Angular Momentum

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There are many ways to approach angular momentum in quantum mechanics. We will approach this by starting from classical physics. In classical mechanics, angular momentum is

$$\vec{\ell} = \vec{r} \times \vec{p}$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

$$= \hat{x}\,(y p_z - z p_y) + \hat{y}\,(z p_x - x p_z) + \hat{z}\,(x p_y - y p_x).$$

Therefore, in quantum mechanics, we can construct the angular momentum operator by

$$\vec{L} = \vec{R} \times \vec{P},$$

where we obtain

$$L_x = Y P_z - Z P_y$$

$$L_y = Z P_x - X P_z$$

$$L_z = X P_y - Y P_x$$

These are quantum mechanics operators

We can check that they are hermitian:

$$L_i^\dagger = L_i \quad \text{for } i = x, y, z.$$

These operators do not commute with each other. We can obtain their commutation relation from the commutation relation of position and momentum operators. Using the Einstein summation convention (where repeated indices are summed over),

$$L_i = \varepsilon_{ijk} R_j P_k, \quad i, j, k \in \{x, y, z\},$$

where ε_{ijk} is the Levi-Civita symbol. Then, the commutation

$$[L_i, L_p] = [\varepsilon_{ijk} R_j P_k, \varepsilon_{pqr} R_q P_r]$$

$$= \varepsilon_{ijk} \varepsilon_{pqr} [R_j P_k, R_q P_r]$$

Recall: $[R_i, P_j] = i\hbar\, \delta_{ij}$ Kronecker delta

$$[A, BC] = [A, B]C + B[A, C]$$

$$= \varepsilon_{ijk} \varepsilon_{pqr} \Big(R_q \underbrace{[R_j P_k, P_r]}_{= i\hbar \delta_{jr} P_k} + \underbrace{[R_j P_k, R_q]}_{= -i\hbar \delta_{kq} R_j} P_r \Big)$$

$$= i\hbar\, \varepsilon_{ijk} \varepsilon_{pqr} (R_q P_k \delta_{jr} - R_j P_r \delta_{kq})$$

$$= i\hbar\, (R_q P_k \varepsilon_{ijk} \varepsilon_{pqj} - R_j P_r \varepsilon_{ijk} \varepsilon_{pkr})$$

From $\varepsilon_{ijk} \varepsilon_{k\ell m} = \delta_{i\ell} \delta_{jm} - \delta_{im} \delta_{j\ell}$

$$= i\hbar\, (- R_k P_k \delta_{ip} + R_i P_p + R_r P_r \delta_{ip} - R_p P_i)$$

$$= i\hbar\, (R_i P_p - R_p P_i)$$

This shows that

$$[L_x, L_y] = i\hbar L_z$$

$$[L_y, L_z] = i\hbar L_x$$

$$[L_z, L_x] = i\hbar L_y$$

$$[L_i, L_j] = i\hbar\, \varepsilon_{ijk} L_k$$

Another compact form is

$$[\vec{a}\cdot\vec{L}, \vec{b}\cdot\vec{L}] = [a_i L_i, b_j L_j]$$

$$= a_i b_j [L_i, L_j]$$

$$= a_i b_j \, i\hbar \, \varepsilon_{ijk} L_k$$

$$\Rightarrow \boxed{[\vec{a}\cdot\vec{L}, \vec{b}\cdot\vec{L}] = i\hbar\,(\vec{a}\times\vec{b})\times\vec{L}}$$

Eigenstates of angular momentum operator

We can learn a lot about the properties of the angular momentum operator just from their commutation relation.

First of all,

$$[L^2, \vec{a}\cdot\vec{L}] = [L_i L_i, a_j L_j]$$

$$= a_j \left(L_i [L_i, L_j] + [L_i, L_j] L_i \right)$$

$$= a_j \left(\underbrace{\varepsilon_{ijk} L_i L_k + \varepsilon_{ijk} L_k L_i}_{=0} \right) i\hbar$$

$$\downarrow$$

$$= 0.$$

Therefore, L^2 commutes with L_x, L_y, and L_z.

