Properties of Fourier Transforms

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2 Linearity property

The Fourier transform of $f_1(x)$ is $F_1(s)$. ie

$$f_1(x) \leftrightharpoons F_1(s)$$

and

$$f_2(x) \leftrightharpoons F_2(s)$$

a and b are constants. The linearity theorem states that

$$af_1(x) \leftrightharpoons aF_1(s)$$

and

$$bf_2(x) \leftrightharpoons bF_2(s)$$

and

$$af_1(x) + bf_2(x) \leftrightharpoons aF_1(s) + bF_2(s)$$

Proof:

let $f(x) = af_1(x) + bf_2(x)$ and let $f(x) \leftrightharpoons F(s)$

From the formula for Fourier transform we have

$$F(s) = \int_{-\infty}^{\infty} f(x)e^{-2\pi isx}\ dx$$

$$F(s) = \int_{-\infty}^{\infty} [af_1(x) + bf_2(x)]e^{-2\pi isx}\ dx$$

We distribute the exponetial and use the linear property of integrals to write

$$F(s) = a\int_{-\infty}^{\infty} f_1(x)e^{-2\pi isx}\ dx + b\int_{-\infty}^{\infty} f_2(x)e^{-2\pi isx}\ dx$$

We see that the 1st term on the right is the Fourier transform of $f_1(x)$ and the 2nd term on the right is Fourier transform of $f_2(x)$. So the Fourier transform is linear because the integration operation is linear.

$$F(s) = aF_1(s) + bF_2(s) \leftrightharpoons f(x) = af_1(x) + bf_2(x)$$

QED

3 Conjugation property

We begin with a quick review of the conjugate of a product of numbers.

Show that the conjugate of a product is the product of the conjugates

$$[(a + ib)(c + id)]^* = (a + ib)^*(c + id)^*$$

We expand the left side and take the conguate

$$[ac + iad + icb + i^2bd]^* = [(ac - bd) + i(ad + cb)]^* = (ac - bd) - i(ad + cb)$$

On the right side we take the conjugate of each factor and then expand

$$(a+ib)^*(c+id)^* = (a-ib)(c-id) = ac - iad - ibc + i^2bd = (ac-bd) - i(ad+bc)$$

comparing the results from the left and right show that they are the same. QED

The conjugation property states that if $f(x) \risingdotseq F(s)$ then

$$f^*(x) \risingdotseq F^*(-s)$$

Proof:

$$f(x) \risingdotseq F(s)$$

$$F(s) = \int_{-\infty}^{\infty} f(x)\ e^{-2\pi sx}\ dx$$

Recall that the conjugate of a product is the product of the conjugates. Therefore taking the conjugate of both sides yields.

$$F^*(s) = \int_{-\infty}^{\infty} f^*(x)\ e^{2\pi sx}\ dx$$

we now replace s with $-s$

$$s \to -s$$

$$F^*(-s) = \int_{-\infty}^{\infty} f^*(x)\ e^{-2\pi sx}\ dx$$

$$F^*(-s) \risingdotseq f^*(x)$$

QED

4 Area under $f(x)$

The area A_x under the curve $f(x)$ is given by

$$A_x = \int_{-\infty}^{\infty} f(x)\ dx$$

The Fourier transform of $f(x)$ is given by

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{-2\pi s x} \ dx$$

The Fourier transform evaluated at $s = 0$ is

$$F(0) = \int_{-\infty}^{\infty} f(x) e^{-2\pi (0) x} \ dx = \int_{-\infty}^{\infty} f(x) \ dx$$

$$A_x = F(s)\big|_{s=0}$$

QED

From this we see that the "DC" value of the signal is simply the area under the curve.

5 Area under $F(s)$

The area A_s under the curve in the frequency domain is given by

$$A_s = \int_{-\infty}^{\infty} F(s) \ ds$$

The inverse Fourier transform is given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{2\pi s x} \ ds$$

Evaluating at $x = 0$ yields

$$f(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{2\pi s (0)} \ ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) \ ds = \frac{1}{2\pi} A_s$$

Therefore

$$A_s = 2\pi f(x)\big|_{x=0}$$

From this we see that the area under the frequency domain curve is 2π times the signal evaluated at zero.

QED

6 Reversal

This property is usually referred to a time reversal, but since we are doing everything in the spatial domain we'll just call it reversal. Of course, there really is no mathematical difference, we are just using x instead of the t.

