习题四

3. $EX_n=\sum_{k=1}^{N}\sigma_k\sqrt{2}E\cos(\alpha_k n-U_k)=0$,

$r_X(n+\tau,n)=EX_{n+\tau}X_n=E\left[\sum_{k=1}^{N}\sigma_k\sqrt{2}\cos(\alpha_k(n+\tau)-U_k)\right]\left[\sum_{k=1}^{N}\sigma_k\sqrt{2}\cos(\alpha_k n-U_k)\right]$

$=\sum_{k=1}^{N}2\sigma_k^2E\cos(\alpha_k(n+\tau)-U_k)\cos(\alpha_k n-U_k)\xrightarrow[和差]{积化}\sum_{k=1}^{N}\sigma_k^2E\cos\alpha_k\tau=\sum_{k=1}^{N}\sigma_k^2\cos\alpha_k\tau=R(\tau),\ (\tau\in Z)$

故$\{X_n,n\in Z\}$为平稳。

6. 因为$\{X(t)\}$为平稳，故 $Var(X(t))=E(X(t)-m)^2=\sigma^2$（常数），从而：

$0=\left(E(X(t)-m)^2\right)'_t=E2(X(t)-m)X'(t)=2cov(X(t),X'(t))$，即$X(t)$与$X'(t)$不相关。

7. (i) 易见$\{X(t)\}$为平稳。$EZ(t)W(t)=EX(t+1)X(t-1)=R_X(2)=4e^{-4}$，$E(Z(t)+W(t))^2=2R_X(0)+2R_X(2)=8(1+e^{-4})$；

(ii) $f_Z(z)=\frac{1}{2\sqrt{2\pi}}\exp(-z^2/8)$，即$Z(t)\sim N(0,4)$，$P(Z(t)<1)=\Phi(0.5)$；

(iii) $(Z(t),W(t))\sim N(0,0,4,4,e^{-4})$，据此可写出$f_{Z,W}(z,w)$。

8. 证明：不妨设ε所有可能的取值为：$a_1,a_2,\cdots a_n$，则：

$P\{Y(t_1+h)\le x_1,\cdots,Y(t_k+h)\le x_k\}=P\{X(t_1+h-\varepsilon)\le x_1,\cdots,X(t_k+h-\varepsilon)\le x_k\}$

$=\sum_{i=1}^{n}P\{\varepsilon=a_i\}P\{X(t_1+h-\varepsilon)\le x_1,\cdots,X(t_k+h-\varepsilon)\le x_k|\varepsilon=a_i\}$

$=\sum_{i=1}^{n}P\{\varepsilon=a_i\}P\{X(t_1-a_i+h)\le x_1,\cdots,X(t_k-a_i+h)\le x_k|\varepsilon=a_i\}$

$=\sum_{i=1}^{n}P\{\varepsilon=a_i\}P\{X(t_1-a_i)\le x_1,\cdots,X(t_k-a_i)\le x_k|\varepsilon=a_i\}$

$=P\{X(t_1-\varepsilon)\le x_1,\cdots,X(t_k-\varepsilon)\le x_k\}=P\{Y(t_1)\le x_1,\cdots,Y(t_k)\le x_k\}$，$(k\in N)$ 故$\{Y(t)\}$为严平稳。

11. 因$\{X(t)\}$为Gauss过程，故$(X(t+\Delta t)-X(t))/\Delta t$服从正态分布。根据有关理论可知$\lim_{\Delta t\to 0}(X(t+\Delta t)-X(t))/\Delta t=X'(t)$亦服从正态分布。又因为$\{X(t)\}$平稳，则根据协方差函数性质4可知$\{X'(t)\}$亦平稳且有：$EX'(t)=0$，$Var(X'(t))=-R_X^{(2)}(0)$。从而得到：$X'(t)\sim N(0,R_X^{(2)}(0))$，进而容易算得：$P\{X'(t)\le a\}=\Phi(a/\sqrt{-R^{(2)}(0)})$。