Recall that we need to construct common eigenstates using commuting variables. Suppose

$$A|a,b\rangle = a|a,b\rangle \quad \text{and} \quad B|a,b\rangle = b|a,b\rangle$$

$$\Rightarrow AB|a,b\rangle = A|a,b\rangle b$$

$$= |a,b\rangle ab$$

$$= B|a,b\rangle a$$

$$= BA|a,b\rangle \quad \forall\, a,b$$

$$\sum_{a,b} AB|a,b\rangle\langle a,b| = \sum_{a,b} BA|a,b\rangle\langle a,b|$$

$$AB = BA \Rightarrow [A,B] = 0$$

Therefore, we can construct a common eigenstate using the operators L^2 and L_z, because they commute.

P/S: We could also choose L_x or L_y instead of L_z. The choice of L_z is by convention and for its convenience when we use spherical coordinates.

Define the common eigenstates $|\beta, m\rangle$:

$$L^2|\beta,m\rangle = \hbar^2\beta|\beta,m\rangle \longrightarrow \hbar^2 \text{ because of units}$$

$$L_z|\beta,m\rangle = \hbar m|\beta,m\rangle$$

By definition,

$$\langle\beta,m|L^2|\beta,m\rangle = \beta\hbar^2\langle\beta,m|\beta,m\rangle = \beta\hbar^2$$

$$= \langle\beta,m|(L_x^2 + L_y^2 + L_z^2)|\beta,m\rangle$$

$$= \underbrace{\langle\beta,m|(L_x^2 + L_y^2)|\beta,m\rangle}_{\geq 0 \text{ because of squared quantity}} + m^2\hbar^2$$

$$\Rightarrow \boxed{\beta \geq m^2}$$

Therefore, the values of m are bounded

Next, by observation

$$[L_z, L_y] = -i\hbar L_x$$

$$[L_z, L_x] = i\hbar L_y$$

$$\Rightarrow [L_z, L_x \pm i L_y] = i\hbar L_y \pm \hbar L_x$$

$$= \pm\hbar (L_x \pm i L_y)$$

When we observe a commutation relation of the form

$$[A, B] = \pm B \quad \text{for} \quad A|a\rangle = |a\rangle a,$$

B is always a ladder operator:

$$[A, B]\,|a\rangle = (AB - BA)\,|a\rangle$$

$$\pm B|a\rangle = A\,B|a\rangle - a\,B|a\rangle$$

$$\Rightarrow A(B|a\rangle) = (a\pm 1)(B|a\rangle).$$

In other words, $B|a\rangle$ is an eigenstate of A and $\propto |a\pm 1\rangle$.
Therefore, we define

$$L_\pm = L_x \pm i L_y$$

and recognise that

$$L_\pm |\beta, m\rangle \propto |\beta, m\pm 1\rangle$$

Since $\beta \geq m^2 \Rightarrow -\sqrt{\beta} \leq m \leq \sqrt{\beta}$, m is bounded from above and from below. Suppose the maximum value of m is ℓ.
Then,

$$L_+ |\beta, \ell\rangle = 0.$$

Otherwise, we have $L_+|\beta, \ell\rangle \propto |\beta, \ell+1\rangle$, which is not allowed.
This means that

$$0 = L_- L_+ |\beta, \ell\rangle$$

$$= (L_x - i L_y)(L_x + i L_y)\,|\beta, \ell\rangle$$

$$= (\underbrace{L_x^2 + L_y^2}_{= L^2 - L_z^2} + i\underbrace{[L_x, L_y]}_{= i\hbar L_z})\,|\beta, \ell\rangle$$

$$= (L^2 - L_z^2 - \hbar L_z)\,|\beta, \ell\rangle$$

$$= \hbar^2(\beta - \ell^2 - \ell)\,|\beta, \ell\rangle,$$

which implies

$$\boxed{\beta = \ell(\ell+1).} \quad \text{—— (1)}$$

We repeat the same for the lower bound. Suppose the min value for m is k.

