习题课

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一、基础知识

1

$$\begin{aligned}
&1) \quad z = 2k\pi - \mathrm{i}\ln(2017),\ k \in \mathbb{Z} \\
&2) \quad z_1 = 2,\ z_2 = 4, z_3 = 3 - \mathrm{i},\ z_4 = 3 + \mathrm{i}
\end{aligned}$$

2

$$a = -4,\ f(z) = 4\mathrm{i}z^2 + \mathrm{i}z + 1$$

3

$$\mathrm{R} = 2,\ \sum_{n=0}^{+\infty} \frac{nz^{n-1}}{2^n} = \frac{2}{(2-z)^2}$$

4

$$f(z) = \sum_{n=0}^{+\infty} \frac{1}{n! \cdot z^{n-3}} = \sum_{m=-3}^{+\infty} \frac{1}{(m+3)! \cdot z^m}$$

5

$$\begin{aligned}
&Case\ 1: \quad |z| < 1,\ N_1 = 2 \\
&Case\ 2: \quad |z| = 1,\ N_2 = 0 \\
&Case\ 3: \quad |z| < 2,\ N_3 = 5 \\
&\text{综上所述：} N = N_3 - N_1 - N_2 = 3
\end{aligned}$$

二、计算复积分

1) $-2+4\mathrm{i}$
2) $\frac{2\pi\mathrm{i}}{e^2}$
3) $10\pi\mathrm{i}$
4) $\frac{\pi\mathrm{i}}{3}(5\cos1-6\sin1)$
5) $\frac{4}{5}\pi e^{\frac{5}{3}}$

三、计算定积分

1) $\frac{10\pi}{21\sqrt{21}}$
2) $\pi(1-e^{-2\sqrt{2}}\cos(2\sqrt{2}))$

四、拉普拉斯变换解初值问题

$$y(t)=te^{3t}(2017+\frac{t^2}{6})h(t)$$

五、

解:

设

$$g(z)=(z-a)^m h(z)$$

其中 $h(z)$ 在 a 点全纯，且 $h(a)\neq 0$, 则有

$$g'(z)=m(z-a)^{m-1}h(z)+(z-a)^m h'(z)$$

$$g'(a)=0\Rightarrow m\neq 1$$

求二阶导可得，

$$g''(z)=m(m-1)(z-a)^{m-2}h(z)+2m(z-a)^{m-1}h'(z)+(z-a)^m h''(z)$$

$$g''(a)=q_1\neq 0\Rightarrow m=2, h(a)=\frac{q_1}{2}, g''(z)=2h(z)+4(z-a)h'(z)+(z-a)^2h''(z)$$

求三阶导可得，

$$g'''(z)=6h'(z)+6(z-a)h''(z)+(z-a)^2h'''(z)$$

$$g'''(a)=q_2\Rightarrow h'(a)=\frac{q_2}{6}$$

所以

$$\begin{aligned}\oint_C \frac{f(z)}{g(z)}dz &= 2\pi\mathrm{i}\ \mathrm{Res}[\frac{f(z)}{g(z)}, a] \\ &= 2\pi\mathrm{i}\frac{\mathrm{d}}{\mathrm{dz}}[\frac{f(z)}{h(z)}]|_{z=a} \\ &= 2\pi\mathrm{i}\frac{f'(a)h(a)-f(a)h'(a)}{h^2(a)} \\ &= \frac{4\pi\mathrm{i}}{3q_1^2}(3p_2q_1 - p_1q_2)\end{aligned}$$

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六、

Proof:

1)

因为

$$f^{(n)}(z) = \frac{n!}{2\pi\mathrm{i}}\int_{|w|=r}\frac{f(w)}{(w-z)^{n+1}}dw$$

由长大不等式得

$$\begin{aligned}|f^{(n)}(0)| &= |\frac{n!}{2\pi\mathrm{i}}\int_{|w|=r}\frac{f(w)}{(w-z)^{n+1}}dw| \\ &\le \frac{n!}{2\pi}\max_{|z|\le r}|f(z)|\int_{|w|=r}\frac{|dw|}{|w|^{n+1}} \\ &\le \frac{n!M(r)}{2\pi}\frac{2\pi r}{r^{n+1}} \\ &= \frac{n!M(r)}{r^n}\end{aligned}$$

2)

假设 $f(z)$ 在 $|z| < \frac{|a_0|r}{|a_0|+M(r)}$ 内有零点 z_0, 令

$$g(z) = \frac{f(z)}{z-z_0} = \frac{f(z)-f(z_0)}{z-z_0}$$

可得

$$\lim_{z\to z_0} g(z) = f'(z_0)$$

由 Riemann 定理可知,g 可解析开拓到 $D(0,r)$.

将 $z=0$ 代入，且由最大模原理可知,

$$\frac{|f(0)|}{|z_0|} = \frac{|a_0|}{|z_0|} = |g(0)| \le \max_{|z|=r}|g(z)| = \max_{|z|=r}\frac{|f(z)|}{|z-z_0|} \le \frac{M(r)}{r-|z_0|}$$

整理得

$$\frac{|a_0|r}{M(r)+|a_0|} \le |z_0|$$

与 $\frac{|a_0|r}{M(r)+|a_0|} > |z_0|$ 矛盾，故假设不成立.

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