14：证明定理4.1的(i)：

必要性：设$\{X_n,n\in Z\}$均值有遍历性并记$\bar{X}_N=\frac{1}{2N+1}\sum_{-N}^{N}X_k$，则有：

$\left(\frac{1}{2N+1}\sum_{0}^{2N}R(\tau)\right)^2=\left[\frac{1}{2N+1}\sum_{k=-N}^{N}cov(X_{-N},X_k)\right]^2=\left[cov(X_{-N},\bar{X}_N)\right]^2\underset{不等式}{\overset{许瓦兹}{\le}}Var(X_{-N})Var(\bar{X}_N)$

$=R(0)E(\bar{X}_N-m)^2\longrightarrow 0.\ (N\to+\infty)$

从而有：$\lim_{N\to+\infty}\frac{1}{2N+1}\sum_{\tau=0}^{2N}R(\tau)=0$，由此易得：$\lim_{N\to+\infty}\frac{1}{N}\sum_{\tau=0}^{N-1}R(\tau)=0$. #

充分性：$E(\bar{X}_N-m)^2=E\left(\frac{1}{2N+1}\sum_{-N}^{N}X_k-m\right)^2=\frac{1}{(2N+1)^2}E\left(\sum_{k=-N}^{N}(X_k-m)\right)^2=\frac{1}{(2N+1)^2}\sum_{i,j=-N}^{N}E(X_i-m)(X_j-m)$

$=\frac{1}{(2N+1)^2}\sum_{i,j=-N}^{N}R(i-j)=\frac{1}{(2N+1)^2}\left[(2N+1)R(0)+2\sum_{0\le j<i\le 2N}R(i-j)\right]=\frac{R(0)}{2N+1}+\frac{2}{(2N+1)^2}\sum_{0\le j<i\le 2N}R(i-j)$

$=A_N+2B_N$. (*)

其中 A_N 显然趋于零，而 B_N 则有

$$|B_N| = \left|\frac{1}{(2N+1)^2}\sum_{0\le j<i\le 2N} R(i-j)\right| = \left|\frac{1}{(2N+1)^2}\sum_{\tau=1}^{2N}(2N+1-\tau)R(\tau)\right|$$

$$= \left|\frac{1}{(2N+1)^2}[2N\times R(1)+(2N-1)\times R(2)+\cdots+1\times R(2N)]\right| = \left|\frac{1}{(2N+1)^2}\left[\sum_{\tau=1}^{2N}R(\tau)+\sum_{\tau=1}^{2N-1}R(\tau)+\cdots+R(1)\right]\right|$$

$$\le \frac{1}{2N+1}\left(\frac{|R(1)|}{1}+\frac{|\sum_1^2 R(\tau)|}{2}+\cdots+\frac{|\sum_1^{2N}R(\tau)|}{2N}\right)\longrightarrow 0.\ (N\to+\infty)$$ (由假设及Cesáro定理)

从而由(※)式可知：$\lim_{N\to+\infty}E(\bar{X}_N-m)^2=0$，即均值遍历性成立。#

15. 证明定理4.3：对于固定的 $\tau\in\mathbb{Z}$，记 $X_{n+\tau}X_n\triangleq Y_n$，则 $EY_n=R_X(\tau)$（常数），且可证明 $\{Y_n\}$ 的协方差仅与时间差有关（见后），即 $Y=\{Y_n,n\in\mathbb{Z}\}$ 为平稳序列。又易见 $X=\{X_n,n\in\mathbb{Z}\}$ 的协方差函数遍历性成立的充要条件是 $Y=\{Y_n,n\in\mathbb{Z}\}$ 的均值遍历性成立。而按题目提示我们有：

$$R_Y(\tau_1)=EY_{n+\tau_1}Y_n-R_X^2(\tau)=EX_{n+\tau_1+\tau}X_{n+\tau_1}X_{n+\tau}X_n-R_X^2(\tau)$$

$$=R_X^2(\tau)+R_X^2(\tau_1)+R_X(\tau_1+\tau)R_X(\tau_1-\tau)-R_X^2(\tau)=R_X^2(\tau_1)+R_X(\tau_1+\tau)R_X(\tau_1-\tau)$$ (由此可见Y平稳)

