Appendix N

Cosmic Microwave Background lensing reconstruction: Quadratic Estimator with the flat-sky approximation

In this appendix, we summarize the of Cosmic Microwave Background (CMB) lensing reconstruction method in Hu & Okamoto (2002). The method summarized here is valid for small angular scales as we use the flat-sky approximation. For general Full-sky treatment with spin-2 spherical harmonics, see Okamoto & Hu (2003).

The intervening matter distribution of the large scale structure of the Universe lenses CMB photon along the direction $\hat{\boldsymbol{n}}$ with lensing potential $\phi(\hat{\boldsymbol{n}})$ given by

$$\phi(\hat{\boldsymbol{n}}) = 2\int d\eta \frac{d_A(\eta_* - \eta)}{d_A(\eta_*)d_A(\eta)}\Phi(d_A(\eta)\hat{\boldsymbol{n}}, \eta), \tag{N.1}$$

where $d_A(\eta)$ is the comoving angular diameter distance corresponding to the comoving distance η, and η_* is the comoving distance to the last scattering surface. Here Φ is the gravitational potential. As a result, temperature ($\Theta(\hat{\boldsymbol{n}}) \equiv \Delta T(\hat{\boldsymbol{n}})/T$) anisotropy and Stokes parameters $Q(\hat{\boldsymbol{n}})$ and $U(\hat{\boldsymbol{n}})$ of CMB is re-mapped as

$$\Theta(\hat{\boldsymbol{n}}) = \tilde{\Theta}(\hat{\boldsymbol{n}} + \nabla\phi(\hat{\boldsymbol{n}}))) \tag{N.2}$$

$$(Q \pm iU)(\hat{\boldsymbol{n}}) = (\tilde{Q} \pm i\tilde{U})(\hat{\boldsymbol{n}} + \nabla\phi(\hat{\boldsymbol{n}})). \tag{N.3}$$

Following the notation of Hu & Okamoto (2002), we denote the unlensed power spectra of $x = \Theta$ (temperature fluctuation), E-, and B-mode polarization as

$$\langle \tilde{x}^*(\boldsymbol{l})\tilde{x}(\boldsymbol{l}')\rangle \equiv (2\pi)^2\delta(\boldsymbol{l} - \boldsymbol{l}')\tilde{C}_l^{xx'} \tag{N.4}$$

$$\langle \phi^*(\boldsymbol{L})\phi(\boldsymbol{L}')\rangle \equiv (2\pi)^2\delta(\boldsymbol{L} - \boldsymbol{L}')L^{-2}C_L^{dd}. \tag{N.5}$$

Note that, as we assume that parity is conserved,

$$\tilde{C}_l^{\Theta B} = \tilde{C}_l^{EB} = 0.$$

On the other hands, observed power spectra are denoted without tilde as

$$\langle x^*(\boldsymbol{l})x(\boldsymbol{l}')\rangle \equiv (2\pi)^2\delta(\boldsymbol{l} - \boldsymbol{l}')C_l^{xx'}. \tag{N.6}$$

These 'observed' power spectra include all the sources of uncertainties such as instrumental white noise and foreground contaminations. In order to quantify the uncertainty of the observed power spectra, we use the Gaussian random detector noise of Knox (1995):

$$C_{l,noise}^{\Theta\Theta} = \left(\frac{T_{CMB}}{\Delta_T}\right)^{-2} e^{l(l+1)\sigma^2/8\ln 2} \tag{N.7}$$

$$C_{l,noise}^{BB} = C_{l,noise}^{EE} = \left(\frac{T_{CMB}}{\Delta_p}\right)^{-2} e^{l(l+1)\sigma^2/8\ln 2}, \tag{N.8}$$

where Δ_T and Δ_p is in the unit of [μK rad]. For *Planck*, we use $\Delta_T = 35.4\mu$K arcmin, and $\Delta_p = 63.1\mu$K arcmin, and set the FWHM of beam $\sigma = 7$ arcmin, which comes from the CMBPol mission study of Zaldarriaga et al. (2008). The "nearly perfect" experiment referred by Hu & Okamoto (2002) has white noise of $\Delta_T = 1\mu$K arcmin, and $\Delta_p = \sqrt{2}\mu$K arcmin, and FWHM of $\sigma = 4$ arcmin.

