1.(2) 令 $z=2e^{i\theta}$, $dz=2ie^{i\theta}d\theta$

$$\int_C \frac{2z-3}{z}dz = \int_C 2\,dz - \int_C \frac{3}{z}dz$$

$$= 8 - \int_{-\pi}^{0} \frac{3}{2e^{i\theta}} 2ie^{i\theta}\, d\theta$$

$$= 8 - \int_{-\pi}^{0} 3i\, d\theta$$

$$= 8-3\pi i$$

~~2(3) $z=e^{i\theta}$, $dz=ie^{i\theta}d\theta$~~

~~$\int_{-i}^{i}|z|dz=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} ie^{i\theta}d\theta=$~~

2.(3) $\int_{-i}^{i}|z|dz=\int_{-i}^{i}1\,dz=i-(-i)=2i$

3(2) 由长大不等式

$$\left|\int_{-i}^{i}(x^2+iy^2)dz\right|\le \pi\sqrt{x^4+y^4}\le\pi$$

其中 $x^4+y^4\le x^2+y^2=1$ (因为 $x^2,y^2\le1$)

4. 在 i 到 $2+i$ 的直线段上 $|z|\ge1 \Rightarrow \frac{1}{|z|^2}\le 1$

故 $\left|\int_i^{2+i}\frac{dz}{z^2}\right|\le 2\times1=2$

第四周作业解答.

高源.

7. 解: 证明: 由 $\lim_{z\to\infty} zf(z)=A$ 可知,

$\forall \varepsilon>0, \exists R_0>0$. 当 $|z|>R_0$ 时. $|zf(z)-A|<\varepsilon$.

又 $\because \int_{C_R} \frac{1}{z}dz = \int_0^{\alpha} \frac{iRe^{i\theta}d\theta}{Re^{i\theta}} = i\alpha$.

取 $R>R_0$. $\left|\int_{C_R} f(z)dz - iA\alpha\right| = \left|\int_{C_R} f(z)dz - \int_{C_R}\frac{A}{z}dz\right|$

$= \left|\int_{C_R} \frac{zf(z)-A}{z}dz\right|$

$\le \int_{C_R} \left|\frac{zf(z)-A}{z}\right| ds$

$= \frac{\varepsilon}{R}\cdot R\alpha = \varepsilon\alpha$. 得证. $\lim_{R\to+\infty}\int_{C_R} f(z)dz = iA\alpha$

8. 解: 证明: 由于 $Q(z)$ 比 $P(z)$ 高 2 次. 则 $\lim_{z\to\infty} \frac{zP(z)}{Q(z)}=0$.

由 7 结论可知: $\lim_{R\to+\infty}\int_{|z|=R} \frac{P(z)}{Q(z)}dz=0$, 得证.

(或者: $\left|\frac{P(z)}{Q(z)}\right| = \frac{1}{|z|^2}\cdot|M(z)|$. 取 R 充分大. $\left|\int_{C_R}\frac{P(z)}{Q(z)}dz\right| \le \frac{M}{R^2}2\pi R = \frac{2\pi M}{R}\to 0$.

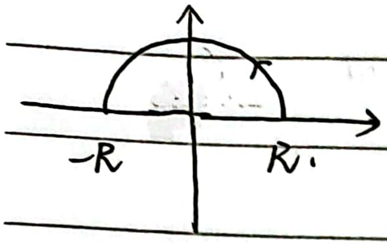
有界, 最大值 M.

$\therefore \lim_{R\to+\infty}\int_{|z|=R}\frac{P(z)}{Q(z)}dz=0$. 得证)

Date. No.

(*)

附加题 Dai 解证明：由 $\lim\limits_{z\to\infty} f(z)=0$ 可知：$\forall \varepsilon>0,\ \exists R_0>0$，当 $|z|>R_0$，$|f(z)|<\varepsilon$



则：$\left|\int_{C_R} f(z)e^{imz}dz\right|$ 由长大不等式

$$\le \int_{C_R}|f(z)|\cdot|e^{imz}|\cdot|dz|$$

由于 $z=Re^{i\theta}$ $(0\le\arg z\le\pi)$，则 $|e^{imz}|=e^{-Rm\sin\theta}$.