由此可知：$|R_Y(\tau_1)|\le R_X^2(\tau_1)+(R_X^2(\tau_1+\tau)+R_X^2(\tau_1-\tau))/2$，故由定理所给条件得到：

$$\left|\frac{1}{N}\sum_{\tau_1=0}^{N-1}R_Y(\tau_1)\right|\le\frac{1}{N}\sum_{\tau_1=0}^{N-1}|R_Y(\tau_1)|\le\frac{1}{N}\sum_{\tau_1=0}^{N-1}\left[R_X^2(\tau_1)+(R_X^2(\tau_1+\tau)+R_X^2(\tau_1-\tau))/2\right]\to 0,\ (N\to+\infty)$$

因此由定理4.1的(i)可知，$Y=\{Y_n,n\in\mathbb{Z}\}$ 的均值遍历性成立，亦即 $X=\{X_n,n\in\mathbb{Z}\}$ 的协方差函数遍历性成立。证毕。

16. 先证明 $\{X_n,n\ge 0\}$ 为平稳序列：

$EX_0=\int_0^1 2x^2dx=2/3$，设 $EX_n=2/3$，则 $EX_{n+1}=E[E(X_{n+1}|X_0,\cdots,X_n)]=E(1-X_n/2)=1-\frac{1}{2}\times\frac{2}{3}=2/3$。

故对 $\forall n\ge 0$，有 $EX_n=2/3$。

又 $EX_0^2=\int_0^1 2x^3dx=1/2$，设 $EX_n^2=1/2$，则 $EX_{n+1}^2=E[E(X_{n+1}^2|X_0,\cdots,X_n)]=E\left[\frac{X_n^2}{12}+\left(\frac{2-X_n}{2}\right)^2\right]=$

$=E(X_n^2-3X_n+3)/3=\frac{1}{3}(\frac{1}{2}-3\times\frac{2}{3}+3)=1/2$，即 $EX_n^2\equiv 1/2$，$(\forall n\ge 0)$

设 $\tau\ge 1$，则 $r_X(n+\tau,n)=EX_{n+\tau}X_n-(\frac{2}{3})^2=E[E(X_{n+\tau}X_n|X_0,\cdots,X_{n+\tau-1})]-\frac{4}{9}=$

$=E[X_nE(X_{n+\tau}|X_0,\cdots,X_{n+\tau-1})]-\frac{4}{9}=E(X_n-\frac{1}{2}X_nX_{n+\tau-1})-\frac{4}{9}=\frac{2}{3}-\frac{1}{2}EX_{n+\tau-1}X_n-\frac{4}{9}=$

$=\frac{2}{3}-\frac{1}{2}\cdot\frac{2}{3}+(-\frac{1}{2})^2EX_{n+\tau-2}X_n-\frac{4}{9}=\cdots\cdots=\frac{2}{3}-\frac{1}{2}\cdot\frac{2}{3}+(-\frac{1}{2})^2\frac{2}{3}+\cdots+(-\frac{1}{2})^{\tau-1}\frac{2}{3}+(-\frac{1}{2})^{\tau}EX_n^2-\frac{4}{9}=$

$=\frac{2}{3}(1+(-\frac{1}{2})+(-\frac{1}{2})^2+\cdots+(-\frac{1}{2})^{\tau-1})+\frac{1}{2}(-\frac{1}{2})^{\tau}-\frac{4}{9}=\frac{1}{18}(-\frac{1}{2})^{\tau}.$

对于一般的 $\tau\in\mathbb{Z}$，可知有 $r_X(n+\tau,n)=(-\frac{1}{2})^{|\tau|}/18=R(\tau)$，从而 $\{X_n,n\ge 0\}$ 为平稳，且因 $\lim_{\tau\to\infty}R(\tau)=0$，故由推论4.2可知 $\{X_n,n\ge 0\}$ 的均值遍历性成立。