$$0 = L_- |\beta, k\rangle$$

$$= L_+ L_- |\beta, k\rangle$$

$$= (L_x + i L_y)(L_x - i L_y)\,|\beta, k\rangle$$

$$= (L_x^2 + L_y^2 - i[L_x, L_y])\,|\beta, k\rangle$$

$$= (L^2 - L_z^2 + \hbar L_z)\,|\beta, k\rangle$$

$$= \hbar^2(\beta - k^2 + k)\,|\beta, k\rangle$$

$$\Rightarrow \beta = k(k-1) \quad \text{—— (2)}$$

Since β is the same for (1) and (2),

$$\Rightarrow \beta = k(k-1) \quad \text{——} \ ②$$

Since β is the same for ① and ②,

$$\ell(\ell+1) = -k(-k+1)$$

$$\Rightarrow \boxed{k = -\ell}$$

Hence, we have $\quad -\ell \leq m \leq \ell$

with $\quad \beta = \ell(\ell+1)$.

For convenience, we relabel the eigenstate such that

$$\boxed{\begin{aligned} L^2 |\ell, m\rangle &= \ell(\ell+1)\hbar^2 |\ell, m\rangle \\ L_z |\ell, m\rangle &= m\hbar |\ell, m\rangle \end{aligned}}$$

Since the ladder operators $L_\pm$ shifts the values of m by ± 1, we must have integer differences between $-\ell$ and ℓ:

$$2\ell \in \mathbb{Z} \longrightarrow \text{integers.}$$

This means that ℓ must be half integer:

$$\boxed{\begin{aligned} &\ell = 0, \quad m = 0 \\ &\ell = \tfrac{1}{2}, \quad m = -\tfrac{1}{2}, \tfrac{1}{2} \\ &\ell = 1, \quad m = -1, 0, 1 \\ &\ \vdots \qquad\qquad \vdots \end{aligned}}$$

Each ℓ has $2\ell+1$ values of m. Therefore, the angular momentum quantisation rules can be obtained just from the commutation relations and algebra.

For completeness, let's derive the exact expression of $L_\pm |\ell, m\rangle$. Suppose

$$L_\pm |\ell, m\rangle = C |\ell, m\pm 1\rangle$$

Since $\quad L_\pm = L_x \pm i L_y$

$$\Rightarrow L_\pm^\dagger = L_x \mp i L_y = L_\mp,$$

then $\quad \langle \ell, m| L_\mp L_\pm |\ell, m\rangle = |C|^2$

$$\Rightarrow |C|^2 = \langle \ell, m| \underbrace{L_\mp L_\pm}_{} |\ell, m\rangle$$

From before, $L_\mp L_\pm = L^2 - L_z^2 \mp \hbar L_z$

$$= \langle \ell, m| (L^2 - L_z^2 \mp \hbar L_z) |\ell, m\rangle$$

$$= \hbar^2 [\ell(\ell+1) - m^2 \mp m]$$

$$C = \hbar \sqrt{\ell(\ell+1) - m(m\pm 1)}$$

Hence,

$$\boxed{L_\pm |\ell, m\rangle = \hbar \sqrt{\ell(\ell+1) - m(m\pm 1)}\ |\ell, m\rangle}$$

PIs: Spin-½ particles are described by Pauli matrices. Recall from QM 1 that the Pauli matrices have the commutation relation:

$$[\vec{a}\cdot\vec{\sigma}, \vec{b}\cdot\vec{\sigma}] = 2i(\vec{a}\times\vec{b})\cdot\vec{\sigma}$$

$$\Rightarrow \left[\vec{a}\cdot\frac{\hbar}{2}\vec{\sigma}, \vec{b}\cdot\frac{\hbar}{2}\vec{\sigma}\right] = i\hbar(\vec{a}\times\vec{b})\cdot\frac{\hbar}{2}\vec{\sigma}.$$

Therefore, $\quad \vec{S} = \frac{\hbar}{2}\vec{\sigma}$

$$\Rightarrow \left[\vec{a}\cdot\frac{\hbar}{2}\vec{\sigma}, \vec{b}\cdot\frac{\hbar}{2}\vec{\sigma}\right] = i\hbar\,(\vec{a}\times\vec{b})\cdot\frac{\hbar}{2}\vec{\sigma}.$$

Therefore, $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$

has the same commutation relation as angular momentum operators. It follows that spins are intrinsic angular momenta

Addition of Angular Momenta

Suppose we have 2 systems with angular momenta. For system 1, we have the eigenstates of J_1^2 and J_{1z}. For system 2, we have the eigenstates of J_2^2 and J_{2z}. To describe the combined systems of 1 and 2, we can use the bases: $|(j_1\ m_1)\ (j_2\ m_2)\rangle$.