$dz=Rie^{i\theta}d\theta \Rightarrow |dz|=Rd\theta$.

$$\therefore |\text{原积分}| \le \int_0^{\pi}|f(z)|\cdot e^{-Rm\sin\theta}\cdot Rd\theta$$

$$=\int_0^{\frac{\pi}{2}}|f(z)|\cdot e^{-Rm\sin\theta}\cdot Rd\theta+\int_{\frac{\pi}{2}}^{\pi}|f(z)|e^{-Rm\sin\theta}\cdot Rd\theta$$

$$=2R\int_0^{\frac{\pi}{2}}|f(z)|e^{-Rm\sin\theta}d\theta$$

由于 $\theta\in[0,\frac{\pi}{2}]$ 时，$\frac{2}{\pi}\theta\le\sin\theta$

$$\therefore |\text{原积分}|\le 2R\int_0^{\frac{\pi}{2}}|f(z)|e^{-Rm\frac{2}{\pi}\theta}d\theta$$

$$=-\frac{m}{\pi}e^{-Rm\frac{2}{\pi}\theta}|f(z)|\Big|_0^{\frac{\pi}{2}}=|f(z)|\left(1-\frac{m}{\pi}\cdot e^{-Rm}\right)$$

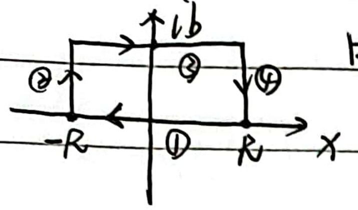
$$<|f(z)|$$

而由(*)式可知：$\forall\varepsilon>0,\ \exists R_0>0$，取 $|z|>R_0$，

则此时 $|\text{原积分}|<|f(z)|<\varepsilon$.

$\therefore \lim\limits_{R\to\infty}\int_{C_R}f(z)e^{imz}dz=0$，得证

附加题2：解：由于在该闭域内 $f(z)=e^{-az^2}$ 解析



由柯西积分定理：$\int_C f(z)\,dz=0.$

即：$\int_{R}^{-R} f(z)\,dz \overset{①}{+} \int_{-R}^{-R+ib} f(z)\,dz \overset{②}{+} \int_{-R+ib}^{R+ib} f(z)\,dz \overset{③}{}$

$+\int_{R+ib}^{R} f(z)\,dz \overset{④}{=} 0.$

对①：$R\to\infty$ 时：$\int_{+\infty}^{-\infty} e^{-az^2}dz = -\int_{-\infty}^{+\infty} e^{-ax^2}dx = -\sqrt{\frac{\pi}{a}}$.（概率积分）

对②：设 $z=-R+iy$. 则：$\int_{-R}^{-R+ib} f(z)\,dz = \int_0^b e^{-a(-R+iy)^2}\,i\,dy = i\int_0^b e^{-aR^2+ay^2+2aRiy}\,dy$

∴ $\left|\int_{-R}^{-R+ib} f(z)\,dz\right| = \int_0^b e^{-aR^2}\cdot e^{ay^2}\,dy$

由于 $\int_0^b e^{ay^2}dy = ye^{ay^2}\Big|_0^b - \underbrace{\int_0^b 2ay^2 e^{ay^2}dy}_{>0}$.

∴ $\int_0^b e^{ay^2}dy < ye^{ay^2}\Big|_0^b = be^{ab^2}$

∴ $\left|\int_{-R}^{-R+ib} f(z)\,dz\right| < e^{-aR^2}\cdot be^{ab^2}$.

当 $R\to+\infty$ 时：$e^{-aR^2}\to 0$. ∴ $\left|\int_{-R}^{-R+ib} f(z)\,dz\right|\to 0$.

