Equations of Motion
COMP 1601

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Typical Mechanics Quantities:

1) Position
2) Velocity
3) Acceleration
4) Force

Position

Position can be described by a simple point in 2D or 3D space (we will work only in 2D)

Physics models take place in a world with dimensions (units of measure) typically:

SI units (kilogram, meter, second)

SAE units: (pound, feet, second)

To work with physics models we need to have dimension in our simulated game world.

Velocity

Velocity is the rate of change of position over time (units: m/sec)

Velocity can be thought to cause a change in position

Velocity can be constant or changing over time

Speed is the magnitude of a Velocity (it has no direction sense of direction)

Acceleration

Acceleration is the rate of change in velocity over time (units m/sec/sec or m/sec^2

Acceleration can be thought to cause a change in velocity, which in turn causes a change in position.

Acceleration can be constant or changing over time

(The magnitude of acceleration does not have a special name.)

Force

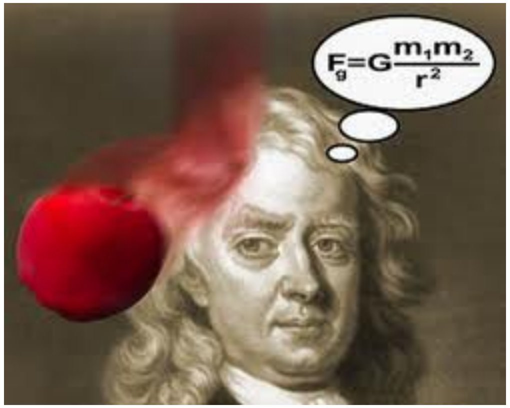
A Force is something that can cause a change in the acceleration of an object with mass.

Units: Newton = 1 kg m/sec^2

Force is related to mass by the Newton's second law: F=ma.

Gravitational Force

Gravitational Force is caused by virtue of objects having mass.



…Force

Gravitational Force is caused by virtue of objects having mass:

$F_G = Gm_1m_2/r^2$ $\quad\quad$ $G = 6.674 \times 10^{-11}$ N-m^2/kg^2

On Earth: $F_G = Gm_1m_{earth}/r^2$

$$a = \frac{F}{m} = \frac{Gm\,mearth}{mr^2} = \frac{Gm_{\;earth}}{r^2} = 9.81\ \frac{m}{sec^2}$$

m = mass, r = distance between centers of masses.

Sample Calculations

Sample Calculations involving gravity

Idea: calculate what should happen based on physics equations of motion and compare that with (Euler approximation-based) simulation code.

Equations of Motion

These equations of motion are typically taught in high school physics. They describe the horizontal and vertical position of an object at time t given knowledge of its initial position, velocity and the constant acceleration it experiences during its travel.

$$x(t) = \tfrac{1}{2} a_x t^2 + v_{x0} t + x_0$$

$$y(t) = \tfrac{1}{2} a_y t^2 + v_{y0} t + y_0$$

Can be evaluated at any time *t* based only on knowing the initial conditions

Equations of Motion

$x(t) = \frac{1}{2} a_x t^2 + v_{x0} t + x_0$

$x(t)$ // horizontal position at time *t*

a_x // horizontal acceleration (constant $a_x=0$)

v_{x0} // initial horizontal velocity

x_0 //initial horizontal position

$y(t) = \frac{1}{2} g t^2 + v_{y0} t + y_0$

$y(t)$ // vertical position at time *t*

$a_y=g$ // vertical acceleration (constant gravity = 9.8 m/sec^2)

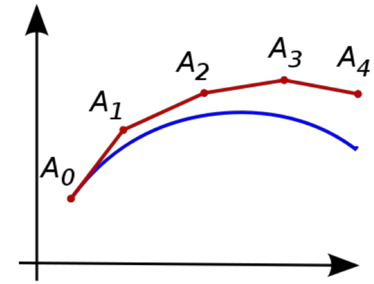
v_{y0} // initial vertical velocity

y_0 //initial vertical position

Euler Approximation of Velocity Integral

Euler Approximation of above

for constant acceleration:



$$v_x(t_2) = v_x(t_1) + a_x \Delta t$$
$$x(t_2) = x(t_1) + v_x(t_2)\Delta t$$
$$v_y(t_2) = v_y(t_1) + a_y \Delta t$$
$$y(t_2) = y(t_1) + v_y(t_2)\Delta t$$

Where $\Delta t = t_2 - t_1$ $\quad a_x = 0$ $\quad a_y = g = 9.8 \frac{m}{\sec \hat{}2}$

In an Euler approximation you compute a new velocity and position after every sampling interval delta-t

The graph shows how an error might accumulate. (The red line shows Euler approximation steps and the blue line the result based on the actual equations of motion)

What is the object's displacement?