However, if we view the 2 systems as 1 combined composite system, the composite system has its own angular momentum

$$\vec{J} = \vec{J}_1 + \vec{J}_2.$$

Using J^2 and J_z, the total angular momentum can also form a complete basis

$$|j_1, j_2; j, m\rangle,$$

where the values of possible j and m depend on j_1 and j_2. How do the 2 bases relate to each other?

Consider

$$J_z = J_{1z} + J_{2z},$$

we have

$$J_z\,|(j_1, m_1)\ (j_2, m_2)\rangle = |(j_1, m_1)\ (j_2, m_2)\rangle\ (m_1+m_2)\hbar$$

Since the maximum value of m_1 is j_1, and the maximum value of m_2 is j_2, it follows that the maximum value of m is j_1+j_2. Since there is only one state in both bases with the maximum m, they must be the same

$$|(j_1\ m_1)\ (j_2\ m_2)\rangle = |j_1, j_2; j=j_1+j_2, m=j_1+j_2\rangle.$$

Starting from here, we can construct $j=j_1+j_2$ and $m<j_1+j_2$ state by applying the lowering operator

$$J_- = J_{1-} + J_{2-}$$

sequentially. For example,

$$J_-\,|j_1, j_2; j, m\rangle = \hbar\sqrt{j(j+1)-m(m-1)}\ |j_1, j_2; j, m\rangle$$

$$\Rightarrow J_-\,|j_1, j_2; j_1+j_2, j_1+j_2\rangle = \hbar\sqrt{2(j_1+j_2)}\ |j_1, j_2; j_1+j_2, j_1+j_2-1\rangle$$

$$= (J_{1-} + J_{2-})\,|(j_1, j_1)\ (j_2, j_2)\rangle$$

$$= \hbar\sqrt{j_1(j_1+1)-j_1(j_1-1)}\ |(j_1, j_1-1)\ (j_2, j_2)\rangle$$

$$+\ \hbar\sqrt{j_2(j_2+1)-j_2(j_2-1)}\ |(j_1, j_1)\ (j_2, j_2-1)\rangle$$

$$\boxed{\Rightarrow |j_1, j_2; j_1+j_2, j_1+j_2-1\rangle = \sqrt{\frac{j_1}{j_1+j_2}}\ |(j_1, j_1-1)\ (j_2, j_2)\rangle + \sqrt{\frac{j_2}{j_1+j_2}}\ |(j_1, j_1)\ (j_2, j_2-1)\rangle}$$

This composite state involves 2 states $|(j_1, j_1-1)\ (j_2, j_2)\rangle$ and $|(j_1, j_2)\ (j_2, j_2-1)\rangle$, which form a 2D subspace. To make it complete, we need the orthogonal state

$$|?\rangle = \sqrt{\frac{j_2}{j_1+j_2}}\ |(j_1, j_1-1)\ (j_2, j_2)\rangle - \sqrt{\frac{j_1}{j_1+j_2}}\ |(j_1, j_1)\ (j_2, j_2-1)\rangle.$$

and $|(j_1,j_2)(j_2,j_2-1)\rangle$, which form a 2D subspace. To make it complete, we need the orthogonal state

$$|?\rangle = \sqrt{\frac{j_2}{j_1+j_2}}\,|(j_1,j_1-1)(j_2,j_2)\rangle - \sqrt{\frac{j_1}{j_1+j_2}}\,|(j_1,j_1)(j_2,j_2-1)\rangle .$$

Which composite state does this correspond to? Since $|(j_1,m_1)(j_2,m_2)\rangle$ is an eigenstate of J_z,

