that $n_K > N$, and in particular $a_{n_K} \in A_{n_K} \subset A_N$. So from $k > K$ we have that $a_{n_k} \in A_N$. So this is a sequence with a limit with elements in A_N; since A_N is closed, we get that $a \in A_N$ as the limit of elements of the set. □

Last time, we finished our discussion of series tests. And we defined a rearrangement- recall:

Definition 32.1. Let $\sum a_n$ be a series, a series $\sum b_k$ will be called a rearrangement of $\sum a_n$ if there exists a bijection $f : \mathbb{N} \to \mathbb{N}$ such that $\forall n \in \mathbb{N}, b_{f(n)} = a_n$.

Today, we will discuss some of the oddities of infinite series. Last time, we saw that if $\sum a_n$ converges absolutely, then the order of summation doesn't matter. So, it seems like it doesn't matter how we choose to sum up our series- we will always get the same result.

This is untrue, though: Consider the series $\sum(-1)^n$. We can write it in two different ways:

$$(-1+1)+(-1+1)+(-1+1)+\cdots = 0+0+0\cdots = 0$$
$$-1+(1-1)+(1-1)+(1-\cdots = -1+0+0+0\cdots = -1$$

So even something as simple as just grouping elements doesn't work when the series doesn't converge!

32.2 Rearrangements

Worse than that, even when the series converges, we have to be careful about things like commuting elements:

A couple of classes ago we also saw that $\sum \frac{(-1)^{n+1}}{n}$ converges. We know that this series only converges conditionally (i.e., $\sum |\frac{(-1)^{n+1}}{n}|$ doesn't converge). Why do we make this distinction? To see it, let's examine this sum more carefully:

$$\begin{aligned} S &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \dots \\ \frac{1}{2}S &= \quad + \frac{1}{2} \qquad - \frac{1}{4} \qquad + \frac{1}{6} - \dots \\ \implies \frac{3}{2}S &= 1 \qquad + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \dots \end{aligned}$$

We know that the order of summation usually doesn't matter with sums, and the sum $\frac{3}{2}S$ contains the same elements in S! How can that be? The answer is when a series converges conditionally- any rearrangement of the sum will result in a series that converges to a different number! In the Pset, you will prove the full result- but we will do some preparations for now.

First, we need to define the following

Definition 32.2. For a series $\sum a_n$, we define two new sequences

$$p_n = \begin{cases} a_n, & a_n \geq 0 \\ 0, & a_n < 0 \end{cases}, q_n = \begin{cases} 0, & a_n \geq 0 \\ a_n, & a_n < 0 \end{cases}$$

We will prove the following statements

Lemma 32.3. *$\sum a_n$ converges absolutely if and only if $\sum p_n, \sum q_n$ both converge.*
If $\sum a_n$ converges only conditionally, then $\sum p_n, \sum q_n$ diverges.

Proof. We will only prove the $\impliedby$ part of the first statement:

We assume that $\sum p_n, \sum q_n$ both converge. We note that $\sum |q_n| = -\sum q_n$ converges as well. So we get that

$$\sum |a_n| = \sum p_n + |q_n|$$

And so it converges as each summand converges, so in particular, $\sum a_n$ converges absolutely- as needed.

Next, if we assume that one sum converges and the other diverges. Say that $\sum p_n$ diverges (converges to ∞), and $\sum q_n = S$. Let $M > 0$, then there some N such that

$$\forall m > N, \sum_{n=1}^{m} p_n > M - S$$

In addition, since $\sum_{n=1}^m q_n$ is a decreasing sequence,we have that

$$\forall m, \sum_{n=1}^{m} q_m \geq S$$

so we get that for $m > N$ we have

$$\sum_{n=1}^{m} a_n = \sum_{n=1}^{m} p_n + \sum_{n=1}^{m} q_n > M - S + S = M$$

So $\sum a_m$ diverges. A similar argument gives that if $\sum q_n$ diverges, we have that $\sum a_n$ diverges to $-\infty$.

This tells us that if it is not true that $\sum p_n \sum q_n$ diverges, then we have that $\sum a_n$ doesn't converge only conditionally. So the contrapositive tells us that if $\sum a_n$ converges only conditionally, then $\sum p_n, \sum q_n$ diverges. $\square$

In the Pset, you will complete the proof and also see that if $\sum a_n$ converges only conditionally, we can rearrange it to converge to any number! You can also arrange so it will converge to $\pm\infty$, but we will not show that.

In order to better help you understand the idea of the proof, we will go over the proof you will build in the pset:

1. Prove that if $\sum a_n$ converges absolutely then $\sum p_n$ and $\sum q_n$ converge. Thus completing the proof of the lemma from class.

2. In this part, you will show that if $\sum a_n$ converges conditionally, then for any $M \in \mathbb{R}$, there is some rearrangement of a_n, b_k, such that $\sum b_k = M$:

   (a) Show that you can find some N_1 such that $\sum_{n=1}^{N_1} p_n > M$

   (b) Show that there is some minimal M_1 such that $\sum_{n=1}^{N_1} p_n + \sum_{n=1}^{M_1} q_n < M$.

   (c) Assume that you have found N_k, M_k such that $\sum_{n=1}^{N_k} p_n + \sum_{n=1}^{M_k} q_n < M$, show that there is some minimal N_{k+1}, M_{k+1} such that

   $$\sum_{n=1}^{N_{k+1}} p_n + \sum_{n=1}^{M_k} q_n > M > \sum_{n=1}^{N_{k+1}} p_n + \sum_{n=1}^{M_{k+1}} q_n$$

   (d) Define b_k according to the above, and conclude that

   $$0 \leq \sum_{n=1}^{N_{k+1}+M_k} b_n - M \leq p_{N_{k+1}}$$

   conclude that if we denote $T_n = \sum_{n=1}^{n} b_k - M$ we have that

   $$T_{N_{k+1}+M_k} \xrightarrow{k\to\infty} 0$$

(e) Similarly shows that

$$q_{M_{k+1}} \leq \sum_{n=1}^{N_{k+1}+M_k} b_k - M \leq 0$$

conclude that

$$T_{N_{k+1}-1+M_{k+1}} \xrightarrow{k\to\infty} 0$$

(f) Show that $\sum b_k = M$. HINT: Use the monotonicity of T_n

32.3 What did we see today?

- We saw that if a series converges absolutely, we can sum it up however we want.
- We saw that if a series converges only conditionally, we should be careful about changing the order of summation!
- We skimmed the proof for the fact that if a series converges conditionally, we can rearrange it to converge to anything!

33 33rd Class- Functional limits

33.1 Review

A couple of comments before we begin:

- The next Pset, number 12, is due, according to the schedule, on Wednesday the 22th. I made it relatively short, so I think it will be fair to keep the submission date as is. It will be released on Sunday.
- The Pset after that, number 13, will be due at the end of class- that is, Dec. 1st. This one will be of regular length- but I will try to include additional questions to practice for the final (these will be graded).
- I remind you, the final will be a take-home exam. Between the 6th and the 9th of December, you will find 3 hours to do the final. Unlike what is written in the syllabus, you can consult your notes. The exam date that is on the course search is not relevant.

Also, before we begin, I want to go over c in Pset 9, which I understand was also a source of difficulties: Recall that $C_0 = [0,1]$, and we define C_n be removing the middle third of C_{n-1}. We define $C = \bigcap\limits_{n=1}^{\infty} C_n$ - this set is known as the Cantor set. We define $f : C \to 2^{\mathbb{Z}}$ in the following way: For $x \in C$, we denote $f(x) = (a_n)$ defined:

$$a_n = \begin{cases} 1, & x \in \text{ the left third of the interval chosen of } C_{n-1} \\ 0, & x \in \text{ the right third of the interval chosen of } C_{n-1} \end{cases}$$

We want to show that f is a bijection.

Proof. First, we note that f is well-defined. If $x \in C$, then for each n, we have that either to the right third or the left third, so the sequence is well defined.

