Elements of DeFi

https://web3.princeton.edu/elements-of-defi/

Professor Pramod Viswanath

Princeton University

Lecture 17

Prediction Markets

Last Lecture: Derivatives and Synthetics

- Derivatives
  - Futures
  - Options
  - Swaps
- Synthetics are tokenized derivatives
  - Wrapped asset-backed tokens
  - CDP based synthetics
  - Perpetuals
  - Options

This lecture: Prediction markets

- Similar to futures markets in TradFi

- Build up to modern prediction markets
  - Proper Scoring Rules
  - Market Scoring Rules
  - Automated Market Makers
  - Polymarket – LOBs and AMMs

Information aggregation

- Markets exist as information aggregators for resource allocation
- Prices serve as distributed signals of relative scarcity
- A high price effectively serves as a bounty – so that individuals with the right knowledge can innovate to solve the problem
- Can something similar be done for events in the future?

Prediction markets

CONTRACT	LAST	CHANGE	HIGH	LOW	OPEN	LAST TRADE
May 25 Corn	451.75	+0.5 ↑	452.75	450	451.75	08:42 AM
May 25 Hard Red Winter Wheat	567.75	+2.75 ↑	570	562	566.25	08:41 AM
May 25 Oats	356.5	-4.75 ↓	361	354.5	361	08:30 AM
May 25 Rough Rice	13.17	-0.01 ↓	13.21	13.17	13.21	09:40 PM
May 25 Soybean Meal	292.6	-1 ↓	294.7	292.2	293.8	08:42 AM
May 25 Soybean Oil	43.82	+1.18 ↑	43.96	42.34	42.64	08:42 AM
May 25 Soybean	1009.25	+8.25 ↑	1009.75	999.75	1001	08:42 AM
May 25 Wheat	534.5	-0.75 ↓	537.25	531.75	536.25	08:42 AM