21: $S(\omega)=\int_{-\infty}^{+\infty}\sigma^2 e^{-\tau^2}e^{-i\omega\tau}d\tau = \sigma^2 e^{-\frac{\omega^2}{4}}\int_{-\infty}^{+\infty} e^{-(\tau^2+i\omega\tau-\frac{\omega^2}{4})}d\tau = \sigma^2 e^{-\frac{\omega^2}{4}}\int_{-\infty}^{+\infty} e^{-\frac{(\tau+\frac{i\omega}{2})^2}{2\times\frac{1}{2}}}d\tau = \sigma^2\sqrt{\pi}\,e^{-\frac{\omega^2}{4}}$.

$S(\omega)$为$\mathbb{R}$上的实的、偶的、非负且可积的函数。

22. 因为: $\cos\omega_0\tau \longleftrightarrow \pi(\delta(\omega+\omega_0)+\delta(\omega-\omega_0))$, $e^{-a|\tau|}\longleftrightarrow \frac{2a}{a^2+\omega^2}$, 故所求谱密度函数 $S(\omega)=\frac{a^2\pi}{2}(\delta(\omega+\omega_0)+\delta(\omega-\omega_0))+\frac{2ab^2}{a^2+\omega^2}$。

23. 由4.3.2节中平方检波的结果可知:

$$R_Y(\tau)=2R_X^2(\tau)=2A^2e^{-2\alpha|\tau|}\cos^2\beta\tau = A^2e^{-2\alpha|\tau|}(1+\cos2\beta\tau)=A^2(e^{-2\alpha|\tau|}+e^{-2\alpha|\tau|}\cos2\beta\tau)$$

由于Fourier变换关系: $e^{-a|\tau|}\longleftrightarrow\frac{2a}{a^2+\omega^2}$, $e^{-a|\tau|}\cos\omega_0\tau\longleftrightarrow\frac{a}{a^2+(\omega+\omega_0)^2}+\frac{a}{a^2+(\omega-\omega_0)^2}$

故可得到$R_Y(\tau)$所对应的谱密度为:

$$S(\omega)=A^2\left(\frac{4\alpha}{4\alpha^2+\omega^2}+\frac{2\alpha}{4\alpha^2+(\omega+2\beta)^2}+\frac{2\alpha}{4\alpha^2+(\omega-2\beta)^2}\right)$$

$$=2\alpha A^2\left[\frac{2}{4\alpha^2+\omega^2}+\frac{1}{4\alpha^2+(\omega+2\beta)^2}+\frac{1}{4\alpha^2+(\omega-2\beta)^2}\right]$$

25: $S(\omega)=\frac{\omega^2}{(\omega^2+1)(\omega^2+3)}=-\frac{1}{2(\omega^2+1)}+\frac{3}{2(\omega^2+3)}$, 故由上题类似的方法可知$S(\omega)$所对应的$R(\tau)=-\frac{1}{4}e^{-|\tau|}+\frac{\sqrt{3}}{4}e^{-\sqrt{3}|\tau|}$, 从而求得$X(t)$的均方值:

$$EX^2(t)=Var(X(t))=R(0)=\frac{\sqrt{3}-1}{4}.\ (假定EX(t)=0)$$

27: $S(\omega)$为: (1) $a\sigma^2\left[\frac{1}{a^2+(\omega+b)^2}+\frac{1}{a^2+(\omega-b)^2}\right]$; (2) $\frac{a\sigma^2\omega}{b}\left(\frac{1}{a^2+(\omega-b)^2}-\frac{1}{a^2+(\omega+b)^2}\right)$;

(3) $a\sigma^2\left[\frac{2+\omega/b}{a^2+(\omega+b)^2}+\frac{2-\omega/b}{a^2+(\omega-b)^2}\right]$;