The reversal property states that if

$$f(x) \leftrightharpoons F(s)$$

then

$$f(-x) \leftrightharpoons F(-s)$$

Proof:

$$F(s) = \int_{-\infty}^{\infty} f(x)\ e^{-2\pi sx}\ dx$$

We replace x with $-x$

$$x \to -x$$

and let

$$F'(s) \leftrightharpoons f(-x)$$

therefore

$$F'(s) = \int_{-\infty}^{\infty} f(-x)\ e^{-2\pi sx}\ dx$$

Notice that the x in the exponential does not change signs. We are taking the Fourier transform of a new function (with reversed x) but the formula for the Fourier transform does not itself change.

Now change the variables: $-x \to \tau$

$$-x = \tau \Rightarrow x = -\tau \Rightarrow dx = -d\tau$$

We change the limits of integration

$$x = -\infty \Rightarrow \tau = \infty$$

$$x = \infty \Rightarrow \tau = -\infty$$

Putting this together simplying, we have the following. *Notice that the minus sign in front of* $d\tau$ *comes outside the integral and is then canceled by reversing the order of integration.*

$$F'(s) = \int_{\infty}^{-\infty} f(\tau)\ e^{-2\pi s(-\tau)}\ (-d\tau) = -\int_{\infty}^{-\infty} f(\tau)\ e^{2\pi s\tau}\ d\tau = \int_{-\infty}^{\infty} f(\tau)\ e^{2\pi s\tau}\ d\tau$$

In order to have the Fourier transform we introduce a minus sign in the exponential but change the sign of s to maintain equality.

$$F'(s) = \int_{-\infty}^{\infty} f(\tau)\ e^{-2\pi(-s)\tau}\ d\tau$$

Therefore, from observation we have

$$F'(s) = F(-s)$$

and

$$f(-x) \leftrightharpoons F(-s)$$

QED

7 Similarity

The similarity theorem states that if

$$f(x) \leftrightharpoons F(s)$$

then

$$f(ax) \leftrightharpoons \frac{1}{|a|} F\left(\frac{s}{a}\right)$$

where a is any real number except for zero.

Proof:

The formula for the Fourier transfrom is

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i s x}\ dx$$

We scale $f(x)$ by a

$$f(x) \to f(ax)$$

Where a is a real, non zero number.

We would like to find

$$F'(s) \leftrightharpoons f(ax)$$

There are two cases: $a > 0$ and $a < 0$

Case I: $a > 0$

$$F'(s) = \int\limits_{-\infty}^{\infty} f(ax)\ e^{-2\pi sx}\ dx$$

let $u = ax \Rightarrow x = \frac{u}{a} \Rightarrow dx = \frac{du}{a}$

The limits of integration do not change when we make the substitution

$$F'(s) = \int\limits_{-\infty}^{\infty} f(u)\ e^{-2\pi s(\frac{u}{a})}\ \frac{du}{a}$$

Now we associate a with s and bring the constant $\frac{1}{a}$ outside the integral to get

$$F'(s) = \frac{1}{a}\int\limits_{-\infty}^{\infty} f(u)\ e^{-2\pi(\frac{s}{a})u}\ du$$

Therefore, by observation,

$$F'(s) = \frac{1}{a}F\left(\frac{s}{a}\right)$$

Case II: $a < 0$

To account for the case where $a < 0$ we take the absolute value of a

$$F'(s) = \frac{1}{|a|}F\left(\frac{s}{a}\right)$$

Therefore

$$f(ax) \leftrightharpoons \frac{1}{|a|}F\left(\frac{s}{a}\right)$$

QED

8 Shift

The shift theorem states that if

$$f(x) \leftrightharpoons F(s)$$

then

$$f(x \pm x_o) \leftrightharpoons e^{\pm 2\pi i x_o} F(s)$$

Proof:

The Fourier transform of $f(x)$ is given by

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x s} \, dx$$

We shift the function to the right by replacing x with $(x - x_o)$.

$$x \to (x - x_o)$$

$$f(x) \to f(x - x_o)$$

We call the Fourier transform of the shifted function $F'(s)$

$$F'(s) = \int_{-\infty}^{\infty} f(x - x_o) \; e^{-2\pi i s x} \, dx$$

let $u = x - x_o \Rightarrow x = u + x_o \Rightarrow dx = du$

$$F'(s) = \int_{-\infty}^{\infty} f(u) \; e^{-2\pi i s (u + x_o)} \, du$$

We separate the exponential

$$F'(s) = \int_{-\infty}^{\infty} f(u) \; e^{-2\pi i s u} \; e^{-2\pi i s x_o} \, du$$

The exponential with x_o is a constant with respect to u so it comes out of the integral. We recognize that what remains in the integral is the Fourier transform of the signal.

$$F'(s) = e^{-2\pi s i x_o} \int_{-\infty}^{\infty} f(u)\ e^{-2\pi i s u}\ du = e^{-2\pi i s x_o} F(s)$$

Therefore

$$f(x \pm x_o) \leftrightharpoons e^{\pm 2\pi i s x_o} F(s)$$

QED

Notice that when the shift is to the right, ie $(x - x_o)$, the exponential has a negative sign and when the shift is to the left, ie $(x + x_o)$, the exponential has a positive sign.