对④：令 $z=R+iy$：则：$\int_{R+ib}^{R} f(z)\,dz = \int_b^0 e^{-a(R+iy)^2}\,i\,dy = -i\int_0^b e^{-aR^2+ay^2-2aRiy}\,dy$

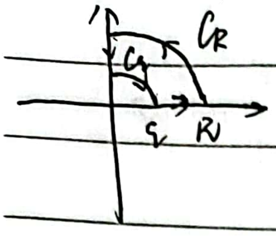
∴ $\left|\int_{R+ib}^{R} f(z)\,dz\right| = \int_0^b e^{-aR^2}\cdot e^{ay^2}\,dy$

同上：可知 $\left|\int_{R+ib}^{R} f(z)\,dz\right| < e^{-aR^2}\cdot be^{ab^2}$.

当 $R\to+\infty$ 时：$\left|\int_{R+ib}^{R} f(z)\,dz\right|\to 0$

综上所述：$\int_{-\infty+ib}^{\infty+ib} f(z)\,dz = \sqrt{\frac{\pi}{a}}$.

附加题3)解：证明：令 $f(z) = z^{s-1} e^{-z}$. 在该闭域内解析.



$\therefore \int_C f(z)\,dz = 0.$

$$\therefore \int_{C_\varepsilon} f(z)\,dz + \int_\varepsilon^R f(z)\,dz + \int_{C_R} f(z)\,dz + \int_{iR}^{i\varepsilon} f(z)\,dz = 0.$$

① ② ③ ④

对①：$\left|\int_{C_\varepsilon} f(z)\,dz\right| < \int_{C_\varepsilon} |f(z)|\,|z'(\theta)|\,d\theta$ $\begin{cases} |f(z)| = \varepsilon^{s-1} \cdot e^{-\varepsilon\cos\theta} \\ |z'(\theta)| = \varepsilon \end{cases}$

取 $z = \varepsilon e^{i\theta}$

$$= \int_{\frac{\pi}{2}}^{0} \varepsilon^s \cdot e^{-\varepsilon\cos\theta}\,d\theta$$

令 $\theta = \frac{\pi}{2} - \theta$ $= \int_{\frac{\pi}{2}}^{0} \varepsilon^s \cdot e^{-\varepsilon\sin\theta}(-d\theta)$ 由于 $\theta \in [0, \frac{\pi}{2}]$ $\sin\theta \geq \frac{2\theta}{\pi}$.

$$\leq \varepsilon^s \int_0^{\frac{\pi}{2}} e^{-\varepsilon\frac{2\theta}{\pi}}\,d\theta$$

$$= \frac{\varepsilon^{s-1}}{2\pi} e^{-\varepsilon\frac{2\theta}{\pi}}\Big|_0^{\frac{\pi}{2}} = \frac{\varepsilon^{s-1}}{2\pi}(e^{-\varepsilon} - 1)$$

由于 $\varepsilon \to 0$ 时：$e^{-\varepsilon} - 1 \sim -\varepsilon$. $\lim\limits_{\varepsilon\to 0}$ 原式 $= \lim\limits_{\varepsilon\to 0} \frac{-2\varepsilon^s}{\pi} = 0.$

对③：$\left|\int_{C_R} f(z)\,dz\right| < \int_{C_R} |f(z)|\,|z'(\theta)|\,d\theta$ $\begin{cases} |f(z)| = R^{s-1} \cdot e^{-R\cos\theta} \\ |z'(\theta)| = R \end{cases}$

$$= \int_0^{\frac{\pi}{2}} R^s \cdot e^{-R\cos\theta}\,d\theta$$

令 $\theta = \frac{\pi}{2} - \theta$

$$= \int_{\frac{\pi}{2}}^{0} R^s \cdot e^{-R\sin\theta}(-d\theta)$$

$$= \int_0^{\frac{\pi}{2}} R^s e^{-R\sin\theta}\,d\theta$$

$$\leq R^s \int_0^{\frac{\pi}{2}} e^{-R\frac{2\theta}{\pi}}\,d\theta$$

$$= \frac{R^{s-1}}{-2\pi} e^{-R\frac{2\theta}{\pi}}\Big|_0^{\frac{\pi}{2}} = -\frac{R^{s-1}}{2\pi}\cdot(e^{-R} - 1)$$

$= \frac{1 - e^{-R}}{2\pi R^{1-s}} \to 0.$ 当 $R \to +\infty$ 时.