- What will be the horizontal displacement of a ball that is kicked off a 850_{m} cliff at a horizontal velocity of $50_{m/s}$?

Displacement along y-component

Account for Vertical motion

- Cliff is 850_m
- Initial speed is $0_{m/s}$
- The vertical displacement is 850_m
- How long will it take to reach the bottom of the cliff?

$$\Delta s = 850_m \qquad a = 9.8_{\frac{m}{s^2}} \qquad v_0 = 0_{\frac{m}{s}}$$

$$s = v_0 t + \frac{at^2}{2}$$

$$850_m = 0t + \frac{9.8_{\frac{m}{s^2}} t^2}{2}$$

$$t^2 = \frac{850_m * 2}{9.8_{\frac{m}{s^2}}} = \frac{1700_m}{9.8_{\frac{m}{s^2}}} = 173.47_{s^2}$$

$$t = 13.17_s$$

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Displacement along the x-component

Compute horizontal displacement

- depends on the amount of time that the object is in the air

$$a = 0_{\frac{m}{s^2}} \quad v_0 = 50_{\frac{m}{s}} \quad t = 13.17_s$$

$$s = v_0 t + \frac{at^2}{2}$$

$$s = 50_{\frac{m}{s}} 13.17_s + \frac{0 * 13.17_s{}^2}{2}$$

$$s = 50_{\frac{m}{s}} 13.17_s = 658.5_m$$

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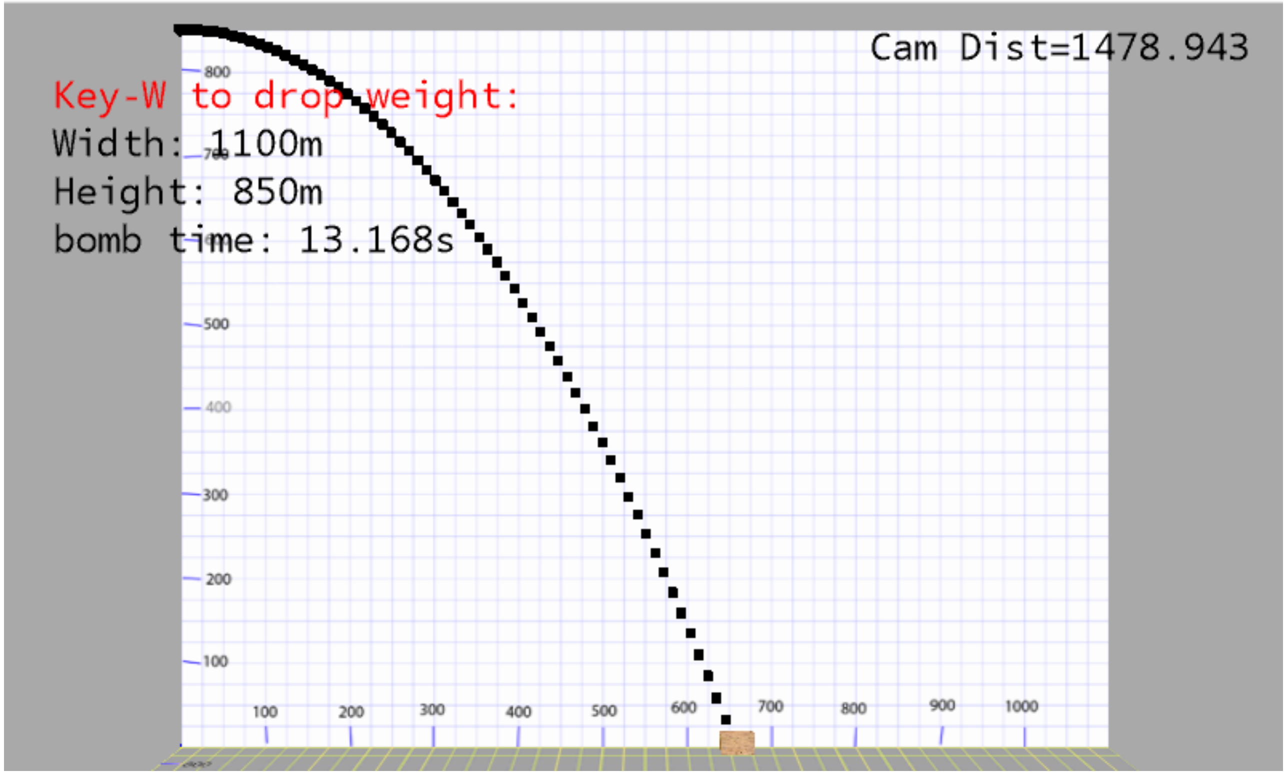
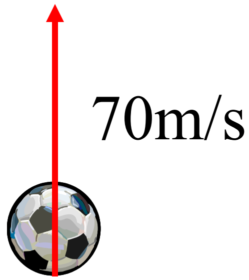
What is the object’s displacement?



