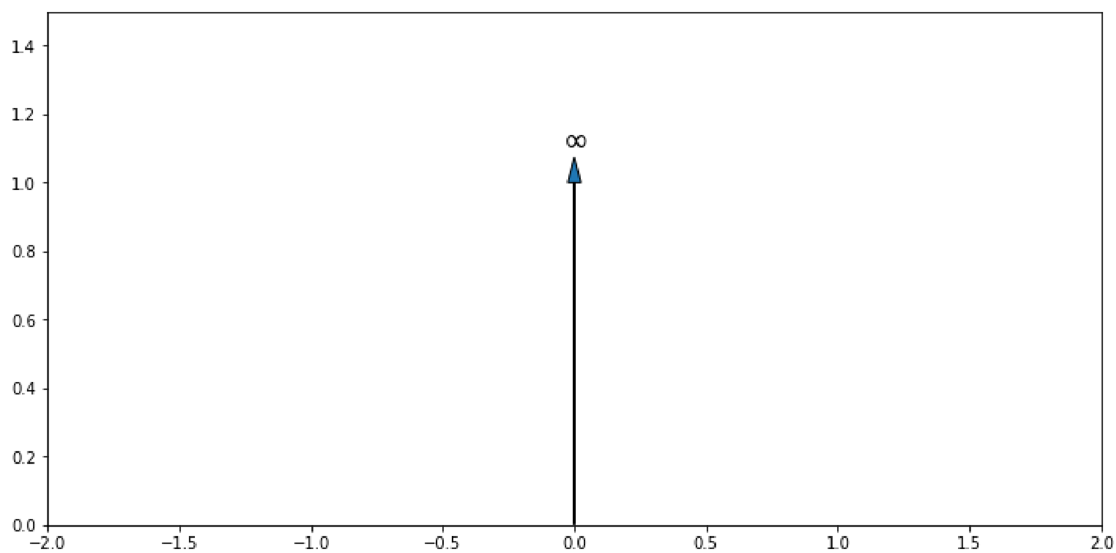
delta function part I

October 3, 2020

1 Delta function $\delta(x)$

The delta function is an integral part of Fourier analysis. Technically speaking, it is not a function. It's typically described as an infintesimally narrow and infinitely tall pulse with the area under the curve equal to one.

$$f(x) = \begin{cases} \infty, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases}$$



The delta function is zero everywhere except at $x = 0$. If has the value of infinitely at $x = 0$. The area under the curve is one.

One way to think of a delta function is as a Gaussian with an infintesimally small σ. In probability theory we use the Gaussian as a probability distribution function. The integral from negative infinity to positive infinity must be equal to one, since a probability greater than one has no meaning. So a very narrow Gaussian fits the bill as a function that is similar to the delta function.

We will normalize a Gaussian function to get a better feel for the delta function. The standard form of a Gaussian function is

$$g(x) = k \exp \frac{x^2}{2\sigma^2}$$

where k is a normalizing constant and σ is a parameter that determines the width. σ is actually the inflection point of curve. i.e. where curvature changes sign, i.e the value of x where the second derivative $\frac{d^2}{dx^2} g(x) = 0$

Our goal is to determine a value for k such that

$$k \int_{-\infty}^{\infty} \exp \frac{-x^2}{2\sigma^2}\ dx = 1$$

We need to evalute the integral and solve for k. This not as easy as it may first appear. There is a trick, however, that makes the integration relatively straight forward. We introduce another function $f(y)$ that is identical to $g(x)$ except that y is the independent variable. So we have two functions:

$$g(x) = k \exp \frac{-x^2}{2\sigma^2}$$

and

$$f(y) = k \exp \frac{-y^2}{2\sigma^2}$$

We have the constraint on both that the integral from positive to negative infinity must be equal to one.

$$\int_{-\infty}^{\infty} g(x)\ dx = 1$$

and

$$\int_{-\infty}^{\infty} f(y)\ dy = 1$$

We can now multiple these two integrals

$$\int_{-\infty}^{\infty} g(x)\ dx \int_{-\infty}^{\infty} f(y)\ dy = 1$$

$$\left[k \int_{-\infty}^{\infty} \exp \frac{x^2}{2\sigma^2}\ dx \right] \left[k \int_{-\infty}^{\infty} \exp \frac{y^2}{2\sigma^2}\ dy \right] = 1$$

Rearranging results in

$$k^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \frac{-x^2}{2\sigma^2} \exp \frac{-y^2}{2\sigma^2} \ dx \ dy = 1$$

We then combine the exponentials

$$k^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \frac{-(x^2+y^2)}{2\sigma^2} \ dx \ dy = 1$$

We can visualize this as a 3 dimensional mound shaped surface over the $x, \ y$ plane. Therefore the double integral represents the volume over the plane. That is, the differential area $dA = dx \ dy$ muliplied by the height of the surface yields the differential volume. Integrating over the entire plane ($-\infty < x < \infty$ and $-\infty < y < \infty$) is the total volume under the surface.

