Problem Set 1

1. Fluid equations in index notation (4 points)

Write down the equations of momentum and energy conservation for a *viscous* fluid in index notation in both Eulerian and Lagrangian forms (so four equations in total). For example, the continuity equation in index notation in Eulerian form is

$$\frac{\partial \rho}{\partial t} + \partial_i(\rho v_i) = 0. \tag{1}$$

2. Index notation (5 points)

Use the index notation, show that
(1) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$.
(2) $\nabla \times (\nabla \phi) = 0$
(3) $\nabla \times (\phi \mathbf{A}) = \nabla \phi \times \mathbf{A} + \phi(\nabla \times \mathbf{A})$
(4) $\nabla \times (\nabla^2 \mathbf{A}) = \nabla^2(\nabla \times \mathbf{A})$
(5) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\nabla \cdot \mathbf{B})\mathbf{A} - (\nabla \cdot \mathbf{A})\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$.
Here ϕ is a scalar, while $\mathbf{A}$ and $\mathbf{B}$ are vectors.
Hint: use the identity $\epsilon_{ijk}\epsilon_{abk} = \delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}$.

3. Bernoulli's principle (4 points)

(1) For a vector field $\mathbf{v}$, show that

$$\mathbf{v} \times (\nabla \times \mathbf{v}) = \frac{1}{2}|\mathbf{v}|^2 - (\mathbf{v} \cdot \nabla)\mathbf{v}. \tag{2}$$

(2) When there is external gravity force $\mathbf{f_g} = -\nabla \Phi$, where Φ is the gravitational potential, the momentum equation becomes

$$\frac{d\mathbf{v}}{dt} = -\frac{\nabla P}{\rho} - \nabla \Phi. \tag{3}$$

Plug in the above vector identity to show that, for an ideal fluid (no dissipation),

$$\frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times (\nabla \times \mathbf{v}) + \nabla \left(\frac{|\mathbf{v}|^2}{2} + \Phi \right) + \frac{\nabla P}{\rho} = 0. \tag{4}$$

(3) Assume that the fluid is in a steady state (i.e., $\partial Q/\partial t = 0$ for any quantity Q) and define

the Bernoulli function

$$B \equiv \frac{|\mathbf{v}|^2}{2} + h + \Phi, \tag{5}$$

where $h = u + P/\rho$ is the specific enthalpy (B is essentially the total specific energy including the fluid's ability to do work). Show that

$$\nabla B = T\nabla s + \mathbf{v} \times \boldsymbol{\omega}, \tag{6}$$

using the fact that $\nabla h = T\nabla s + \nabla P/\rho$ from thermodynamics. This is known as the *Crocco's theorem*, which describes the spatial variation of the Bernoulli function.
(4) Project Eq. 4 onto the velocity vector $\mathbf{v}$ (i.e., dot product with $\mathbf{v}$) and show that the material derivative of B vanishes in steady-state flows, i.e.,

$$\frac{dB}{dt} = (\mathbf{v} \cdot \nabla)B = 0, \tag{7}$$

In other words, the Bernoulli function is constant (or conserved) *along streamlines* (which does not mean that B is constant *everywhere*! c.f. Eq. 6).

4. Potential flow (irrotational flow) (1 point)

Assume the fluid has zero vorticity ($\boldsymbol{\omega} = \nabla \times \mathbf{v} = 0$) everywhere, its velocity can then be expressed as $\mathbf{v} = \nabla\phi$, where ϕ is a scalar field, or the velocity potential (not to be confused with the gravitational potential Φ in the previous problem). In the absence of gravity, show that

$$\frac{\partial \phi}{\partial t} + \frac{|\mathbf{v}|^2}{2} + h = \text{const.} \tag{8}$$

In other words, the Bernoulli function of a steady-state potential flow is constant everywhere in space! Note that Eq. 8 also applies to *unsteady* ($\partial/\partial t \neq 0$) potential flows. Potential flows allow analytic descriptions of fluids using the potential theory, but they cannot describe flows near solid surfaces where viscocity becomes important. This leads to the *d'Alembert's paradox* which states that a steady potential flow experiences zero drag force as it passes a solid body.