Differentiation in space

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We begin by stating that if a function $f(x)$ has a Fourier transform $F(s)$

$$f(x) \leftrightharpoons F(s)$$

then

$$\frac{d}{dx} f(x) \leftrightharpoons 2\pi i s F(s)$$

and that in general

$$\frac{d^k}{dx^k} f(x) \leftrightharpoons (2\pi i s)^k F(s)$$

Proof:

The inverse Fourier transform of $F(s)$ is given by

$$f(x) = \int\limits_{-\infty}^{\infty} F(s) e^{2\pi i s x} \, ds$$

Because $F(s)$ is not a funciton of x, taking the derivative of both sides yields is

$$\frac{d}{dx} f(x) = \int\limits_{-\infty}^{\infty} F(s) \frac{d}{dx} e^{2\pi i s x} \, ds = \int\limits_{-\infty}^{\infty} F(s) e^{2\pi i s x} (2\pi i s) \, ds$$

The term $(2\pi i s)$ comes from the chain rule. We associate this factor with $F(s)$ giving us

$$\frac{d}{dx} f(x) = \int\limits_{-\infty}^{\infty} 2\pi i s F(s) e^{2\pi i s x} \, ds$$

Therefore

$$\frac{d}{dx} f(x) \leftrightharpoons 2\pi i s F(s)$$

QED

If we take take the derivative of this we get another $(2\pi is)$ factor. If we take the kth derivative we get $(2\pi ks)^k$. Therefore

$$\frac{d^k}{dx^k} f(x) \leftrightharpoons (2\pi is)^k F(s)$$

QED