Vertical trajectory

- Vertical trajectory occurs when the projectile angle is 90^o to horizontal
  - Kicking a ball straight up at 70m/s
- How long will the ball be in the air?
- What height will the ball reach?

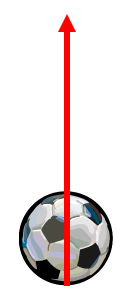


How long will the ball be in the air?

- The ball will travel up, slowing down, until it stops and then it will start a free fall.
- Total time = up time +down time

- The ball will stop when its velocity is 0m/s

- Corollary - the ball will be in the air twice as long as it takes it to stop



How long will it take the ball to stop?

$$v_y = 70_{\frac{m}{s}} \quad a = -9.8_{\frac{m}{s^2}}$$

$$v = v_0 + at$$

$$0_{\frac{m}{s}} = 70_{\frac{m}{s}} - 9.8_{\frac{m}{s^2}} t$$

$$t = \frac{70_{\frac{m}{s}}}{9.8_{\frac{m}{s^2}}} \approx 7.14_s$$

The ball will be in the air twice as long – 14.28s (7.14* 2)

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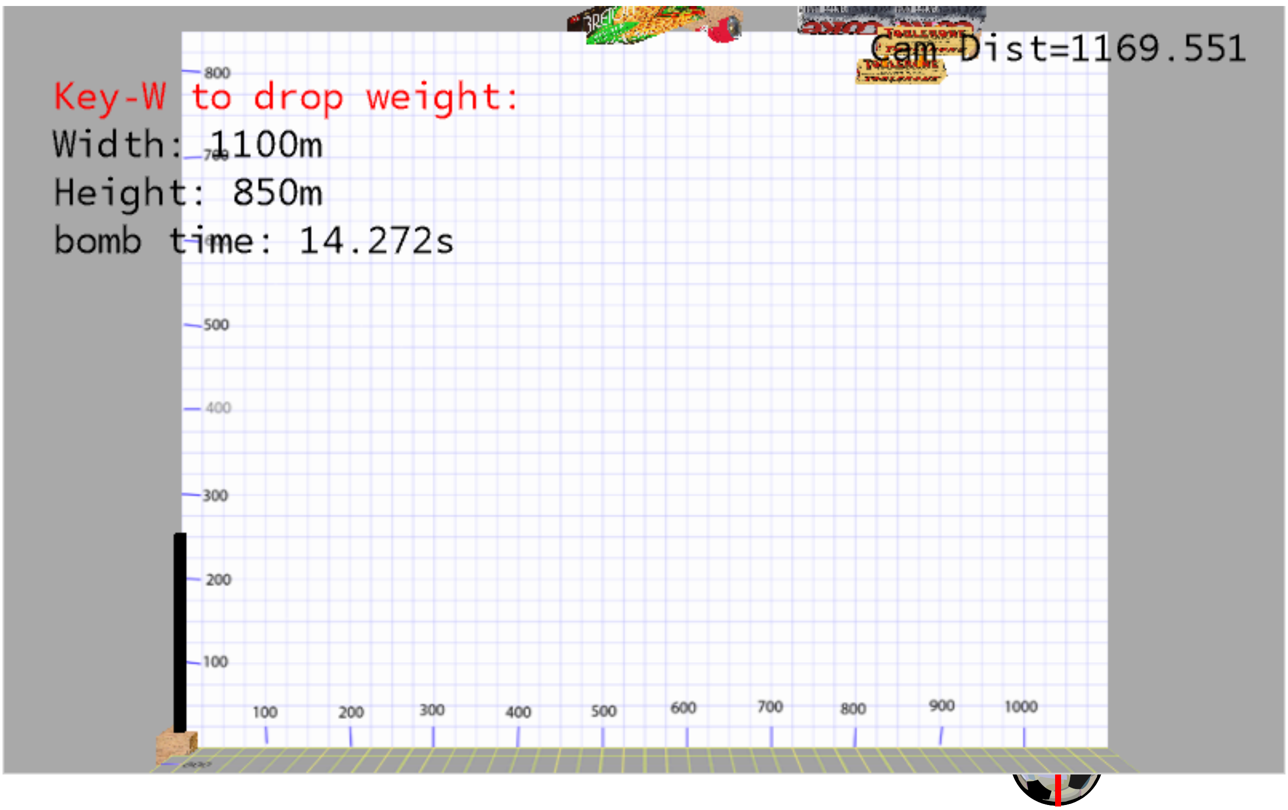
Vertical trajectory