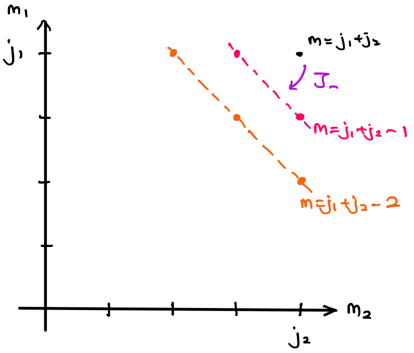
$$J_z|?\rangle = (j_1+j_2-1)\hbar\,|?\rangle .$$

Since $m=j_1+j_2-1$ can only happen for $j=j_1+j_2$ and $j=j_1+j_2-1$, we must have

$$|?\rangle = |j_1,j_2;j_1+j_2-1,j_1+j_2-1\rangle .$$

We then continue to apply J_- sequentially and construct the other states.

While the description of the procedure sounds complicated, it is clearer if we view this pictorally



We first identify the starting point at the upper right corner, to construct the $j=j_1+j_2$, $m=j_1+j_2$ state.

Then, we apply J_- to construct the $j=j_1+j_2$, $m=j_1+j_2-1$ state

Notice that 2 points are involved, so there are 2 degrees of freedom. Therefore, the state that is orthogonal to $j=j_1+j_2$, $m=j_1+j_2-1$ state in this $m=j_1+j_2-1$ sector must be the $j=j_1+j_2-1$, $m=j_1+j_2-1$ state. From the $m=j_1+j_2-1$ sector, we apply J_- again and obtain the $j=j_1+j_2$, $m=j_1+j_2-2$ and $j=j_1+j_2-1$, $m=j_1+j_2-2$ states. Since there are 3 points in this sector, the state that is orthogonal to these 2 constructed states must be $j=j_1+j_2-2$, $m=j_1+j_2-2$. If we repeat this procedure and travel down the m values, the we construct all the $|j_1,j_2;j,m\rangle$ states in terms of $|(j_1,m_1)(j_2,m_2)\rangle$ states.

When do we stop? When we exhaust all degrees of freedom in $|(j_1,m_1)(j_2,m_2)\rangle$. For (j_1,m_1), we have $(2j_1+1)$ states. For (j_2,m_2), we have $(2j_2+1)$ states. In total, we will have $(2j_1+1)(2j_2+1)$ states.

For the number of states in $|j_1,j_2;j,m\rangle$, for each j, we have $2j+1$ different m states using the properties of J^2 and J_z. Therefore,

$$(2j_1+1)(2j_2+1) = \sum_{j=j_{min}}^{j_1+j_2}(2j+1)$$

$$= \sum_{j=j_{min}}^{j_1+j_2}\left[(j+1)^2 - j^2\right]$$

$$= (j_1+j_2+1)^2 - j_{min}^2$$

$$\Rightarrow j_{min}^2 = (j_1+j_2+1)^2 - (2j_1+1)(2j_2+1)$$

$$= j_1^2+j_2^2+1+2j_1+2j_2+2j_1j_2 - 4j_1j_2 - 2j_1 - 2j_2 - 1$$

$$= j_1^2 - 2j_1j_2 + j_2^2$$

$$= (j_1-j_2)^2$$

$$\boxed{j_{min} = |j_1 - j_2|}$$

because j_{min} cannot be negative.

In summary, we now have a procedure to relate

$$|(j_1 m_1)(j_2 m_2)\rangle \text{ to } |j_1, j_2; j, m\rangle ,$$

where $j = j_1 + j_2, \; j_1 + j_2 - 1, \dots, |j_1 - j_2|$

and $m = j, j-1, \dots, -j+1, -j$

The coefficients in

$$|j_1, j_2; j, m\rangle = \sum_{m_1, m_2} |(j_1, m_1)(j_2, m_2)\rangle \underbrace{\langle (j_1, m_1)(j_2, m_2) | j_1, j_2; j, m\rangle}_{\text{Clebsch-Gordan coefficients}}$$

are well-studied and tabulated. One property of the C-G coefficients follows immediately from our discussion:

$$\langle (j_1, m_1)(j_2, m_2) | j_1, j_2; j, m\rangle = 0$$

unless $m = m_1 + m_2$ and $|j_1 - j_2| \leq j \leq j_1 + j_2$

Example: 2 spin-½ system

Let's apply this to 2 spin-½ system. First of all, we identify

$$\left|\tfrac{1}{2}, \tfrac{1}{2}; 1, 1\right\rangle = \left|\left(\tfrac{1}{2}, \tfrac{1}{2}\right)\left(\tfrac{1}{2}, \tfrac{1}{2}\right)\right\rangle .$$