$\therefore \int_{\varepsilon}^{R} f(z) + \int_{R}^{\varepsilon} f(z)\,dz = 0$. 当 $\varepsilon \to 0$, $R \to +\infty$.

No.

Date.

有: $\int_0^{+\infty} \underset{\text{取 } z=x}{z^{s-1} e^{-z}\,dz} + \int_{+\infty}^{0} \underset{\text{取 } z=iy}{z^{s-1} e^{-z}\,dz} = 0$

$\therefore \int_0^{+\infty} x^{s-1} e^{-x}\,dx = \int_0^{+\infty} (iy)^{s-1} \cdot e^{-iy}(i\,dy)$

$\Rightarrow \int_0^{+\infty} x^{s-1} e^{-x}\,dx = i^s \int_0^{+\infty} y^{s-1} e^{-iy}\,dy$

$= e^{i\frac{\pi}{2}s}\left[\int_0^{+\infty} y^{s-1} (\cos y - i\sin y)\,dy\right]$

$\therefore e^{-i\frac{\pi s}{2}} \Gamma(s) = \int_0^{+\infty} y^{s-1} \cos y\,dy - i\int_0^{+\infty} y^{s-1} \sin y\,dy$

$\Rightarrow \begin{cases} \text{Re 左} = \Gamma(s) \cdot \cos\frac{\pi s}{2} = \int_0^{+\infty} t^{s-1} \cos t\,dt. = \text{Re 右}. \\ -\text{Im 左} = \Gamma(s) \cdot \sin\frac{\pi s}{2} = -\int_0^{+\infty} t^{s-1} \sin t\,dt. = -\text{Im 右}. \end{cases}$

$\therefore \begin{cases} \int_0^{+\infty} t^{s-1} \cos t\,dt = \Gamma(s) \cos\frac{\pi s}{2} \\ \int_0^{+\infty} t^{s-1} \sin t\,dt = \Gamma(s) \sin\frac{\pi s}{2}. \end{cases}$ 得证.

P71-72

9. 解：$\int_{|z|=1} \frac{e^z}{z} dz = 2\pi i \cdot e^z \Big|_{z=0} = 2\pi i$.

证：由于 $\tilde{\theta} = -\theta$ 时，原式 $= \int_0^{-\pi} e^{\cos\theta} \cos(\sin\theta)(-d\theta) = \int_{-\pi}^{0} e^{\cos\theta} \cos(\sin\theta) d\theta$

$\therefore \int_0^{\pi} e^{\cos\theta} \cos(\sin\theta) d\theta + \int_{-\pi}^{0} e^{\cos\theta} \cos(\sin\theta) d\theta = 2\int_0^{\pi} e^{\cos\theta} \cos(\sin\theta) d\theta$

$\therefore$ 原式 $= \frac{1}{2} \int_{-\pi}^{\pi} e^{\cos\theta} \cos(\sin\theta) d\theta$

$= \frac{1}{2} \int_{-\pi}^{\pi} e^{\cos\theta} \frac{e^{i\sin\theta} + e^{-i\sin\theta}}{2} d\theta = \frac{1}{2} \int_{-\pi}^{\pi} \frac{e^{\cos\theta + i\sin\theta} + e^{\cos\theta - i\sin\theta}}{2} d\theta$

令 $z = e^{i\theta}$. $dz = ie^{i\theta} d\theta \Rightarrow d\theta = \frac{dz}{iz}$.

$\therefore$ 原式 $= \frac{1}{4} \int_{|z|=1} \left(\frac{e^z + e^{\bar{z}}}{iz}\right) dz$

$= \frac{1}{4i} \cdot 2\pi i \left(e^z \Big|_{z=0} + e^{\bar{z}} \Big|_{z=0} \right)$

$= \frac{1}{4i} \cdot 2\pi i \cdot (1+1) = \pi$. 得证.