How high will the ball go?

$$v_y = 70_{\frac{m}{s}} \quad a = -9.8_{\frac{m}{s^2}} \quad t = 7.14_s$$

$$h = v_0 t + \tfrac{1}{2} a t^2$$

$$h = 70_{\frac{m}{s}} * 7.14_s - \frac{9.8_{\frac{m}{s^2}} * (7.14_s)^2}{2}$$

$$h = 499.8_m - \frac{499.6_m}{2} = 250_m$$

Example – javelin throwing

- An athlete throws a javelin at 30^o degree angle giving it a speed of 70m/s
- How far did he throw the javelin?
  - First compute the time that the javelin is in the air
  - Using the time compute the distance

Time that the javelin in the air

- The javelin is in the air is twice as long as it takes it to reach its maximum height

$$v_y = 70_{\frac{m}{s}} \sin(30) = 35_{\frac{m}{s}} \quad a = -9.8_{\frac{m}{s^2}}$$

$$v = v_0 + at$$

$$0_{\frac{m}{s}} = 35_{\frac{m}{s}} - 9.8_{\frac{m}{s^2}} t$$

$$t = \frac{35_{\frac{m}{s}}}{9.8_{\frac{m}{s^2}}} \approx 3.57_s$$

$$\text{air time} = 2t = 7.14_s$$

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How far was the javelin thrown?

- The throw distance depends on the horizontal displacement (along the x-component)