In order to quantify the effect of weak lensing, we first Taylor expand equation (N.2) and equation (N.2):

$$\Theta(\hat{\boldsymbol{n}}) = \tilde{\Theta}(\hat{\boldsymbol{n}}) + (\nabla\tilde{\Theta}(\hat{\boldsymbol{n}})) \cdot (\nabla\phi(\hat{\boldsymbol{n}})) + \cdots \tag{N.9}$$

$$(Q \pm iU)(\hat{\boldsymbol{n}}) = (\tilde{Q} \pm i\tilde{U})(\hat{\boldsymbol{n}}) + (\nabla(\tilde{Q} \pm i\tilde{U})(\hat{\boldsymbol{n}})) \cdot (\nabla\phi(\hat{\boldsymbol{n}})) + \cdots, \tag{N.10}$$

and Fourier transform them[1]. Then, the re-mapping effect of gravitational lensing in Fourier space becomes a mode coupling between $\tilde{\Phi}$, $\tilde{E}$, $\tilde{B}$ and lensing potential ϕ:

$$\Theta(\boldsymbol{l}) = \tilde{\Theta}(\boldsymbol{l}) - \int \frac{d^2 l_1}{(2\pi)^2} \int d^2 l_2 \delta^D(\boldsymbol{l} - \boldsymbol{l}_{12}) \boldsymbol{l}_1 \cdot \boldsymbol{l}_2 \phi(\boldsymbol{l}_2) \tilde{\Theta}(\boldsymbol{l}_1) \tag{N.13}$$

$$\begin{aligned} E(\boldsymbol{l}) \pm iB(\boldsymbol{l}) &= \tilde{E}(\boldsymbol{l}) \pm i\tilde{B}(\boldsymbol{l}) - \int \frac{d^2 l_1}{(2\pi)^2} \int d^2 l_2 \delta^D(\boldsymbol{l} - \boldsymbol{l}_{12}) \boldsymbol{l}_1 \cdot \boldsymbol{l}_2 \phi(\boldsymbol{l}_2) \\ &\quad \times (\tilde{E}(\boldsymbol{l}_1) \pm i\tilde{B}(\boldsymbol{l}_1)) e^{\pm 2i(\varphi_{l_1} - \varphi_l)}. \end{aligned} \tag{N.14}$$

Decomposing the equations for each Fourier mode of Θ, E, and B, the change due to weak

[1]Fourier transform with the flat sky approximation is defined as

$$\Theta(\hat{\boldsymbol{n}}) = \int \frac{d^2 l}{(2\pi)^2} \Theta(\boldsymbol{l}) e^{i\boldsymbol{l}\cdot\boldsymbol{n}} \tag{N.11}$$

$$(Q \pm iU)(\hat{\boldsymbol{n}}) = -\int \frac{d^2 l}{(2\pi)^2} \left[E(\boldsymbol{l}) \pm iB(\boldsymbol{l})\right] e^{2l\varphi_l} e^{i\boldsymbol{l}\cdot\boldsymbol{n}}, \tag{N.12}$$

where $\varphi_l \equiv \cos^{-1}(\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{l}})$ is an azimuthal angle of $\boldsymbol{l}$.

lensing is (Hu, 2000):

$$
\begin{aligned}
\Theta(\boldsymbol{l}) &= \tilde{\Theta}(\boldsymbol{l}) + \int \frac{d^2 l'}{(2\pi)^2} W(\boldsymbol{l}, \boldsymbol{l'}) \tilde{\Theta}(\boldsymbol{l'}) && \text{(N.15)} \\
E(\boldsymbol{l}) &= \tilde{E}(\boldsymbol{l}) + \int \frac{d^2 l'}{(2\pi)^2} W(\boldsymbol{l}, \boldsymbol{l'}) \left[\tilde{E}(\boldsymbol{l'}) \cos(2\varphi_{l'l}) - \tilde{B}(\boldsymbol{l'}) \sin(2\varphi_{l'l})\right] && \text{(N.16)} \\
B(\boldsymbol{l}) &= \tilde{B}(\boldsymbol{l}) + \int \frac{d^2 l'}{(2\pi)^2} W(\boldsymbol{l}, \boldsymbol{l'}) \left[\tilde{B}(\boldsymbol{l'}) \cos(2\varphi_{l'l}) + \tilde{E}(\boldsymbol{l'}) \sin(2\varphi_{l'l})\right], && \text{(N.17)}
\end{aligned}
$$

where

$$
W(\boldsymbol{l}, \boldsymbol{l'}) \equiv -\boldsymbol{l'} \cdot (\boldsymbol{l} - \boldsymbol{l'}) \phi(\boldsymbol{l} - \boldsymbol{l'}).
$$