(4) $\frac{2a\sigma^2}{a^2+\omega^2}+\frac{2a\sigma^2(a^2-\omega^2)(1-4a^2)}{(a^2+\omega^2)^2}+\frac{4a^3\sigma^2(a^4-4a^2\omega^2+\omega^4)}{(a^2+\omega^2)^4}$ 。

28: $R(\tau)$等于: (1) $\frac{5}{7}e^{-2|\tau|}-\frac{13}{70}e^{-5|\tau|}$; (2) $\frac{1}{4}e^{-|\tau|}(1+|\tau|)$;

(3) $\sum_{k=1}^{N}\frac{a_k}{2b_k}e^{-b_k|\tau|}$; (4) $\frac{a\sin b\tau}{\pi\tau}$;

(5) $\frac{b^2}{\pi\tau}(\sin2a\tau-\sin a\tau)$。

补充题答案: $S(\omega)=\frac{2a}{a^2+\omega^2}\longleftrightarrow R(\tau)=e^{-a|\tau|}$; $S(\omega)=\frac{4k^3}{(k^2+\omega^2)^2}\longleftrightarrow R(\tau)=(1+k|\tau|)e^{-k|\tau|}$;

$S(\omega)=\frac{4k\omega^2}{(k^2+\omega^2)^2}\longleftrightarrow R(\tau)=(1-k|\tau|)e^{-k|\tau|}$; $R(\tau)=\max(1-|\tau|/T,0)\longleftrightarrow S(\omega)=\frac{4\sin^2(\omega T/2)}{T\omega^2}$;

$R(\tau)=e^{-\frac{k}{2}\tau^2}\longleftrightarrow S(\omega)=\sqrt{\frac{2\pi}{k}}e^{-\omega^2/2k}$; $R(\tau)=e^{-a|\tau|}\cos\omega_0\tau\longleftrightarrow S(\omega)=\frac{a}{a^2+(\omega+\omega_0)^2}+\frac{a}{a^2+(\omega-\omega_0)^2}$。

4.3

31. 设$\{X_n, n\in \mathbb{Z}\}$的均值为0，协方差为$R(\tau)$，

(1) 设$\hat{X}^* = aX_n$为X_{n+1}的最佳预报，则据投影定理应有：$E(X_{n+1}-\hat{X}^*)bX_n =$

$= bE(X_{n+1}X_n - aX_n^2) = 0$. $(\forall b\in \mathbb{R})$. 不妨设$b\neq 0$. 则有：$R(1) - aR(0) = 0$, $\Rightarrow a = \dfrac{R(1)}{R(0)}$；

(2) 类似可求出：$a = \dfrac{(R(0)-R(2))R(1)}{R^2(0)-R^2(1)}$，$b = \dfrac{(R(0)-R(1))R(2)}{R^2(0)-R^2(1)}$；

(3) $\hat{X}_{n+1}^{(2)}$的均方误差较小，且有：

$$E[X_{n+1}-\hat{X}_{n+1}^{(2)}]^2 - E[X_{n+1}-\hat{X}_{n+1}^{(1)}]^2 = \frac{-(R^2(1)-R(0)R(2))^2}{R^2(0)-R^2(1)};$$

(4)

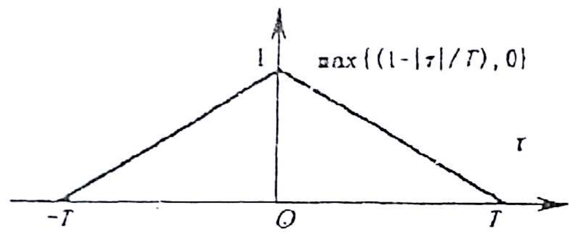
$$a = \frac{R(0)R(k)-R(N-k)R(N)}{R^2(0)-R^2(N)},\quad b = \frac{R(0)R(N-k)-R(k)R(N)}{R^2(0)-R^2(N)};$$