Let $x, y \in C$ and assume that $f(x) = f(y) = (a_n)$. We define the sequence of nested intervals that correspond to a_n. If $a_n = 1$, we chose the left third; if $a_n = 0$, we chose the right third. We note that $I_n \supset I_{n+1}$, we note that $\ell_n = \frac{1}{3^n}$, where ℓ_n is the length of the intervals. So, we have a sequence of nested intervals with length going to 0, so their intersection contains a single point. Since $x, y \in \bigcap_{n \in \mathbb{N}} I_n$, we get that $x = y$- so f is injective.

Surjectivity is achieved similarly: if we have some sequence (a_n), we can define

$$I_n = \begin{cases} \text{the left third of } I_{n-1}, a_n = 1 \\ \text{the right third of } I_{n-1}, a_n = 0 \end{cases}$$

Again, this is a sequence of nested intervals with length going to 0, containing a single point - x, then $x \in C$, and $f(x) = (a_n)$. □

Last time, we finished our discussion series. Today, we will start discussing our final chapter of this course- functions! First, let's recall what a function is precisely

Definition 33.1. Given two sets A, B a function $f : A \to B$ (read: f from A to B) is a rule, or mapping, that for each $x \in A$ assigns a *single* element in B. We denote that element $f(x)$. A is called the domain of f, and we also define the *range* of f by

$$\operatorname{Ran}(f) = \{y \in B \mid \exists x \in A, f(x) = y\}$$

33.2 Function limit

The first notion we will want to define is the idea of a limit of functions. The motivation for this is, of course, the notion of continuity. Continuous function - is a function that has no "gaps" somehow. Similar to the idea that $\mathbb{R}$ is complete, it has no gaps. So we will want to say that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at c if we have that the limit of the function is equal to the value at the point:

$$\lim_{x \to c} f(x) = f(c)$$

But how can we define it? We only know how to define a limit for sequences!

. One way to define it is to say the following:

We say that $\lim\limits_{x \to c} f(x) = L$, if $\lim\limits_{n \to \infty} f(c - \frac{1}{n}) = L$. But does it capture what we want from a continuous function? Discuss!

There are several problems here: First, it is one-sided- but that is relatively mild. But what happens if the values on this sequence converge, but other sequences don't converge to the same limit?

To illustrate this, consider the Dirichlet function

$$g(x) = \begin{cases} 1, x \in \mathbb{Q} \\ 0, x \in \mathbb{I} \end{cases}$$

This is truly pathological! If we consider any sequence of rational points x_n approaching $c = \frac{1}{2}$ we will get that $\lim g(x_n) = 0$, and if we chose a sequence of irrational points y_n approaching $\frac{1}{2}$, we will have $\lim g(y_n) = 1$!

What can we conclude from this exercise? If we want to ensure it has the same meaning as we think it should have, the limit of a function should not depend on how we approach the point! This is similar to the subsequential limits! The sequence has a limit if and only if any subsequence converges to the same number- here, in analogy, any sequence from the line approaching the point should converge to the same number! So we get

Definition 33.2 (Sequential limit)**.** Let $f : A \to \mathbb{R}$, $A \subset \mathbb{R}$, we say that $\lim_{x\to x_0} f(x) = L$ for x_0 limit point of $Dom(f)$, if for every sequence (x_n) such that $\forall n \in \mathbb{N}, x_n \in Dom(f)$, and $x_n \to x_0$ we have

$$\lim_{n\to\infty} f(x_n) = L$$

What does this definition remind you of? It is closely related to the idea of limit points! We can define continuity in the language of ϵ-neighborhoods:

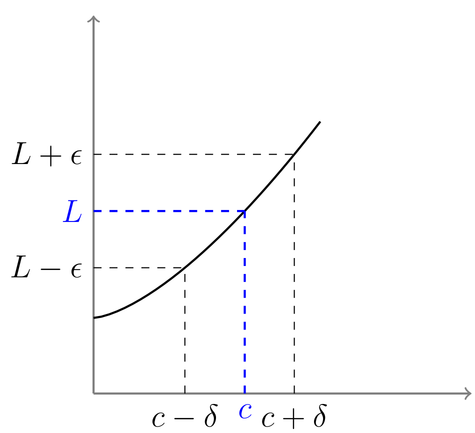
Definition 33.3 (Topological limit)**.** We say that the limit of $f : A \to \mathbb{R}$ at a point x_0 -limit point of $Dom(f)$, is L, if

$$\forall\epsilon > 0\exists\delta > 0 \text{ s.t. } x \in V_\delta(x_0) \implies f(x) \in V_\epsilon(L)$$

which can also be written as

$$\forall\epsilon > 0, \exists\delta > 0, \text{ s.t. } |x - x_0| < \delta \implies |f(x) - L| < \epsilon$$

Let's draw this idea:



So, we need to show that these two definitions are indeed the same

Theorem 33.4. *We have that* $\lim\limits_{x\to x_0} f(x) = L$ *in the topological sense if and only if* $\lim\limits_{x\to x_0} f(x) = L$ *in the sequential sense.*

Proof. $\underline{\implies}$ Assume that $\lim\limits_{x\to x_0} f(x) = L$ is topological sense, and consider (x_n) such that $x_n \in A$ and $x_n \to x_0$. Let $\epsilon > 0$, then we have some $\delta > 0$, such that

$$|x - x_0| < \delta \implies |f(x) - L| < \epsilon$$

Since $x_n \to x_0$, we have some $N \in \mathbb{N}$ such that $\forall n > N$ we have

$$|x_n - x_0| < \delta$$

and so we have that for all $n > N$

$$|f(x_n) - L| < \epsilon$$

and so $f(x_n) \to L$- as needed.

$\underline{\impliedby}$ Assume that $\lim\limits_{x\to x_0} f(x) = L$ in the sequential sense. Assume by way of contradiction that it is not the topological limit. So we have that

$$\exists \epsilon > 0, \forall \delta > 0, \text{ s.t. } -(|x - x_0| < \delta \implies |f(x) - L| < \epsilon)$$

in particular, let ϵ be from the above, we can choose $\delta = \frac{1}{n}$, so we have that

$$|x - x_0| < \frac{1}{n} \implies |f(x) - L| < \epsilon$$

is false for all $n \in \mathbb{N}$. For each n there is some x_n such that

$$|x_n - x_0| < \frac{1}{n} \wedge |f(x_n) - L| > \epsilon$$

So we have that $x_n \to x_0$, but $f(x_n)$ doesn't converge to L in contradiction to the assumption! □

33.3 What did we see today?

- We started talking about functional limits.
- We gave two definitions of functional limits and saw they are the same!

34 34th Class- Algebra of functional limits

34.1 Review

Last time, we started talking about functional limits! We saw two different definitions: topological and sequential. Let's recall them:

Definition 34.1 (Sequential limit)**.** Let $f : A \to \mathbb{R}$, $A \subset \mathbb{R}$, we say that $\lim_{x\to x_0} f(x) = L$ for x_0 limit point of $Dom(f)$, if for every sequence (x_n) such that $\forall n \in \mathbb{N}, x_n \in Dom(f)$, and $x_n \to x_0$ we have

$$\lim_{n\to\infty} f(x_n) = L$$

Definition 34.2 (Topological limit)**.** We say that the limit of $f : A \to \mathbb{R}$ at a point x_0 -limit point of $Dom(f)$, is L, if

$$\forall \epsilon > 0 \exists \delta > 0 \text{ s.t. } x \in V_\delta(x_0) \implies f(x) \in V_\epsilon(L)$$

which can also be written as

$$\forall \epsilon > 0, \exists \delta > 0, \text{ s.t. } |x - x_0| < \delta \implies |f(x) - L| < \epsilon$$

And we saw that they were equivalent.

34.2 Examples

Let's do some examples to warm up:

Examples 34.3.
- Show that for $f(x) = 2x^2 + 1$ we have that $\lim_{x\to x_0} f(x) = f(x_0)$ in the sequential and the topological sense.
- Show that for $f(x) = \sqrt{x}$ we have that $\lim_{x\to a} f(x) = f(a)$ in the topological sense, for $a \geq 0$.