Note that lensed E-mode and B-mode polarization are mixed. That is why the most of lens reconstruction signal comes from the estimator using EB cross-correlation (see, Figure N.2).

How do we reconstruct the lensing potential from the observed CMB anisotropy? The key is that weak lensing also mixes different wave-modes and the mode-mixing strength are proportional to the Fourier transform of lensing potential $\phi(\boldsymbol{l})$. Let us first quantify the coupling strength. For fixed lensing potential, we define the mode coupling strength as

$$
\langle x(\boldsymbol{l}) x'(\boldsymbol{l'}) \rangle_{CMB} = f_\alpha(\boldsymbol{l}, \boldsymbol{l'}) \phi(\boldsymbol{L}). \tag{N.18}
$$

Here, $\boldsymbol{L} = \boldsymbol{l} + \boldsymbol{l'}$, and x is one of anisotropy variables, $\{\Theta,\ E,\ B\}$, and α denotes the xx' pairing. We calculate the coupling strength f_α by using equations (N.15), (N.16) and (N.17) [2]:

$$
\begin{aligned}
f_{\Theta\Theta}(\boldsymbol{l}_1, \boldsymbol{l}_2) &= \tilde{C}_{l_1}^{\Theta\Theta}(\boldsymbol{L} \cdot \boldsymbol{l}_1) + \tilde{C}_{l_2}^{\Theta\Theta}(\boldsymbol{L} \cdot \boldsymbol{l}_2) && \text{(N.19)} \\
f_{\Theta E}(\boldsymbol{l}_1, \boldsymbol{l}_2) &= \tilde{C}_{l_1}^{\Theta E} \cos(2\varphi_{\boldsymbol{l}_1 \boldsymbol{l}_2})(\boldsymbol{L} \cdot \boldsymbol{l}_1) + \tilde{C}_{l_2}^{\Theta E}(\boldsymbol{L} \cdot \boldsymbol{l}_2) && \text{(N.20)} \\
f_{\Theta B}(\boldsymbol{l}_1, \boldsymbol{l}_2) &= \tilde{C}_{l_1}^{\Theta E} \sin(2\varphi_{\boldsymbol{l}_1 \boldsymbol{l}_2})(\boldsymbol{L} \cdot \boldsymbol{l}_1) && \text{(N.21)} \\
f_{EE}(\boldsymbol{l}_1, \boldsymbol{l}_2) &= \left[\tilde{C}_{l_1}^{EE}(\boldsymbol{L} \cdot \boldsymbol{l}_1) + \tilde{C}_{l_2}^{EE}(\boldsymbol{L} \cdot \boldsymbol{l}_2)\right] \cos(2\varphi_{\boldsymbol{l}_1 \boldsymbol{l}_2}) && \text{(N.22)} \\
f_{EB}(\boldsymbol{l}_1, \boldsymbol{l}_2) &= \left[\tilde{C}_{l_1}^{EE}(\boldsymbol{L} \cdot \boldsymbol{l}_1) - \tilde{C}_{l_2}^{BB}(\boldsymbol{L} \cdot \boldsymbol{l}_2)\right] \sin(2\varphi_{\boldsymbol{l}_1 \boldsymbol{l}_2}) && \text{(N.23)} \\
f_{BB}(\boldsymbol{l}_1, \boldsymbol{l}_2) &= \left[\tilde{C}_{l_1}^{BB}(\boldsymbol{L} \cdot \boldsymbol{l}_1) + \tilde{C}_{l_2}^{BB}(\boldsymbol{L} \cdot \boldsymbol{l}_2)\right] \cos(2\varphi_{\boldsymbol{l}_1 \boldsymbol{l}_2}) && \text{(N.24)}
\end{aligned}
$$