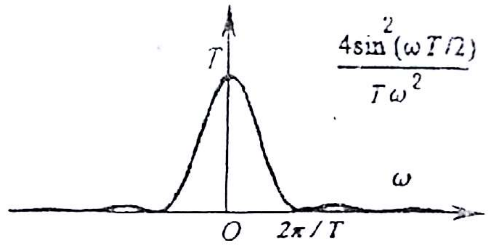
(5) $a = b = \dfrac{\sum_{k=0}^{N} R(k)}{R(0)+R(N)}$。

附表2 常见的相关函数与谱密度函数

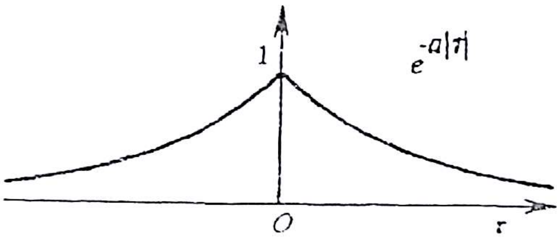
	$R(\tau)$	$S(\omega)$

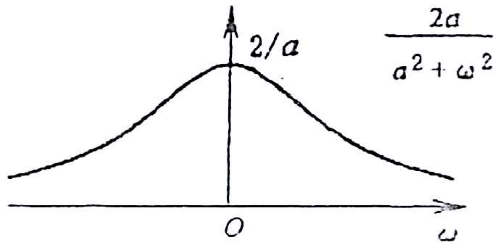
(1)



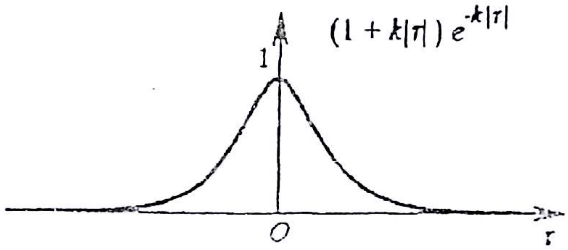


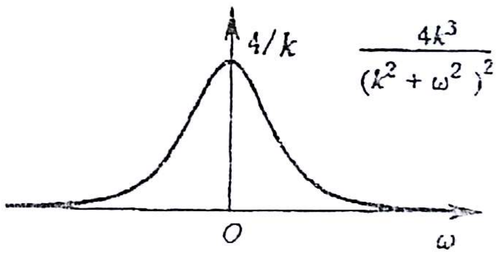
(2)



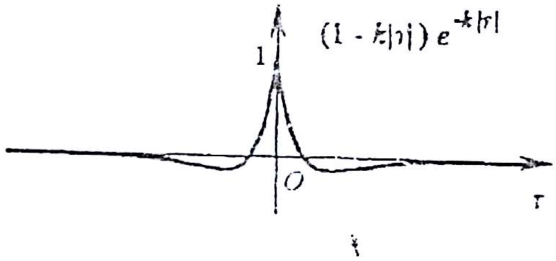


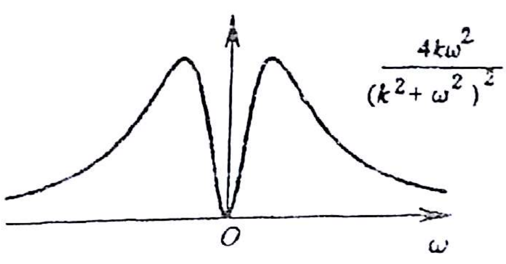
(3)



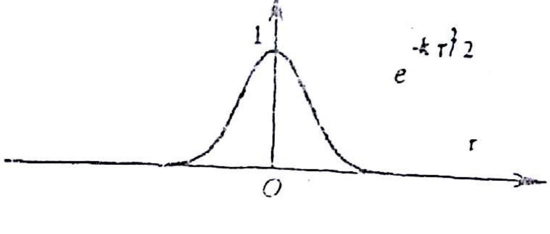


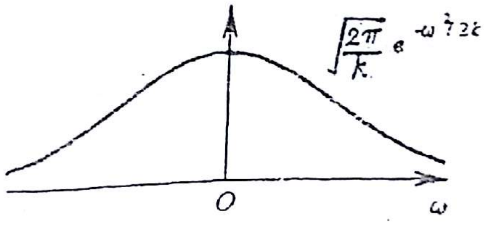
(4)