Proof.
- Sequential: Here it is quite easy: let $x_n \to x_0$, then we have, by algebra of limits

$$\lim f(x_n) = \lim 2x_n^2 + 1 = 2 \lim x_n^2 + \lim 1 = 2[\lim x_n]^2 + 1 = 2x_0 + 1 = f(x_0)$$

as needed.

Topological:Let $x_0 \in \mathbb{R}$, and $\epsilon > 0$, Then we note that

$$|f(x) - f(x_0)| = |2x^2 + 1 - (2x_0^2 + 1)| = |2(x^2 - x_0^2)| = 2|x - x_0||x + x_0|$$

We note that if we assume that $|x - x_0| < \delta$ we will have

$$|x + x_0| = |x - x_0 + 2x_0| < \delta + 2|x_0|$$

So we choose $\delta > 0$ such that

$$\delta < \min\{1, \frac{\epsilon}{2(1 + 2|x_0|})\}$$

we will get that

$$|f(x) - f(x_0)| = 2|x - x_0||x + x_0| < 2\delta(\delta + 2|x_0|) \leq 2(1 + 2|x_0|)\delta < \epsilon$$

as needed.

- Say that $a \neq 0$ then we note that we have, for $x > 0$

$$|\sqrt{x} - \sqrt{a}| = |\sqrt{x} - \sqrt{a}| \frac{|\sqrt{x} + \sqrt{a}|}{|\sqrt{x} + \sqrt{a}|} = \frac{|x - a|}{\sqrt{x} + \sqrt{a}}$$

So we get that if $\epsilon > 0$, we let $\delta = \min\{\sqrt{a}\epsilon, |a|\}$, then we have that if

$$|x - a| < \delta$$

Then $x > 0$ and we have that

$$|f(x) - f(a)| = |\sqrt{x} - \sqrt{a}| = \frac{|x-a|}{\sqrt{x}+\sqrt{a}} \leq \frac{\delta}{\sqrt{a}} < \epsilon$$

as needed.

If $a = 0$, we note that we can only take 0 as a limit point of $Dom(f)$- or we can only take limits of $x \geq 0$. And then we have that if $|x| < \epsilon^2$

$$|f(x) - 0| = |\sqrt{x}| < \epsilon$$

as needed.

□

Example 34.4. Define $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, by $f(x) = x^2 \sin(\frac{1}{x})$, show that $\lim_{x\to 0} f(x) = 0$- according to both definitions.

Proof. Sequential: Let $x_n \to 0$, and let $\epsilon > 0$ then there is some $N \in \mathbb{N}$ such that

$$\forall n > N, |x_n| < \sqrt{\epsilon}$$

Then we have that

$$|f(x_n)| = |x_n^2 \sin(\frac{1}{x_n})| \leq |x_n|^2 < \epsilon$$

as needed.

Topological: Let $\epsilon > 0$, choose $\delta = \sqrt{\epsilon}$, then we have that if $|x - 0| < \delta = \sqrt{\epsilon}$ then we have that

$$|f(x) - 0| = |x^2 \sin(\frac{1}{x})| \leq |x|^2 < \epsilon$$

as needed. □

From here on out, we will not distinguish between the definitions except for one question in the Pset.

34.3 Algebra of limits

The fact the two definitions are the same gives us the following corollary

Theorem 34.5 (Algebra of limits)**.** *Let f, g be function defined on the same domain $A \subset \mathbb{R}$, assume $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = \ell$, then*

1. $\forall k \in \mathbb{R}, \lim_{x\to x_0} kf(x) = kL.$

2. $\lim_{x \to x_0} f(x) + g(x) = L + \ell$.

3. $\lim_{x \to x_0} f(x)g(x) = L\ell$.

4. $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{L}{\ell}$ *, if* $\ell \neq 0$

Proof. This is immediate from the definition of sequential limits and the algebra of limits for sequences. For example, to prove the last item. Let (x_n) be a sequence in A such that $x_n \to x_0$. Since $\lim g(x_n) \to \ell$, and $\ell \neq 0$, there is some some $N > 0$ such that $\forall n > N x_n \neq 0$, so we can assume that $\forall n, x_n \neq 0$. Then, by the algebra of limits, we have that

$$\lim \frac{f(x_n)}{g(x_n)} = \frac{\lim f(x_n)}{\lim g(x_n)} = \frac{L}{\ell}$$

as needed. □

Another corollary from these definitions is

Corollary 34.6. *Let* $f : A \to \mathbb{R}$*, and* x_0 *limit point of* A*, if there are* $(x_n), (y_n)$ *sequences in* A *with* $\forall n, x_n \neq x_0 \neq y_n$*, such that*

$$\lim x_n = \lim y_n = x_0, \lim f(x_n) \neq \lim f(y_n)$$

Then the limit $\lim_{x \to x_0} f(x)$ *doesn't exists.*

Let's do an example:

Example 34.7. Consider $f(x) = \sin(\frac{1}{x})$, then $\lim_{x \to 0} f(x)$ doesn't exists.

Proof. We can consider the sequence $x_n = \frac{1}{2\pi n}$. We have that $\lim x_n = 0$, and we have that

$$\lim f(x_n) = \lim \sin(\frac{1}{\frac{1}{2\pi n}}) = \lim \sin(2\pi n) = \lim 0 = 0$$

On the other hand, we can consider $y_n = \frac{1}{2\pi n + \frac{\pi}{2}}$, which also have $\lim y_n = 0$, but

$$\lim f(y_n) = \lim \sin(\frac{1}{\frac{1}{2\pi n + \frac{\pi}{2}}}) = \lim \sin(2\pi n + \frac{\pi}{2}) = \lim 1 = 1$$

And we conclude that $\lim_{x \to 0} f(x)$ doesn't exist. □

34.4 What did we see today?

- We saw the algebra of functional limits.
- We defined continuous functions and saw their algebra.

35 35th Class- Continuous functions

35.1 Review

Last time, we talked about the algebra of function limits and gave some examples. Today, we are going to start talking about continuous functions!

35.2 Continuous functions

We all know the definition of a continuous function as a function that we don't need to lift the pen of the paper to draw it. But we want t a more precise notion. So, we define the notion of a continuous function

Definition 35.1. We say that a function $f : A \to \mathbb{R}$ is continuous at $x_0 \in A$ if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

We say that f is continuous at $S \subset A$ if f is continuous at every $x \in S$.

An immediate implication of the algebra of limits gives us

Theorem 35.2 (Algebra of continuous functions)**.** *Let f, g be function defined on the same domain $A \subset \mathbb{R}$, assume both are continuous at $x_0 \in A$, then*

1. *$\forall k \in \mathbb{R}$ the function kf defined as $[kf](x) = kf(x)$, is continuous at x_0*
2. *The function $f + g$ defined as $[f + g](x) = f(x) + g(x)$ is continuous at x_0.*
3. *The function fg defined as $[fg](x) = f(x)g(x)$ is continuous at x_0.*
4. *The function $\frac{f}{g}$ defined as $[\frac{f}{g}](x) = \frac{f(x)}{g(x)}$ is continuous at x_0, given that $g(x_0) \neq 0$.*

Furthermore, we have the following theorem:

Theorem 35.3. *If f is continuous at x_0 and g is continuous at $f(x_0)$ then $g \circ f$, defined as $g(f(x))$, is continuous at x_0.*

Proof. Let x_n be a sequence in $Dom(f)$, such that $x_n \to x_0$. Since f is continuous at x_0 then $\lim f(x_n) = f(x_0)$. Denote $y_n = f(x_n), y_0 = f(x_0)$, then $\lim y_n = y_0$. Since g is continuous at y_0 we have that $\lim g(y_n) = g(y_0)$, in other words we have

$$\lim g(f(x_n)) = g(f(x_0))$$

as needed. □

In the pset, you will explore this further.

35.3 Extreme value theorem

We start with a first glance into the connection between continuity and compactness:

Theorem 35.4. *Let $f : A \to \mathbb{R}$ be continuous, then if $K \subset A$ is compact, then $f(K)$ is compact.*

Proof. For this, we will use the definition of sequential compactness that you proved in Pset 10. So let (y_n) be a sequence in $f(K)$. We need to find a subsequence (y_{n_k}) that converges to something in $f(K)$.