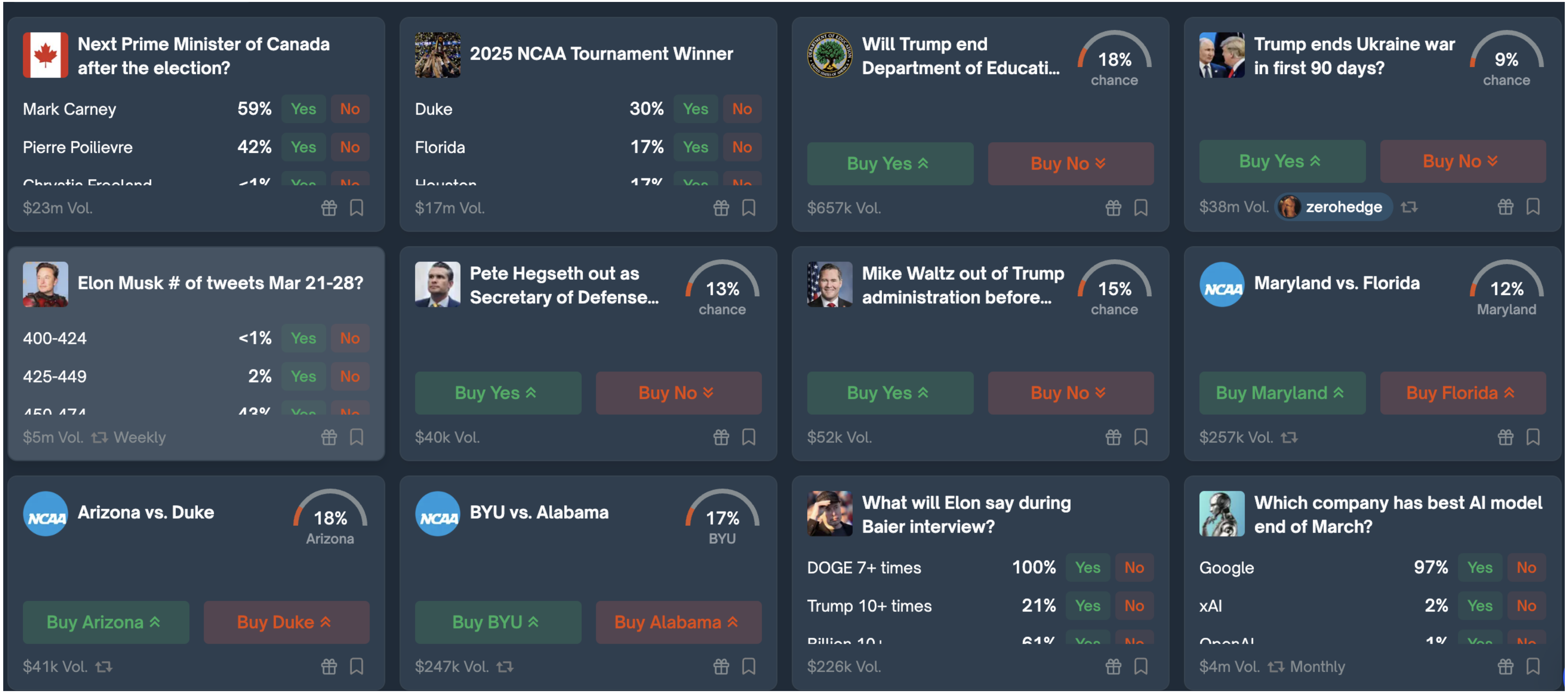
Prediction markets

- In the 2000’s - used internally in tech companies to predict future events
  - HP used it internally to predict printer sales
  - Google, Microsoft used it to predict whether a product would ship on time

- Found to be more precise than domain experts in most instances

- Also proposed by DARPA as a way to share information across intelligence agencies – FBI, CIA, NSA

Prediction markets



Prediction markets

Prediction markets

Goal

- Elicit information from a group of individuals – by effectively placing a bounty

- Need to be incentive compatible

Best strategy of trader should be to express their true beliefs

Design 1: Proper Scoring Rules

- Suppose an event A has N possible outcomes
- A trader has a belief over what these outcomes might be
- Represent their true belief with the probabilities –
$$\vec{p} = [p_1 \;\; p_2 \;\; p_3 \ldots p_N]$$
- They report their belief as $\vec{r}$, which may not be $= \vec{p}$
- Let $S_i(\vec{r})$ be the bounty for the trader reporting their belief as $\vec{r}$ if i is the event that ends up happening

Goal : set a reward function $S_i(\vec{r})$ such that the trader reports their true belief.

Design 1: Proper Scoring Rules

Goal : set a reward function $S_i(\vec{r})$ such that the trader reports their true belief $\vec{p}$

How do you translate this condition into mathematics?

$$\vec{p} = argmax_{\vec{r}} \sum_i p_i S_i(\vec{r}) \; \ldots where \; \sum_i r_i = 1$$

The strategy that gives the most expected profit is to report the truth

Design 1: Proper Scoring Rules

Examples :

$$
\begin{aligned}
\text{Quadratic} \quad s_i &= \quad a_i + br_i - b\sum_j r_j^2/2, \\
\text{Spherical} \quad s_i &= \quad a_i + b\,r_i/(\sum_j r_j^2)^{1/2}, \\
\text{Logarithmic} \quad s_i &= \quad a_i + b\log(r_i), \\
\text{Power Law} \quad s_i &= \quad a_i + b\alpha \int_0^{r_i} \rho_i^{\alpha-2} d\rho_i - b\sum_j r_j^\alpha
\end{aligned}
$$

Design 1: Proper Scoring Rules

What is right about this design?

- Incentive compatible

What is wrong about this design?

- Not easy to combine opinions of multiple traders

Design 2: Market Scoring Rules

Algorithm –

1. Start with a proper scoring rule S_i

2. Market maker starts the market at $t = 0$ with their beliefs p_0

3. At time step t, a trader updates $\vec{p}_{t-1}$ to $\vec{p}_t$, and is promised a reward $S_i(\vec{p}_t) - S_i(\vec{p}_{t-1})$

4. When the event happens, distribute the payouts to all traders

Design 2: Market Scoring Rules

What is right about this design?

- Incentive compatible
- Easy to combine opinions of multiple traders

What is wrong about this design?

- No human has an explicit probability distribution in their head
- The trader has to report the chances over all possible events – this can be solved – how?

Design 2.5: Log Market Scoring Rule

The trader has to report the chances over all possible events

- This can be solved with $S_i(\vec{p}) = \ln(p_i)$ – only rule where reward of an event depends on the probability you assign it – called the logarithmic market scoring rule – LMSR

- Also easy to express conditional belief without changing the prior probability

Design 3: Automated Market Makers

- How do we convert a market scoring rule into a market maker?
- Traders should be able to buy/sell shares, instead of reporting their beliefs
- Every possible outcome i has a share price $\pi_i(\vec{q})$ where $\vec{q}$ is the vector of all share sold so far, and $\pi_i(\vec{q}) \geq 0, \sum_i \pi_i(\vec{q}) = 1$
- Each share awards \$1 if the corresponding outcome ends up happening

Design 3: Automated Market Makers

- How do we set $\pi_i(\vec{q})$ so that trader faces same incentives as LMSR?