9 Frequency shift

The frequency shift states that if $f(x) \leftrightharpoons F(s)$ then

$$e^{\pm 2\pi i s_o x} f(x) \leftrightharpoons F(s \mp s_o)$$

Proof:
The Fourier transform of $f(x)$ is

$$F(s) = \int_{-\infty}^{\infty} f(x)\ e^{-2\pi i s x}\ dx$$

We refer to the Fourier transform of the phase shifted function $F'(s)$

$$e^{2\pi i s_o x} f(x) \leftrightharpoons F'(s)$$

We apply the Fourier transform formula to the phase shifted signal

$$F'(s) = \int_{-\infty}^{\infty} e^{2\pi i s_o x} f(x)\ e^{-2\pi i s x}\ dx$$

and combine the exponentials

$$F'(s) = \int_{-\infty}^{\infty} f(x)\ e^{-2\pi i (s - s_o) x}\ dx$$

By observation

$$F'(s) = F(s - s_o) \leftrightharpoons e^{2\pi i s_o x} f(x)$$

Notice that if the frequency is shifted to the left the equation has the opposite sign. Therefore

$$F'(s) = F(s \mp s_o) \leftrightharpoons e^{\pm 2\pi i s_o x} f(x)$$

QED

10 Convolution in space (or time)

Before we show the convolution theorem we will state the definition of convolution. We will use $*$ to indicate convolution. $f(x)$ convolved with $g(x)$ is written as

$$f(x) \ * \ g(x) = \int\limits_{-\infty}^{\infty} f(\tau) g(x - \tau) \ d\tau$$

τ is dummy variable that's used to keep things straight in the integral

Given two functions, each with it's corresponding Fourier transforms:

$$f(x) \leftrightharpoons F(s)$$
$$g(x) \leftrightharpoons G(s)$$

The convolution theorem states that

$$f(x) \ * \ g(x) \leftrightharpoons F(s) \ G(s)$$

That is, convolution in the spatial domain is equivalent to multiplication in the spatial frequency domain.

Proof:

let $h(x) = f(x) \ * \ g(x)$

$$h(x) \leftrightharpoons H(s)$$

The Fourier transform of $h(x)$ is given by

$$H(s) = \int\limits_{-\infty}^{\infty} h(x) \ e^{-2\pi s x} \ dx$$

Substituting the equation for $h(x)$ into the formula yields

$$H(s) = \int\limits_{-\infty}^{\infty} \left[\int\limits_{-\infty}^{\infty} f(\tau) g(x - \tau) \ d\tau \right] e^{-2\pi i s x} \ dx$$

let $x-\tau=\lambda \Rightarrow x=\lambda+\tau \Rightarrow dt=d\lambda$.

The range of integration is same after substitution

$$H(s)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau)g(\lambda)\ d\tau\ e^{-2\pi is(\lambda+\tau)}\ d\lambda$$

We separate the exponential

$$H(s)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau)g(\lambda)\ d\tau\ e^{-2\pi is\lambda}\ e^{-2\pi is\tau}\ d\lambda$$

We rearrange by grouping factors with τ together and grouping factors with λ together to yield

$$H(s)=\int_{-\infty}^{\infty} f(\tau)\ e^{-2\pi is\tau}\ d\tau \int_{-\infty}^{\infty} g(\lambda)e^{-2\pi is\lambda}\ d\lambda$$

From this we see that the integral associated with τ is the Fourier transform of $f(x)$ and the integral associated with λ is the Fourier transform of $g(x)$.

$$H(s)=F(s)G(s) \leftrightharpoons f(x)\ *\ g(x)$$

QED

This property is useful in computing convolutions. We can multiply the Fourier transforms of two functions and take the inverse Fourier transform to find the convolution $h(x)$

11 Mulitplication in space

Also known as convolution in spatial frequency.