10. 解：(2) $\int_C \frac{e^z}{1+z^2} dz = \int_{|z+i|=1} \frac{e^z}{(z+i)(z-i)} dz$

$= 2\pi i \cdot \frac{e^z}{z-i} \Big|_{z=-i}$

$= -\pi e^{-i}$

11. 解：奇点：$z=0$.（二阶） $z=-1$. $z=1$（一阶）

① $r>1$. 原式 $= 2\pi i \left[\left(\frac{1}{(z+1)(z-1)}\right)' \Big|_{z=0} + \frac{1}{z^2(z+1)} \Big|_{z=1} + \frac{1}{z^2(z-1)} \Big|_{z=-1} \right]$

奇点均在域内

$= 2\pi i \cdot \left(0 + \frac{1}{2} - \frac{1}{2}\right) = 0$.

②$0<r<1$ 时：原式 $=2\pi i\left[\frac{1}{(z+1)(z-1)}\right]'\Big|_{z=0}=0.$ Date. No.

只有 $z=0$ 在内

综上：原积分 $=0.$

12. 解：(2) 奇点：$z=3.\ z=-3.\ z=-i.$ (一阶).

$|z|=\frac{10}{3}>3$. 故均在内. $f(z)=\frac{-z}{(z^2-9)(z+i)}=\frac{-z}{(z+3)(z-3)(z+i)}$

$\therefore$ 原式 $=2\pi i\left[\frac{-z}{(z+3)(z+i)}\Big|_{z=3}+\frac{-z}{(z-3)(z+i)}\Big|_{z=-3}+\frac{-z}{(z+3)(z-3)}\Big|_{z=-i}\right]$

$=2\pi i\cdot\left(\frac{-3}{6(3+i)}+\frac{3}{6(-3+i)}+\frac{i}{-10}\right)$

$=0$

13. 解：(1) 证：由柯西积分公式可知：$f(z_0)=\frac{1}{2\pi i}\int_c\frac{f(z)}{z-z_0}dz.$

取 $f(z)=2z^2-z+1$. 则 $f(z_0)=\frac{1}{2\pi i}\int_c\frac{2z^2-z+1}{z-z_0}dz$

$\therefore g(z_0)=2\pi i\,f(z_0)$

$\therefore g(1)=2\pi i f(1)=(2-1+1)\cdot 2\pi i=4\pi i.$ 得证.

(2) $|z_0|>2$ 时，积分闭域内无奇点，全解析.

故由柯西积分定理可知：$g(z_0)=0.$

14. 解：奇点：$[(z+i)(z-i)]^2\Rightarrow z=i\quad z=-i.$

$\therefore$ 原积分 $=2\pi i\cdot\left[\frac{z^2}{(z+i)^2}\right]'\Big|_{z=i}$

$=2\pi i\cdot\frac{2iz}{(z+i)^3}\Big|_{z=i}$

$=2\pi i\cdot\frac{1}{4i}$

$=\frac{\pi}{2}$

1). 解：证明：由题可知 $\dfrac{P'(z)}{P(z)} = \dfrac{1}{z-a_1}+\cdots+\dfrac{1}{z-a_n} = \sum_{i=1}^{n} \dfrac{1}{z-a_i}$.

不妨设C内有零点k个，分别为 $a_{(1)}, a_{(2)}, \cdots a_{(k)}$. $k<n$

(k) 表示按下标从小至大排序. $(1)\neq 1$.

由柯西积分公式：原式 $=\dfrac{1}{2\pi i}\sum_{i=1}^{n}\int_C \dfrac{1}{z-a_i}dz$

$$= \frac{1}{2\pi i}\left[2\pi i \times \left(1\Big|_{z=a_{(1)}} + 1\Big|_{z=a_{(2)}}+\cdots+1\Big|_{z=a_{(k)}}\right) + \underline{0\times(n-k)}\right]$$

另外 $n-k$ 个零点不在C内. 则积分为0.

$= k$.

$\therefore \dfrac{1}{2\pi i}\int_C \dfrac{P'(z)}{P(z)}dz = k$ 即C内零点个数，得证.