For every $n \in \mathbb{N}$ we have that $y_n \in f(K)$, and so there is some $x_n \in K$ such that $f(x_n) = y_n$. Since K is compact, we have a subsequence x_{n_k} that converges to a limit x_0 in K. So we have that $x_{n_k} \to x_0$, since f is continuous we get that $\lim f(x_{n_k}) = f(x_0)$, and since $x_0 \in K$ we have that $f(x_0) \in f(K)$. So $y_{n_k} = f(x_{n_k})$ is a subsequence of y_n in $f(K)$ and $y_{n_k} \to f(x_0)$, as needed. $\square$

So, continuous function preserves compact sets!

So now we give the following natural definition

Definition 35.5. A function f is bounded if the set $\{f(x) \mid x \in Dom(f)\}$ is a bounded set.

As a corollary, we get the following well-known theorem:

Theorem 35.6 (Extreme value theorem)**.** *If $f : K \to \mathbb{R}$ is continuous on a compact set K, then f attains a maximum and minimum value. That is, there are $x_0, x_1 \in K$ such that $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in K$.*

Proof. By the previous part, $f(K)$ is compact, So in particular bounded, so $\sup f(K) \in \mathbb{R}$. Moreover, $f(K)$ is closed, and so $\sup f(K) \in f(K)$. So there are some $M \in f(K)$ such that

$$\forall x \in K, f(x) \leq M$$

Since $M \in f(K)$ there is some $x_1 \in K$ such that $f(x_1) = M$. Similarly, we can conclude for the minimum. $\square$

35.4 Intermediate Value Theorem

Continuous functions preserve other properties as well! For example, connectedness! This idea has a lot to do with "not lifting the pen from the paper"- the property with which we started the discussion of continuous functions with. So we define:

Definition 35.7. We say that a set $E \subset \mathbb{R}$ is connected if whenever $a < c < b$ with $a, b \in E$ then $c \in E$.

This definition makes a lot of sense- a set is connected if it contains any interval between any two points in it. We note that on $\mathbb{R}$, only intervals (possibly unbounded) or a single point are connected.

This gives a new way to phrase the intermediate value theorem:

Theorem 35.8 (Intermediate Value Theorem). *Continuous functions preserve connected sets!*

In other words, for $f : I \to \mathbb{R}$, *- I interval, and* $a, b \in I$ *such that* $a < b$, *for all* $\min(f(a), f(b)) < y < \max(f(a), f(b))$ *there is some* $x \in (a, b)$ *such that* $f(x) = y$

Next time, we will see the proof and some applications of this theorem

35.5 What did we see today?

- We saw the extreme value theorem.
- We saw the intermediate value theorem

36 36th Class- Applications of IVT

36.1 Review

Last time, We talked about the intermediate value theorem. But we still need to prove it! We will also see some more uses of it, including an algorithm based on this theorem!

36.2 IVT

First, we recall the definition of a connected set:

Definition 36.1. We say that a set $E \subset \mathbb{R}$ is connected if whenever $a < c < b$ with $a, b \in E$ then $c \in E$.

Recall the statement of the Intermediate Value Theorem

Theorem 36.2 (Intermediate Value Theorem). *Continuous functions preserve connected sets!*

In other words, for $f : I \to \mathbb{R}$ *continuous, and* I *interval, and* $a, b \in I$ *such that* $a < b$, *for all* $\min(f(a), f(b)) < y < \max(f(a), f(b))$ *there is some* $x \in (a, b)$ *such that* $f(x) = y$

Proof. We start by showing that two statements are equivalent:

First, if f preserves connectedness and $Dom(f) = I$ an interval, then $f(I)$ is connected, and in particular, if $a < b$ such that $a, b \in I$, then $f(a), f(b) \in f(I)$, then any point between $f(a), f(b)$ belongs to $f(I)$-and we get the second statement.

Second, if the second statement is true, we want to show that $f(I)$ is connected. If $f(I)$ is a single point, we are done- as a single point is connected. Otherwise, we have at least 2 points in $f(I)$, so let $a, b \in f(I)$ be such that $a < c < b$. Since $a, b \in f(I)$ we have some $x_0, x_1 \in I$ such that $f(x_0) = a, f(x_1) = b$. So we can apply IVT to the interval $[\min(x_0.x_1), \max(x_0, x_1)]$, to get that there is some $x \in I$ such that $f(x) = c$.

So we have that $c \in f(I)$, and we conclude that $f(I)$ is connected. Now, we prove the second statement. WLOG, we assume that $f(a) < y < f(b)$. We define

$$S = \{x \in [a, b] \mid f(x) < y\}$$

We have that $a \in S$, and since S is bounded by b (from definition), it has a finite supremum. Denote $x_0 = \sup S$, since $[a, b]$ is closed $x_0 \in [a, b]$. From the property of sup, we have that

$$\forall n \in \mathbb{N}, \exists x_n \in S, x_0 - \frac{1}{n} < x_n < x_0$$

So we have $x_n \to x_0$, and $f(x_n) < y$ so we get that

$$f(x_0) = \lim f(x_n) \leq y$$

Consider $y_n = \min\{b, x_0 + \frac{1}{n}\}$, then we have that $\lim y_n = x_0$ (as $x_0 \leq y_n \leq x_0 + \frac{1}{n}$). Since $y_n > x_0$ we have that $y_n \notin S$, and so $f(y_n) \geq y$. In particular

$$f(x_0) = \lim f(y_n) \geq y$$

So we conclude $f(x_0) = y$- as needed.

□

This is very nice, but why is it useful? Let's do some examples

Example 36.3. Let $f : [0, 1] \to [0, 1]$ be continuous, then f has a fixed point! That is $\exists x_0 \in [0, 1]$ such that $f(x_0) = x_0$!

Fixed points are very useful when trying to analyze properties of unknown functions.

Proof. A fixed point is the same as saying that the graph $(x, f(x))$ intersects (x, x). so we consider $g(x) = f(x) - x$. We note that g is also continuous (by the algebra of continuous functions) on $[0, 1]$. We have that

$$g(0) = f(0) - 0 = f(0) \geq 0$$
$$g(1) = f(1) - 1 \leq 0$$

By IVT we have some $x_0 \in [0, 1]$ such that $g(x_0) = 0$ which implies $f(x_0) = x_0$. □

Example 36.4. Let $y > 0$ and $m \in \mathbb{N}$, then $\sqrt[m]{y}$ exists and is positive.

Proof. We consider $f(x) = x^m$ - which is continuous (you will show this in the Pset). Denote $b = \max\{1, y\}$; we note that then we have that $y \leq b^m$. So we have that

$$0 = 0^m = f(0) < y \leq b^m = f(b)$$

If $y = b^m$, then $\sqrt[m]{y} = b$ (and then $y = 1$). Otherwise, the last inequality is strict, and by IVT, we have some $x \in (0, b)$ such that $f(x) = y$- as needed. □

Does the converse of IVT hold? No!

Example 36.5. Consider $f(x) = \begin{cases} \sin(\frac{1}{x}), x \neq 0 \\ 0, x = 0 \end{cases}$. Then $f(x)$ is not continuous and has the intermediate value property on $[0, 1]$

Proof. By the example of last time, f is not continuous (as $\lim\limits_{x \to 0} f(x)$ doesn't exist). But it is easy to see it has the intermediate value property on $[0, 1]$ □

36.3 What did we see today?

- We saw some application of the intermediate value theorem.

37 37th Class- Bisection and Uniform continuity

37.1 Review

Before we begin, let's say a few more words about the final:

- There will be true or false questions, which we haven't got a lot of practice in, so I wrote some for you in additional problems. It is very similar to question 1 in Pset 13- only in the pset you need to justify. You don't need to justify anything in these questions- write true or false.
- There will be questions proving things from definition- so note that you can't rely on things that were proven later for these questions.
- Some of the questions will be very closely related to the HW, so I suggest you go over the solutions of the Psets.
- The additional problems I promised will be posted tonight.
- Please send me questions to review in our last class!