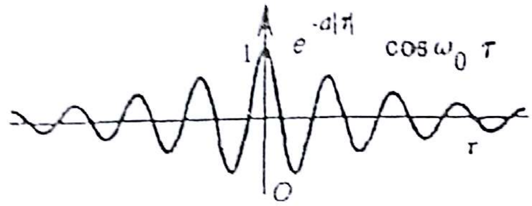
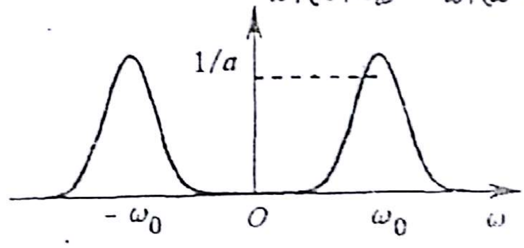
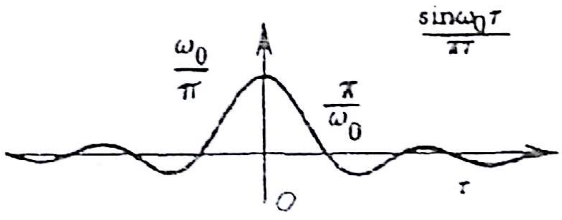
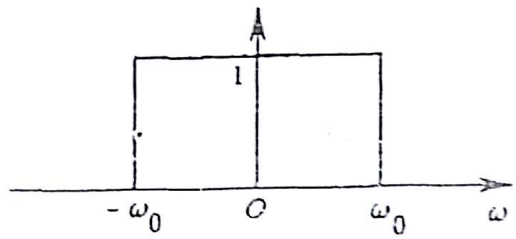
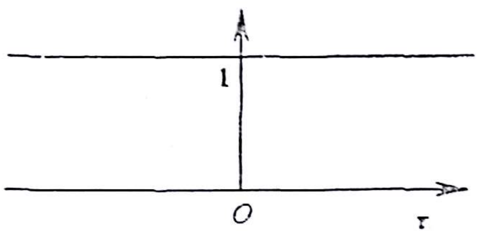
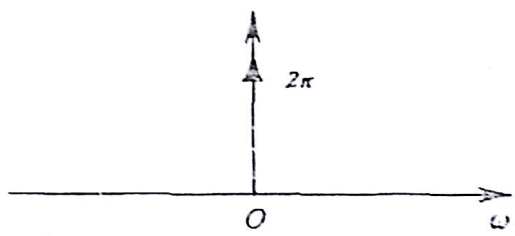
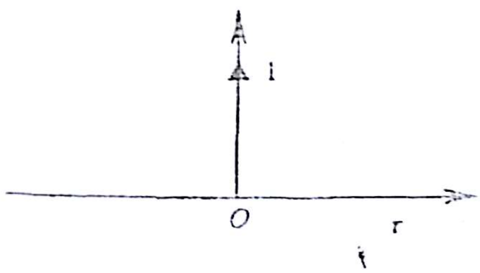
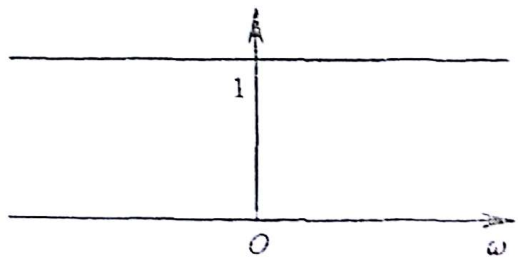


(5)





续附表 2

	$R(\tau)$	$S(\omega)$
(6)		$\frac{a}{a^2+(\omega+\omega_0)^2}+\frac{a}{a^2+(\omega-\omega_0)^2}$ 
(7)		
(8)		
(9)		
(10)