Math 302 , Problem Set 3 Due February 6,2025 6 P.M.

We can define addition in the natural numbers recursively. That is we define $n+m$ by

$$n+1=s(n),$$

where s is the successor function in the Peano axioms and

$$n+s(m)=s(n+m).$$

Now use the Peano axioms (and especially the fifth axiom which is the axiom of induction) to do problems 1 through 7 establishing that addition is associative and commutative.

1. Prove that for any natural number $n \in \mathbb{N}$ the set of m for which we have defined $n+m$ is an inductive set, so that we have defined $n+m$ for all$n, m \in \mathbb{N}$.

2. Prove by induction of n that for each $n \in \mathbb{N}$, we have

$$n+1=1+n.$$

3. Prove that for any $m \in \mathbb{N}$

$$(m+1)+1=m+(1+1).$$

4. Use 3. as the base case to prove by induction on l that for any $l \in \mathbb{N}$,

$$(m+1)+l=m+(1+l).$$

5. Use 4. as the base case to prove by induction on n that for any $l, n, m \in \mathbb{N}$

$$(m+n)+l=m+(n+l).$$

This is called associativity for addition.

6. Prove that for any $m, n \in \mathbb{N}$ so that $m \neq n$ there is $l \in \mathbb{N}$ so that either $m+l=n$ or $n+l=m$. Hint: Apply the well ordering principle to $\{m, n\}$. From the proof of the well ordering principle there is a largest consecutive set of natural numbers A so that $A \cap \{m, n\}=\emptyset$. Let m be the smaller of $\{m, n\}$. Show that $A \cup m+\mathbb{N}$ is an inductive set. Thus $n \in m+\mathbb{N}$.

7. Show that $\forall n, m \in \mathbb{N}$

$$n=m=m+n.$$

This is called commutativity of addition

We can also define multiplication of natural numbers recursively. For any natural number n,we define

$$n \times 1 = n.$$

For the recursive set, we define

$$n \times s(m) = n \times m + n.$$

Multiplication is thus defined for all natural numbers

Challenge Problem 1 Prove multiplication is commutative

Challenge Problem 2 Prove multiplication is associative

Challenge Problem 3 Prove multiplication distributes under addition.

I'm not requiring that you do the two challenge problems. You'd probably find them tedious. But I hope maybe I've convinced you that you could do them if it were a matter of life and death. From this point on, we will pretend that you know all basic facts about arithmetic of natural numbers, and rational numbers. [But not real numbers because you actually don't.]

For the remainder of this assignment do Exercise 7 from Section 1.1 of Cranks