For two functions with their cooresponding Fourier transforms

$$f(x) \leftrightharpoons F(s)$$
$$g(x) \leftrightharpoons G(s)$$

The convolution in spatial frequency theorem states that

$$f(x)g(x) \leftrightharpoons \frac{1}{2\pi}\Big|F(s)\ *\ G(s)\Big|$$

To prove the theorm above we consider the Fourier transform (FT) of the product of functions in the spatial domain

$$\text{FT}\Big[f(x)g(x)\Big] = \frac{1}{2\pi}\Big[F(s)\ *\ G(s)\Big]$$

Taking the inverse Fourier transform (IFT) of both sides gives us

$$f(x)g(x) = \text{IFT}\Big[\frac{1}{2\pi}\Big(F(s)\ *\ G(s)\Big)\Big]$$

Therefore proving that this is true is the same as proving the theorem.

Let $H(s) = \frac{1}{2\pi}\big(F(s)\ *\ G(s)\big) \leftrightharpoons h(x) = f(x)g(x)$

so

$$h(x) = \text{IFT}\Big[H(s)\Big] = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} H(s)e^{2\pi isx}\ ds$$

We express the convolution directly from the defintion

$$H(s) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} F(\lambda)G(s-\lambda)\ d\lambda$$

We substitute this into the equation for $h(x)$

$$h(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} F(\lambda)G(s-\lambda)\ d\lambda\right]e^{2\pi isx}\ ds$$

Rearranging yields

$$h(x) = \left(\frac{1}{2\pi}\right)^2\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} F(\lambda)G(s-\lambda)\ d\lambda\ e^{2\pi isx}\ ds$$

Next we multiply by $\frac{e^{2\pi i\lambda x}}{e^{2\pi i\lambda x}}$

$$h(x) = \left(\frac{1}{2\pi}\right)^2\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} F(\lambda)G(s-\lambda)\ d\lambda\ e^{2\pi isx}\ ds\ \frac{e^{2\pi i\lambda x}}{e^{2\pi i\lambda x}}$$

We associate the exponential in the numerator with $F(\lambda)$ and the exponetial in the denominator with $G(s-\lambda)$.

$$h(x) = \left(\frac{1}{2\pi}\right)^2\int\limits_{-\infty}^{\infty} F(\lambda)\ e^{2\pi i\lambda x}\ d\lambda\int\limits_{-\infty}^{\infty} G(s-\lambda)\ e^{2\pi isx}\ e^{-2\pi i\lambda x}\ ds$$

We combine the two exponetials associated with $G(s-\lambda)$

$$h(x) = \left(\frac{1}{2\pi}\right)^2 \int\limits_{-\infty}^{\infty} F(\lambda)\ e^{2\pi i\lambda x}\ d\lambda \int\limits_{-\infty}^{\infty} G(s-\lambda)\ e^{2\pi i(s-\lambda)x}\ ds$$

now we let $u = s - \lambda \Rightarrow du = ds$. Substituing these into the equation and associating a $\frac{1}{2\pi}$ with each integral gives us

$$h(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} F(\lambda)\ e^{2\pi i\lambda x}\ d\lambda\ \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} G(u)\ e^{2\pi iux}\ du$$

In this form we can see that the 1st integral is the IFT of $F(s)$ and the 2nd is the IFT of $G(s)$

Therefore we have shown that a convolution in spatial frequency is equivilent to a multiplication in space

$$h(x) = f(x)\ g(x) \leftrightharpoons \frac{1}{2\pi}\Big|F(s)\ *\ G(s)\Big|$$

QED

12 Differentiation in space

For a function $f(x)$ with a Fourier transform $F(s)$

$$f(x) \leftrightharpoons F(s)$$

$$\frac{d}{dx}f(x) \leftrightharpoons 2\pi isF(s)$$

And in general

$$\frac{d^k}{dx^k}f(x) \leftrightharpoons (2\pi is)^k F(s)$$

Proof:

The signal function $f(x)$ is expressed as an inverse Fourier transform:

$$f(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} F(s)\ e^{2\pi isx}\ ds$$

Taking the derivative with respect to x of both sides yeilds

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left[\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} F(s)\ e^{2\pi isx}\ ds\right]$$

Because only the exponetial is a function of x all else is treated as a constant. Applying the chain rule when taking the derivative gives us

$$\frac{d}{dx}f(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} F(s)(2\pi is)\ e^{2\pi isx}\ ds$$

Bringing the constant outside the integrals let's us see that the derivative of the inverse Fourier transform is simply the inverse Fourier transform multiplied by $2\pi is$

$$\frac{d}{dx}f(x) = (2\pi is)\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} F(s)\ e^{2\pi isx}\ ds$$

Therefore

$$\frac{d}{dx}f(x) \leftrightharpoons 2\pi isF(s)$$

QED