$$a_x = 0_{\frac{m}{s^2}} \quad t = 7.14_s$$

$$v_x = 70_{\frac{m}{s}} \cos(30) = 60.62_{\frac{m}{s}}$$

$$s_x = v_x t + \tfrac{1}{2} a_x t^2$$

$$s_x = 60.62_{\frac{m}{s}} * 7.14_s + \tfrac{1}{2} 0 t^2$$

$$s_x = 432.8_m$$

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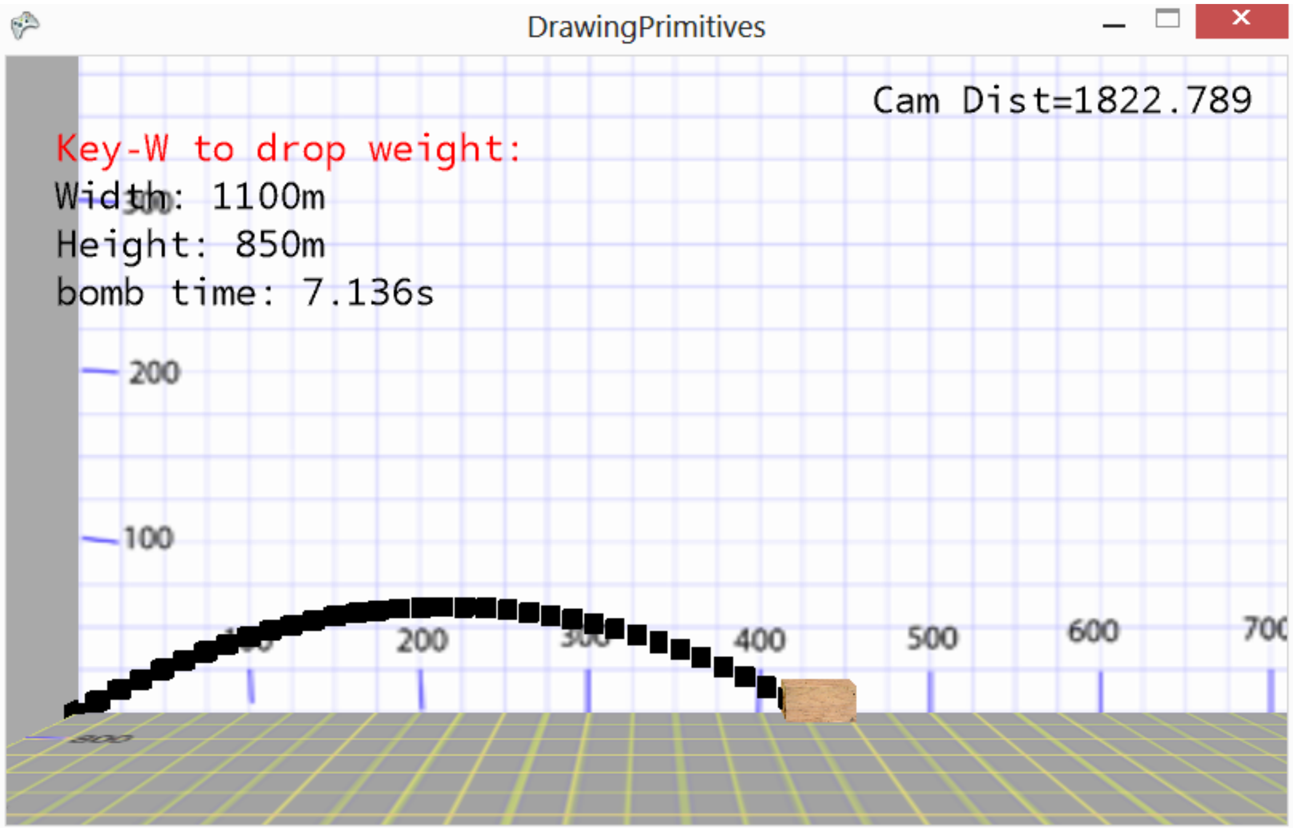
How far was the javelin thrown?



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$$s_x = 60.62_{\frac{m}{s}} * 7.14_s + \tfrac{1}{2} 0 t^2$$

$$s_x = 432.8_m$$

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Time that the javelin in the air

- This time the javelin is thrown at 30º degree angle giving it a speed of 70m/s off an 850m cliff
- The javelin reaches top (0 vertical velocity) in 3.57 seconds
- Next compute how high it went.

$$v_y = 70_{\frac{m}{s}} \sin(30) = 35_{\frac{m}{s}} \quad a = -9.8_{\frac{m}{s^2}}$$

$$v = v_0 + at$$

$$0_{\frac{m}{s}} = 35_{\frac{m}{s}} - 9.8_{\frac{m}{s^2}} t$$

$$t = \frac{35_{\frac{m}{s}}}{9.8_{\frac{m}{s^2}}} \approx 3.57_s$$

$$\text{air time} = 2t = 7.14_s$$

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How high will the javelin go?

$$v_y = 70_{\frac{m}{s}} * \sin(30) = 35_{\frac{m}{s}} \quad a = -9.8_{\frac{m}{s^2}} \quad t = 3.57_s$$

$$h = v_0 t + \tfrac{1}{2} a t^2$$

$$h = 35_{\frac{m}{s}} * 3.57_s - \frac{9.8_{\frac{m}{s^2}} * (3.57_s)^2}{2}$$

$$h = 124.95_m - \frac{124.9_m}{2} = 62.5_m$$

- Javelin reaches a height of 62.5m above its launch height.
- Height from ground is 850 + 62.5 = 912.5m.

How long for the Javelin to fall to ground?

Vertical motion

- Cliff is 850_m
- Initial speed is $0_{m/s}$
- Total vertical displacement is 912.5_m
- How long will it take to reach the bottom of the cliff?

$$\Delta s = 912.5_m \qquad a = 9.8_{\frac{m}{s^2}} \qquad v_0 = 0_{\frac{m}{s}}$$

$$s = v_0 t + \frac{at^2}{2}$$

$$912.5_m = 0t + \frac{9.8_{\frac{m}{s^2}} t^2}{2}$$

$$t^2 = \frac{912.5_m * 2}{9.8_{\frac{m}{s^2}}} = \frac{1825_m}{9.8_{\frac{m}{s^2}}} = 186.225_{s^2}$$

$$t = 13.65_s$$

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How far was the javelin thrown?