However, we cannot use equation (N.18) as an estimator of ϕ, as $\langle \phi \rangle = 0$ due to the statistical isotropy. Instead, we estimate the deflection field $d(\hat{\boldsymbol{n}})$ by taking a weighted average over

[2] There is a typo in Table 1 of Hu & Okamoto (2002). For $\alpha = \Theta E$, $\cos(\varphi_{\boldsymbol{l}_1 \boldsymbol{l}_2})$ in the equation has to be $\cos(2\varphi_{\boldsymbol{l}_1 \boldsymbol{l}_2})$.

multipole moments. The estimate suggested by Hu & Okamoto (2002) is

$$d_\alpha(\boldsymbol{L}) = \frac{A_\alpha(L)}{L}\int \frac{d^2 l_1}{(2\pi)^2} x(\boldsymbol{l}_1)x'(\boldsymbol{l}_2)F_\alpha(\boldsymbol{l}_1,\boldsymbol{l}_2). \tag{N.25}$$

Here, we introduce a normalization factor

$$A_\alpha(L) = L^2\left[\int \frac{d^2 l_1}{(2\pi)^2} f_\alpha(\boldsymbol{l}_1,\boldsymbol{l}_2)F_\alpha(\boldsymbol{l}_1,\boldsymbol{l}_2)\right]^{-1}, \tag{N.26}$$

so that

$$\langle d_\alpha(\boldsymbol{L})\rangle_{CMB} \equiv L\phi(\boldsymbol{L}) \tag{N.27}$$

is satisfied. By minimizing the variance $\langle d_\alpha(L)d_\alpha(L)\rangle$, Hu & Okamoto (2002) has calculate the minimum variance filter $F_\alpha(\boldsymbol{l}_1,\boldsymbol{l}_2)$ as:

$$F_\alpha(\boldsymbol{l}_1,\boldsymbol{l}_2) = \frac{C_{l_1}^{x'x'}C_{l_2}^{xx}f_\alpha(\boldsymbol{l}_1,\boldsymbol{l}_2) - C_{l_1}^{xx'}C_{l_2}^{xx'}f_\alpha(\boldsymbol{l}_2,\boldsymbol{l}_1)}{C_{l_1}^{xx}C_{l_2}^{x'x'}C_{l_1}^{x'x'}C_{l_2}^{xx} - \left(C_{l_1}^{xx'}C_{l_2}^{xx'}\right)^2}, \tag{N.28}$$

For $\alpha = \Theta\Theta$, EE and BB, F_α is reduced to

$$F_\alpha(\boldsymbol{l}_1,\boldsymbol{l}_2) = \frac{f_\alpha(\boldsymbol{l}_1,\boldsymbol{l}_2)}{2C_{l_1}^{xx}C_{l_2}^{xx}}, \tag{N.29}$$

and if $\tilde{C}_l^{xx'} = 0$ as in the case of $\alpha = \Theta B$ and EB,

$$F_\alpha(\boldsymbol{l}_1,\boldsymbol{l}_2) = \frac{f_\alpha(\boldsymbol{l}_1,\boldsymbol{l}_2)}{C_{l_1}^{xx}C_{l_2}^{x'x'}}. \tag{N.30}$$

What about the noise matrix of lensing reconstruction? The noise matrix for lensing reconstruction,

$$\left\langle d_\alpha^*(\boldsymbol{L})d_\beta(\boldsymbol{L}')\right\rangle = (2\pi)^2\left[C_L^{dd} + N_{\alpha\beta}(L)\right]\delta^D(\boldsymbol{L}-\boldsymbol{L}'), \tag{N.31}$$

is calculated as

$$\begin{aligned} N_{\alpha\beta}(L) &= \frac{A_\alpha(L)A_\beta(L)}{L^2}\int \frac{d^2 l_1}{(2\pi)^2}F_\alpha(\boldsymbol{l}_1,\boldsymbol{l}_2) \\ &\quad \times\left[F_\beta(\boldsymbol{l}_1,\boldsymbol{l}_2)C_{l_1}^{x_\alpha x_\beta}C_{l_2}^{x'_\alpha x'_\beta} + F_\beta(\boldsymbol{l}_2,\boldsymbol{l}_1)C_{l_1}^{x_\alpha x'_\beta}C_{l_2}^{x'_\alpha x_\beta}\right]. \end{aligned} \tag{N.32}$$

