MATH 302 HW3

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Proof 1 *Let* $n \in \mathbb{N}$ *be fixed. We need to prove that the set*

$$S_n = \{m \in \mathbb{N} : n + m \text{ is defined}\}$$

is an inductive set. By the axiom of induction (Peano's fifth axiom), this will show that $S_n = \mathbb{N}$*, meaning* $n + m$ *is defined for all* $m \in \mathbb{N}$*.*

We need to prove two conditions:

1. $0 \in S_n$ *(base case)*
2. *For all* $k \in S_n$*,* $s(k) \in S_n$ *(inductive step)*

Base Case: *We need to show that* $n + 0$ *is defined. By the recursive definition of addition, we have:*

$$n + 0 = n$$

This is well-defined since $n \in \mathbb{N}$*. Therefore,* $0 \in S_n$*.*

Inductive Step: *Let* $k \in S_n$*, meaning* $n + k$ *is defined. We need to show that* $n + s(k)$ *is defined. By the recursive definition of addition:*

$$n + s(k) = s(n + k)$$

Since $n + k$ *is defined (by our inductive hypothesis), and* s *is a function defined for all natural numbers (by the Peano axioms),* $s(n + k)$ *is defined. Therefore,* $n + s(k)$ *is defined, so* $s(k) \in S_n$*.*

By the axiom of induction, $S_n = \mathbb{N}$*. Since* n *was arbitrary, this proves that* $n + m$ *is defined for all* $n, m \in \mathbb{N}$*.*

Proof 2 *We proceed by induction on* n*.*

Base Case: *Show that* $0 + 1 = 1 + 0$

$$\begin{aligned} 0 + 1 &= s(0) \text{ (by definition of addition)} \\ 1 + 0 &= 1 \text{ (by definition of addition)} \\ &= s(0) \end{aligned}$$

Therefore, $0 + 1 = 1 + 0$*.*

Inductive Step: *Assume $k+1=1+k$ for some $k\in\mathbb{N}$. We need to prove that $s(k)+1=1+s(k)$.*

$$\begin{aligned}
s(k)+1 &= s(s(k)) \text{ (by definition of addition)}\\
1+s(k) &= s(1+k) \text{ (by definition of addition)}\\
&= s(k+1) \text{ (by inductive hypothesis)}\\
&= s(s(k))
\end{aligned}$$

Therefore, $s(k)+1=1+s(k)$.

By the principle of induction, $n+1=1+n$ for all $n\in\mathbb{N}$.

Proof 3 *Let $m\in\mathbb{N}$ be arbitrary.*

$$\begin{aligned}
(m+1)+1 &= s(m+1) \text{ (by definition of addition)}\\
&= s(s(m)) \text{ (by definition of addition)}\\
m+(1+1) &= m+s(1) \text{ (by definition of addition)}\\
&= s(m+1) \text{ (by definition of addition)}\\
&= s(s(m))
\end{aligned}$$

Therefore, $(m+1)+1=m+(1+1)$ for all $m\in\mathbb{N}$.

Proof 4 *We proceed by induction on l, using Problem 3 as our base case.*

Base Case: *When $l=1$, we have already proven in Problem 3 that $(m+1)+1=m+(1+1)$ for all $m\in\mathbb{N}$.*

Inductive Step: *Assume that for some $k\in\mathbb{N}$, $(m+1)+k=m+(1+k)$ for all $m\in\mathbb{N}$. We need to prove $(m+1)+s(k)=m+(1+s(k))$.*

$$\begin{aligned}
(m+1)+s(k) &= s((m+1)+k) \text{ (by definition of addition)}\\
&= s(m+(1+k)) \text{ (by inductive hypothesis)}\\
&= m+s(1+k) \text{ (by definition of addition)}\\
&= m+(1+s(k)) \text{ (by definition of addition)}
\end{aligned}$$

By the principle of induction, $(m+1)+l=m+(1+l)$ for all $l\in\mathbb{N}$.

Proof 5 *We proceed by induction on n, using Problem 4 as our base case.*

Base Case: *When $n=1$, we have proven in Problem 4 that $(m+1)+l=m+(1+l)$ for all $m,l\in\mathbb{N}$.*

Inductive Step: *Assume that for some $k\in\mathbb{N}$, $(m+k)+l=m+(k+l)$ for all $m,l\in\mathbb{N}$. We need to prove $(m+s(k))+l=m+(s(k)+l)$.*

$$\begin{aligned}
(m+s(k))+l &= s(m+k)+l \text{ (by definition of addition)}\\
&= s((m+k)+l) \text{ (by definition of addition)}\\
&= s(m+(k+l)) \text{ (by inductive hypothesis)}\\
&= m+s(k+l) \text{ (by definition of addition)}\\
&= m+(s(k)+l) \text{ (by definition of addition)}
\end{aligned}$$

By the principle of induction, $(m + n) + l = m + (n + l)$ for all $m, n, l \in \mathbb{N}$.

Proof 6 *Let $m, n \in \mathbb{N}$ such that $m \neq n$. By the well-ordering principle of natural numbers, we can let m be the smaller of $\{m, n\}$. Then there exists $l \in \mathbb{N}$ such that either:*

1) $m + l = n$, or 2) $n + l = m$

Since m is the smaller number, case (2) is impossible. Therefore, $m + l = n$ for some $l \in \mathbb{N}$.

Now, let $A = \{i \in \mathbb{N} : m + i \in [m, n]\}$. We claim A is an inductive set:

1) $0 \in A$ since $m + 0 = m \in [m, n]$ 2) If $k \in A$ and $m + k < n$, then $m + s(k) = s(m + k) \leq n$, so $s(k) \in A$

Therefore, by induction, $l \in A$ and is the least number such that $m + l = n$.

Proof 7 *We proceed by induction on n.*

Base Case: *When $n = 0$, for any $m \in \mathbb{N}$:*

$$0 + m = m \text{ (by definition of addition)}$$
$$m + 0 = m \text{ (by definition of addition)}$$

Therefore, $0 + m = m + 0$.

Inductive Step: *Assume that for some $k \in \mathbb{N}$, $k + m = m + k$ for all $m \in \mathbb{N}$. We need to prove $s(k) + m = m + s(k)$.*

$$\begin{aligned} s(k) + m &= s(k + m) && \text{(by definition of addition)} \\ &= s(m + k) && \text{(by inductive hypothesis)} \\ &= m + s(k) && \text{(by definition of addition)} \end{aligned}$$

By the principle of induction, $n + m = m + n$ for all $n, m \in \mathbb{N}$.