Last time, we saw some application of the intermediate value theorem. Today, we will talk about an algorithm based on this theorem and start talking about uniform continuity.

37.2 Bisection

Now, we will introduce an algorithm to locate a root of a function! This is called the bisection method. In fact, we already used the bisection method- in the proof of Heine Borel! But now we will prove it in this setting:

Theorem 37.1. *Let $f : [a, b] \to \mathbb{R}$ a continuous function, such that $f(a)f(b) \leq 0$. Then there is a point $c \in [a, b]$, and an algorithm that after N steps produces a sequences $(c_n)_{n=1}^{N} \in [a, b]$ such that*

$$f(c) = 0$$
$$\forall 1 \leq n \leq N, |c - c_n| < \frac{b-a}{2^n}$$

in other words, to get within $\epsilon > 0$ error of the root, we need $\log_2(\frac{b-a}{\epsilon})$ steps.

What do we mean by an algorithm? In this context, we mean a computable sequence - that is, it requires us to evaluate the function at certain points.

Proof. If $f(a) = 0$ or $f(b) = 0$, we take c as that point and $c_n = c$ as the sequence.

If $f(a)f(b) < 0$, then WLOG assume $f(a) < 0 < f(b)$. Then we define $p_0 = a, q_0 = b$.

By IVT, we have some $a < c < b$ such that $f(c) = 0$. We define $c_1 = \frac{b+a}{2}$, and we have

$$|c - c_1| < q_0 - c_1 = b - \frac{b+a}{2} = \frac{b-a}{2}$$

Then we define recursively: assume that we have $p_k < q_k$, such that $f(p_k) < 0 < f(q_k)$ and we have that

$$|q_k - p_k| = \frac{b-a}{2^k}$$

then

- Define $c_{k+1} = \frac{p_k+q_k}{2}$.
- Compute $f(c_{k+1})$,
- If $f(c_{k+1}) \leq 0$, define $p_{k+1} = c_{k+1}, q_{k+1} = q_k$. If $f(c_{k+1}) > 0$, then we define $p_{k+1} = p_k, q_{k+1} = c_{k+1}$.
- In all cases, we have that

$$|c_{k+1} - p_k| = \frac{1}{2}|q_k - p_k| = \frac{b-a}{2^{k+1}}$$
$$|q_k - c_{k+1}| = \frac{1}{2}|q_k - p_k| = \frac{b-a}{2^{k+1}}$$

  In particular, by IVT, this implies that there is some $p_{k+1} \leq c < q_{k+1}$ such that $f(c) = 0$. So we conclude

$$|c - c_{k+1}| < \frac{b-a}{2^{k+1}}$$

And so we get that desired sequence. It is easy to see that $\lim p_n = \lim q_n = \lim c_n = c$- and we have all the required properties. □

This is a swift way to find a root of a function! There are some disadvantages- this only allows us to find a single root of a function - we can think about what happens when there are multiple roots- but we will not deal with this in detail.

Let's give an example.

Example 37.2. We will want to compute the value of $\sqrt{2}$ with one digit. So our error is $\epsilon = \frac{1}{10}$, so we will need at least

$$N > \log_2(b-a) - \log_2 \frac{1}{10} = \log_2 10 + \log_2(b-a)$$

We recall that $2^3 = 8, 2^4 = 16$, so $\log_2 10$ is slightly bigger than 3, so we will take $\log_2(10) \approx 4$.

We consider the function $f(x) = x^2 - 2$. In other words, we want to find the numerical value of $\sqrt{2}$. So we now that $f(1) = 1^2 - 2 = -1 < 0$, and $f(2) = 2^2 - 2 = 4 - 2 = 2 > 0$. So our interval is $[1, 2]$.

So we have $b - a = 1$, so we will need about 4 step to get what we want!

So we start computing:

- $c_1 = \frac{3}{2}$, and we have $f(\frac{3}{2}) = \frac{9}{4} - 2 = \frac{1}{4} > 0$. So our new interval is $[1, \frac{3}{2}]$.
- $c_2 = \frac{5}{4}$, and we have that $f(c_2) = \frac{25}{16} - 2 < 0$. So our new interval is $[\frac{5}{4}, \frac{3}{2}]$.
- $c_3 = \frac{11}{8}$, and we have that $f(c_3) = \frac{121}{64} - 2 < 0$. So our new interval is $[\frac{11}{8}, \frac{3}{2}]$.
- $c_4 = \frac{23}{16} = 1.4375$, and we have that $\sqrt{2} \approx 1.414$, as needed!

37.3 Uniform continuity

As you recall, one of the emphases we had at the beginning of the course was the order of quantifiers. So, let's consider the definition of continuity on the entire domain $Dom(f)$:

$$\forall x_0 \in Dom(f), \forall \epsilon > 0 \exists \delta > 0, \forall x \in Dom(f), |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$$

That tells me that for every point x_0, and every ϵ I have some δ, so if I am δ-close to x_0, $f(x)$ have to be ϵ close to $f(x_0)$.

What happens if we change the order of quantifiers?

$$\forall \epsilon > 0, \exists \delta > 0. \forall x_0, x \in Dom(f), |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$$

What is the difference? TPS

Definition 37.3. We say that a function $f : B \to \mathbb{R}$ is uniformly continuous on $A \subset B$ if

$$\forall \epsilon > 0, \exists \delta > 0. \forall x_0, x \in A, |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$$

It tells us that throughout the domain (or some subset), we can choose δ uniformly! That is, in a way that does not depend on the point we check!

Let's give some examples!

Example 37.4. Consider $f(x) = \frac{1}{x^2}$, and show that it is uniformly continuous on $[a, \infty)$, for $a > 0$

Proof. First, we will want to show it is continuous on $(0, \infty)$ and then examine the proof carefully to see why we can say that it is uniformly continuous on $[a, \infty)$.

Let $x_0 > 0, \epsilon > 0$, then we note that

$$f(x) - f(x_0) = \frac{1}{x^2} - \frac{1}{x_0^2} = \frac{x_0^2 - x^2}{x_0^2 x^2} = \frac{(x - x_0)(x + x_0)}{x_0^2 x^2}$$

We note that

$$|x + x_0| \leq |x - x_0| + 2|x_0|$$

And we have that if $|x - x_0| < \delta' |x_0|$

$$|x| \geq ||x - x_0| - |x_0|| = (1 - \delta')|x_0|$$

So we get

$$|f(x) - f(x_0)| = |\frac{(x - x_0)(x + x_0)}{x_0^2 x^2}| \leq \frac{\delta'(\delta' + 2)}{x_0^2(1 - \delta')^2}$$

So we can choose some δ' such that

$$\frac{(\delta')^2 + 2\delta'}{(\delta')^2 - 2\delta' + 1} < \epsilon x_0^2$$

(since when $\delta' \to 0$,we have that this function goes to 0)

So, it is continuous.

But throughout the process, we used x_0 quite a lot. How do we rectify that?

Well, in the uniform continuity, we consider $x, y \geq a$ for some $a > 0$. So we can use to bond the problematic term:

$$\frac{x + y}{x^2 y^2} = \frac{1}{xy^2} + \frac{1}{x^2 y} < \frac{1}{a^3} + \frac{1}{a^3}$$

So we can write

$$|f(x) - f(y)| = |\frac{(x - y)(x + y)}{y^2 x^2}| < |x - y|\frac{2}{a^3}$$

So by choosing $|x - y| < \frac{\epsilon a^3}{2}$, we get the desired result . □

Remark 37.5. Note that we have a very explicit dependence on a. Even worse, as $a \to 0$, we get that our bound is getting worse. And it is completely useless at $a = 0$.

37.4 What did we see today?

- We saw the bisection method
- We started talking about uniform continuity.
- We started talking about examples.

38 38th Class- Uniform continuity -continued

38.1 Review

Last time, we defined uniform continuity and were in the middle of the examples.