- Cost of changing $\vec{q}$ to $\vec{q} + \vec{z}$ is

$$\int_0^{\mathbf{z}} \sum_{i=1}^{k} \pi_i(\mathbf{q} + \mathbf{x}) d\mathbf{x}.$$

- This integral should be path independent – why?
- Think of a trader changing $\vec{q}$ to $\vec{q} + \vec{z}$ and back

Design 3: Automated Market Makers

- We have seen these kind of functions in physics
- To enforce path independence, we enforce that

$$\int_0^{\mathbf{z}} \sum_{i=1}^{k} \pi_i(\mathbf{q}+\mathbf{x})d\mathbf{x} \quad = \quad C(\mathbf{q}+\mathbf{z}) - C(\mathbf{q})$$

Where $C(.)$ is called the cost function

- To replicate LMSR, turns out that we need to set the cost function to

$$C(\mathbf{q}) = \ln\left(\sum_{i=1}^{k} e^{q_i}\right) \Longrightarrow \pi_i(\mathbf{q}) \ = \frac{e^{q_i}}{\sum_{j=1}^{k} e^{q_j}}$$

Design 3: Automated Market Makers

$$C(\mathbf{q}) = \ln\left(\sum_{i=1}^{k} e^{q_i}\right) \Rightarrow \pi_i(\mathbf{q}) = \frac{e^{q_i}}{\sum_{j=1}^{k} e^{q_j}}$$

- Why is this the same as LMSR?

- In LMSR, changing belief from $\pi_i(\vec{q})$ to $\pi_i(\vec{q}+\vec{z}\,)$would reward the trader with the following amount

$$S_i(\pi_i(\vec{q}+\vec{z}\,)) - S_i\big(\pi_i(\vec{q})\big) = \ln\left(\frac{e^{q_i+z_i}}{\sum_{j\in X} e^{q_j+z_j}}\right) - \ln\left(\frac{e^{q_i}}{\sum_{j\in X} e^{q_j}}\right)$$

Design 3: Automated Market Makers

In LMSR, changing belief from $\pi_i(\vec{q})$ to $\pi_i(\vec{q}+\vec{z}\,)$would reward the trader with the following amount

$$S_i(\pi_i(\vec{q}+\vec{z}\,)) - S_i\big(\pi_i(\vec{q})\big) = \ln\left(\frac{e^{q_i+z_i}}{\sum_{j\in X} e^{q_j+z_j}}\right) - \ln\left(\frac{e^{q_i}}{\sum_{j\in X} e^{q_j}}\right)$$

$$= z_i - \Bigg[\underbrace{\ln\left(\sum_{j\in X} e^{q_j+z_j}\right) - \ln\left(\sum_{j\in X} e^{q_j}\right)}_{\text{This expression is the difference in the function } C(.) \text{ that we used to define the AMM!}}\Bigg]$$

z_i is the number of shares the trader holds – equal to the payoff if outcome is i

This expression is the difference in the function $C(.)$ that we used to define the AMM!

Design 3: Automated Market Makers

We have studied AMMs before as CFMMs – what is the connection?

The cost function of an AMM can be written in terms of its bonding curve $\psi(\mathrm{x}, \mathrm{y})$ as

$$C(\vec{q}) = \inf\{c \in R: \psi\left(c - q_x, c - q_y\right) \geq \psi(x_0, y_0)\}$$

Similarly, we can go from the cost function to the bonding curve

Design 3: Automated Market Makers

What is right about this design?

- Incentive compatible
- Easy to combine opinions of multiple traders
- Intuitive - does not need traders to have a probability distribution in mind – they can buy individual shares in an outcome

What is wrong about this design?

- Requires an initial investment that would end up in a loss to pay for bounties given out

Design 4: Polymarket – LOB || AMM

AMMs are affordable only for niche markets – we expect less volume in trading – can set up with small initial capital

What if the outcome is highly speculated on? – e.g. presidential elections, sports, etc.

Use a limit order book on binary outcomes – "Yes" and "No"

Design 4: Polymarket – LOB || AMM

Idea - If trader Y places a limit order for 1 “Yes” share at \$0.6, trader N places a limit order for 1 “No” share at \$0.4 -> match them

When outcome is announced, the winner gets \$1 per share

This mechanism is used in Polymarket

For niche markets, option to set up AMM instead

Design 4: Polymarket – LOB || AMM

What is right about this design?

- Incentive compatible
- Easy to combine opinions of multiple traders
- Intuitive – can buy/sell shares in an outcome
- If platform has enough attention, no capital required to set up new market

Open Problems

What is wrong about this design?

- Setting up market for large outcome spaces is difficult (not just “Yes” or “No” events) – e.g. weather
- Aggregate information across different AMMs to boost liquidity

Conclusion

We saw how prediction markets aggregate information

Four stages

1. Proper scoring rules
2. Market scoring rule
3. Automated market makers
4. LOBs or AMMs – decide based on market volume

LECTURE ENDS