The thrown distance depends on the horizontal displacement (along the x-component)

It travels horizontally for the time it is in the air

$$a_x = 0_{\frac{m}{s^2}} \quad t = 3.57 + 13.65 = 17.22_s$$

$$v_x = 70_{\frac{m}{s}} \cos(30) = 60.62_{\frac{m}{s}}$$

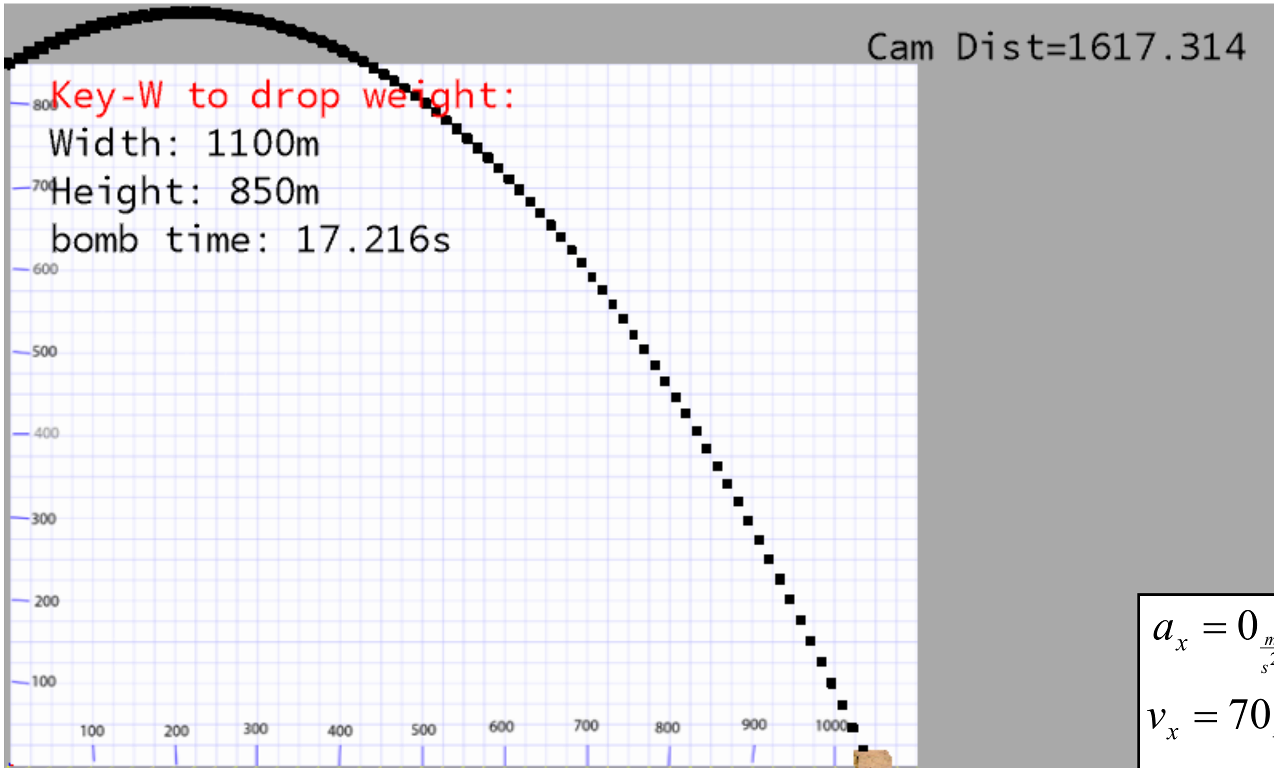
$$s_x = v_x t + \tfrac{1}{2} a_x t^2$$

$$s_x = 60.62_{\frac{m}{s}} * 17.22_s + \tfrac{1}{2} 0 t^2$$

$$s_x = 1043.88_m$$

Show Demo

How far was the javelin thrown?



$$a_x = 0_{\frac{m}{s^2}} \quad t = 3.57 + 13.65 = 17.22_s$$

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$$s_x = 1043.88_m$$