Note that for the minimum variance filter,

$$N_{\alpha\alpha}(L) = A_\alpha(L).$$

Now we have five different estimators where $\alpha = \Theta\Theta$, ΘE, ΘB, EE, EB, and $N_{\alpha\beta}$ is a 5 by 5 matrix. We linearly combine these estimators

$$d_{\rm mv}(\boldsymbol{L}) = \sum_\alpha w_\alpha(L)d_\alpha(\boldsymbol{L}) \tag{N.33}$$

with coefficient determined by

$$w_\alpha = \frac{\sum_\beta \left(N^{-1}\right)_{\alpha\beta}}{\mathrm{Tr}\left(N^{-1}\right)}.$$

Then, final noise power spectrum for lensing reconstruction is

$$N_{\mathrm{mv}}(L) = \frac{1}{\sum_{\alpha\beta}\left(N^{-1}\right)_{\alpha\beta}}. \tag{N.34}$$

Finally, as our analysis in Chapter 6 is based on the convergence field $\kappa \equiv \nabla^2\phi/2$, the noise power spectrum in equation (N.34) has to be properly rescaled as following. The deflection field d is defined to be $d = l\phi$ in equation (N.27), therefore, the convergence power spectrum is related to the deflection field power spectrum as

$$C_l^{\kappa\kappa} = \frac{1}{4} l^4 C_l^{\phi\phi} = \frac{1}{4} l^2 C_l^{dd}. \tag{N.35}$$

The noise power spectrum of convergence field scales in the same way:

$$N_l^\kappa = \frac{1}{4} l^2 N_{\mathrm{mv}}(l). \tag{N.36}$$

Figure N.1 and N.2 show two examples of noise power spectra. In those figures, we show the noise power spectra of each of five estimators, $d_{\Theta\Theta}$, $d_{\Theta E}$, $d_{\Theta B}$, d_{EE}, d_{EB} as well as that of the minimum variance linear combination d_{mv} (*orange solid line*).

Figure N.1 shows the noise power spectra for *Planck* mission. As Planck satellite is expected to observe EE and ΘE correlation but probably not to observe the EB and BB correlation, even minimum variance noise power spectrum exceeds the lensing potential power spectrum. Therefore, we cannot measure the lensing potential power spectrum for a single wavenumber for Planck. Instead, we may have to constrain a few parameters which depends on the whole shape of the lensing potential power spectrum.

On the other hand, Figure N.2 shows the noise power spectra for the *nearly perfect* experiment suggested by Hu & Okamoto (2002). For this case, the instrumental noise is so small that we can detect the B-mode polarization from CMB lensing down to $l \sim 1000$ (Figure 2 of Hu & Okamoto (2002)), and we can detect the lensing potential for individual wave mode until $l \sim 1000$. Note that the best estimator is d_{EB} provide the best estimator. It is because B-mode polarization in this analysis is sorely from weak lensing of E-mode polarization as we ignore the primordial tensor mode, Since we need to first reconstruct the lensing potential map in order to cross correlate them with the lens galaxies, we use latter case for our analysis in Chapter 6.

