# Noether's Theorem

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## Review of Lagrangian Mechanics

Conservation laws are closely related to symmetry. To understand that, it is easy to start from the Lagrangian formalism in classical mechanics.

The Lagrangian mechanics builds upon the calculus of variation. Suppose you want to optimise a functional

$$S = \int dt\, L(t, q(t), \dot{q}(t)),$$

where a functional is an object that takes in a function and give a number. For example, the time taken to travel between 2 points in space depends on the path (a function), making it a functional.

Just like in normal calculus, finding the optimum value means finding a stationary point. This means that a small variation around the optimum function (say, the path) does not vary the value of the functional (say, time).

$$0 = \delta S$$

$$= \int dt \left( \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right)$$

We use integration by path:

$$\frac{\partial L}{\partial \dot{q}} \delta \dot{q} = \frac{d}{dt}\left( \frac{\partial L}{\partial \dot{q}} \delta q \right) - \frac{d}{dt}\left( \frac{\partial L}{\partial \dot{q}} \right) \delta q$$

$$= \frac{\partial L}{\partial \dot{q}} \delta q \Big|_{\text{boundary}} - \int dt \left[ \frac{\partial L}{\partial q} - \frac{d}{dt}\left( \frac{\partial L}{\partial \dot{q}} \right) \right] \delta q$$

If the boundary is fixed, $\delta q \big|_{\text{boundary}} = 0$,

we then have

$$0 = \int dt\, \delta q \left[ \frac{\partial L}{\partial q} - \frac{d}{dt}\left( \frac{\partial L}{\partial \dot{q}} \right) \right] \quad \forall\ \delta q(t)$$

We must have

$$\boxed{\frac{d}{dt}\left( \frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q}}$$

Euler-Lagrange equation

Turns out, Newton's equation can be cast as a problem for calculus of variation. Without loss of generality, consider 1D. Newton's equation is $\frac{dp}{dt} = -\frac{\partial V}{\partial x}$,

where $V(x)$ is the potential energy for a conservative force.

Then:
$$\frac{d}{dt}(m\dot{x}) = \frac{\partial}{\partial x}(-V)$$

$$\frac{d}{dt}\left( \frac{\partial}{\partial \dot{x}}\left(\tfrac{1}{2} m \dot{x}^2\right) \right) = \frac{\partial}{\partial x}(-V)$$

If we define

$$L = \tfrac{1}{2} m \dot{x}^2 - V(x), \longrightarrow \text{Lagrangian}$$

then the Euler-Lagrange equation

$$\frac{d}{dt}\left( \frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x}$$

is the Newton's equation. We call

$$S[q(t), \dot{q}(t)] = \int dt\, L(t, q(t), \dot{q}(t))$$

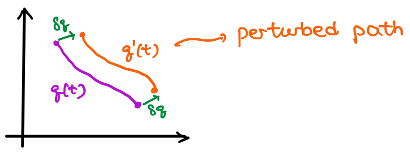
action. Why do we want to do this? Because instead of dealing with vectors in the Newton's equation, we now only deal with scalars in the Lagrangian. Furthermore, since $S$ is just a number, action. Why do we want to do this? Because instead of dealing with vectors in the Newton's equation, we now only deal with scalars in the Lagrangian. Furthermore, since $S$ is just a number, we can use any representation in the Lagrangian, so we speak of generalised coordinates $q(t)$.

## Noether's Theorem

With the Lagrangian formalism, we can better understand Noether's theorem.

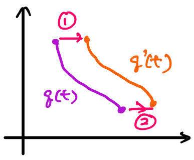
> Noether's theorem states that for every continuous symmetry, there is a corresponding conserved quantity

We first view this physically. Consider an optimised path $q(t)$. If we perturb the path slightly by $\delta q$, since $q(t)$ is optimised, we must have $\delta S = 0$




If a system has a particular symmetry (say translation symmetry), then the dynamics and system remain unchanged under the symmetry transformation (say translation). If the dynamics is the same, then its action must be the same:

$$S[q(t)] = S[q'(t)]$$




Since $\delta S = 0$,

we must have

$$S[q(t)] = S_{①} + S[q'(t)] - S_{②}$$

-ve sign because ② goes in the opposite direction

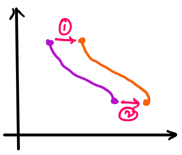
$$\Rightarrow S_{①} = S_{②}$$

It follows that some quantity $S_{①} = S_{②}$ is conserved. Therefore, if there is a continuous symmetry, there is a conserved quantity. This is Noether's theorem.

PS: In the above, our argument uses differential changes. That is why the theorem emphasises continuous symmetry. This does not work for discrete symmetry.

## Conservation of Momentum

As a concrete example, consider a system with translation symmetry. When deriving the Lagrangian formalism, we assumed that the end points are fixed. If the boundary changes, then the boundary term does not vanish.




Since ① and ② changes the boundary, we have

$$0 = \delta S$$

$$= \frac{\partial L}{\partial \dot{x}} \delta x \Big|_{\text{boundary}} + [\text{Euler-Lagrange equation part}]$$

where $\delta x$ is the translation. We then have

$$\frac{\partial L}{\partial \dot{x}}\Big|_{①} = \frac{\partial L}{\partial \dot{x}}\Big|_{②}.$$

Since $\frac{\partial L}{\partial \dot{x}} = m\dot{x} = p \equiv$ momentum,

we have $P_{\text{initial}} = P_{\text{final}}$.

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Translation symmetry leads to the conservation of momentum.

