Newtonian (Polar)

$$F_g = \frac{GMm}{r^2}$$

$$F = \cancel{m}a = \frac{GM\cancel{m}}{r^2} \qquad a = \frac{GM}{r^2}$$

$$\vec{a} = \langle a_x, a_y \rangle = \langle \ddot{x}, \ddot{y} \rangle$$

$$x = r\cos\theta \qquad \frac{dx}{dt} = \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$

$$y = r\sin\theta \qquad \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

$$\ddot{x} = \ddot{r}\cos\theta - \dot{r}\dot{\theta}\sin\theta - \dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta$$

$$\ddot{y} = \ddot{r}\sin\theta + \dot{r}\dot{\theta}\cos\theta + \dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta$$

$$a_r = \vec{a}\cdot\hat{r}$$

$$\hat{r} = \langle \cos\theta, \sin\theta \rangle$$

$$a_r = \ddot{x}\cos\theta + \ddot{y}\sin\theta$$

$$= \ddot{r}\cos^2\theta - \cancel{\dot{r}\dot{\theta}\sin\theta\cos\theta} - \cancel{\dot{r}\dot{\theta}\sin\theta\cos\theta} - \cancel{r\ddot{\theta}\sin\theta\cos\theta} - r\dot{\theta}^2\cos^2\theta$$

$$+ \ddot{r}\sin^2\theta + \cancel{\dot{r}\dot{\theta}\sin\theta\cos\theta} + \cancel{\dot{r}\dot{\theta}\sin\theta\cos\theta} + \cancel{r\ddot{\theta}\sin\theta\cos\theta} - r\dot{\theta}^2\sin^2\theta$$

$$= \ddot{r} - r\dot{\theta}^2$$

$$ma_r = F_g \qquad \ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2}$$

$$L = mr^2\omega = mr^2\dot{\theta} \qquad \tau = \vec{r}\times\vec{F} = 0 \qquad \tau = \frac{dL}{dt} = 0$$

constant

$$L = mr^2\dot{\theta}$$

$$\frac{L}{m} = r^2\dot{\theta} = h$$

(1) $r_0 =$ ____

(2) $v_{r0} = 0.0$

(3) $v_{t0} = 1.0$

(4) $h = rv_t$ or $r^2\dot{\theta}$

$$\ddot{r} = r^2\dot{\theta} - \frac{GM}{r^2}$$

(1) $\dot{\theta} = \frac{h}{r^2}$ (2) $\theta \mathrel{+}= \dot{\theta}\, dt$

(3) $\ddot{r} = h - \frac{GM}{r^2}$ (4) $\dot{r} \mathrel{+}= \left(h - \frac{GM}{r^2}\right) dt$ (5) $r \mathrel{+}= \dot{r}\, dt$

Schwarzschild Metric

$r_s = \frac{2GM}{c^2}$

$$ds^2 = \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) - c^2\left(1 - \frac{r_s}{r}\right)dt^2$$

for null geodesics

$$ds^2 = -c^2 dt^2 + \underbrace{dx^2 + dy^2 + dz^2}_{dq^2 \to (\text{total distance})^2}$$

speed of photon (aka light) $= c$

so $q = ct$

$dq^2 = c^2 dt^2$

$$= -c^2 dt^2 + c^2 dt^2 = 0$$

$$ds^2 = 0 = \underbrace{g_{\mu\nu}}_{\text{(metric tensor)}} dx^\mu dx^\nu = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \quad \dot{x} = \frac{dx}{d\lambda}$$

curvature of space time

Euler Lagrange

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

$L = ds^2$

$q = r, \phi$

step along light ray

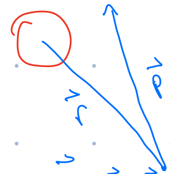
$$\Rightarrow \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$\Rightarrow \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$$

Solve for null-geodesic stepper

$L = \frac{1}{2} g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$ $\quad f(r) = 1 - \frac{r_s}{r}$ $\quad$ symmetry $\Rightarrow \theta = \frac{\pi}{2}$

$$L = \frac{1}{2}\left[f(r)^{-1}\dot{r}^2 + r^2\dot{\phi}^2 - f(r)c^2\dot{t}^2\right]$$



$\vec{r}$ $\quad \vec{p}$ $\quad |\vec{p}| = \frac{E}{c}$

$\vec{L} = \vec{r}\times\vec{p}$ $\quad \vec{L} = \frac{E}{c}(\vec{r}\times\vec{n})$

$\frac{\partial L}{\partial \phi} = 0$ so $\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0$

$\frac{\partial L}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left(\frac{1}{2}r^2\dot{\phi}^2\right) = r^2\dot{\phi}$ $\quad \frac{d}{d\lambda}(r^2\dot{\phi}) = 0$ $\quad r^2\dot{\phi} = L$ = constant $\quad \boxed{\dot{\phi} = \frac{L}{r^2}}$

$\frac{\partial L}{\partial \dot{r}} = \frac{\dot{r}}{f(r)}$ $\quad f'(r) = \frac{r_s}{r^2}$ $\quad \frac{\partial L}{\partial r} = r\dot{\phi}^2 - \frac{\dot{r}^2}{2f(r)^2}\cdot f'(r) - \frac{f'(r)c^2\dot{t}^2}{2}$

$\frac{\partial L}{\partial \dot{t}} = -f(r)c^2\dot{t}$ $\quad \frac{\partial L}{\partial t} = 0$ $\quad \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{t}}\right) = 0$ $\quad \frac{\partial L}{\partial \dot{t}}$ = constant $\quad E = f(r)\dot{t}$ $\quad \dot{t} = \frac{E}{f(r)}$

$$r\dot{\phi}^2 - \frac{\dot{r}^2 f'(r)}{2f(r)^2} - \frac{E^2 f'(r)c^2}{2(f(r))^2} = \frac{d}{d\lambda}\left(\frac{\dot{r}}{f(r)}\right) = \frac{f(r)\ddot{r} - \dot{r}\left(\frac{r_s}{r^2}\cdot\dot{r}\right)}{(f(r))^2} = \frac{\ddot{r}}{f(r)} - \frac{\dot{r}^2(f'(r))}{(f(r))^2}$$

$$\ddot{r} = f(r)r\dot{\phi}^2 - \frac{\dot{r}^2 f'(r)}{2f(r)} + \frac{\dot{r}^2 f'(r)}{f(r)} - \frac{E^2 f' c^2}{f(r)}$$

$$= f(r)r\dot{\phi}^2 + \frac{\dot{r}^2 f' - 2E^2 f' c^2}{2f}$$

$$0 = f^{-1}\dot{r}^2 + r^2\dot{\phi}^2 - \frac{\cancel{f}c^2E^2}{f^{\cancel{2}}} = \dot{r}^2 + fr^2\dot{\phi}^2 - c^2E^2$$

$\boxed{\dot{r} = \pm\sqrt{c^2E^2 - fr^2\dot{\phi}^2}}$ $\longrightarrow$ dont really need $\ddot{r}$ anymore

$$= \pm\sqrt{1 - \left(1 - \frac{r_s}{r}\right)r^2\cdot\left(\frac{b}{r^2}\right)^2} = \pm\sqrt{1 - \left(1 - \frac{r_s}{r}\right)\cdot\frac{b^2}{r^2}}$$