We apply J_-:

$$J_- \left|\tfrac{1}{2}, \tfrac{1}{2}; 1, 1\right\rangle = J_- \left|\left(\tfrac{1}{2}, \tfrac{1}{2}\right)\left(\tfrac{1}{2}, \tfrac{1}{2}\right)\right\rangle$$

$$\hbar\sqrt{1(1+1) - 1(1-1)} \left|\tfrac{1}{2}, \tfrac{1}{2}; 1, 0\right\rangle = \hbar\sqrt{\tfrac{1}{2}\left(\tfrac{1}{2}+1\right) - \tfrac{1}{2}\left(\tfrac{1}{2}-1\right)} \Big(\left|\left(\tfrac{1}{2}, -\tfrac{1}{2}\right)\left(\tfrac{1}{2}, \tfrac{1}{2}\right)\right\rangle$$
$$+ \left|\left(\tfrac{1}{2}, \tfrac{1}{2}\right)\left(\tfrac{1}{2}, -\tfrac{1}{2}\right)\right\rangle \Big)$$

$$\boxed{\left|\tfrac{1}{2}, \tfrac{1}{2}; 1, 0\right\rangle = \frac{1}{\sqrt{2}}\left[\left|\left(\tfrac{1}{2}, -\tfrac{1}{2}\right)\left(\tfrac{1}{2}, \tfrac{1}{2}\right)\right\rangle + \left|\left(\tfrac{1}{2}, \tfrac{1}{2}\right)\left(\tfrac{1}{2}, -\tfrac{1}{2}\right)\right\rangle\right]}$$

In our usual notation, this is

$$|s=1, m=0\rangle = \frac{|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle}{\sqrt{2}}$$

The state that is orthogonal to $|s=1, m=0\rangle$ is

$$|s=0, m=0\rangle = \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} .$$

We apply J_- once more:

$$J_- \left|\tfrac{1}{2}, \tfrac{1}{2}; 1, 0\right\rangle = \frac{1}{\sqrt{2}} J_- \Big[\underbrace{\left|\left(\tfrac{1}{2}, -\tfrac{1}{2}\right)\left(\tfrac{1}{2}, \tfrac{1}{2}\right)\right\rangle + \left|\left(\tfrac{1}{2}, \tfrac{1}{2}\right)\left(\tfrac{1}{2}, -\tfrac{1}{2}\right)\right\rangle}_{\text{can only apply to this, otherwise } 0} \Big]$$

$$\hbar\sqrt{2} \left|\tfrac{1}{2}, \tfrac{1}{2}; 1, -1\right\rangle = \frac{\hbar}{\sqrt{2}} \, 2 \left|\left(\tfrac{1}{2}, -\tfrac{1}{2}\right)\left(\tfrac{1}{2}, -\tfrac{1}{2}\right)\right\rangle$$

$$\boxed{\left|\tfrac{1}{2}, \tfrac{1}{2}; 1, -1\right\rangle = \left|\left(\tfrac{1}{2}, -\tfrac{1}{2}\right)\left(\tfrac{1}{2}, -\tfrac{1}{2}\right)\right\rangle}$$

In summary, we have

$$\left.\begin{aligned} \ell = 1, \; m = -1, &\quad |\downarrow\downarrow\rangle \\ m = 0, &\quad \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} \\ m = 1, &\quad |\uparrow\uparrow\rangle \end{aligned}\right\} \text{ Triplet state}$$

$$\ell = 0, \; m = 0, \quad \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} \rightarrow \text{singlet state}$$

as we have learned in quantum mechanics.

P/S: The Clebsch-Gordan coefficients are related to as we have learned in quantum mechanics.

P/S: The Clebsch-Gordan coefficients are related to the Wigner 3j-symbol.