Next we represent this function in cylindrical corrdinates with the following substitutions

$$r^2 = x^2 + y^2$$

and the differential area in rectangular coordinates to the differential area in cyclindrical coordinates (Jacobian)

$$dA = dx \ dy = r \ dr \ d\theta$$

$$k^2 \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} \exp \frac{-r^2}{2\sigma^2} \ r \ dr \ d\theta = 1$$

Let

$$u = \frac{-r^2}{2\sigma^2} \Rightarrow du = \frac{-r}{\sigma^2} \ dr \Rightarrow dr = -\frac{\sigma^2}{r} \ du$$

Note that the limits of integration change with this substitution. Replacing the exponential with u and substuting dr into the double integral yields

$$k^2 \int_{\theta=0}^{2\pi} \int_{u=0}^{-\infty} \exp u \ r \left(-\frac{\sigma^2}{r} \ du \right) \ d\theta = 1$$

The r's cancel (This is the reason for putting things in cylindrical coordinates). Pulling constants out of the integral gives us

$$-k^2\sigma^2 \int\limits_{\theta=0}^{2\pi} d\theta \int\limits_{u=0}^{-\infty} \exp u \; du = 1$$

Integration of $d\theta$ from 0 to 2π gives us 2π and the integral of e^u is e^u so

$$-2\pi k^2\sigma^2 \left[\exp \frac{-r^2}{2\sigma^2} \right]_{r=0}^{\infty} = -2\pi k^2\sigma^2(0-1) = 1$$

Evaluating the integral yields

$$2\pi k^2\sigma^2 = 1$$

Solving for k gives us

$$k = \frac{1}{\sqrt{2\pi\sigma^2}}$$

Therefore

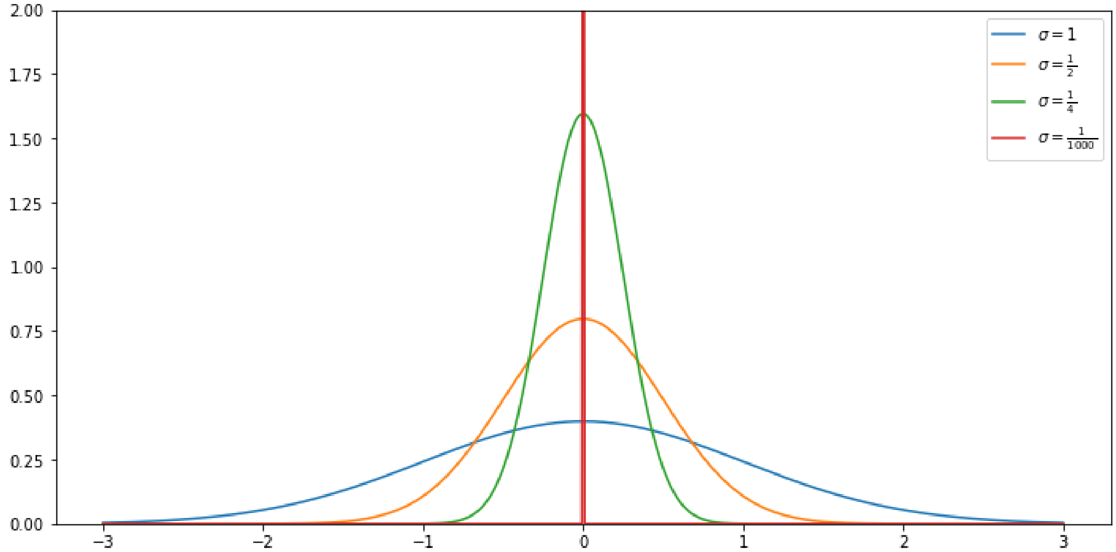
$$g(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-x^2}{2\sigma^2}\right)$$

QED

With some effort we have convinced ourselves that the derived value of k will ensure that the area under the Gaussian will be 1 when the limits of integration are positive and negative infinity. This meets one of the requirements of the delta function. Now we think of the delta function as

$$\delta(x) = \lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-x^2}{2\sigma^2}\right)$$

From the derivation above we know that the area under this curve is one for arbitrarily small σ. From the equation we see that the maximum value of a Gaussian gets larger as σ gets smaller. Therefore, as $\sigma \to 0$ the maximum value of the Gaussian must approach infinity.



The graph above shows Gaussian functions with smaller and smaller σ.