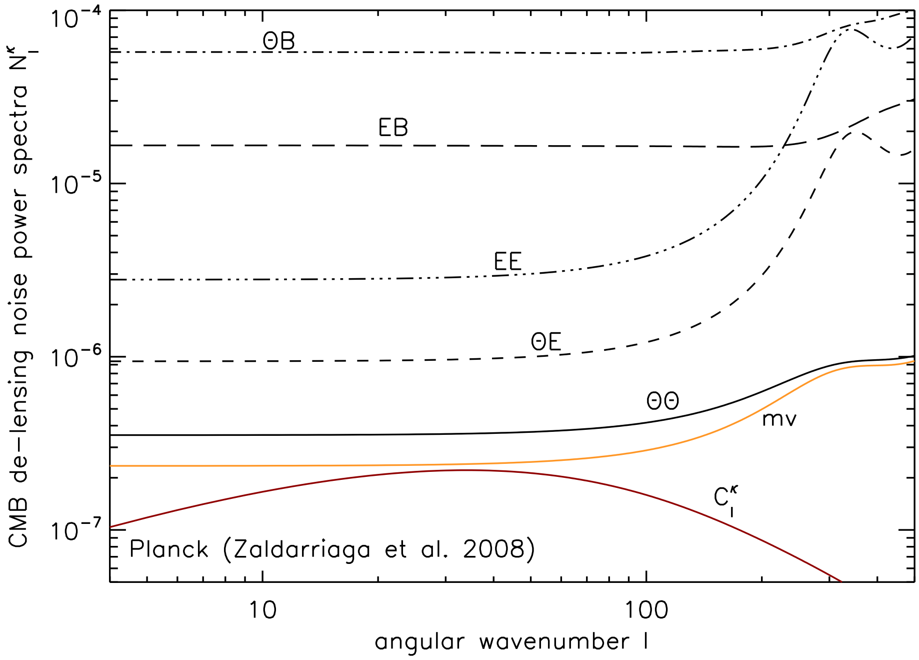

Figure N.1: Noise power spectra of Cosmic Microwave Background lensing reconstruction. We show the noise power spectrum of $d_{\Theta\Theta}$ (*solid line*), $d_{\Theta E}$ (*dashed line*), $d_{\Theta B}$ (*dot-dashed line*), d_{EE} (*dots-dashed line*), d_{EB} (*long-dashed line*), and minimum variance estimator $d_{\rm mv}$ (*orange solid line*). For comparison, we also show the convergence power spectrum C_l^κ (*red line*). Noise power spectrum is calculated for Planck satellite: $\Delta_T = 35.4$ μKarcmin, $\Delta_P = 63.1$ μKarcmin, and $\sigma = 7$ arcmin as described in Appendix A of Zaldarriaga et al. (2008).

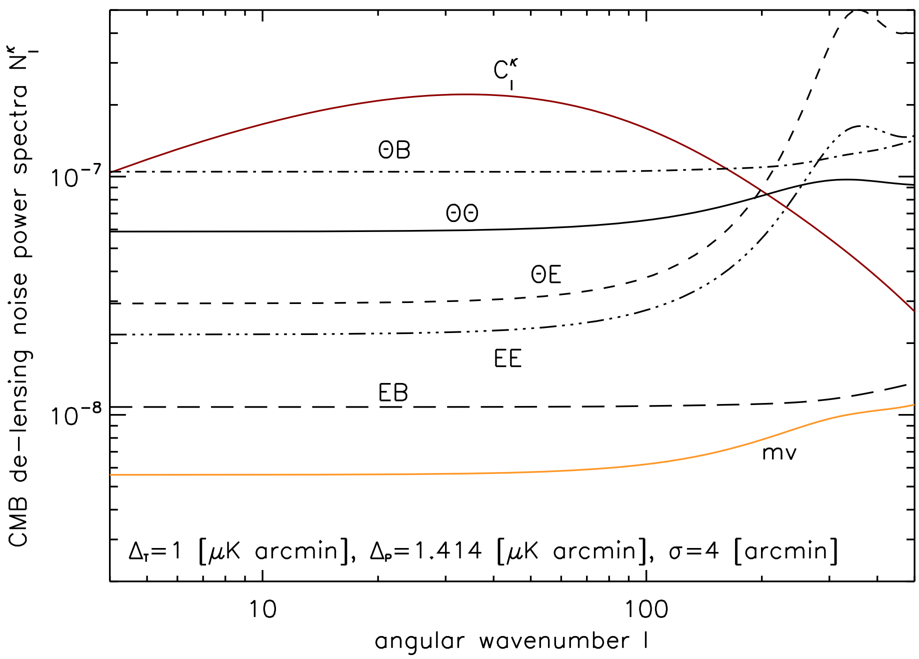

Figure N.2: Same as Figure N.1, but for the *nearly perfect* experiment quoted by Hu & Okamoto (2002). We use $\Delta_T = 1$ μKarcmin, $\Delta_P = \sqrt{2}$ μKarcmin, and $\sigma = 4$ arcmin. In Chapter6, we estimate the noise power spectrum (N_l^κ) by minimum variance estimator (*orange solid line*).