P/S: We see that if we move the end points, we change the action by $p\,\delta x$. We speak of momentum $p$ as a generator of translation. This is how we identify the momentum operator in quantum mechanics.

We can also obtained the conservation of momentum from the Euler-Lagrange equation. If a system has a translation symmetry,

$$\frac{\partial L}{\partial x} = 0.$$

From the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}p = 0 = \frac{\partial L}{\partial x}$$

$$\Rightarrow p = \text{constant}$$

Recall that the Lagrangian formalism is not restricted to just position $x$ and momentum $p$. In general, we can use the generalised coordinate $q$ and generalised momentum

$$p = \frac{\partial L}{\partial \dot{q}}$$ → unfortunately, we use $p$ to represent generalised momentum too

The Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$

then implies that if there is a symmetry in the generalised coordinates, the generalised momentum is conserved. For example, if $q$ is the rotational angle, $p$ is the angular momentum. Rotational symmetry then implies the conservation of angular momentum.

## Conservation of Energy

What about a time translation symmetry? Consider

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q}\dot{q} + \underbrace{\frac{\partial L}{\partial \dot{q}}\frac{d\dot{q}}{dt}}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q}$$

By Euler-Lagrange equation,

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

$$= \frac{\partial L}{\partial t} + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q}\right)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q} - L\right) = -\frac{\partial L}{\partial t}$$

Therefore, if the system has a time translation symmetry

$$\frac{\partial L}{\partial t} = 0,$$

then

$$\mathcal{H} \equiv \frac{\partial L}{\partial \dot{q}}\dot{q} - L$$

is conserved. What is $\mathcal{H}$? If $q$ is position $x$,

$$L = \frac{1}{2}m\dot{x}^2 - V(x)$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\Rightarrow \mathcal{H} = m\dot{x} \times \dot{x} - \frac{1}{2}m\dot{x}^2 + V(x)$$

$$= \frac{1}{2}m\dot{x}^2 + V(x).$$ → This is why we have -ve sign in $\frac{\partial L}{\partial t}$ above, to get positive

$$= \frac{1}{2}m\dot{x}^2 + V(x).$$

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which is the Hamiltonian of the system.

PIS: If we observe the differential behaviour,

$$dL = \frac{\partial L}{\partial t}dt + \frac{\partial L}{\partial q}dq + \frac{\partial L}{\partial \dot{q}}d\dot{q}.$$

$$= \frac{\partial L}{\partial t}dt + \frac{\partial L}{\partial q}dq + p\,d\dot{q}.$$

Then,

$$d\mathcal{H} = d(p\dot{q} - L)$$

$$= \dot{q}\,dp + p\,d\dot{q} - \frac{\partial L}{\partial t}dt - \frac{\partial L}{\partial q}dq - p\,d\dot{q}$$

$$= \dot{q}\,dp - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial t}dt.$$

Therefore, the natural variable for $\mathcal{H}$ is $p$, not $\dot{q}$. That is why we write

$$\mathcal{H} = \frac{p^2}{2m} + V(x)$$

rather than with $\dot{x}$. In fact, the Hamiltonian is a Legendre transform of the Lagrangian. You probably have also seen Legendre transforms in your thermodynamics course, when you switch between different thermodynamic potentials

## Quantum Mechanics

Noether's theorem also works in quantum mechanics. A symmetry transformation leaves the system unchanged. Therefore,

$$\mathcal{U}^{+}\mathcal{H}\,\mathcal{U} = \mathcal{H},$$

where $\mathcal{U}$ is a unitary matrix. This implies that

$$\mathcal{H}\,\mathcal{U} = \mathcal{U}\mathcal{H}$$

$$\Rightarrow [\mathcal{U}, \mathcal{H}] = 0$$

From the Heisenberg's equation of motion,

$$\frac{d\mathcal{U}}{dt} = \frac{\partial \mathcal{U}}{\partial t} + \underbrace{\frac{1}{i\hbar}[\mathcal{U},\mathcal{H}]}_{=0}$$

$= 0$ because we are considering a coordinate transformation, which should not depend on time explicitly

$$= 0$$

Therefore, $\mathcal{U}$ is the conserved quantity.

## Gauge Symmetry

We learn from electromagnetism that under the gauge transformation

$$\vec{A}' = \vec{A} + \vec{\nabla}\lambda \quad \text{and} \quad V' = V - \frac{\partial \lambda}{\partial t},$$

the electric field $\vec{E}$ and magnetic field $\vec{B}$ are unchanged. What is the conserved quantity for this continuous symmetry?

Gauge symmetry actually leads to the continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0,$$

which is the conservation of charge. The derivation of this is beyond the scope of this module.

P.S: Deriving this involves field theory, where the action integrates over space and time, and we look at the Lagrangian density instead of the Lagrangian. Then, we look at the consequence of an unchanged action under the gauge transformation. Note that the action is also unchanged if the Lagrangian density differs by a divergence.

P.S: For those who are familiar with relativity, you may have noticed that field theory is needed, since the continuity equation is a divergence in 4-dimensions

$$\partial_\mu j^\mu = 0 ,$$

where $\partial_\mu$ is the 4-gradient, $j^\mu$ is the 4-current, and we used the Einstein summation convention.

Here, $$\partial_\mu = \left( \frac{1}{c}\frac{\partial}{\partial t} \quad \vec{\nabla} \right)$$

and $$j^\mu = \begin{pmatrix} c\rho \\ \vec{J} \end{pmatrix}$$