Recall that

Definition 38.1. We say that a function $f : B \to \mathbb{R}$ is uniformly continuous on $A \subset B$ if

$$\forall \epsilon > 0, \exists \delta > 0. \forall x_0, x \in A, |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$$

And we started the example:

Example 38.2. Consider $f(x) = \frac{1}{x^2}$, and show that it is uniformly continuous on $[a, \infty)$, for $a > 0$

Proof. First, we will want to show it is continuous on $(0, \infty)$ and then scrutinize the proof to see why we can say that it is uniformly continuous on $[a, \infty)$.

Let $x_0 > 0, \epsilon > 0$, then we note that

$$f(x) - f(x_0) = \frac{1}{x^2} - \frac{1}{x_0^2} = \frac{x_0^2 - x^2}{x_0^2 x^2} = \frac{(x - x_0)(x + x_0)}{x_0^2 x^2}$$

Last time, we bounded

$$\frac{(x + x_0)}{x^2} \leq \frac{\delta + 2}{(1 - \delta)^2}$$

but that required us to take $|x - x_0| < \delta |x_0|$ - which doesn't help with uniform continuity. So we need a different bound for uniform continuity! In the uniform continuity, we consider $x, y \geq a$ for some $a > 0$. So we can use to bond the problematic term:

$$\frac{x + y}{x^2 y^2} = \frac{1}{xy^2} + \frac{1}{x^2 y} < \frac{1}{a^3} + \frac{1}{a^3}$$

So we can write

$$|f(x) - f(y)| = |\frac{(x - y)(x + y)}{y^2 x^2}| < |x - y| \frac{2}{a^3}$$

So by choosing $|x - y| < \frac{\epsilon a^3}{2}$, we get the desired result . □

Remark 38.3. Note that we have a very explicit dependence on a. Even worse, as $a \to 0$, we get that our bound is getting worse. And it is utterly useless at $a = 0$.

This is not a problem with our proof- this is actually true in general!

Example 38.4. $f(x) = \frac{1}{x^2}$ is not uniformly continuous on $(0, \infty)$

Proof. We will show it quite directly: Consider $y = x + \delta$ (we can also take $\frac{\delta}{2}$, but this is slightly easier). then we have

$$f(x + \delta) - f(x) = \frac{(x + \delta - x)(x + \delta + x)}{x^2 (x + \delta)^2} = \frac{\delta(2x + \delta)}{x^2 (x + \delta)^2}$$

So again, to make our life easier, we can take $x = \delta$

$$f(2\delta) - f(\delta) = \frac{\delta^2 3}{\delta^4 4} = \frac{3}{4} \frac{1}{\delta^2}$$

So if $\delta < \frac{1}{2}$, we get that

$$f(2\delta) - f(\delta) = \frac{3}{4} \frac{1}{\delta^2} > \frac{3}{4} \frac{1}{\frac{1}{4}} = 3$$

This, of course, can't be small! Note that we didn't choose a specific point- we showed that if you give me a small δ, I can find 2 points such they are δ close, but their image is 3 apart in particular not as close as you want.

In other words, we showed there is an $\epsilon > 0$, such that for all $\delta > 0$, we can find 2 points that are δ close, but their image is more than ϵ apart! □

Let's do another example:

Example 38.5. the function $f(x) = x^2$ is uniformly continuous on $[-a, a]$

Proof. Let $\epsilon > 0$, then we note that

$$f(x) - f(y) = x^2 - y^2 = (x - y)(x + y)$$

Then we have that $|x + y| < 2a$, for $x, y \in [-a, a]$, and so if we have that $|x - y| < \frac{\epsilon}{2a}$ we have

$$|f(x) - f(y)| < |x - y||x + y| < 2a\frac{\epsilon}{2a} = \epsilon$$

as needed. □

So now we want a criterion for failing to be uniformly continuous:

Theorem 38.6. *A function $f : A \to \mathbb{R}$ is not uniformly continuous on A if and only if there is some $\epsilon_0 > 0$ and two sequence $(x_n), (y_n)$ in A such that*

$$|x_n - y_n| \to 0 \wedge |f(x_n) - f(y_n)| \geq \epsilon_0$$

Proof. $\underline{\Longleftarrow:}$ We note that this is almost by the negation of the definition. We need to show that

$$\exists\epsilon > 0.\forall\delta > 0\exists x, y, |x - y| < \delta \wedge |f(x) - f(y)| \geq \epsilon$$

So for that ϵ_0, let $\delta > 0$, then there is some $N \in \mathbb{N}$ such that

$$|x_n - y_n| < \delta \wedge |f(x_n) - f(y_n)| \geq \epsilon_0$$

As needed.

$\underline{\Longrightarrow}$ By the negation of uniform continuity, we have some $\epsilon > 0$ such that for all $\delta > 0$ we have points x, y that are δ-close, but their image is ϵ -apart. Take then $\delta_n = \frac{1}{n}$, and let x_n, y_n the points generated from this property. These sequences have the required characteristics. □

Let's do an example:

Example 38.7. The function $f(x) = \sin(\frac{1}{x})$ is continuous at any point on $(0, 1)$ but not uniformly continuous.

Proof. The fact that it is continuous, we get since $\frac{1}{x}$ is continuous at $(0, 1)$, and $\sin(x)$ is continuous for nay $x \in \mathbb{R}$. So, as a composition of two continuous functions, it is continuous. To show that it is uniform, we can take

$$x_n = \frac{1}{\frac{\pi}{2} + 2n\pi}, y_n = \frac{1}{\frac{3\pi}{2} + 2n\pi}$$

Then we have that

$$|x_n - y_n| = \frac{\pi}{(\frac{3\pi}{2} + 2n\pi)(\frac{\pi}{2} + 2n\pi)} \to 0$$

But we have that

$$|f(x_n) - f(y_n)| = |\sin(\frac{\pi}{2} + 2n\pi) - \sin(\frac{3\pi}{2} + 2n\pi)| = |1 - (-1)| = 2$$

So, by the criterion, we get that it doesn't converge uniformly! $\square$

We note that this was very close to the proof that $\sin(\frac{1}{x})$ is not continuous at 0. In fact, the connection is not accidental since we have the following theorem:

Theorem 38.8. *A continuous function on a compact set K is uniformly continuous on K.*

Proof. Let $f : K \to \mathbb{R}$ a continuous function, where $K \subset \mathbb{R}$ is compact.

By way of contradiction, assume that f is not uniformly continuous. So we know there is some $\epsilon > 0$ and two sequences $(x_n), (y_n)$ such that

$$|x_n - y_n| \to 0 \wedge \forall n, |f(x_n) - f(y_n)| \geq \epsilon$$

Since K is compact, by Bolzano-Weierstrass, we have that (x_n) has some convergent subsequence x_{n_k}. So we write

$$x_{n_k} \to x$$

We note that by the algebra of limits, we have

$$\lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} (y_{n_k} - x_{n_k}) + x_{n_k} = \lim_{k\to\infty} y_{n_k} - x_{n_k} + \lim_{k\to\infty} x_{n_k} = 0 + x$$

where we used that $|x_{n_k} - y_{n_k}| \to 0$. So we get since f is continuous:

$$\lim_{k\to\infty} f(x_{n_k}) = f(\lim_{k\to\infty} x_{n_k}) = f(x)$$

$$\lim_{k\to\infty} f(y_{n_k}) = f(\lim_{k\to\infty} y_{n_k}) = f(x)$$

So we get that

$$|f(x_{n_k}) - f(y_{n_{k)}}| \to 0$$

But we assumed that

$$\forall n, |f(x_n) - f(y_n)| \geq \epsilon$$

A contradiction! So, we conclude that f has to be uniformly continuous. $\square$

38.2 What did we see today?

- We saw some more examples and counter-examples
- We talked about the connection between uniform continuity and compact sets!

39 39th Class- Extensions and overview

39.1 Review

Last time, we discussed uniform continuity and saw examples and non-examples of it.

39.2 Extensions

Why do we care about uniform continuity? We care about this because it allows us to extend a function!

Definition 39.1. Let $f : A \to \mathbb{R}$, and $A \subset B$. We call $\tilde{f} : B \to \mathbb{R}$ an extension of f if

$$\forall x \in A, f(x) = \tilde{f}(x)$$

Example 39.2. As we saw we can extend the function $f(x) = x^2 \sin(\frac{1}{x})$ defined on $(0, 1]$ to a function $\tilde{f} : [0, 1] \to \mathbb{R}$ by setting:

$$\tilde{f}(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & 0 < x \leq 1 \\ 0, & x = 0 \end{cases}$$

And we have that $\tilde{f}$ is continuous

So, we will want to prove the following theorem:

Theorem 39.3. *Let A be set such that $\overline{A}$ is compact (that is, $\overline{A}$ bounded). Then a function $f : A \to \mathbb{R}$ is uniformly continuous on A, if and only if it can be extended to a continuous function $\tilde{f} : \overline{A} \to \mathbb{R}$.*

in order to prove this fact, we will need a preparation lemma

Lemma 39.4. *If f is uniformly continuous on A and (a_n) is a Cauchy sequence in A, then $f(a_n)$ is also Cauchy.*

Proof. Let (a_n) be Cauchy in A, and let f be uniformly continuous on A, and let $\epsilon > 0$. Since f is uniformly continuous we have that there is some $\delta > 0$ such that

$$\forall x, y \in A, |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Since (a_n) is Cauchy, there is some N such that

$$\forall n, m > N, |a_n - a_m| < \delta$$

That implies that

$$\forall n, m > N, |f(a_n) - f(a_m)| < \epsilon$$

So $f(a_n)$ is Cauchy. $\square$

Now we can prove our theorem

Proof. $\underline{\Longleftarrow:}$ Assume that f can be extended to a continuous function $\tilde{f}$ on $\overline{A}$. Since $\tilde{f}$ is continuous on a compact set- $\overline{A}$, we have that $\tilde{f}$ is uniformly continuous. Since f and $\tilde{f}$ are the same on A, we get that f is uniformly continuous.

$\underline{\Longrightarrow:}$ Suppose that f in uniformly continuous. Let a be a limit point of A. We will show that if we have two sequences $(a_n), (b_n)$ in A that converge to a, then $f(a_n), f(b_n)$ converge and $\lim f(a_n) = \lim f(b_n)$.

First, this is enough - since then, we can define $\tilde{f}(a) = \lim f(a_n)$- and by the property, any sequence that converges to a in $\overline{A}$ will have the same limit (since we already showed that any sequence in $\overline{A}$ could be approximated by a sequence from A).

So first, we note that since $a_n \to a$, it is Cauchy. Since f is uniformly continuous, we get that $f(a_n)$ is Cauchy. So, it converges. Similarly, we get that $f(b_n)$ converges.

To show that they converge to the same limit, we define

$$c_n = \begin{cases} a_n, & n \text{ is odd} \\ b_n, & n \text{ is even} \end{cases}$$

Note that since $a_n \to a, b_n \to a$, we have that that c_n Cauchy and converges to a. This tells us that $f(c_n)$ is also Cauchy, and in particular, any subsequence of it has to converge to the same limit. In particular, we have that $\lim f(a_n) = \lim f(b_n)$ as needed. □

One example of this use is the following

Example 39.5. The function $f : \mathbb{Q} \cap [0,1] \to \mathbb{R}$ defined by $f(x) = a^x$, for $a > 1$ can be extended to a continuous function on $\tilde{f} : [0,1] \to \mathbb{R}$

Proof. We need to show that f is uniformly continuous on the domain (which is this weird set $[0,1] \cap \mathbb{Q}$). So we start by writing

$$|f(x+h) - f(x)| = |a^{x+h} - a^x| = |a^x||a^h - 1|$$

We note that since $a > 1$, and $x = \frac{p}{q}$, for $p \leq q$, and $p, q \in \mathbb{Z}$ we can write

$$a^p < a^q \qquad\qquad a^{\frac{p}{q}} < a^1$$

So we get that

$$|f(x+h) - f(x)| < a|a^h - 1|$$

Let $\epsilon > 0$. Now recall that you showed in Pset 8, Question 3 that $\lim a^{\frac{1}{n}} = 1$. So there is some $N \in \mathbb{N}$ such that if $n > N$ we have

$$|a^{\frac{1}{N}} - 1| < \frac{\epsilon}{a}$$

So by taking $|h| < \frac{1}{N}$ we have that

$$|f(x+h) - f(x))| < \epsilon$$

Since our h is independent of x, we have that f is uniformly continuous and, therefore, can be extended to all of $\mathbb{R}$! □

39.3 Overview

So, what have you seen this semester?

In the beginning, I tried to tell you that math is a language - and this course will teach you how to communicate with this language. Did we succeed? That is up to you to decide, but let's try and see things from a bird's-eye view and see what we have learned. So, take some time and consider what topics we have covered this semester.

1. The first chapter of the course can be called Preliminaries: Logic and Set Theory
   - We started with logic: how to write arguments in a formal language. We learned about truth tables and quantifiers and how to prove statements in standard form. Recall how crucial it was to limit definitions.
   - Then we started talking about sets and set operations. This was informed somewhat by our discussion about logic and the connections between the two. Recall how crucial it was to $\limsup$ and $\liminf$.
2. The next chapter dealt with building our notions of numbers.
   - We introduced the natural numbers and showed that their structure allows induction proofs. We build the integers by forcing a group structure.
   - This idea led us to the notion of a field with a group structure with addition and multiplication. So, we proved some basic properties of a field from the axioms and saw what happens if we have an order relation on top of this.
   - we introduced the notions of supremum and infimum. This allowed us to build the real numbers by adding the completeness axiom.
   - Finally, we explored the consequences of completeness, like the Archimedean property, nested interval property, and so on.
3. We had a short interlude dealing with cardinality and the Cantor argument.
4. Next, we explored the notion of sequences and limits.
   - We talked about the definition of limits and their algebra. What it means to converge to $\pm\infty$.
   - Then we started talking about comparing sequences and monotonicity, and we developed the notion of Cauchy sequences.
   - This new language allowed us to describe the real numbers in more clarity as limits of Cauchy sequences,
   - We also discussed subsequences and subsequential limits and proved different notions with them
5. This discussion let us define some basic concepts in the topology of the real line
   - We talked about open and closed sets.
   - We understood what compact sets are and proved the Heine Borel theorem.

6. Then we introduced the idea of a series
   - We talked about how to define series.
   - We developed different tests: comparison, root, ratio, and alternating to see if a series converges or diverges.
   - We saw how rearranging the series can affect certain series but not others.
7. Finally, in the last chapter, we talked about functional limits and continuity.
   - We discussed what it means for a function to have a limit in the topology and sequential sense.
   - We introduced the notion of continuous functions and so what they preserve
   - Finally, we discussed uniform continuity.

40 40th Class- Overview and Review

40.1 Final logistics

- In the final, you can use any course-related material: course notes, the Psets, and their solutions. It can be the ones I provided, what you have written (in class or at home), or someone else who took the course. Please do not use any other material. You can not collaborate on this.

- The exam will be 3 hours- which you may take as you please between the 6th and the 9th. Since I couldn't find any good tech solution- please pledge that you did it within the time limit.

- Please let me know if there is a need for a review session during study days.

- Rigor of solution: If you are asked a question about the limit- don't justify your solution using field properties. If you want to use a theorem, please either state it in full (including all the assumptions), use its name, or make it explicit enough so it will be clear what you mean (e.g., in a field question, don't write $a^2 = a \implies a(a-1) = 0 \implies a = 0, 1$ write more as we will see later).

- For your convenience, I will publish a single pdf file with all the course notes. So, the numbering of the theorems will be continuous, making it easier to understand what you want to refer to.

40.2 Overview

So, what have you seen this semester?

In the beginning, I tried to tell you that math is a language - and this course will teach you how to communicate with this language. Did we succeed? That is up to you to decide, but let's try and see things from a bird's-eye view and see what we have learned. So, take some time and consider what topics we have covered this semester.

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   - We talked about the definition of limits and their algebra. What it means to converge to $\pm\infty$.
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   - We introduced the notion of continuous functions and so what they preserve
   - Finally, we discussed uniform continuity.

40.3 Additional problems - selected solutions

1. For the following statements, write truth or false- no need to prove:
   (a) If every proper subsequence of (a_n) converges then (a_n) converges.- True
   (b) If there is a subsequence of (a_n) that diverges, then (a_n) diverges. -True
   (c) Every bounded subset of $\mathbb{R}$ has an infimum and supremum. - True
   (d) There is no bijection between the Cantor set and $\mathbb{Q} \cap [0,1]$ -True

(e) The sequence $\frac{2n+1}{3n-2}$ is Cauchy. - True

(f) If (x_n) is bounded and diverges, then there exists two subsequences of (x_n) that converge to different limits.- True

(g) The series $\sum \frac{n^4-2n^2-n+3}{n^6}$ converges.- True

(h) the set $\{\frac{3n-2}{n^2}\} \cup \{0\}$ is compact. - True

(i) The image of $[-1, 1]$ under a continuous function is always an open interval. -False

(j) The function $f(x) = \frac{1}{x-4}$ is continuous on $(4, \infty)$.- True

(k) The function $f(x) = \frac{1}{x-4}$ is uniformly continuous on $(4, \infty)$.- False

2. Let g be continuous on an interval A and let F be the set of points where g fails to be injective:

$$F = \{x \in A \mid \exists x \neq y \in A, g(x) = g(y)\}$$

Show F is either empty or uncountable. Hint: try drawing the case where $F \neq \emptyset$, and then use the IVT to justify the intuition.

Proof. Assuming that F is not empty, then we have at least two elements $x, y \in F$ such as $x < y$. Try to draw what is going on here!

If $\forall x \leq z \leq y$, we have that $f(z) = f(x)$ we get that $[x, y] \subset F$, and then F is uncountable. Otherwise, there is some $z \in [x, y]$ such that $f(x) \neq f(z)$. WLOG assume $f(z) > f(x)$. Then for every $f(x) < w < f(z)$, by IVT, as f continuous and $[x, z]$ is compact, we have some $c_1 \in [x, z]$ such that $f(c_1) = w$. since $f(x) = f(y)$, then $f(y) < w < f(z)$, by IVT, since f is continuous and $[z, y]$ is compact, we have some $c_2 \in [z, y]$ such that $f(c_2) = w$. So we have that $\forall f(x) < w < f(z) \exists c_1 \in [x, z], c_2 \in [z, y], f(c_1) = f(c_2)$. So we get that the cardinality of F is at least the cardinality of $[f(x), f(z)]$, and in particular uncountable. $\square$

3. Show that if $\mathbb{F}$ is a a field and $a \in \mathbb{F}$ is such that $a^2 = a$ then $a = 0$ or $a = 1$ (note: use only field properties).

Proof. Let $\mathbb{F}$ be a field, and $a \in \mathbb{F}$, such that $a^2 = a$. Then, we can write

$$\begin{aligned}
&a^2 + (-a) + a = a^2 + 0 = a^2 = a = a + 0 = a + (-a) + a \\
&a^2 + (-a) = a + (-a) = 0 \\
&a(a + (-1)) = 0
\end{aligned}$$

where we also used the distributive property.

If $a \neq 0$, then we can write

$$0 = a^{-1}0 = a^{-1}a(a + (-1)) = (a^{-1}a)(a + (-1)) = 1(a + (-1)) = a + (-1)$$

So we get that

$$\begin{aligned}
&0 + 1 + (-1) = 0 + 0 = 0 = a + (-1) = a + (-1) + 1 + (-1) \\
&1 = 0 + 1 = a + (-1) + 1 = a + 0 = a
\end{aligned}$$

And so $a = 1$. $\square$

4. Show that the function $f(x) = \frac{x}{1-|x|}$ is a bijection between $(-1, 1)$ and $\mathbb{R}$ (hint: prove the $f(x)$ and x have the same sign).

Proof. We start with the hint- if $x > 0$ then $1 - |x| = 1 - x > 0$,and $f(x) > 0$, if $x \leq 0$ then $1 - |x| = 1 + x > 0$ and so $f(x) \leq 0$. We want to show that f is injective. So if $f(x) = f(y)$ then we have $\frac{x}{1-|x|} = \frac{y}{1-|y|}$which implies

$$\begin{aligned} x(1 - |y|) &= y(1 - |x|) \\ x - y &= x|y| - y|x| \end{aligned}$$

Since $f(x) = f(y)$, they have the same sign, and so x, y have the same sign, and so we have

$$x|y| - y|x| = \begin{cases} xy - yx, & x, y > 0 \\ -xy + yx, & x, y \leq 0 \end{cases} = 0$$

And we conclude that $x = y$.

For surjectivity, we consider if $x > 0$, then $f(x) = y > 0$, and

$$\begin{aligned} y &= \frac{x}{1 - x} \\ y - yx &= x \\ y &= x(1 + y) \\ \frac{y}{1 + y} &= x \\ \frac{y}{1 + |y|} &= x \end{aligned}$$

and if $x \leq 0$, we have $y \leq 0$, and we have

$$\begin{aligned} y &= \frac{x}{1 + x} \\ y + yx &= x \\ y &= x(1 - y) \\ \frac{y}{1 - y} &= x \\ \frac{y}{1 + |y|} &= x \end{aligned}$$

So we get that if $y \in (-1, 1)$, then $f(\frac{y}{1+|y|}) = y$, as needed. □

5. Prove the following is a tautology, for any $1 \leq i < j \leq N$ (hint: use the results from the mid-semester assignment

$$[(A_1 \implies A_2) \wedge \cdots \wedge (A_{n-1} \implies A_n) \wedge (A_n \implies A_1)] \implies (A_i \iff A_j)$$

Proof. We denote the following:

$$
\begin{aligned}
B &= (A_1 \implies A_2) \wedge \cdots \wedge (A_{i-1} \implies A_i) \\
C &= (A_i \implies A_{i+1}) \wedge \cdots \wedge (A_{j-1} \implies A_j) \\
D &= (A_j \implies A_{j+1}) \wedge \cdots \wedge (A_{n-1} \implies A_n) \wedge (A_n \implies A_1)
\end{aligned}
$$

Then we need to show that

$$(B \wedge C \wedge D) \implies (A_i \iff A_j)$$

In the mid-semester assignment, we saw that

$$
\begin{aligned}
B &\implies (A_1 \implies A_i) \\
C &\implies (A_i \implies A_j) \\
D &\implies (A_j \implies A_1)
\end{aligned}
$$

are tautologies. We also saw that if $B \implies C$ is a tautology, then $B \wedge D \implies C \wedge D$ is a tautology.

So we conclude that since $B \implies (A_1 \implies A_i)$ is a tautology we have that

$$(B \wedge C \wedge D) \implies ((A_1 \implies A_i) \wedge C \wedge D)$$

is a tautology.

Similarly, we can do the other expressions and get that

$$(B \wedge C \wedge D) \implies ((A_1 \implies A_i) \wedge (A_i \implies A_j) \wedge (A_j \implies A_1))$$

is a tautology, since $A \wedge B$ is the same as $B \wedge A$, we get that

$$(B \wedge C \wedge D) \implies ((A_i \implies A_j) \wedge (A_j \implies A_1) \wedge (A_1 \implies A_i))$$

is a tautology. Again, using the result from the mid-semester assignment, we get that

$$((A_j \implies A_1) \wedge (A_1 \implies A_i)) \implies (A_j \implies A_i)$$

is a tautology. We also showed there that if $B \implies C$ and $C \implies D$ are tautologies, then $B \implies D$ is a tautology. So, we conclude that

$$(B \wedge C \wedge D) \implies ((A_i \implies A_j) \wedge (A_j \implies A_i))$$

is a tautology. The RHS is the definition of $A_i \iff A_j$- as needed. $\square$